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On a Theorem of Ganter

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We prove a generalization of a theorem of Ganter concerning the embedding of partial Steiner systems into Steiner systems. As an application we discuss a further version of the problem of Rosenfeld on embedding graphs into strongly regular graphs.

1. Introduction

An $S(2,k,v)$ *Steiner system* is a k-uniform hypergraph ($k > 2$) on v points with a set of blocks (edges) such that every two points are contained in exactly one block; k and v are the *parameters* of the system. For example, all the finite spaces are Steiner systems with the lines as blocks. More details about Steiner systems can be found in the standard literature (for example, see [7]). It is well known that the necessary conditions for the existence of an $S(2,k,v)$ Steiner system are:

$$v-1 \equiv 0 \quad (k-1)$$
$$v(v-1) \equiv 0 \quad (k(k-1))$$

The famous existence conjecture proved by Wilson [9] shows that these conditions are also sufficient if v is sufficiently large.

Consider a subsystem of an $S(2,k,v)$ Steiner system, that is, the collection of some blocks, it is trivial that in this system every pair of points is contained in at most one block, i.e. every pair of blocks share at most one point in common.

Definition. *A k-partial Steiner system is a k-uniform hypergraph where every pair of blocks share at most one point in common.*

The basic embedding problem asks if one can embed a given object into an object with described properties, provided some minimum necessary conditions are met. One of the most famous embedding questions in combinatorics is whether one may embed a partial Steiner system into a Steiner system of the same block size (when the set of vertices can be enlarged). If so we call the Steiner system an *extension* of the partial Steiner system. Partial

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results of this interesting problem were given by Treash [8], Ganter [3] and Quackenbush [6] considering special values of k. Finally, with the aid of Wilson's theorem, Ganter gave a positive answer for the general case [4].

Theorem 1.1. (Ganter) *Any partial Steiner system can be embedded in a Steiner system.*

The first goal of this paper is to give a generalization of Ganter's theorem in the case when not one but many extensions of the same parameters are required for a given partial Steiner system (Theorem 1.2). The main difficulty of this is to guarantee the non-isomorphism of the constructions. Secondly, we discuss some interesting applications concerning the question of Rosenfeld on the problem of embedding graphs into strongly regular graphs.

Theorem 1.2. *Let A be a k-partial Steiner system and n an arbitrary positive integer. For some v one can find n non-isomorphic $S(2,k,v)$ Steiner systems each of which is an extension of A.*

2. Preliminaries

Let A be a partial Steiner system with P as the set of points and B as the set of blocks; the cardinality of P is called the *order* of A. Let S be a Steiner system. Write $A \subset S$ if S is an extension of A. The *intersecting graph* of A is the graph with B as the set of vertices and two vertices are adjacent iff the corresponding blocks intersect. If $A \subset S$ then it is obvious that the intersecting graph of A is an induced subgraph of that of S. We say that four blocks of a partial Steiner system form a *complete quadrilateral* if any pair of them intersect and no three have a point in common, the intersecting points are called *vertices* of the quadrangle; if only five pairs intersect we call the configuration a *near-complete quadrilateral.*

The main idea of the proof of Theorem 1.2 is to find a Steiner system containing a partial Steiner system A such that this Steiner system can be transformed into new ones which have different numbers of complete quadrilaterals, this property guarantees that they are non-isomorphic. On the other hand, this transformation preserves A as a subsystem.

In the Steiner system (P,B) a set of disjoint blocks which partition P is called a *parallel class,* the system is *resovable* if its blocks can be partitioned into parallel classes and such a partition is a *resolution* of the system. In a partial Steiner system we call a collection of some pairwise disjoint blocks a *partial parallel class* by the same terminology. Let πA be a partition of blocks of a partial Steiner system A into partial parallel classes $\{\pi_1, \pi_2, \ldots, \pi_t\}$ and S an extension of A. Furthermore, suppose that there is a resolution πS of the blocks of S into parallel classes so that each π_j is contained in a class of this resolution and the classes containing π_i and π_j are different whenever $i \neq j$. In this case we say that the resolution πS *preserves* the partial parallel classes π_i. The following theorem, proved by Linner [5], is a stronger version of Ganter's theorem where the extension is required to preserve given partial parallel classes of the partial Steiner system.

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Theorem 2.1. (Linner) *Any partial Steiner system along with a partition of its blocks into partial parallel classes can be embedded in a resolvable Steiner system which preserves these classes.*

Following the proof given by Linner, one can deduce that if A has block size p^β with p prime, then the required extension S could be found of the order p^α for some α. (The proof of this fact relies on a long count but no new idea is needed, therefore we ignore this.) A graph is *strongly regular* with parameters (v,k,λ,μ) if it is a regular graph with valency k on v vertices; moreover, any two adjacent vertices have precisely λ common neighbours, and any two non-adjacent vertices have precisely μ common neighbours. It is easy to see that the intersecting graph of a Steiner system is strongly regular [2]. If not specially mentioned, all graphs considered in this paper are simple. Finally, we denote by K_n the clique on n vertices.

3. Proof of Theorem 1.2

To prove Theorem 1, first we need to construct arbitrarily many common-parametered extensions of A. The following trivial propositions provide some routine techniques which will help us to do that.

Proposition 3.1. *Let S be an $S(2,k,v)$ Steiner system and S' a $S(2,v,p)$ Steiner system. Defining on every block of S' a Steiner system isomorphic to S we obtain an $S(2,k,p)$ Steiner system.*

Proposition 3.2. *Let S be an $S(2,k,v)$ Steiner system and P be subset of points such that a subsystem of blocks contained in P is an $S(2,k,|P|)$ Steiner system. Replace this subsystem by another $S(2,k,|P|)$ Steiner system the resulting system is still an $S(2,k,v)$ Steiner system.*

Lemma 3.1. *Let A be a k-partial Steiner system, then there is a prime number p such that A can be embedded in an $S(2,k,p)$ Steiner system.*

Proof. By Ganter's theorem one can embed A in an $S(2,k,v)$ Steiner system for some v. Wilson's theorem mentioned in the introduction shows that there is an $S(2,v,p)$ Steiner system for every $p = v(v-1)t+1$ where t is sufficently large. By Dirichlet's theorem we can choose p to be a prime. Proposition 3.1 completes the proof. □

The proof of Theorem 1.2 consists of two steps. In the first step we prove the theorem in the case $n = 2$. The second step discusses the general case using the result of the first step.

First step

Let A be the given partial Steiner system, by Lemma 3.1 there is an $S(2,k,p)$ Steiner system S such that $A \subset S$ and p is a prime. Consider the affine space Q of dimension d and rank p. Fix x and y two points of Q and L the line through x and y. Fix a line L' through x differing from L and denote by P the plane containing L and L'. Let u and v

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be two points on L' distinct from x. Put on L' a copy of S such that x, u and v are in the same block, say b. On each plane P' through L' except P take three lines $L_1^{P'}, L_2^{P'}$ and $L_3^{P'}$ such that $L_1^{P'}, L_2^{P'}, L_3^{P'}$ and L' form a complete quadrilateral with x, u and v as three of its six vertices. Define on the $L_i^{P'}$ a Steiner system isomorphic to S such that the vertices of the above complete quadrilateral are contained in the same block if they are in the same line; this is clearly possible. Note that these blocks along with b also form a complete quadrilateral with the same vertices. Denote by C_{xuv} the set of complete quadrilateral obtained in this way for fixed x, u and v.

On every remaining line of the space define a Steiner system isomorphic to S. Hence by Proposition 3.2 we have a $(2, k, p^d)$ Steiner system. Let us denote this by S_0. We call a line containing a block the 'father' of this block (each block has exactly one 'father').

Let T be a transformation on the blocks of S_0 such that:

$$T(l) = \begin{cases} l - x + y, & \text{if } l \text{ is a block in } P \text{ but not in } L,\ x \in l, y \notin l \\ l - y + x, & \text{if } l \text{ is a block in } P \text{ but not in } L,\ y \in l, x \notin l \\ l & \text{otherwise.} \end{cases}$$

Let S_1 be $T(S_0)$, i.e. the system obtained from S_0 by the transformation. Propositions 3.1 and 3.2 yield that S_1 is a Steiner system, too. We shall prove that S_0 is not isomorphic to S_1. (It is clear that both of them have A as a subsystem.) Let us denote by v_i the number of complete quadrilaterals of S_i; $i = 0, 1$. It is sufficient to prove that $v_0 > v_1$.

Denote by Γ_{01} the set of complete quadrilaterals (l_1, l_2, l_3, l_4) of S_0 where the transformed blocks $T(l_j), j = 1, 2, 3, 4$ do not form a complete quadrilateral in S_1. Conversely, Γ_{10} is the set of quadruples of blocks (l_1, l_2, l_3, l_4) of S_0 which do not form a complete quadrilateral but the $T(l_j)$ do. Let γ_{hj} be the cardinality of Γ_{hj}. So γ_{01} and γ_{10} are the numbers of complete quadrilaterals we lose and gain by the transformation.

Consider Γ_{01}. Note that every complete quadrilateral in C_{xuv} constructed above belongs to Γ_{01}. Thus $\gamma_{01} \geq \beta$, where β is the number of planes through L' other than P.

Consider Γ_{10}. Observe that if a quadruple of blocks belongs to Γ_{10} then these blocks must form a near-complete quadrilateral. Furthermore, one of the two blocks which do not intersect must contain x and the other must contain y, and at least one of them must be contained in P. (T has no effect on blocks not contained in P.)

Consider the fathers of these blocks. Since they are lines of an affine space they must be in the same plane. Hence only two cases may occur:

(1) All these lines are contained in P.
(2) Exactly one of them is contained in P.

In case (2) one can easily see by the elementary properties of affine spaces that the only line contained in P must contain both x and y and therefore must be L. But in this case, due to the definition of T, all of the blocks of the considered quadruple remain the same after the transformation and hence can not form a complete quadrilateral in S_1. So, only case (1) may occur. Observe that there are $p^2 + p$ lines in P, and each line contains $p(p-1)/k(k-1)$ blocks. Hence, the number of quadruples of blocks in P is $\binom{\alpha}{4}$ where $\alpha = (p^2 + p)p(p-1)/k(k-1)$. Choose $d \geq 18$; we have

$$\gamma_{10} \leq \binom{\alpha}{4} < p^{16} < \beta = \frac{p^d - p}{p^2 - p} - 1 \leq \gamma_{01}$$

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The inequality $\gamma_{10} < \gamma_{01}$ implies $v_0 > v_1$, which completes this step. One can reduce d by a more detailed consideration, here we are interested in the existence only.

Second step

Choose $n-1$ disjoint subspaces A_i of dimension d in the affine space $AF(p,m)$ with m sufficiently large to guarantee the existence of the disjoint subspaces. On each line not contained in the A_i define a Steiner system isomorphic to S giving a system $\mathscr{X}$. In each A_i let x_i, y_i, L_i and P_i be the distinguished points, lines and planes as considered in the first step. Define on every line of P_i a Steiner system isomorphic to S and call the resulting subsystem $\mathscr{P}_i$. Let $\mathscr{P}'_i = T(\mathscr{P}_i)$. Let μ_i (μ'_i) be the number of complete quadrilaterals in the system $\mathscr{X} \cup \mathscr{P}_i$ ($\mathscr{X} \cup \mathscr{P}'_i$) which have exactly one block contained in $\mathscr{P}_i$ ($\mathscr{P}'_i$). Since T is idempotent, we can suppose that $\mu_i \geq \mu'_i$. Completing each A_i with the subsystem $\mathscr{P}_i$ to a system isomorphic to S_0, we obtain a Steiner system in the whole space. Let us denote this by $\mathscr{X}_0$. Let $\mathscr{X}_i$ be the Steiner system obtained from $\mathscr{X}_0$ by replacing $\mathscr{P}_j$ by $\mathscr{P}'_j$, $j = 1, 2, \ldots, i-1$ for every $i = 1, 2, \ldots n-1$. We shall prove that $v(\mathscr{X}_i) > v(\mathscr{X}_{i+1})$, $i = 0, 1, \ldots, n-2$; this inequality implies that the $\mathscr{X}_i$ are non-isomorphic. Note that in $\mathscr{X}_i$ the subsystems defined on $A_j, j = 1, 2 \ldots, i-1$ are isomorphic to S_1. Observe that:

$$v(\mathscr{X}_i) = \alpha_i + \beta_i + \gamma_i + \mu_i$$

where α_i is the number of complete quadrilaterals of $\mathscr{X}_i$ with none of the blocks contained in A_i, β_i is the number of complete quadrilaterals of $\mathscr{X}_i$ which have exactly one block in A_i, but not in P_i, and γ_i is the number of complete quadrilaterals of $\mathscr{X}_i$ all of the blocks of which are contained in A_i.

Due to the construction, we have $\alpha_i = \alpha_{i+1}$, $\beta_i = \beta_{i+1}$ and $\mu_i \geq \mu_{i+1}$. The first step yields $\gamma_i > \gamma_{i+1}$ which proves the required inequality and hence completes our proof. □

The following corollary is important.

Corollary 3.1. *The Steiner systems $\mathscr{X}_i$ constructed above have non-isomorphic intersecting graphs.*

Proof. Let G_i be the intersecting graph of $\mathscr{X}_i$ and $\xi(G_i)$ the number of K_4 in G_i. We will compare the numbers $\xi(G_i)$ ($i = 0, 1, 2, ..., n-1$). Due to the definition of intersecting graphs, four vertices span a clique iff their corresponding blocks are pairwise intersecting. So these blocks should be in one of the following situations:

(1) They share one point in common.
(2) Three of them share one point in common and the fourth meets each.
(3) They form a complete quadrilateral.

An easy counting shows that the numbers of quadruples of blocks in (1) and (2) depend only upon the parameters of the system. (In fact, they are $v\binom{(v-1)/(k-1)}{4}$ and $(1/3)\binom{(v-1)/(k-1)}{2}(k-1)^2(k-2)$ in every $S(2,k,v)$ Steiner system, respectively.)

Since the $\mathscr{X}_i$ have the same parameters, we have to compare the numbers of the complete quadrilaterals only. By the last conclusion in the proof of Theorem 1.2, we have $v(\mathscr{X}_i) > v(\mathscr{X}_{i+1})$, thus $\xi(G_i) > \xi(G_{i+1})$ which yields that the G_i are non-isomorphic. □

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Remark. Corollary 3.2 is slightly stronger then Theorem 1.2 because the non-isomorphism of the intersecting graphs implies that of the systems, but the converse statement is not always true. For example, every projective plane $PG(2,p)$ has the same intersecting graph K_{p^2+p+1}; on the other hand, it is well known that there are non-isomorphic planes of the same order.

4. An application

Is it true that every graph can occur as an induced subgraph of a strongly regular graph? This question was asked by Rosenfeld and answered by Bollobás and Thomason [1]. Using the famous Weil's theorem proving the Riemann's hypothesis on the curves over finite fields, they showed that every Paley graph of sufficiently large order contains every graph of given order. A simpler proof can be found in [10]. Exploiting the constructions discussed in Section 3, here we show a new method to handle this problem and a further version of it. First, note that every graph can be seen as an intersecting graph of a partial Steiner system. Corollary 3.2 and the note at the end of Section 2 immediately imply the following, which is a generalization of Bollobás and Thomason's theorem (this idea is suggested by Brouwer [2]).

Theorem 4.1. *Let G be an arbitrary graph and n a positive integer. There is a family of n non-isomorphic strongly regular graphs with the same parameters, each of which contains G as an induced subgraph.*

When a graph is considered as the intersecting graph of a hypergraph, then some interesting properties of the graph can be conveniently derived from the construction of the hypergraph. This simple idea, combined with the theory of Steiner systems, leads us to some more surprising results. We say that a graph G' *blows up* the independent sets of G if G is an induced subgraph of G', and every independent set of vertices of G is a subset of a maximum independent set of G'. (It may be a nice exercise for interested readers to show that every graph has an extension with this property.) Regarding the difficulty of determining a maximum independent set of vertices in a graph, the following theorem shows some of the strength of the method.

Theorem 4.2. *Let G be an arbitrary graph and n a positive integer. There is a family of n non-isomorphic strongly regular graphs $G_1, G_2, \dots G_n$ with the same parameters, each of which blows up the independent sets of G.*

Proof. As mentioned above, G can be seen as an intersecting graph of a partial Steiner system A. Some vertices of G are independent iff their corresponding blocks form a partial parallel class. On the other hand, a parallel class of a Steiner system forms a maximum independent set in its intersecting graph. So it suffices to construct a Steiner system S containing A with the property that every maximal partial parallel class of A can be extended to a parallel class of S $(*)$. If such a Steiner system S can be found with order equal to the rank of some affine spaces, then by using the construction in the proof of

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Theorem 1.2 we can produce an arbitrary large family of Steiner systems with property (∗). The intersecting graphs of these systems satisfy Theorem 4.2 due to Corollary 3.2 and the mentioned property.

To construct the system S, note that we can suppose that the block size of A is a prime number p (if the original block size is not a prime we enlarge the blocks by disjoint sets – this operation does not change the intersecting graph of the system). For every vertex v of G denote by $m(v)$ the number of maximal independent sets containing v. Let B_v be the block corresponding to v in A. Take $m(v)$ disjoint sets $B_v^1, B_v^2, \dots B_v^{m(v)}$ of size $p^2 - p$ satisfying the condition that the sets $\cup_{i=1}^{m(v)} B_v^i, v \in V(G)$ are pairwise disjoint and disjoint from the point set of A.

Let $C_v^i = B_v \cup B_v^i$, hence $|C_v^i| = p^2$, we call B_v the 'son' of C_v^i for every $i = 1, 2, \dots, m(v)$. The blocks C_v^i $(v \in V(G); i = 1, 2, \dots, m(v))$ form a partial Steiner system C with block size p^2. Let $I_1, I_2, \dots I_l$ be the maximal independent sets of G, where $I_j = \{v_{j1}, v_{j2}, \dots, v_{ji_j}\}$. Due to the construction the set of blocks of C can be partitioned into l partial parallel classes $\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_l$ where $\mathscr{C}_j$ has i_j blocks with their sons being $B_{j1}, B_{j2}, \dots, B_{ji_j}$. We call this class the father of the maximal partial parallel class $B_{j1}, \dots, B_{ji_j}$ of A.

Linner's theorem and the note following that in Section 2 say that we can find a resovable Steiner system S' of order p^α which is an extension of C and which preserves the partial parallel classes $\mathscr{C}_j$. On each block of S' define an affine plane such that every block of A is a line, by Proposition 3.1 the resulting system S is a $S(2, p, p^\alpha)$ Steiner system containing A as a subsystem. It is easy to verify that this is the system with property (∗) we wanted. Since there are affine spaces of rank p^α the method discussed in Section 3 can be used to construct n non-isomorphic Steiner systems $\mathscr{X}_1, \mathscr{X}_2, \dots, \mathscr{X}_n$ on the point set of an affine space. S plays the same role in the construction as in the proof of Theorem 1.2, so a copy of S is put on almost every line of the space. Observe that an affine space can be covered by distinct lines and we can choose these lines such that each contains a copy of S. This shows that $\mathscr{X}_i$ also satisfies (∗) completing the proof. □

5. A question

Let G be an arbitrary graph and denote by $B(G)$ the smallest order of a strongly regular graph which contains G as an induced subgraph. Of course, $B(G)$ depends mainly on the construction of G, and it indicates the difficulty of extending G to a strongly regular graph. We would like to determine the value $B(n) = \max B(G)$ where G runs over the set of graphs of order n. It is not difficult to show that $B(n) \geq cn^2$ [11], but we do not know the right order of magnitude of $B(n)$.

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