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Concentrated sets

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A non-enumerable set of real numbers, Z, is *concentrated* provided that there exists an enumerable set K such that if G is open and $G \supset K$ then $Z - G$ is at most enumerable. Z is said to be universally concentrated if we can take for K any enumerable set H for which $\bar{H} \supset Z$.

We show that if Z is concentrated then Z is not analytic, if Z is universally concentrated then Z is not the complement of an analytic set. We also show that if Z is concentrated and f is a Lebesgue measurable function then $f^{-1}(Z)$ is a Lebesgue measurable set. These results combine to provide a second example which disproves a conjecture of Goldman (3), that if a set X is such that for every Lebesgue measurable function f, the set $f^{-1}(X)$ is a Lebesgue measurable set, then X is a Borel set. Such a set X need not be even analytic or the complement of an analytic set. The falsity of Goldman's conjecture has been shown by Davies (6) who based his argument on properties of an analytic non-Borel set due to Kuratowski.

1. *A second category concentrated set.*

THEOREM 1. *Z is universally concentrated if and only if it is non-enumerable and meets any first category set in an at most enumerable subset.*

Suppose that Z is universally concentrated and H is a first category set. Then $H = \bigcup_{i=1} H_i$ where $\bar{H}_i$ is non-dense. Let K_i be an enumerable set with $K_i \subset \mathscr{C}(\bar{H}_i)$ and $\bar{K}_i \supset \mathscr{C}(\bar{H}_i)$. Then $\bar{K}_i$ contains the whole real line. In particular $\bar{K}_i \supset Z$ and thus any open set G such that $G \supset K_i$, is such that $Z - G$ is enumerable. Take $G = \mathscr{C}(\bar{H}_i)$ then $Z \cap H_i \subset Z \cap \bar{H}_i = Z - G$ is enumerable. Hence $Z \cap H$ is enumerable.

On the other hand, if Z meets any first category set in an at most enumerable subset, let K be an enumerable set with $\bar{K} \supset Z$. Let G be an open set such that $G \supset K$. Then $\bar{K} - G$ is a non-dense closed set. Hence $Z \cap (\bar{K} - G)$ is at most enumerable and since $\bar{K} \supset Z$ we conclude that $Z - G$ is enumerable. Thus Z is concentrated about any enumerable set K for which $\bar{K} \supset Z$. Hence Z is universally concentrated.

COROLLARY. *A universally concentrated set is of the second category.*

It cannot be of the first category since it is non-enumerable and meets every first category set in an at most enumerable set.

Remark. We assume the validity of the continuum hypothesis. With this assumption the existence of such sets has been established (see (5) (1)*a*).

THEOREM 2. (*a*) *Any concentrated set is non-analytic.*

(*b*) *Any universally concentrated set is not the complement of an analytic set.*

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(*a*) Any analytic set A contains a perfect subset Y over the subsets of which it is possible to define a non-atomic regular finite outer measure Γ (see (2)), with $\Gamma(Y) > 0$. This measure is extended to all subsets of A by defining $\Gamma_*(M) = \Gamma(M \cap Y)$ for $M \subset A$. Then Γ_* is a non-atomic regular finite outer measure defined over all the subsets of A with $0 < \Gamma_*(A) < \infty$. Next let Z be a set concentrated about K where K is the points $k_1, k_2, k_3, \ldots$, and let Θ be a non-atomic regular finite outer measure defined over all the subsets of Z with $\Theta(Z) < \infty$. We assume that $\Theta(Z) > 0$.

Then $\Theta(\{k_n\}) = 0$. Write $I_n(j)$ for the open interval

$$\left(k_n - \frac{1}{j}, k_n + \frac{1}{j}\right).$$

As $j \to \infty$, $I_n(j)$ decreases to k_n and $\Theta(I_n(j) \cap Z) \to 0$. Thus we can find an integer j_n such that

$$\Theta(I_n(j_n) \cap Z) < \Theta(Z)/2^{n+1}.$$

Now $\bigcup_{n=1}^{\infty} I_n(j_n)$ is an open set and it contains K. Then $Z - \bigcup_{n=1}^{\infty} I_n(j_n)$ is enumerable and its Θ measure is zero. Thus

$$\Theta(Z) \leqslant \sum_{n=1}^{\infty} \Theta(I_n(j_n) \cap Z) < \tfrac{1}{2}\Theta(Z),$$

which is a contradiction. Thus no such Θ exists. Thus Z is non-analytic.

(*b*) If Z is a universally concentrated set then it is a second category set. If further Z is the complement of an analytic set then

$$Z = (F \cup K_1) - K_2,$$

where F is a closed set and K_1, K_2 are first category sets. For if Z is the complement of an analytic set then we can write

$$\mathscr{C}Z = (G \cup K_3) \cap \mathscr{C}K_4,$$

where G is open, K_3, K_4 are first category sets (see e.g. G. Aumann, *Reelle Funktionen*). But then

$$\begin{aligned} Z &= (\mathscr{C}G \cap \mathscr{C}K_3) \cup K_4 \\ &= (\mathscr{C}G \cup K_4) \cap (\mathscr{C}K_3 \cup K_4) \\ &= (\mathscr{C}G \cup K_4) \cap \mathscr{C}(K_3 \cap \mathscr{C}K_4) \\ &= (F \cup K_1) - K_2, \end{aligned}$$

where $F = \mathscr{C}G$ is closed, $K_1 = K_4$ is of first category and $K_2 \subset K_3$ is also of first category. But then $Z \cap K_1$ and $Z \cap K_2$ are both at most enumerable and Z differs from a closed set by adjoining or removing enumerably many points. Thus Z is a Borel set. This is impossible as Z is not even analytic.

2. *The set* $f^{-1}(Z)$.

THEOREM 3. *If Z is a concentrated set and f is a Lebesgue measurable function then $f^{-1}(Z)$ is a Lebesgue measurable set.*

Let f be a Lebesgue measurable function defined over a measurable set M. For each positive integer k write $M_k = M \cap [-k, k]$. Given $\epsilon > 0$, there is a closed subset E_k of M_k such that (using Λ to denote linear measure)

(*a*) $\Lambda(M_k - E_k) < \epsilon/2^k$,

(*b*) f is continuous relative to E_k.

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We now regard the integer k as fixed. Suppose Z is concentrated about X and that the points of X are $x_1, x_2, x_3, \ldots$. Write

$$J_n(j) = \left(x_n - \frac{1}{j}, x_n + \frac{1}{j}\right).$$

Then, as $j \to \infty$, $f^{-1}(J_n(j))$ decreases to $f^{-1}(x_n)$ and thus

$$\Lambda(f^{-1}(J_n(j)) \cap E_k - f^{-1}(x_n) \cap E_k) \to 0 \quad \text{as} \quad j \to \infty.$$

Choose j_n so large that

$$\Lambda(f^{-1}(J_n(j_n)) \cap E_k - f^{-1}(x_n) \cap E_k) < \epsilon/2^n \tag{1}$$

for $n = 1, 2, 3, \ldots$. Thus $f^{-1}(Z) \cap [-k, k]$ consists of

(i) a subset of $M_k - E_k$;

(ii) a union of subsets of E_k whose total measure (by(1)) is less than ϵ;

(iii) the sets $f^{-1}(x_n) \cap E_k$, $n = 1, 2, 3, \ldots$;

(iv) the sets $f^{-1}(h) \cap E_k$ for all

$$h \in Z - \bigcup_{n=1}^{\infty} J_n(j_n).$$

Since f is continuous relative to the closed set E_k each of the sets in (iii) and (iv) is closed. There are enumerably many sets in (iii) by definition and in (iv) since Z is concentrated about X. The sets in (i) and (ii) have total outer linear measure less than 2ϵ. Thus $f^{-1}(Z) \cap [-k, k]$ differs from an F_σ set by a set of outer measure less than 2ϵ. Since this is true for every $\epsilon > 0$, $f^{-1}(Z) \cap [-k, k]$ is Lebesgue measurable. This is true for every positive integer k, hence $f^{-1}(Z)$ is a Lebesgue measurable set.

An example of a function f which is not Borel measurable but is such that $f(g(\ ))$ is Lebesgue measurable for every Lebesgue measurable function g.

Let K be a concentrated set. Define f by

$$f(x) = 1 \quad \text{if} \quad x \in K,$$
$$f(x) = x \quad \text{if} \quad x \notin K.$$

Since $f^{-1}(1) = \{1\} \cup K$, $f(\ )$ is not Borel measurable. But for any Borel set B

$$f^{-1}(B) = B - K \quad \text{if} \quad 1 \notin B$$
$$= B \cup K \quad \text{if} \quad 1 \in B.$$

If g is a Lebesgue measurable function

$$g^{-1}(f^{-1}(B)) = g^{-1}(B) - g^{-1}(K) \quad \text{or} \quad g^{-1}(B) \cup g^{-1}(K).$$

Since $g^{-1}(K)$ is Lebesgue measurable by the argument above, and $g^{-1}(B)$ is Lebesgue measurable since g is Lebesgue measurable, we conclude that $f(g(\ ))$ is Lebesgue measurable.

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