Proc. Camb. Phil. Soc. (1966), **62**, 197
With 1 *text-figure*
PCPS 62–32
Printed in Great Britain

Quadratic forms over $\mathbf{R}(t)$

BY JAMES T. KNIGHT
Trinity College, Cambridge

(*Received* 2 *August* 1965)

Abstract. A theory of quadratic forms is developed over $\mathbf{R}(t)$, the field of rational functions with real coefficients, instead of over an algebraic number field: some results are analogous to the classical ones, but many, in particular the non-finiteness of the class number, whose more detailed treatment will be the subject of a later paper, are not analogous.

Classical work on quadratic forms has been mainly confined to global fields; the present exposition studies them over the field $\mathbf{R}(t)$ of rational functions with real coefficients, in a single indeterminate t. The work arose from an attempt to see what lay behind the theorem (Proposition 2·1 below) of Witt ((2)).

We label with $\mathbf{C} \cup \infty$ in the following way the set of all valuations on $\mathbf{R}(t)$ which are trivial on $\mathbf{R}$: if the valuation has prime element $t-\alpha$, we label it α ($\alpha \in \mathbf{R}$); if it has prime element $(t-\beta)(t-\bar{\beta})$ we label it β ($\beta \in \mathbf{C}-\mathbf{R}$); and if it has prime element t^{-1} we label it ∞. In each case we denote the prime element by π: note the sign convention.

We denote the completion of $\mathbf{R}(t)$ with respect to the valuation $\mathfrak{p}$ by $F_{\mathfrak{p}}$ and call it a *series field*. It will turn out, as might be expected, that somehow or other F_∞ corresponds to the real, and F_α ($\alpha \in \mathbf{C}$) to the p-adic, numbers, in the classical case over $\mathbf{Q}$.

The set of valuations $\mathbf{C}$ is a Dedekind set for $\mathbf{R}(t)$; its Dedekind domain is $\mathbf{R}[t]$, a principal ideal domain. We shall denote rings of integers, both in $\mathbf{R}(t)$ and in series fields, by $\mathfrak{o}$.

For quadratic forms, we shall use the notation of (1): and shall sometimes refer to (1) to shorten proofs.

We see that the place corresponding to the valuation α is the intrinsic homomorphism:

$$a(t) \to a(\alpha), \quad (a(t) \in \mathbf{R}(t))$$

so that the residue class field of α is $\mathbf{R}$ or $\mathbf{C}$; we call F_α real or complex accordingly.

In this paper we shall deal with equivalence over series fields and over $\mathbf{R}(t)$, and integral equivalence over series fields: and shall add a few remarks about integral equivalence over $\mathbf{R}(t)$. In a later paper we shall deal with binary integral equivalence over $\mathbf{R}(t)$, and in particular evaluate the number of classes in a genus.

1. *Equivalence over series fields*

PROPOSITION 1·1. *Let F be a series field; if F is complex, $\dot{F}/\dot{F}^2$ has 2 elements, $\dot{F}^2$ and $\pi\dot{F}^2$; if F is real, $\dot{F}/\dot{F}^2$ has 4 elements, $\dot{F}^2$, $-\dot{F}^2$, $\pi\dot{F}^2$, $-\pi\dot{F}^2$.*

https://doi.org/10.1017/S030500410003975X Published online by Cambridge University Press

Proof. Suppose F is complex and suppose $\alpha \in F$ with ord α even. Then (mod $\dot{F}^2$) α may be supposed to be a unit; and by taking places there is a unit ξ such that $|\xi^2 - \alpha| < 1$, whence by Newton's iteration there is a unit η such that $\eta^2 = \alpha$, so that $\alpha \in \dot{F}^2$; if ord α is odd, $\alpha \in \pi\dot{F}^2$ clearly.

The real case is similar.

PROPOSITION 1·2. *Let V be a regular n-dimensional quadratic space over a complex series field F; if $n \geqslant 3$, V is isotropic; so that the discriminant dV completely characterizes V.*

Proof. $V \cong \langle \alpha_1 \rangle \perp \ldots \perp \langle \alpha_n \rangle$ say, with, by Proposition 1·1, $\alpha_i = 1$ or π; and since $n \geqslant 3$, $\alpha_1 = \alpha_2$ say; so that $\alpha_1 + \alpha_2 i^2 = 0$ and V is isotropic.

Thus the well-known characterization by discriminant and Hasse algebra becomes valid: and by, (1) 57:8 and 58:8, we have the result.

PROPOSITION 1·3. *Over a real series field F, there are 4 similarity classes of quaternion algebras, which under tensor product form the four-group, so that they also comprise the Hasse algebras.*

Proof. Write (α, β) for the similarity class of the quaternion algebra $(\alpha, \beta | F)$; since $(\alpha\xi^2, \beta\eta^2) = (\alpha, \beta)$, we need only consider, by Proposition 1·1, the values $1, -1, \pi, -\pi$ for α and β. We put $1, -1, P, -P$ for $(1, -1)$, $(-1, -1)$, $(\pi, -1)$, $(-\pi, -1)$ respectively, and readily compile the following table of (α, β) versus α and β:

Table 1

	1	−1	π	$-\pi$
1	1	1	1	1
−1	1	−1	P	$-P$
π	1	P	P	1
$-\pi$	1	$-P$	1	$-P$

The following table giving $A \otimes B$ versus A and B is equally easy to construct:

Table 2

	1	−1	P	$-P$
1	1	−1	P	$-P$
−1	−1	1	$-P$	P
P	P	$-P$	1	−1
$-P$	$-P$	P	−1	1

The theorem is now clear.

For regular binary spaces $V \cong \langle \epsilon_1 \rangle \perp \langle \epsilon_2 \rangle$ we now construct the following table of the discriminant dV, the Hasse algebra SV, and the elements non-trivially represented, $Q\dot{V}$ (mod $\dot{F}^2$).

For brevity we denote the sets occurring in the column $Q\dot{V}$ by 1, 1π, $1-\pi$, etc.

https://doi.org/10.1017/S030500410003975X Published online by Cambridge University Press

Table 3

ϵ_1	ϵ_2	dV	SV	$Q\dot{V}$
1	1	1	1	1
1	-1	-1	-1	F
1	π	π	P	$1, \pi$
1	$-\pi$	$-\pi$	$-P$	$1, -\pi$
-1	-1	1	-1	-1
-1	π	$-\pi$	-1	$-1, \pi$
-1	$-\pi$	π	-1	$-1, -\pi$
π	π	1	P	π
π	$-\pi$	-1	-1	F
$-\pi$	$-\pi$	1	$-P$	$-\pi$

PROPOSITION 1·4. *Over a real series field, 2 regular binary spaces are isometric iff they represent the same elements.*

Proof. Table 3 and (1), 58:4.

PROPOSITION 1·5. *Let V be a regular quadratic space over a real series field; then $Q\dot{V}$ is one of the 9 sets $1, -1, \pi, -\pi, 1\pi, 1-\pi, -1\pi, -1-\pi, F$.*

Proof. By induction, and Table 3, $\langle\pi\rangle\perp\langle\pi\rangle\perp\ldots\perp\langle\pi\rangle$ represents exactly π; similarly for $-\pi$, 1, -1; so that without loss of generality we may suppose $V \cong \langle\epsilon_1\rangle\perp\ldots\perp\langle\epsilon_r\rangle$; $\epsilon_i \neq \epsilon_j, i \neq j; r \leqslant 4$.

If $r \geqslant 3$, $1, -1$ or $\pi, -\pi$ must figure together among the ϵ_i, so that $Q\dot{V} = F$ by Table 3.

If $r = 2$, Table 3 gives the result.

If $r = 1$, the result is trivial.

Let $V \cong \langle\epsilon_1\rangle\perp\ldots\perp\langle\epsilon_n\rangle$ with each $\epsilon_i = 1, -1, \pi$ or $-\pi$; cancel all pairs $1, -1$ and $\pi, -\pi$. Then we call the dimension n, and the $\epsilon_1, \ldots, \epsilon_r$ remaining, the *characteristics* of V.

PROPOSITION 1·6. *2 spaces U and V over a real series field are isometric iff they have the same characteristics: in particular, the characteristics are well defined.*

Proof. $\Leftarrow$: trivial from Table 3.

$\Rightarrow$:

(i) Note Witt's theorem: if U and V are isometric regular spaces, and U_1 and V_1 are isometric regular subspaces of them, then U_1^* and V_1^* are isometric, where * denotes orthogonal complementation.

(ii) Now let U and V have characteristics n; $\epsilon_1, \ldots, \epsilon_r$ and n; $\eta_1, \ldots, \eta_s$. Suppose $r > s$; (i) then shows that

$$\langle\epsilon_1\rangle\perp\ldots\perp\langle\epsilon_r\rangle \quad \text{and} \quad \langle\eta_1\rangle\perp\ldots\perp\langle\eta_s\rangle\perp\langle 1\rangle\perp\langle -1\rangle\perp\ldots$$

are isometric; but the latter represents F and the former, by the construction of the characteristics and Proposition 1·5, does not; thus $r = s$.

Suppose $\epsilon_i \neq \eta_j \; \forall i, j$; then Proposition 1·5 again gives us a contradiction and using (i) the result follows by induction.

https://doi.org/10.1017/S030500410003975X Published online by Cambridge University Press

We can improve on this characterization: let $V \cong \langle \epsilon_1 \rangle \perp \ldots \perp \langle \epsilon_n \rangle$ and define:

$$\left.\begin{aligned} s &= \text{no. of } 1\text{'s} - \text{no. of } -1\text{'s} \\ p &= \text{no. of } \pi\text{'s} - \text{no. of } -\pi\text{'s} \end{aligned}\right\} \text{figuring among the } \epsilon_i.$$

PROPOSITION 1·7. *n, s, p constitute a complete set of invariants for a regular quadratic space V over a real series field: in particular s and p are well defined. There exists V with prescribed n, s, p iff $|s| + |p| \leqslant n$ and $s + p = n \bmod 2$. There are $(n+1)^2$ distinct regular spaces V of dimension n.*

Proof. Let $V \cong \langle \epsilon_1 \rangle \perp \ldots \perp \langle \epsilon_n \rangle$; since s and p are not affected by replacing $\pi, -\pi$ by $1, -1$, and neither is isometry, we may suppose that $\pi, -\pi$ does not figure among the ϵ_i. Then Proposition 1·6 tells us that this reduced set, considered as a combination with repetitions, is canonical; if $p \geqslant 0$, it contains p π's and no $-\pi$'s; if $p \leqslant 0$, it contains no π's and $-p$ $-\pi$'s; it contains $\frac{1}{2}(n+s-|p|)$ 1's and $\frac{1}{2}(n-s-|p|)$ -1's; so that n, s, p determine $\epsilon_1, \ldots, \epsilon_n$; and the converse is trivial: so that n, s, p constitute a complete set of invariants for V.

The next assertion is now trivial.

Finally the number of possible V is

$$\sum_{|p| \leqslant n} (n - |p| + 1) = (n+1)^2.$$

The following figure in (s, p) space (Fig. 1) is a square of side $n\sqrt{2}$; any V of dimension n falls on this square; and then at the vertices, on the edges, and in the interior, $Q\dot{V}$ takes the values shown:

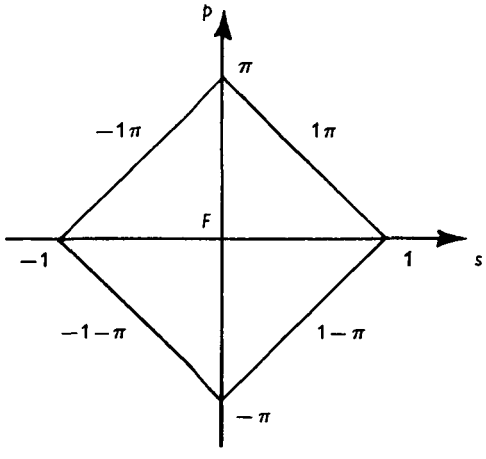


Fig. 1

We recall the definition of:

O_n the orthogonal group,
O_n^+ the subgroup of rotations,
O_n' the kernel of the spinor norm on O_n^+,
Ω_n the derived group of O_n,

for a space V of dimension n (see, e.g. (1), chapters IV, V).

https://doi.org/10.1017/S030500410003975X Published online by Cambridge University Press

PROPOSITION 1·8. *Over a series field* F, $O'_n = \Omega_n$.

Proof. For the complex case, use Proposition 1·2 and (1), p. 141.

For the real case, note first by Proposition 1·5 that any space V with $\dim V = n \geqslant 2$ contains a subspace U with $\dim U = 2$ and $Q\dot{U} = Q\dot{V}$; and (1), p. 141, shows that we may suppose $n \geqslant 2$ without loss of generality.

Let $\tau_{y_1} \ldots \tau_{y_r} \in O'_n(V)$, factored into symmetries.

Let $z_1, \ldots, z_r \in U$ with $Q(z_i) = Q(y_i)$; then as in (1), 55:3, it is enough to show $\tau_{z_1} \ldots \tau_{z_r} \in \Omega_n(V)$.

$$Q(z_1) \ldots Q(z_r) = Q(y_1) \ldots Q(y_r) \in \dot{F}^2; \quad \text{thus} \quad \tau_{z_1} \ldots \tau_{z_r}|_U \in O'_2(U);$$

and by (1), p. 141, $\tau_{z_1} \ldots \tau_{z_r}|_U \in \Omega_2(U)$. Thus

$$\tau_{z_1} \ldots \tau_{z_r}|_U = \tau_{w_1} \tau_{w_2} \tau_{w_1} \tau_{w_2} \ldots|_U, w_i \in U;$$

but τ_{z_i}, τ_{w_i} leave U^* unchanged; thus

$$\tau_{z_1} \ldots \tau_{z_r} = \tau_{w_1} \tau_{w_2} \tau_{w_1} \tau_{w_2} \ldots \in \Omega_n(V).$$

2. *Equivalence over* $\mathbf{R}(t)$

Throughout, when we speak of a quadratic form over $\mathbf{R}(t)$ *at* α we mean the form over $\mathbf{R}$ obtained by putting α for t in the coefficients: i.e. the form whose matrix is obtained by applying the place α to the matrix of the given form.

'$\widetilde{\forall}$' means 'for all except a finite number of'.

Proposition 2·1. is due, in a more general form, to Witt ((2)):

PROPOSITION 2·1. *A non-singular quadratic form in* 3 *or more variables over* $\mathbf{R}(t)$ *is isotropic iff* $\widetilde{\forall}\, \alpha \in \mathbf{R}$, *the form at* α *is isotropic over* $\mathbf{R}$

whence may be deduced Proposition 2·2:

PROPOSITION 2·2. 2 *non-singular quadratic forms over* $\mathbf{R}(t)$ *with the same dimension and discriminant are equivalent iff* $\widetilde{\forall}\, \alpha \in \mathbf{R}$, *the forms at* α *are equivalent over* $\mathbf{R}$.

This shows that dimension, discriminant, and signature of the form at α $\widetilde{\forall}\, \alpha \in \mathbf{R}$ form a complete set of invariants for a non-singular form.

Propositions 2·1 and 2·2 seem a little analogous to the Hasse–Minkowski theorem in the classical case: but there is a more exact analogue:

PROPOSITION 2·3. *Let* V *be a regular quadratic space over* $\mathbf{R}(t)$; *then*

$$V \text{ is isotropic iff } V_{\mathfrak{p}} \text{ is isotropic } \quad \forall\, \mathfrak{p} \in \mathbf{C} \cup \infty$$
$$\text{iff } V_{\mathfrak{p}} \text{ is isotropic } \quad \forall\, \mathfrak{p} \in \mathbf{C}.$$

Proof. It is only necessary to show that if $V_{\mathfrak{p}}$ is isotropic $\forall\, \mathfrak{p} \in \mathbf{C}$, then V is isotropic. Let $n = \dim V$.

If $n = 1$, $V_{\mathfrak{p}}$ is never isotropic.

If $n = 2$, by (1), 42:9 $dV_{\mathfrak{p}} = -1$ $\forall\, \mathfrak{p} \in \mathbf{C}$; and thus $dV = -1$, so that V is isotropic.

If $n \geqslant 3$, let $V \cong \langle \epsilon_1 \rangle \perp \ldots \perp \langle \epsilon_n \rangle$, $\epsilon_i \in \mathbf{R}(t)$.

$\widetilde{\forall}\, \alpha \in \mathbf{R}$

$$V_\alpha \overset{*}{\cong} \langle \epsilon_1(\alpha) \rangle \perp \ldots \perp \langle \epsilon_n(\alpha) \rangle$$
$$\overset{*}{\cong} \langle \pm 1 \rangle \perp \ldots \perp \langle \pm 1 \rangle$$

https://doi.org/10.1017/S030500410003975X Published online by Cambridge University Press

with each sign occurring, since V_α is isotropic; thus the form at α is isotropic, and 2·1 shows that V is isotropic.

We obtain as usual (e.g. (1), 66:3, 66:4):

PROPOSITION 2·4. *Let U and V be regular quadratic spaces over $\mathbf{R}(t)$; then*

$$\begin{aligned} V \text{ represents } U \text{ iff } &\quad \forall\, \mathfrak{p} \in \mathbf{C}, V_\mathfrak{p} \text{ represents } U_\mathfrak{p} \\ \text{iff } &\quad \forall\, \mathfrak{p} \in \mathbf{C} \cup \infty, V_\mathfrak{p} \text{ represents } U_\mathfrak{p}. \end{aligned}$$

PROPOSITION 2·5. *Let U and V be as in Proposition 2·4; then*

$$\begin{aligned} U \text{ is isometric to } V \text{ iff } &\quad \forall\, \mathfrak{p} \in \mathbf{C}, U_\mathfrak{p} \text{ is isometric to } V_\mathfrak{p} \\ \text{iff } &\quad \forall\, \mathfrak{p} \in \mathbf{C} \cup \infty, U_\mathfrak{p} \text{ is isometric to } V_\mathfrak{p}. \end{aligned}$$

It now follows that we can specify a regular space over $\mathbf{R}(t)$ by dV_α $\forall\, \alpha \in \mathbf{C}-\mathbf{R}$ and by integers s_α, p_α corresponding to V_α $\forall\, \alpha \in \mathbf{R}$, which by Proposition 1·7 satisfy $s_\alpha + p_\alpha \equiv \bmod 2$ and $|s_\alpha| + |p_\alpha| \leqslant n$.

PROPOSITION 2·6. *There exists a regular space V of dimension n over $\mathbf{R}(t)$ with specified dV_α, s_α, p_α as above iff*

(i) $dV_\alpha = 1\ \widetilde{\forall}\, \alpha \in \mathbf{C}-\mathbf{R}$,

(ii) $p_\alpha = 0\ \widetilde{\forall}\, \alpha \in \mathbf{R}$,

(iii) $\forall\, \alpha > \beta > \gamma \ni p_\beta \neq 0$ *and* $p_\xi = 0$ *for* $\xi \in [\gamma, \alpha] - \beta$,

$$s_\alpha = s_\beta + p_\beta,$$

$$s_\gamma = s_\beta - p_\beta.$$

Proof. For $\Rightarrow$, let $V \cong \langle \epsilon_1 \rangle \perp \ldots \perp \langle \epsilon_n \rangle$ ($\epsilon_i \in \mathbf{R}(t)$); then (i), (ii), (iii) are obvious.

For $\Leftarrow$ we construct suitable ϵ_i thus: use the value of s_α at large α to fix the sign of their leading coefficients, and move down the real line until the first β such that $p_\beta \neq 0$; if $p_\beta > 0$, insert a factor $t-\beta$ at p_β of the ϵ_i with positive leading coefficients; if $p_\beta < 0$, similarly at the ϵ_i with negative leading coefficients: (iii) allows us to do this. For subsequent β repeat this, using the sign of ϵ_i at β rather than the sign of its leading coefficient; in each case (iii) shows that this is possible; and (ii) shows that we need do this only a finite number of times. Finally, for each of the finite number (by (i)) of α with $dV_\alpha = \pi_\alpha$, insert π_α at ϵ_1.

We now look for an analogue of the Hilbert Reciprocity Law: we define

$$|\pm 1| = 1, |\pm P| = -1; \quad \text{then put} \quad \left(\frac{\alpha, \beta}{\mathfrak{p}}\right) = \left|\left(\frac{a, \beta}{F_\mathfrak{p}}\right)\right|$$

$\forall\, \mathfrak{p} \in \mathbf{C} \cup \infty$ and $\alpha, \beta \in \mathbf{R}(t)$.

PROPOSITION 2·7. $\prod_{\mathfrak{p} \in \mathbf{C} \cup \infty} \left(\frac{a, \beta}{\mathfrak{p}}\right) = 1\ \forall\, \alpha, \beta \in \mathbf{R}^\cdot(t)$.

Proof. $\widetilde{\forall}\, \mathfrak{p} \in \mathbf{C} \cup \infty\ (\alpha, \beta)/\mathfrak{p} = 1$, so that the product is well defined.

Since

$$\left(\frac{\alpha, \beta\delta}{\mathfrak{p}}\right) = \left(\frac{\alpha, \beta}{\mathfrak{p}}\right)\left(\frac{\alpha, \delta}{\mathfrak{p}}\right),$$

it is enough to suppose α, β are irreducible polynomials or constants.

https://doi.org/10.1017/S030500410003975X Published online by Cambridge University Press

If $\mathfrak{p} \in \mathbf{C}-\mathbf{R}$, $(\alpha, \beta/\mathfrak{p}) = 1$; and if $\partial\alpha, \partial\beta$ are even $(\alpha, \beta/\infty) = 1$. Hence, if neither α nor β is linear we have the result.

If they are both linear, we tabulate $(\alpha, \beta/\mathfrak{p})$, for all values of $\mathfrak{p}$ for which it might not be 1:

Table 4

α	β	$(\alpha, \beta/a)$	$(\alpha, \beta/b)$	$(\alpha, \beta/\infty)$	Product
$t-a$	$t-a$	-1	—	-1	1
$t-a$	$a-t$	1	—	1	1
$a-t$	$a-t$	-1	—	-1	1
$t-a$	$t-b$	1	-1	-1	1
$t-a$	$b-t$	-1	-1	1	1
$a-t$	$t-b$	1	1	1	1
$a-t$	$b-t$	-1	-1	1	1
	$(a > b)$				

Suppose finally α linear and β quadratic or constant: we make a table:

Table 5

α	β	$(\alpha, \beta/a)$	$(\alpha, \beta/\infty)$	Product
$t-a$	+	1	1	1
$a-t$	+	1	1	1
$t-a$	−	-1	-1	1
$a-t$	−	-1	-1	1

and deduce the result.

This is a much weaker result than the classical analogue, and is not worth deducing the trivial law of quadratic reciprocity from.

PROPOSITION 2·8. *Let V be a regular quadratic space over $\mathbf{R}(t)$ of dimension $\geqslant 2$; then V has a regular binary subspace U with $Q\dot{U} = Q\dot{V}$.*

Proof. The idea of the proof is: for each $\alpha \in \mathbf{R}$, we find where V_α lies on Fig. 1, and place U_α in a similar position: more precisely, we construct U_α for each $\alpha \in \mathbf{R}$ as in the following table:

Table 6

$p_\alpha(V)$	$s_\alpha(V)$	$Q\dot{V}_\alpha$	$p_\alpha(U)$	$s_\alpha(U)$	$Q\dot{U}_\alpha$
0	n	1	0	2	1
0	$-n$	-1	0	-2	-1
0	$> -n, < n$	F_α	0	0	F_α
$> 0, < n$	$n-p_\alpha$	1π	1	1	1π
$> 0, < n$	$p_\alpha-n$	-1π	1	-1	-1π
$> 0, < n$	$n-p_\alpha, > p_\alpha-n$	F_α	0	0	F_α
$< 0, > -n$	$n+p_\alpha$	$1-\pi$	-1	1	$1-\pi$
$< 0, > -n$	$-n-p_\alpha$	$-1-\pi$	-1	-1	$-1-\pi$
$< 0, > -n$	$< n+p_\alpha,$ $> -n-p_\alpha$	F_α	0	0	F_α
n	0	π	2	0	π
n	0	$-\pi$	-2	0	$-\pi$

https://doi.org/10.1017/S030500410003975X Published online by Cambridge University Press

If $\alpha \in \mathbf{C}-\mathbf{R}$, we put $dU_\alpha = dV_\alpha$. The reader may check, using Proposition 2·6 twice, that such a U exists.

Proposition 2·4 and the third and sixth columns, and the fact that over a complex series field a regular binary space is universal, shows that $Q\dot{U} = Q\dot{V}$.

Clearly by section 1, for all $\alpha \in \mathbf{C}$ there is a representation $\sigma_\alpha : U_\alpha \to V_\alpha$, whence by Proposition 2·4 there is a representation $\sigma : U \to V$. Here $\dim \sigma U \leqslant 2$; and since $Q(\sigma\dot{U}) = Q\dot{U}$, we cannot have $\dim \sigma U = 0$ or 1, for then $Q\dot{U}$ would have at most 2 elements (mod $\mathbf{R}^\cdot(t)^2$). Thus $\dim \sigma U = 2$ and σU is the required subspace.

PROPOSITION 2·9. *Over* $\mathbf{R}(t)$, $O'_n = \Omega_n$.

Proof. As in Proposition 1·8, using Proposition 2·8.

In the classical case, Proposition 2·9, and consequently Proposition 2·8, are false: see e.g. (1), p. 295.

3. *Integral equivalence over series fields*

We recall the theory of Jordan splittings of lattices on a quadratic space over a complete discrete-valued field F in which 2 is a unit: any lattice has an orthogonal basis and the components of this basis may be grouped in a well-defined way to give an orthogonal splitting $L = L_1 \perp L_2 \perp \ldots \perp L_l$ in which each L_i is modular and $\mathfrak{s}L_i \supset \mathfrak{s}L_{i+1}\ \forall\, i$; this is called a *Jordan splitting* of L; if $L = K_1 \perp \ldots \perp K_s$ is such another, then

$$s = t, \quad \dim L_i = \dim K_i, \quad \mathfrak{s}L_i = \mathfrak{s}K_i \quad \forall\, i;$$

and if the space is scaled so that K_i and L_i are unimodular, then there is a linear injection $\phi : FL_i \to FK_i$ with $\phi L_i \subseteq K_i$ and $\forall\, x, y \in L_i$, $B(x,y) \equiv B(\phi x, \phi y) \bmod \pi$.

For the proofs of these assertions, see (1), section 91C.

Since our transformations will now all be unimodular, the discriminant, etc., will be defined modulo $\mathfrak{u}^2$, where $\mathfrak{u}$ is the group of units: and any lattice can be put in the form

$$\langle \alpha_1 \rangle \perp \ldots \perp \langle \alpha_n \rangle$$

with

$$\alpha_i = \pm \pi^{r_i} \quad \text{in the real case and}$$

$$\alpha_i = \pi^{r_i} \quad \text{in the complex case.}$$

PROPOSITION 3·1. *Let* $K \cong \langle \alpha_1 \rangle \perp \ldots \perp \langle \alpha_n \rangle$ *and* $L \cong \langle \beta_1 \rangle \perp \ldots \perp \langle \beta_n \rangle$ *be lattices on quadratic spaces over a series field, with* α_i, β_i *reduced as above; then* K *and* L *are isometric iff the* β_i *are a permutation of the* α_i.

Proof. $\Leftarrow$ is obvious.

For $\Rightarrow$ without loss of generality we may suppose $K = L$; then we may group the basis elements to give Jordan splittings:

$$L = L_1 \perp \ldots \perp L_l = K_1 \perp \ldots \perp K_t.$$

The introductory remarks then show that sign apart the β_i are a permutation of the α_i, so that in the complex case, we have the result.

For the real case, we suppose by scaling that K_i and L_i are unimodular; let

$$L_i = \mathfrak{o}x_1 + \ldots + \mathfrak{o}x_r \quad \text{with} \quad Q(x_i) = \pm 1;$$

then the determinant $d(x_1, \ldots, x_r) \in \mathfrak{n}$.

https://doi.org/10.1017/S030500410003975X Published online by Cambridge University Press

Let ϕ be the linear injection of the introductory remarks; then

$$\phi L_i = \mathfrak{o}\phi x_1 + \ldots + \mathfrak{o}\phi x_r;$$

but $d(\phi x_1, \ldots, \phi x_r) \equiv d(x_1, \ldots, x_r) \bmod \pi$ so that $\mathfrak{v}\phi L_i = \mathfrak{v}K_i$, and thus by (1), 82:11*a*, $\phi L_i = K_i$.

Let $^-: \mathfrak{o} \to \mathbf{R}$ be the place of the valuation on the series field; and for $x \in L_i$, $x = \Sigma\lambda_j x_j$, $\lambda_j \in \mathfrak{o}$, we define $\bar{x} = \bar{\lambda}_j \bar{x}_j$ with $\bar{x}_j$ indeterminate; similarly in K_i we define $\bar{y} = \Sigma\bar{\mu}_j \overline{\phi x_j}$ with $\overline{\phi x_j}$ indeterminate.

Let us define $\bar{\phi}\bar{x}_j = \overline{\phi x_j}$ and extend this definition by linearity; then $\bar{\phi}\bar{x} = \overline{\phi x}$ clearly.

Define (well) $\bar{B}(\bar{x}, \bar{y}) = \overline{B(x, y)}$; this makes $\bar{\phi}$ an isometry between $\bar{L}_i$ and $\bar{K}_i$, quadratic spaces over $\mathbf{R}$.

Now let $y_1, \ldots, y_r$ be another free basis for K_i, with $Q(y_i) = \pm 1$, and define $\tilde{y} = \Sigma\bar{\mu}_j\tilde{y}_j$ similarly, and $\tilde{B}(\tilde{x}, \tilde{y}) = \overline{B(x, y)}$; then if we put $\sigma: \bar{y} \to \tilde{y}$, it may readily be checked that σ is a well-defined isometry between $\bar{K}_i$ and $\tilde{K}_i$.

Thus $\tilde{L}_i$ and $\tilde{K}_i$ have the same signature; and taking the orthogonal basis in each case, we have the result.

4. *Integral equivalence over* $\mathbf{R}(t)$

We can define, as in (1), section 102, using $\mathbf{C}$ as our Dedekind set, cls L, spn L, gen L, etc; and we can readily show, using 102·9, that if V is isotropic, spn L = gen L. We may also show, using the methods of (3), that 2 lattices in the same genus are semi-equivalent; but when we try to show that the number of classes in a genus is finite, we have trouble, essentially because the classical theory is concerned with finite-dimensional vector spaces over Galois fields, but here we have finite-dimensional vector spaces over $\mathbf{R}$. In fact it will be shown in a subsequent paper ((4)) that the number of classes in a genus is seldom finite.

I should like to thank my supervisor, Dr J. W. S. Cassels, for suggesting the problem and reading the manuscript.

REFERENCES

(1) O'Meara, O. T. *Introduction to quadratic forms*. Springer-Verlag, 1963.
(2) Witt, E. *J. Reine Angew. Math.* **176** (1937), 31–44, Satz 22.
(3) Jones, B. W. *The arithmetic theory of quadratic forms*. Carus Monograph No. 10, 1950.
(4) To appear in *Proc. Cambridge Philos. Soc.*

https://doi.org/10.1017/S030500410003975X Published online by Cambridge University Press