FLOW OF A SECOND GRADE FLUID OVER A STRETCHING SURFACE WITH NEWTONIAN HEATING

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ABSTRACT

This article describes the boundary layer flow and heat transfer in a second grade fluid over a stretching sheet. Heat transfer analysis is carried out in the presence of a Newtonian heating. The partial differential systems have been transformed into the ordinary differential systems by appropriate relations. Homotopy analysis method (HAM) is used for the solutions. Graphical and tabulated results are presented to see the significance of influential parameters on the velocity and temperature fields. It is seen that temperature profiles and heat transfer rate significantly increase by increasing the conjugate parameter (γ) for Newtonian heating.

Keywords: Heat transfer, Second grade fluid, Newtonian heating, Stretching surface.

1. INTRODUCTION

The boundary layer flows of non-Newtonian fluids have many practical applications. Such flows have been attracted much by the researchers during the last few decades. Several engineering processes such as paper production, glass fiber, wire drawing crystal growing, drawing of plastic films *etc*. involve such flows. These processes are quite common in manufacturing of foods, slurry transporting, petroleum in drilling and polymer processing. A variety of non-Newtonian fluid models have been proposed by the researchers due to their different rheological characteristics. The model of second grade fluid has been studied much in view of its simplicity. A host of recent researchers have contributed towards the flow analysis of second grade fluids under the various configurations [1-10]. Prandtl noticed that viscous interactions are not that significant in determining the flow field far away from the boundary. However in a thin region near a solid boundary, the viscous interactions have a significant effect on the fluid motion. Prandtl's boundary layer theory made the calculations easy and took into account the viscosity. This theory is applicable on the calculation of friction drag of bodies in a flow for example the drag of a flat plate at zero incidence, the friction drag of ship, an airfoil, the body of an airplane or a turbine blade. The majority of the heat transfer takes place within the boundary layer allowing the equations to be simplified in the flow field. This theory has been applied significantly to simply the constitutive equations for flow and heat transfer of various non-Newtonian fluids. For instance Mushtaq *et al*. [11] analyzed the mixed convection boundary layer flow of second grade fluid past a vertical stretching sheet. Numerical solutions for three-dimensional boundary layer flow of second grade fluid have been provided by Nazar and Latip [12]. Combined effects of viscous dissipation and thermal radiation on the boundary layer flow of second grade fluid have been investigated by Hayat *et al*. [13]. Hayat and Nawaz [14] examined the characteristics of heat transfer in the three-dimensional boundary layer flow of second grade fluid. Partial slip effects on the magnetohydrodynamic boundary layer flow and heat transfer of second grade fluid have been addressed by Abel *et al*. [15].

It is established that wall to ambient temperature distributions can be presented by four common heating processes (see Merkin [16]). When the heat transfer rate from the bounding surface with a finite heat capacity is proportional to the local surface, the correspond-

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ing heating processes is called Newtonian heating which is usually termed as conjugate heat flow. It is not a realistic assumption that the coupled conduction-convection effects are inter-related since conjugate heat transfer processes in many applications involve such inter-relation. This conjugate heat flow situation occurs in various important engineering devices such as heat exchanger where the conduction in solid tube wall is largely affected by the convection in the fluid flowing over it, conjugate heat transfer around fins where the conduction within the fin and the convection in the fluid surrounding it must be simultaneously examined in order to obtain the vital design information and also in convection flows when the bounding surfaces absorb heat by solar radiation. Recent contributions which have taken into consideration the boundary conditions for Newtonian heating include those of Lesnic *et al*. [17-19], Pop *et al*. [20] and Salleh *et al*. [21]. In [21], the boundary layer flow of viscous fluid over a stretched surface is studied in the regime of Newtonian heating. The numerical solution of the differential system is obtained by Keller-box method. Chaudhary and Jain [22] examined the influence of Newtonian heating on the unsteady boundary layer flow of a viscous fluid. An exact solution for the velocity and temperature fields has been derived in this study. Mixed convection boundary layer flow with Newtonian heating is studied by Salleh *et al*. [23]. Numerical solution of the entire differential system is developed by Keller-box method. The purpose of applying Newtonian heating boundary conditions is to introduce a dimensionless conjugate parameter (γ) (defined below) which measures the strength of surface heating. It has been observed in [21] that obtained solution is dependent upon this parameter γ and the Prandtl number Pr. There exists a critical value γ_c, associated with the parameter γ for a fixed Prandtl number Pr, such that physically acceptable solution is possible only for $\gamma < \gamma_c$. Therefore we are interested in obtaining the analytic solutions for a wide range of Prandtl number Pr and $\gamma < \gamma_c$.

Numerical solutions of the nonlinear differential equations give discontinuous points on a curve. Therefore it is often costly and time consuming to get a complete curve of results. Perturbation techniques require the existence of small/large parameter in the differential equations for the development of analytic solutions. Unfortunately most of the problems in science and engineering do not contain such parameter. This restriction doesn't exist in non perturbative methods such as δ–expansion method, Adomain decomposition method and homotopy perturbation method (HPM). However these perturbative and non perturbative methods do not provide us ways to adjust and control the convergence of the obtained solutions. Liao [24] developed an approximate analytic method, based on the homotopy of topology, valid for all nonlinear problems known as homotopy analysis method (HAM). This method is independent upon the existence of small/large parameters in the differential systems. Further it provides us a simple way to adjust the convergence rate of the obtained series solutions. It also gives us huge freedom to choose the initial guess and the linear operator for various problems. This method has been successfully applied to various nonlinear problems including [25-35]. The effect of Newtonian heating (NH) on the flow and heat transfer in the non-Newtonian fluids has not been reported so far. Hence it seems worthwhile attempt to present such study for flow of second grade. Homotopy approach is utilized for the construction of solution expressions. The current computations also include the analysis for constant wall temperature (CWT) and constant heat flux (CHF) cases. Physical quantities of interest are examined.

2. MODEL FORMULATION

We consider the steady boundary layer flow of an incompressible second grade fluid induced by a stretching surface. The sheet is stretched with a velocity $u_w(x) = ax$, (where a is a real number). The heat transfer in the presence of Newtonian heating is further considered. The cauchy stress tensor for a second grade fluid model is given by

$$\boldsymbol{\tau} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 , \tag{1}$$

where p is the pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity, α_1 and α_2 are the material fluid parameters and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and second Rivlin Ericksen tensors respectively defined as

$$\mathbf{A}_1 = (\text{grad}\,\mathbf{V}) + (\text{grad}\,\mathbf{V})^t , \tag{2}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\text{grad}\,\mathbf{V}) + (\text{grad}\,\mathbf{V})^t \mathbf{A}_1 , \tag{3}$$

where $\mathbf{V}$ denotes the velocity field, grad the gradient operator and $d/dt(= \partial/\partial t + (\mathbf{V}\cdot\nabla)\mathbf{V})$ the material derivative.

For thermodynamic second grade fluid model the material parameters obey the following conditions

$$\mu \geq 0 , \ \alpha_1 \geq 0 , \ \alpha_1 + \alpha_2 = 0 . \tag{4}$$

Invoking the above values in the momentum equation

$$\rho \frac{d\mathbf{V}}{dt} = \nabla.\boldsymbol{\tau} , \tag{5}$$

and applying boundary layer approximation the governing equations for the flow and heat transfer of an incompressible second grade fluid are (see Cortell [5] and Salleh *et al*. [16])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{6}$$

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$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+\frac{\alpha_1}{\rho}\left[\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+u\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}\right], \tag{7}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}. \tag{8}$$

In above expressions u and v are the velocity components in the x– and y– directions, α_1 the second grade fluid parameter, T the fluid temperature, k the thermal conductivity of the fluid, c_p is specific heat, $\nu=(\mu/\rho)$ the kinematic viscosity and ρ the density of the fluid.

The boundary conditions are [16] 0921060964

$$u=u_w(x)=ax,\ v=0,\ \frac{\partial T}{\partial y}=h_s T\ \ \text{(NH) at } y=0,$$
$$u=0,\ T\to T_\infty\ \text{ as } y\to\infty. \tag{9}$$

Here h_s is heat transfer coefficient, $u_w = ax$ is the velocity of the stretching sheet, T is fluid temperature and T_∞ is the ambient temperature. We use the following transformations [16]

$$u=axf'(\eta),\ v=-\sqrt{a\nu}f(\eta),$$
$$\theta(\eta)=\frac{T-T_\infty}{T_\infty}\ \text{(NH)(Newtonian heating)},$$
$$\eta=y\sqrt{\frac{a}{\nu}},\ \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}\ \text{(CWT)},$$
$$\theta(\eta)=\left(\frac{k}{q_w}\right)(T-T_\infty)\sqrt{\left(\frac{a}{\nu}\right)}\ \text{(CHF)}. \tag{10}$$

Using above transformations, Eqs. (7) ~ (9) can be written as

$$f'''+ff''-f'^2+\alpha(2ff'''-f''^2-ff'''')=0, \tag{11}$$

$$\theta''+\Pr f\theta'=0, \tag{12}$$

$$f(0)=0,\ f'(0)=1,\ f'(\infty)=0, \tag{13}$$

$$\theta'=-\gamma[1+\theta(0)]\ \text{(NH) at}\ \eta=0\quad \theta=0\ \text{at}\ \eta\to\infty$$
$$\theta=1\ \text{(CWT) at}\ \eta=0\quad \theta=0\ \text{at}\ \eta\to\infty$$
$$\theta'=-1\ \text{(CHF) at}\ \eta=0\quad \theta=0\ \text{at}\ \eta\to\infty, \tag{14}$$

where Pr is the Prandtl number, γ is the conjugate parameter for Newtonian heating and α is the second grade parameter. These quantities are given by

$$\Pr=\frac{\mu c_p}{k},\quad \gamma=h_s\sqrt{\frac{\nu}{a}},\quad \alpha=\frac{\alpha_1 a}{\mu} \tag{15}$$

It is worth mentioning here that the exact solution of Eq. (11) satisfying the boundary conditions (14) is [5]

$$f(\eta)=\left[\frac{1-\exp(-K\eta)}{K}\right],\ K=\frac{1}{\sqrt{1+\alpha}}. \tag{16}$$

After substituting Eq. (16) into dimensionless energy Eq. (12) the resulting differential equation subject to the boundary conditions (14) (for NH case) can be solved numerically by symbolic computation software MATHEMATICA. An exact solution of Eq. (12) for CWT case is

$$\theta(\eta)=\exp\left(-\frac{\eta}{K}\right)\frac{{}_1F_1\left[\frac{\Pr}{K^2};\ 1+\frac{\Pr}{K^2};-\frac{\Pr}{K^2}\exp(-\eta K)\right]}{{}_1F_1\left[\frac{\Pr}{K^2};\ 1+\frac{\Pr}{K^2};-\frac{\Pr}{K^2}\right]} \tag{17}$$

where ${}_1F_1$ are the confluent hypergeometric functions.

The skin friction coefficient C_f and local Nusselt number Nu_x are

$$C_f=\frac{\tau_w}{\rho u_w^2},\quad Nu_x=\frac{xq_w}{k(T_w-T_\infty)}, \tag{18}$$

where the wall skin friction τ_w and the heat transfer q_w from the plate are given by

$$\tau_w=\left[\mu\frac{\partial u}{\partial y}+\alpha_1\left\{2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y\partial x}+u\frac{\partial^2 u}{\partial x\partial y}\right\}\right]_{y=0}, \tag{19}$$

$$q_w=-k\left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{20}$$

The above expressions along with Eq. (10) may read

$$(\mathrm{Re}_x)^{1/2}C_f=f''(1+3\alpha f')\big|_{\eta=0}, \tag{21}$$

$$Nu/\mathrm{Re}_x^{1/2}=\gamma\left[1+\frac{1}{\theta(0)}\right]\ \text{(For NH)}, \tag{22}$$

$$Nu/\mathrm{Re}_x^{1/2}=-\theta'(0)\ \text{(For CWT)}, \tag{23}$$

$$Nu/\mathrm{Re}_x^{1/2}=\frac{1}{\theta(0)}\ \text{(For CWT)}. \tag{24}$$

where $\mathrm{Re}_x=(ax^2/\nu)$ is the local Reynolds number.

3. SOLUTIONS BY HOMOTOPY APPROACH

We write the initial guesses and auxiliary linear operators as

$$f_0(\eta)=1-e^{-\eta},\quad \theta_0(\eta)=\frac{\gamma\exp(-\eta)}{1-\gamma};\ \gamma\neq 1,\text{(NH)}$$
$$\theta_0(\eta)=\exp(-\eta)\quad \text{(CWT) and (CHF)} \tag{25}$$

$$L_f=f'''-f',\quad L_\theta=\theta''-\theta, \tag{26}$$

with

$$L_f(A_1+A_2e^{\eta}+A_3e^{-\eta})=0,\quad L_\theta(A_4e^{\eta}+A_5e^{-\eta})=0, \tag{27}$$

where A_i $(i=1-5)$ denote the arbitrary constants.

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3.1 Zeroth Order Deformation Problems

The problems at this order are

$$(1-p)L_f\left[\hat{f}(\eta;p)-f_0(\eta)\right]=ph_f\,\mathbf{N}_f\left[\hat{f}(\eta;p),\hat{\theta}(\eta,\ p)\right], \tag{28}$$

$$(1-p)L_\theta\left[\hat{\theta}(\eta;\ p)-\theta_0(\eta)\right]=ph_\theta\,\mathbf{N}_\theta\left[\hat{f}(\eta;p),\ \hat{\theta}(\eta,\ p)\right], \tag{29}$$

$$\hat{f}(0;\ p)=0\,,\ \hat{f}'(0;\ p)=1\,,\ \hat{f}'(\infty;\ p)=0\,, \tag{30}$$

$$\hat{\theta}'(0,\ p)=-\gamma\left[1+\theta(0,\ p)\right],\ \hat{\theta}(\infty,\ p)=0\,,\ \text{(NH)} \tag{31}$$

$$\hat{\theta}(0,\ p)=1\,,\ \hat{\theta}(\infty,\ p)=0\,,\quad \text{(CWT)} \tag{32}$$

$$\hat{\theta}'(0,\ p)=-1\,,\ \hat{\theta}(\infty,\ p)=0\,,\quad \text{(CHF)} \tag{33}$$

$$\begin{aligned}&\mathbf{N}_f\left[f(\eta,\ p)\,,\ \theta(\eta,\ p)\right]\\&=\frac{\partial^3 f(\eta,\ p)}{\partial\eta^3}-f(\eta,\ p)\frac{\partial^2 f(\eta,\ p)}{\partial\eta^2}-\left(\frac{\partial f(\eta,\ p)}{\partial\eta}\right)^2\\&+\alpha\left[\begin{array}{c}2\dfrac{\partial f(\eta,p)}{\partial\eta}\dfrac{\partial^3 f(\eta,p)}{\partial\eta 3}-\left(\dfrac{\partial f^2(\eta,p)}{\partial\eta^2}\right)^2\\-f(\eta,p)\dfrac{\partial^4 f(\eta,p)}{\partial\eta^4}\end{array}\right],\end{aligned} \tag{34}$$

$$\mathbf{N}_\theta\left[\theta(\eta,p),f(\eta,p)\right]=\frac{\partial^2\theta(\eta,\ p)}{\partial\eta^2}+\Pr f(\eta,\ p)\frac{\partial\theta(\eta,p)}{\partial\eta}, \tag{35}$$

where p is an embedding parameter, h_f and h_θ are the non-zero auxiliary parameters and $\mathbf{N}_f$ and $\mathbf{N}_\theta$ are the nonlinear operators. For $p = 0$ and $p = 1$ we have

$$\begin{aligned}&f(\eta;0)=f_0(\eta)\,,\ \theta(\eta,0)=\theta_0(\eta)\ \text{and}\\&f(\eta;1)=f(\eta)\,,\ \theta(\eta,1)=\theta(\eta)\end{aligned} \tag{36}$$

and $f(\eta,\ p)$ and $\theta(\eta,\ p)$ vary from $f_0(\eta)$, $\theta_0(\eta)$ to $f(\eta)$ and $\theta(\eta)$ when p varies from 0 to 1. Taylor series gives

$$f(\eta,p)=f_0(\eta)+\sum_{m=1}^{\infty}f_m(\eta)p^m\,, \tag{37}$$

$$\theta(\eta,p)=\theta_0(\eta)+\sum_{m=1}^{\infty}\theta_m(\eta)p^m\,, \tag{38}$$

$$f_m(\eta)=\frac{1}{m!}\frac{\partial^m f(\eta;p)}{\partial\eta^m}\bigg|_{p=0}\,,\ \theta_m(\eta)=\frac{1}{m!}\frac{\partial^m\theta(\eta;p)}{\partial\eta^m}\bigg|_{p=0}\,. \tag{39}$$

The series in Eqs. (36) and (37) are strongly dependent upon h_f and h_θ. The values of h_f and h_θ are selected by a processes that the series converge at $p = 1$. Hence

$$f(\eta)=f_0(\eta)+\sum_{m=1}^{\infty}f_m(\eta)\,, \tag{40}$$

$$\theta(\eta)=\theta_0(\eta)+\sum_{m=1}^{\infty}\theta_m(\eta)\,. \tag{41}$$

3.2 Mth-Order Deformation Problems

The problems at this order are

$$L_f\left[f_m(\eta)-\chi_m f_{m-1}(\eta)\right]=h_f\mathbf{R}_f^m(\eta)\,, \tag{42}$$

$$L_\theta\left[\theta_m(\eta)-\chi_m\,\theta_{m-1}(\eta)\right]=h_\theta\,\mathbf{R}_\theta^m(\eta)\,, \tag{43}$$

$$f_m(0)=f_m'(0)=f_m'(\infty)=0\,, \tag{44}$$

$$\theta_m'(0)+\gamma\theta_m(0)=\theta_m(\infty)=0\ \text{(NH)} \tag{45}$$

$$\theta_m(0)=\theta_m(\infty)=0\quad\text{(CWT)} \tag{46}$$

$$\theta_m'(0)=\theta_m(\infty)=0\qquad\text{(CHF)} \tag{47}$$

$$\begin{aligned}\mathbf{R}_f^m(\eta)&=f_{m-1}'''(\eta)+\sum_{k=0}^{m-1}\left[f_{m-1-k}f_k''-f_{m-1-k}'f_k'''\right]\\&+\alpha\sum_{k=0}^{m-1}\left[2f_{m-1-k}'f'''-f_{m-1-k}''f_k''-f_{m-1-k}f_k''''\right],\end{aligned} \tag{48}$$

$$\mathbf{R}_\theta^m(\eta)=\theta_{m-1}''+\Pr\sum_{k=0}^{m-1}\theta_{m-1-k}'\,f_k\,, \tag{49}$$

$$\chi_m=\begin{cases}0,\ m\le 1\,,\\1,\ m>1\end{cases} \tag{50}$$

and the series solutions are

$$f_m(\eta)=f_m^*(\eta)+A_1+A_2\,e^{\eta}+A_3\,e^{-\eta}\,, \tag{51}$$

$$\theta_m(\eta)=\theta_m^*(\eta)+A_4\,e^{\eta}+A_5\,e^{-\eta}, \tag{52}$$

in which f_m^* and θ_m^* indicate the special solutions and the values of $A_i (i = 1 - 5)$ are obtained by using the boundary conditions (44) ~ (47)

$$\begin{aligned}&A_2=A_4=0\,,\qquad A_3=\frac{\partial f_m^*(\eta)}{\partial\eta}\bigg|_{\eta=0}\,,\qquad A_1=-A_3-f_m^*(0)\,,\\&A_5=\frac{1}{1-\gamma}\left(\theta_m^{*\prime}(\eta)\big|_{\eta=0}-\gamma\theta_m^*(\eta)\big|_{\eta=0}\right),\qquad\text{(NH)}\\&A_5=-\theta_m^*(\eta)\big|_{\eta=0}\qquad\text{(CWT)}\\&A_5=\theta_m^{*\prime}(\eta)\big|_{\eta=0}\qquad\text{(CHF)}\end{aligned} \tag{53}$$

Note that the solutions of the problems involving Eqs. (42) ~ (50) have been solved by using the symbolic computation software MATHEMATICA when m = 1, 2, 3, …..

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4. CONVERGENCE OF HOMOTOPY SOLUTIONS

The explicit analytic expressions in Eqs. (40) and (41) have auxiliary parameters $\hbar_f$ and $\hbar_\theta$. These parameters provide the convergence region and rate of approximations for the derived solutions. For an appropriate range of values of these parameters, the $\hbar_f$ and $\hbar_\theta$-curves are plotted at 25th-order of approximations for NH, CWT and CHF cases in the Figs. 1 and 2(a) ~ 2(c) respectively. It is found that the admissible values of $\hbar_f$ and $\hbar_\theta$ are $-1.5 \le \hbar_f \le -0.1$, $-1.7 \le \hbar_\theta \le -0.5$ (NH), $-1.8 \le \hbar_\theta \le -0.3$ (CWT) and $-1.75 \le \hbar_\theta \le -0.75$ (CHF) respectively. We have also plotted the residual error for the function θ at 15th-order of approximations for some fixed values of parameters. It is quite obvious that obtained results are correct up to six decimal places. It is impossible to achieve such accuracy in the other approximate analytic and numerical methods. For $\hbar_f = -0.8$ and $\hbar_\theta = -1.4$ we obtain the convergent series solutions in the whole region of $\eta(0 \le \eta < \infty)$.

5. GRAPHICAL RESULTS AND DISCUSSION

This section displays the influence of key parameters on the velocity, temperature, coefficient of skin friction and local Nusselt number. Table 1 indicates that 5th-order of approximations gives convergent series solutions for the velocity. However the convergence for the temperature function is obtained at 15th-order of approximations. Keeping this fact in mind we have plotted graphs for the function f' and θ at 5th and 15th-order of approximations respectively. Figure 3 is displayed to analyze the comparison of the present analytic results with the exact solutions given in Eq. (15). An excellent agreement is found between the two solutions for different values of α. In the Figs. 5 ~ 7, the temperature profiles obtained by HAM and numerical solutions are plotted versus η for different values of α, Pr and γ. Figure 5 shows the behavior of second grade parameter α on the temperature θ(η). It is seen that an increase in the second grade parameter α decreases the temperature and thermal boundary layer thickness. It is observed that a rapid increase in the normal stress differences reduces the temperature and thermal boundary layer thickness. Figure 6 shows the effect of Prandtl number Pr on the temperature field θ(η). Obviously a large Prandtl number has a relatively lower thermal diffusivity which acts to decay the thermal boundary layer thickness. An increase in conjugate parameter γ on the temperature is noticed in Fig. 7. It is found that an increase in γ significantly increases the temperature θ. It follows from the fact that a slight increase in γ increases the Newtonian heating which therefore enhances the temperature and the thermal boundary layer thickness. Further, it is also

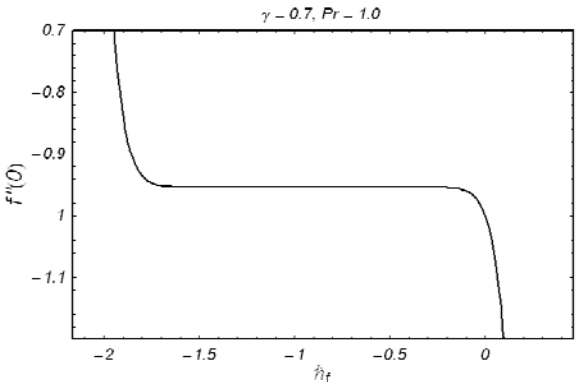


Fig. 1 *h*-curve for the function *f*

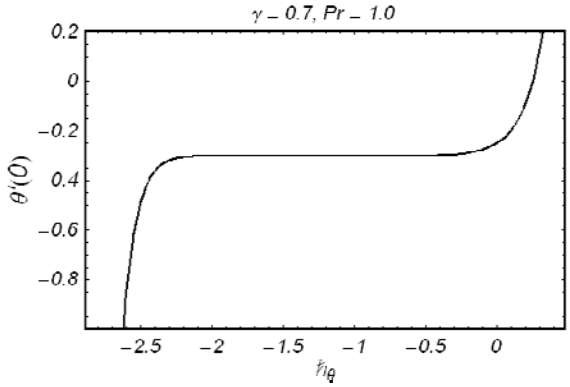


Fig. 2(a) *h*-curve for the θ for NH case

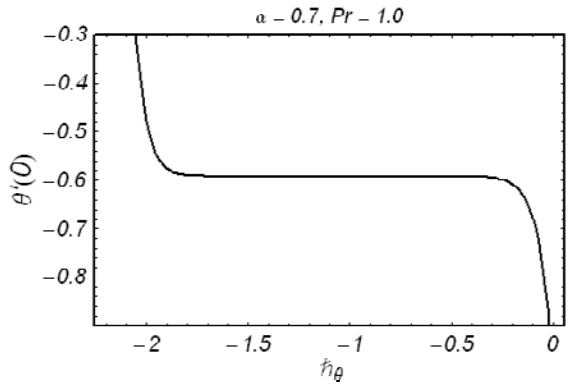


Fig. 2(b) *h*-curve for the θ for CWT case

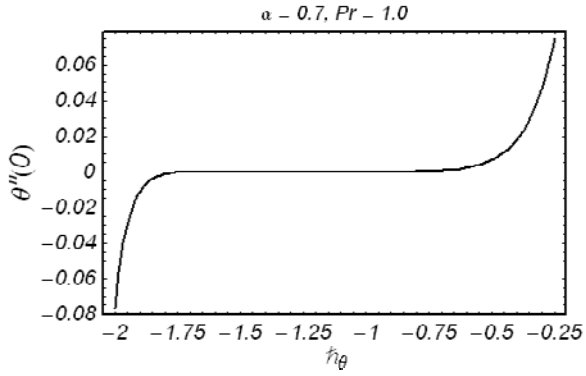


Fig. 2(c) *h*-curve for the θ for CHT case

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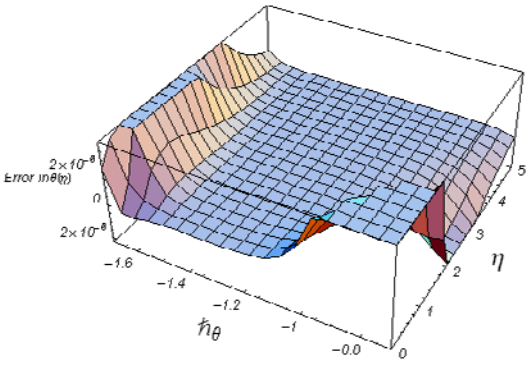


Fig. 3 Three-dimensional $h_\theta \sim \eta$ — curve for the residual error at 15th-order of approximations

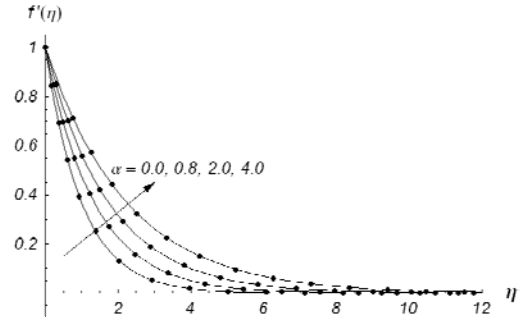


Fig. 4 Influence of second grade parameter α on f'. Solid line: Exact solution (Eq. (15)). Filled circles: 5th-order HAM solution

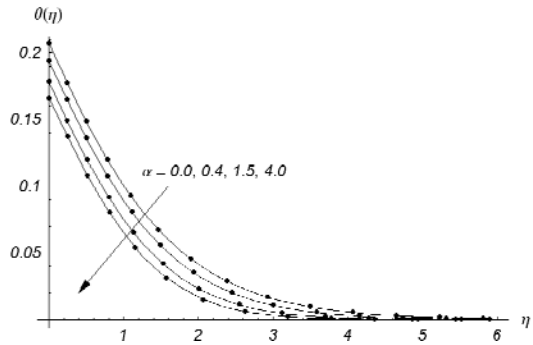


Fig. 5 Influence of second grade parameter α on θ when Pr = 1 and γ = 0.1. Solid line: Numerical solution. Filled circles: 15th-order HAM solution

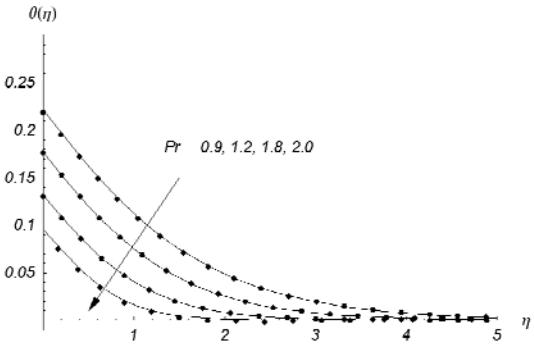


Fig. 6 Influence of Prandtl number Pr on θ when α = γ = 0.1. Solid line: Numerical solution. Filled circles: 15th-order HAM solution

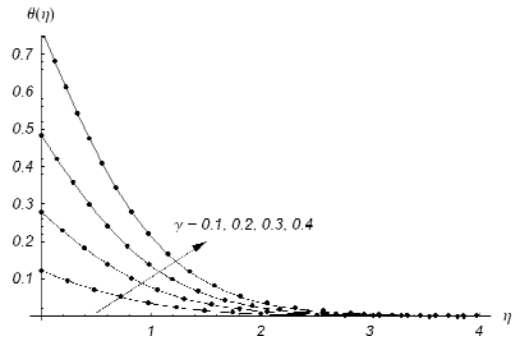


Fig. 7 Influence of conjugate parameter γ on θ when α = 0.1 and Pr =1. Solid line: Numerical solution. Filled circles: 15th-order HAM solution

Table 1 Convergence of the homotopy solutions for different order of approximation when α = 0.1, Pr = 1.0, γ = 0.1 and $\hbar_f = -0.8$, $\hbar_\theta = -1.4$

Order of approximation	$-f''(0)$	$-\theta'(0)$
5	0.953466	0.1202993
10	0.953463	0.1203431
15	0.953463	0.1203383
20	0.953463	0.1203383
25	0.953463	0.1203383
30	0.953463	0.1203383
35	0.953463	0.1203383
40	0.953463	0.1203383

obvious from Figs. 5 ~ 7 that analytic and numerical solutions are in very good agreement. It has been seen that HAM can handle the large values of the embedding parameters which include the Prandtl number Pr and the conjugate parameter γ. In fact the results get better as we increase the values of these parameters.

To gain insight towards the effects of Pr and γ on the local Nusselt number Fig. 8 is presented. This Fig. depicts that the heat transfer rate is appreciably increased when Pr and γ are increased. Table 1 is prepared to analyze the convergence rate of the present series solutions. This table indicates that 10th-order of approximations yields a convergent series solution for velocity field and 30th-order approximations is required for a convergent series solution for temperature θ(η). In Table 2, we have computed the numerical values of local Nusselt number for various values of second grade parameter α. It is revealed that the magnitude of local Nusselt number is an increasing function of second grade parameter α.

6. CONCLUDING REMARKS

The effect of Newtonian heating on the boundary layer flow and heat transfer in a second grade fluid over a stretching sheet is investigated by homotopy analysis method. The main results of this study are:

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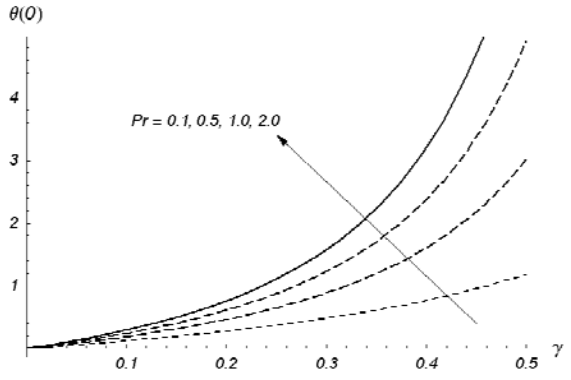


Fig. 8 Influence of conjugate parameter γ and Prandtl number Pr on θ(0).

Table 2 Values of $(\mathrm{Re}_x)^{1/2} Nu_x$ for different values of α when γ = 0.1, Pr = 1.0 and $\hbar_f = \hbar_\theta = -0.6$

α	ϒ(1 + 1 / θ(0))
0.0	0.581977
0.1	0.591684
0.2	0.600229
0.3	0.607824
0.4	0.614631
0.5	0.620774
1.0	0.644427
1.2	0.651592
1.5	0.660724

- The series solutions for velocity and temperature fields are in a very good agreement with the exact and numerical solutions respectively.
- Convergence for f is achieved at 15th-order of approximations while such convergence for θ is obtained later *i.e.* at 30th-order of approximations.
- An increase in the second grade parameter α results in an increase in the velocity and the boundary layer thickness. However the temperature and the thermal boundary layer thickness are found to decrease upon increasing α.
- As expected, the larger values of Prandtl number Pr reduces the temperature and the thermal boundary layer thickness.
- An appreciable increase in the magnitude of θ(0) is shown for large values of Pr and γ. The temperature profiles are also increased by increasing γ.

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