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Infinite regular electrical networks

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An infinite regular two-dimensional network is composed of unit resistances joining adjacent nodes. What is the resistance of the whole network measured between two different nodes? Three cases are considered, in which the meshes of the network are square, hexagonal or triangular. Some extensions of the problem are also treated. Asymptotic approximations are given for large distances.

1 Introduction

My interest in this subject was aroused by a question in the Archimedeans' Problems Drive 1996, set by McCaughan & McCaughan [1]. This showed part of an infinite network of hexagonal meshes, such as shown in Figure 1, in which each link between adjacent nodes is an electrical conductor of unit resistance, and asked for the resistance of the whole network between a pair of adjacent nodes, such as O and X. This is an easy problem, but it suggests the more difficult one of finding the resistance of the network between non-adjacent nodes, such as A and Y or A and D.

Analogous problems for a square mesh were examined by Trier [2], following van der Pol & Bremmer [3]. The nodes are then at the integer points (m, n) in Figure 2, and the method is to consider the flow of unit current from the origin O to infinity. The fall of potential from O to an adjacent node, such as A, is then $\frac{1}{4}$. Similarly, if unit current flows from infinity to A, the potential drop from O to A is again $\frac{1}{4}$. Thus unit current from O to A through the network causes a total potential fall of $\frac{1}{2}$, which is therefore the resistance of the network.

The idea of considering current flow from the origin to infinity still applies to the problem of finding the resistance between non-adjacent nodes, but it is necessary to solve for the potential distribution in the whole network. The solution for the square mesh is given by Trier, by van der Pol and Bremmer, and also by Doyle & Snell [4], whose interest is in the relation between random walks and current flow in networks. Further results are given in the next section, where the problem is generalized to the case in which the resistances are different in the m and n directions.

2 Rectangular mesh

In Figure 3 the resistances are ρ in the m direction and σ in the n direction. When unit current enters at O, the potential $v(m, n)$ at the node (m, n) satisfies the current balance

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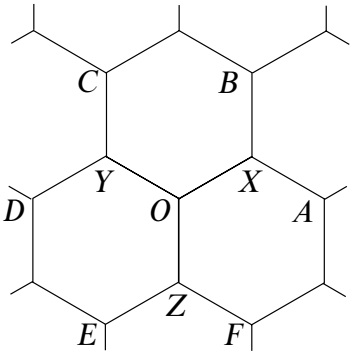


FIGURE 1. Hexagonal mesh.

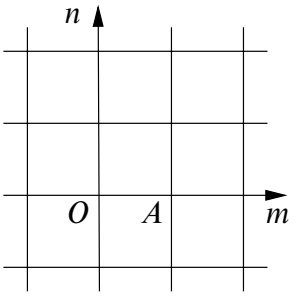


FIGURE 2. Square mesh.

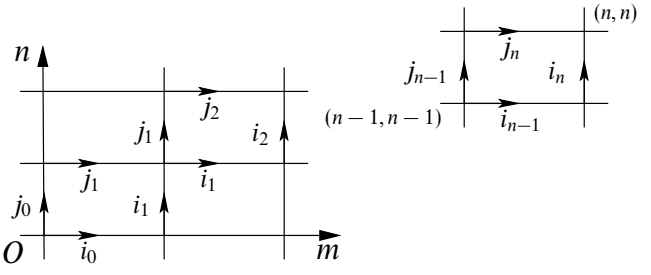


FIGURE 3. Rectangular mesh.

equation

$$\rho^{-1}\{v(m-1,n)-2v(m,n)+v(m+1,n)\}+\sigma^{-1}\{v(m,n-1)-2v(m,n)+v(m,n+1)\}=\begin{cases}-1, & m=n=0\\ 0, & \text{otherwise.}\end{cases} \tag{2.1}$$

The currents in the links are required to vanish as m and n tend to infinity. To derive the solution, it is convenient to introduce at each node a large leakage resistance ϵ^{-1} to earth at zero potential. This adds $\epsilon v(m,n)$ to the right side of (2.1) and, in the case $\rho=\sigma$, makes

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(2.1) equivalent to the equation studied by van der Pol and Bremmer. The Fourier series

$$V(\theta,\phi;\epsilon)=\sum_{m,n=-\infty}^{\infty} v(m,n;\epsilon)\exp\{i(m\theta+n\phi)\} \tag{2.2}$$

is now convergent, and satisfies

$$\{\rho^{-1}(e^{i\theta}-2+e^{-i\theta})+\sigma^{-1}(e^{i\phi}-2+e^{-i\phi})-\epsilon\}V=1. \tag{2.3}$$

Hence

$$V(\theta,\phi;\epsilon)=\{2\rho^{-1}(1-\cos\theta)+2\sigma^{-1}(1-\cos\phi)+\epsilon\}^{-1} \tag{2.4}$$

and

$$v(m,n;\epsilon)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\exp\{i(m\theta+n\phi)\}}{2\rho^{-1}(1-\cos\theta)+2\sigma^{-1}(1-\cos\phi)+\epsilon}\,d\theta\,d\phi. \tag{2.5}$$

When $\epsilon\to 0$, $v(m,n;\epsilon)$ becomes unbounded, but the difference $v(0,0;\epsilon)-v(m,n;\epsilon)$ tends to the limit

$$v(0,0)-v(m,n)=\frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi}\frac{1-\cos m\theta\cos n\phi}{\rho^{-1}(1-\cos\theta)+\sigma^{-1}(1-\cos\phi)}\,d\theta\,d\phi, \tag{2.6}$$

which is the required solution of the current balance equation (2.1).

An alternative form is obtained by writing

$$\{\rho^{-1}(1-\cos\theta)+\sigma^{-1}(1-\cos\phi)\}^{-1}=\int_0^{\infty}\exp\{-\left(\rho^{-1}(1-\cos\theta)+\sigma^{-1}(1-\cos\phi)\right)t\}\,dt, \tag{2.7}$$

when the θ and ϕ integrations give

$$v(0,0)-v(m,n)=\frac{1}{2}\int_0^{\infty}\exp\left(-\frac{t}{\rho}-\frac{t}{\sigma}\right)\left\{I_0\left(\frac{t}{\rho}\right)I_0\left(\frac{t}{\sigma}\right)-I_m\left(\frac{t}{\rho}\right)I_n\left(\frac{t}{\sigma}\right)\right\}dt \tag{2.8}$$

where $I_k(s)$ is the modified Bessel function. The relation

$$I_{k-1}(s)+I_{k+1}(s)=2I_k'(s) \tag{2.9}$$

shows that (2.8) satisfies the current balance equation, and the recurrence

$$I_{k-1}(s)-I_{k+1}(s)=\frac{2k}{s}I_k(s) \tag{2.10}$$

leads to

$$n\sigma\{v(m-1,n)-v(m+1,n)\}=m\rho\{v(m,n-1)-v(m,n+1)\}, \tag{2.11}$$

which may be described as a radial current flow equation.

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Equation (2.11) enables one of the adjacent nodes to be eliminated from the current balance at (m,n). When $v(m-2,n)$ is thus removed from the current balance at $(m-1,n)$, we have

$$\left(\frac{2}{\rho}+\frac{2}{\sigma}\right)nv(m-1,n)=\frac{2n}{\rho}v(m,n)+\frac{m+n-1}{\sigma}v(m-1,n-1)+\frac{n-m+1}{\sigma}v(m-1,n+1), \tag{2.12}$$

provided that $(m,n)\neq(1,0)$. Similar equations may be derived for $v(m+1,n), v(m,n-1)$ and $v(m,n+1)$. When these are inserted in equation (2.1) for $v(m,n)$, the relation between the node (m,n) and its diagonal neighbours is found to be

$$\begin{aligned}8mnv(m,n)=&(m+n)\{(m+n-1)v(m-1,n-1)+(m+n+1)v(m+1,n+1)\}\\&-(m-n)\{(m-n-1)v(m-1,n+1)+(m-n+1)v(m+1,n-1)\},\end{aligned} \tag{2.13}$$

independently of ρ and σ, provided $m^2+n^2\neq 1$.

In particular, equation (2.13) shows that

$$4nv(n,n)=(2n-1)v(n-1,n-1)+(2n+1)v(n+1,n+1), \tag{2.14}$$

from which

$$v(n-1,n-1)-v(n,n)=\frac{K}{2n-1}, \tag{2.15}$$

where K depends on ρ and σ, and hence

$$v(0,0)-v(n,n)=K\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right) \tag{2.16}$$

for $n=1,2,3,\ldots$. K may be evaluated from the integral (2.6), but it can also be inferred from asymptotic considerations, as follows.

If m and n are large, equation (2.1) may be approximated by Laplace's equation by writing

$$x=r\cos\theta=m\rho^{1/2},\quad y=r\sin\theta=n\sigma^{1/2}. \tag{2.17}$$

Then as $r\to\infty$, equation (2.1) becomes

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{2}{4!}\left(\rho\frac{\partial^4 v}{\partial x^4}+\sigma\frac{\partial^4 v}{\partial y^4}\right)+\frac{2}{6!}\left(\rho^2\frac{\partial^6 v}{\partial x^6}+\sigma^2\frac{\partial^6 v}{\partial y^6}\right)+\cdots=0, \tag{2.18}$$

in which higher derivatives are required to vanish more rapidly than lower ones. In order to obtain unit current outflow in terms of m and n, the leading term in the solution of (2.18) as $r\to\infty$ must be

$$v\sim-\frac{(\rho\sigma)^{1/2}}{2\pi}\log r. \tag{2.19}$$

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When $m = n$, $r = n(\rho + \sigma)^{1/2}$, and from (2.16) as $r \to \infty$

$$\begin{aligned} v(0,0) - v(n,n) &= \frac{1}{2}K\{\log(4n) + \gamma + O(n^{-2})\} \\ &= \frac{1}{2}K\left\{\log\left(\frac{4r}{(\rho+\sigma)^{1/2}}\right) + \gamma + O(r^{-2})\right\}, \end{aligned} \tag{2.20}$$

where γ is Euler's constant. Thus

$$K = (\rho\sigma)^{1/2}/\pi. \tag{2.21}$$

Equations (2.1) and (2.11) combine to show that there is no current flow across the diagonal $m = n$. Consequently (see Figure 3),

$$\left.\begin{aligned} \sigma^{-1}\{v(n,n-1) - v(n,n)\} = i_n = \rho^{-1}\{v(n,n) - v(n+1,n)\} \\ \rho^{-1}\{v(n-1,n) - v(n,n)\} = j_n = \sigma^{-1}\{v(n,n) - v(n,n+1)\} \end{aligned}\right\}, \tag{2.22}$$

so that

$$\rho i_{n-1} + \sigma i_n = \sigma j_{n-1} + \rho j_n = v(n-1,n-1) - v(n,n) = \frac{K}{2n-1}. \tag{2.23}$$

If $\rho \geqslant \sigma$ this shows that

$$i_0 = \frac{K}{\rho}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\left(-\frac{\sigma}{\rho}\right)^{n-1} = \frac{1}{\pi}\tan^{-1}\left(\left(\frac{\sigma}{\rho}\right)^{1/2}\right), \tag{2.24}$$

and hence

$$j_0 = \frac{1}{2} - i_0 = \frac{1}{\pi}\tan^{-1}\left(\left(\frac{\rho}{\sigma}\right)^{1/2}\right). \tag{2.25}$$

If $\rho \leqslant \sigma$, j_0 comes from (2.23) and i_0 follows. As with K, both i_0 and j_0 can also be found from the integral (2.6). Equations (2.1) and (2.11) enable values of the potential fall to be built up from those with $m = n$ and the cases $(m,n) = (1,0)$ and $(0,1)$.

A check may be obtained from the generating function for the potential drop when $n = 0$. This can be evaluated from (2.6) as

$$\begin{aligned} \sum_{m=0}^{\infty}\{v(0,0) - v(m,0)\}x^m &= \frac{x}{\pi}\left(\frac{\rho\sigma}{(1-x)^2\sigma - 4\rho x}\right)^{1/2}\tan^{-1}\left(\frac{\{(1-x)^2\sigma - 4\rho x\}^{1/2}}{\rho(1+x)}\right) \\ &= \frac{\rho x}{\pi}f(x)\left\{\tan^{-1}\left(\left(\frac{\sigma}{\rho}\right)^{1/2}\right) - 2\left(\frac{\sigma}{\rho}\right)^{1/2}\int_0^x \frac{f(y)}{1-y}\,dy\right\}, \end{aligned} \tag{2.26}$$

where

$$f(x) = \{(1-x)^2 - 4(\rho/\sigma)x\}^{-1/2} = \sum_{m=1}^{\infty} P_m(1 + 2\rho/\sigma)x^m, \tag{2.27}$$

and $P_m(t)$ is the Legendre polynomial.

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Equation (2.18) can also be used to find the asymptotic expansion of the potential. It is convenient to work with

$$z = x + iy, \quad \bar{z} = x - iy \tag{2.28}$$

as independent variables, in terms of which the appropriate solution has the form

$$-\frac{4\pi}{(\rho\sigma)^{1/2}}v \sim \log(z\bar{z}) + a + b(z^{-2} + \bar{z}^{-2}) + (c_1 z\bar{z} + c_0)(z^{-4} + \bar{z}^{-4}) + (d_1 z\bar{z} + d_0)(z^{-6} + \bar{z}^{-6}) + (e_2 z^2\bar{z}^2 + e_1 z\bar{z} + e_0)(z^{-8} + \bar{z}^{-8}) + \cdots. \tag{2.29}$$

The current balance equation leaves $a, b, c_0, d_0, e_0, \ldots$ undetermined, but gives the other coefficients in terms of them. Apart from a, all the coefficients may be obtained from the radial current flow equation, from which

$$b = \frac{1}{12}(\rho - \sigma), \quad c_1 = -\frac{1}{24}(\rho + \sigma), \quad c_0 = -\frac{3}{160}(\rho + \sigma)^2 - \frac{7}{240}(\rho - \sigma)^2,$$
$$d_1 = \frac{1}{15}(\rho^2 - \sigma^2), \quad e_2 = -\frac{5}{192}(\rho + \sigma)^2. \tag{2.30}$$

A check on these results, and the value of a, come from the asymptotic expansion of (2.16). Thus

$$v(0,0) - v(m,n) = \frac{(\rho\sigma)^{1/2}}{2\pi}\left\{\log\left(\frac{4r}{(\rho+\sigma)^{1/2}}\right) + \gamma + \frac{b\cos 2\theta + c_1 \cos 4\theta}{r^2} + \frac{e_0 \cos 4\theta + d_1 \cos 6\theta + e_2 \cos 8\theta}{r^4} + O(r^{-6})\right\}. \tag{2.31}$$

The method used to derive (2.16) is also applicable to the leaky mesh introduced earlier. The radial flow equation is unaltered from (2.11), but in the relation (2.13) between the node (m,n) and its diagonal neighbours, the left side must be replaced by

$$8wmnv(m,n;\epsilon), \tag{2.32}$$

where

$$w = 1 + \frac{1}{2}(\rho + \sigma)\epsilon + \frac{1}{8}\rho\sigma\epsilon^2. \tag{2.33}$$

The recurrence for the diagonal $m = n$ is therefore

$$4nwv(n,n;\epsilon) = (2n-1)v(n-1,n-1;\epsilon) + (2n+1)v(n+1,n+1;\epsilon). \tag{2.34}$$

The general solution of this recurrence relation is

$$v(n,n;\epsilon) = AP_{n-\frac{1}{2}}(w) + BQ_{n-\frac{1}{2}}(w), \tag{2.35}$$

where P, Q denote Legendre functions. Since $w > 1$, $P_{n-\frac{1}{2}}(w)$ is unbounded as $n \to \infty$,

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whereas $Q_{n-\frac{1}{2}}(w) \to 0$. Hence

$$v(n,n;\epsilon) = BQ_{n-\frac{1}{2}}(w). \tag{2.36}$$

The coefficient B is found from

$$\begin{aligned} v(0,0;\epsilon) &= \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \frac{d\theta\, d\phi}{2\rho^{-1}(1-\cos\theta)+2\sigma^{-1}(1-\cos\phi)+\epsilon} \\ &= \frac{1}{\pi}\int_0^\pi \frac{d\theta}{\{2\rho^{-1}(1-\cos\theta)+\epsilon\}^{1/2}\{2\rho^{-1}(1-\cos\theta)+4\sigma^{-1}+\epsilon\}^{1/2}}. \end{aligned} \tag{2.37}$$

The substitution

$$\tan\frac{1}{2}\theta = \left(\frac{\epsilon}{4\rho^{-1}+\epsilon}\right)^{1/2}\tan\psi \tag{2.38}$$

leads to

$$v(0,0;\epsilon) = \frac{(\rho\sigma)^{1/2}}{\pi}\int_0^{\frac{1}{2}\pi} \frac{d\psi}{\{(2+\frac{1}{2}\epsilon\rho)(2+\frac{1}{2}\epsilon\sigma)\cos^2\psi+\epsilon(\rho+\sigma+\frac{1}{4}\epsilon\rho\sigma)\sin^2\psi\}^{1/2}}. \tag{2.39}$$

Also

$$Q_{-\frac{1}{2}}(w) = \sqrt{2}\int_0^{\frac{1}{2}\pi} \frac{d\psi}{\{(w+1)\cos^2\psi+(w-1)\sin^2\psi\}^{1/2}}, \tag{2.40}$$

so that

$$B = (\rho\sigma)^{1/2}/\pi. \tag{2.41}$$

The method used to evaluate $v(0,0;\epsilon)$ shows that $v(1,0;\epsilon)$ can be expressed in terms of an elliptic integral of the third kind as

$$\begin{aligned} v(1,0;\epsilon) &= (1+\frac{1}{2}\epsilon\rho)v(0,0;\epsilon) \\ &\quad -\frac{\epsilon\rho}{\pi}\left(\frac{\rho\sigma}{(4+\epsilon\rho)(4+\epsilon\rho)}\right)^{1/2}\int_0^{\frac{1}{2}\pi} \frac{d\psi}{\left(1-\frac{\sin^2\psi}{1+\frac{1}{4}\epsilon\rho}\right)\left(1-\frac{\sin^2\psi}{(1+\frac{1}{4}\epsilon\rho)(1+\frac{1}{4}\epsilon\sigma)}\right)^{1/2}}. \end{aligned} \tag{2.42}$$

In the (non-leaky) square mesh case, we can also find the potential fall along the diagonal $m = n$ by using the modified form of Neumann's integral [5, § 2.6 (3)], namely

$$I_k^2(s) = \frac{1}{\pi}\int_0^\pi I_0(2s\cos\psi)\cos 2k\psi\, d\psi, \tag{2.43}$$

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where k is an integer. Use of this in (2.8), with $\rho = \sigma = 1$, gives

$$\begin{aligned} v(0,0) - v(n,n) &= \frac{1}{2\pi} \int_0^\infty e^{-2t} \int_0^\pi I_0(2t\cos\psi)(1-\cos 2n\psi)\, d\psi\, dt \\ &= \frac{1}{2\pi} \int_0^\pi \frac{1-\cos 2n\psi}{2\sin\psi}\, d\psi \\ &= \frac{1}{2\pi} \int_0^\pi (\sin\psi + \sin 3\psi + \cdots + \sin(2n-1)\psi)\, d\psi \\ &= \frac{1}{\pi}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right), \qquad n > 1. \end{aligned} \tag{2.44}$$

The square mesh has symmetry about the diagonals $m = \pm n$, so that the terms in $\cos(4k+2)\theta$ must be absent from the asymptotic expansion (2.31). This may be extended as

$$\begin{aligned} 2\pi\{v(0,0) - v(m,n)\} &= \log(2\sqrt{2}r) + \gamma - \frac{1}{12}\cos 4\theta\, r^{-2} \\ &\quad - \left(\frac{3}{40}\cos 4\theta + \frac{5}{48}\cos 8\theta\right) r^{-4} - \left(\frac{57}{112}\cos 8\theta + \frac{35}{72}\cos 12\theta\right) r^{-6} \\ &\quad - \left(\frac{217}{320}\cos 8\theta + \frac{45}{8}\cos 12\theta + \frac{1925}{384}\cos 16\theta\right) r^{-8} + O(r^{-10}). \end{aligned} \tag{2.45}$$

As a numerical example, the neglect of r^{-10} in (2.45) gives

$$v(0,0) - v(5,0) \simeq 0.512899,$$

which may be compared with the exact value

$$v(0,0) - v(5,0) = \frac{401}{4} - \frac{940}{3\pi} \simeq 0.512902. \tag{2.46}$$

3 Cubic lattice

The solution for the square mesh network is easily modified to deal with the cubic lattice, which has nodes (l,m,n) for all integers l,m,n, each node being connected to its six neighbours by a unit resistance. When unit current flows through the lattice from the origin to infinity, the potential fall is given by

$$v(0,0,0) - v(l,m,n) = \frac{1}{2\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{1-\cos l\theta\cos m\phi\cos n\psi}{3-\cos\theta-\cos\phi-\cos\psi}\, d\theta\, d\phi\, d\psi, \tag{3.1}$$

$$= \frac{1}{2}\int_0^\infty e^{-3t}\{I_0^3(t) - I_l(t)I_m(t)I_n(t)\}\, dt. \tag{3.2}$$

The form (3.2) leads to the pair of radial flow equations

$$\begin{aligned} l^{-1}\{v(l-1,m,n) - v(l+1,m,n)\} &= m^{-1}\{v(l,m-1,n) - v(l,m+1,n)\} \\ &= n^{-1}\{v(l,m,n-1) - v(l,m,n+1)\}. \end{aligned} \tag{3.3}$$

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For the cubic lattice, the potential remains finite at infinity. Doyle & Snell [4] quote a result equivalent to

$$\int_0^\infty e^{-3t} I_0^3(t)dt = \frac{1}{16\sqrt{6}\pi^3}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right)$$
$$\simeq 0.505462. \tag{3.4}$$

Thus the resistance of the cubic lattice between different nodes varies only from $\frac{1}{3}$, when the nodes are adjacent, to just over $\frac{1}{2}$ when they are far apart.

The asymptotic form of the potential when l, m, n are large can be expressed as a sum of terms of the form

$$r^{2k}\phi(l, m, n)$$

where $r^2 = l^2 + m^2 + n^2$, and ϕ is a solution of

$$\frac{\partial^2\phi}{\partial l^2} + \frac{\partial^2\phi}{\partial m^2} + \frac{\partial^2\phi}{\partial n^2} = 0 \tag{3.5}$$

that is symmetrical in l, m, n and an even function of each. The coefficients in the expansion are determined by the radial flow equations (3.3). Taking $v = 0$ at infinity, we obtain

$$4\pi v(l, m, n) = r^{-1} + \frac{1}{4}r^{-7}S_4 + \frac{63}{286}r^{-9}S_4 - \frac{5}{88}r^{-11}S_6 + \frac{231}{416}r^{-13}S_8 + O(r^{-7}), \tag{3.6}$$

where

$$\begin{aligned} S_4 &= l^4 + m^4 + n^4 - 3(m^2n^2 + n^2l^2 + l^2m^2), \\ S_6 &= l^6 + m^6 + n^6 - \frac{15}{2}(m^4n^2 + m^2n^4 + n^4l^2 + n^2l^4 + l^4m^2 + l^2m^4) + 90l^2m^2n^2, \\ S_8 &= l^8 + m^8 + n^8 - 14(m^6n^2 + m^2n^6 + n^6l^2 + n^2l^6 + l^6m^2 + l^2m^6) \\ &\quad + 35(m^4n^4 + n^4l^4 + l^4m^4). \end{aligned} \tag{3.7}$$

4 Hexagonal mesh

The argument given for the square mesh in the introduction shows that the resistance of the hexagonal mesh of unit resistances between adjacent nodes is $\frac{2}{3}$. Since the current from O to infinity divides equally at X in Figure 1, it also shows that the resistance between O and A is 1.

The nodes of the hexagonal mesh can be divided into two classes: nodes in the same class are separated by an even number of links, but those in different classes by an odd number. One class may be eliminated from the problem. In Figure 1 the current balance at O gives

$$v(X) + v(Y) + v(Z) - 3v(O) = -i(O), \tag{4.1}$$

where $i(O)$ is the current (if any) entering at O. When $v(X), v(Y)$ and $v(Z)$ are eliminated by use of the current balance at the respective nodes, the result is

$$v(A) + v(B) + v(C) + v(D) + v(E) + v(F) - 6v(O) = -j(O), \tag{4.2}$$

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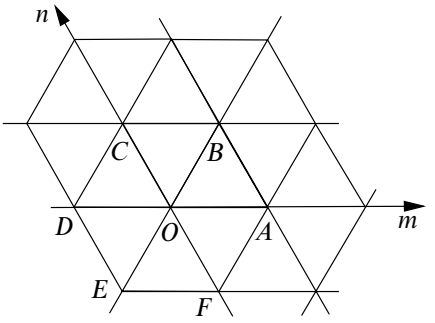


FIGURE 4. Triangular mesh.

where

$$j(O) = 3i(O) + i(X) + i(Y) + i(Z). \tag{4.3}$$

Thus the problem of finding the potential distribution in the hexagonal mesh network reduces to that for the triangular mesh shown in Figure 4.

5 Triangular mesh

The triangular mesh network may be regarded as the projection of the cubic lattice, treated in § 3, on to the plane $l + m + n = 0$. In this projection, nodes that lie on the same line normal to this plane are connected by perfect conductors. This idea leads to the specification of the triangular mesh by oblique coordinates (m, n) inclined at $2\pi/3$, as shown in Figure 4. With nodes at all integers m, n, the current balance equation at (m, n) is

$$v(m-1,n) + v(m+1,n) + v(m,n-1) + v(m,n+1) + v(m-1,n-1) + v(m+1,n+1) - 6v(m,n) = \begin{cases} -1, & m = n = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

The solution (3.2) for the cubic lattice suggests that

$$v(0,0) - v(m,n) = \frac{1}{2}\int_0^\infty e^{-3t} \sum_{k=-\infty}^{\infty} \{I_k^3(t) - I_k(t) I_{m+k}(t) I_{n+k}(t)\}\, dt. \tag{5.2}$$

This satisfies equation (5.1) and has the required symmetry properties

$$v(m,n) = v(n,m) = v(m, m-n). \tag{5.3}$$

The radial flow condition derived from (5.2) is

$$n\{v(m-1,n) - v(m+1,n)\} - m\{v(m,n-1) - v(m,n+1)\} = (m-n)\{v(m-1,n-1) - v(m+1,n+1)\}. \tag{5.4}$$

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The fall of potential to the node $(m, 0)$ can be calculated by the use of (2.43) combined with

$$\sum_{k=-\infty}^{\infty} I_{m+k}(t)\cos 2k\psi = \exp(t\cos 2\psi)\cos 2m\psi. \tag{5.5}$$

Thus

$$\begin{aligned} v(0,0) - v(m,0) &= \frac{1}{2\pi}\int_0^\infty \int_0^\pi \exp(-(3-\cos 2\psi)t) I_0(2t\cos\psi)(1-\cos 2m\psi)\, d\psi\, dt \\ &= \frac{1}{2\pi}\int_0^\pi \frac{1-\cos 2m\psi}{\{(3-\cos 2\psi)^2 - (2\cos\psi)^2\}^{1/2}}\, d\psi \quad \text{as} \\ &= \frac{1}{4\pi}\int_0^\pi \frac{1-\cos 2m\psi}{(4-\cos 2\psi)^{1/2}\sin\psi}. \end{aligned} \tag{5.6}$$

Values of the integral (5.6) can be obtained from the generating function

$$\begin{aligned} \sum_{m=0}^{\infty}\{v(0,0) - v(m,0)\}x^m &= \frac{x+x^2}{2\pi(1-x)}\int_0^\pi \frac{\sin\psi\, d\psi}{(4-\cos^2\psi)^{1/2}(1-2x\cos\psi+x^2)} \\ &= \frac{x}{\pi(1-x)(1-14x+x^2)^{1/2}}\tan^{-1}\left(\frac{(1-14x+x^2)^{1/2}}{3^{1/2}(1+x)}\right). \end{aligned} \tag{5.7}$$

This may be expressed as

$$\sum_{m=0}^{\infty}\{v(0,0) - v(m,0)\}x^m = xg(x)\left\{\frac{1}{6} - \frac{2\sqrt{3}}{\pi}\int_0^x g(y)\, dy\right\}, \tag{5.8}$$

where

$$g(x) = (1-x)^{-1}(1-14x+x^2)^{-1/2} = \sum_{m=0}^{\infty}\sum_{n=0}^{m} P_n(7)x^m. \tag{5.9}$$

From the values at $(m, 0)$ and equations (5.1) and (5.4), values of the potential drop can be built up at all nodes (m, n).

For the triangular mesh, the asymptotic form of the potential is obtained by approximating to the network by a conducting sheet of specific resistance $1/\sqrt{3}$, referred to the length of each side. It is now appropriate to work with variables

$$z = re^{i\theta} = m + n\exp\left(\frac{2}{3}\pi i\right), \bar{z} = re^{-i\theta} = m + n\exp\left(-\frac{2}{3}\pi i\right), \tag{5.10}$$

in terms of which the current balance equation (5.1) becomes

$$\frac{\partial^2 v}{\partial z\partial\bar{z}} + \frac{1}{4}\frac{\partial^4 v}{\partial z^2\partial\bar{z}^2} + \frac{1}{6!}\left(\frac{\partial^6 v}{\partial z^6} + 20\frac{\partial^6 v}{\partial z^3\partial\bar{z}^3} + \frac{\partial^6 v}{\partial\bar{z}^6}\right) + \cdots = 0. \tag{5.11}$$

This shows that the asymptotic solution has the form

$$\begin{aligned} 4\pi\sqrt{3}\{v(0,0) - v(m,n)\} \sim \log(z\bar{z}) + a + (b_1 z\bar{z} + b_0)(z^{-6} + \bar{z}^{-6}) \\ + (c_2 z^2\bar{z}^2 + c_1 z\bar{z} + c_0)(z^{-12} + \bar{z}^{-12}) + \cdots. \end{aligned} \tag{5.12}$$

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Although $a, b_0, c_0, \ldots$ are not determined by (5.11), they can be derived, except for a, from the radial flow equation (5.4). The results can be checked, and a found, from the asymptotic expansion of $v(0,0) - v(m,0)$.

Equation (5.6) can be expressed as

$$2\pi\sqrt{3}\{v(0,0) - v(m,0)\} = \int_0^{\frac{1}{2}\pi} \frac{1 - \cos 2m\psi}{\sin\psi}\, d\psi - \int_0^{\frac{1}{2}\pi} (1 - \cos 2m\psi) F(\psi)\, d\psi, \tag{5.13}$$

where

$$F(\psi) = \left\{1 - \left(1 + \frac{1}{3}\sin^2\psi\right)^{-\frac{1}{2}}\right\} \operatorname{cosec}\psi. \tag{5.14}$$

The first integral in (5.13) was treated in (2.44), and the substitution

$$\cos\psi = 2\sin\chi \tag{5.15}$$

gives

$$\int_0^{\frac{1}{2}\pi} F(\psi) d\psi = \int_0^{\frac{1}{6}\pi} \frac{d\chi}{2\cos\chi + \sqrt{3}} = \frac{1}{2}\log\left(\frac{4}{3}\right). \tag{5.16}$$

Repeated integration by parts shows that as $m \to \infty$

$$\int_0^{\frac{1}{2}\pi} F(\psi)\cos 2m\psi\, d\psi \sim \sum_{k=1}^{\infty} \frac{(-1)^k}{(2m)^{2k}} F^{(2k-1)}(0). \tag{5.17}$$

In this way, the asymptotic form of $v(m,n)$ as $r \to \infty$ is obtained as

$$\begin{aligned} 2\pi\sqrt{3}\{v(0,0) - v(m,n)\} = \log(2\sqrt{3}r) + \gamma - \frac{1}{30}\frac{\cos 6\theta}{r^4} - \frac{5}{84}\frac{\cos 6\theta}{r^6} \\ - \frac{7}{20}\frac{\cos 12\theta}{r^8} - \frac{215}{66}\frac{\cos 12\theta}{r^{10}} + O(r^{-12}), \end{aligned} \tag{5.18}$$

where

$$r^2 = m^2 - mn + n^2, \quad \tan\theta = \frac{n\sqrt{3}}{2m - n}. \tag{5.19}$$

As a numerical example, when r^{-10} is neglected in (5.8)

$$v(0,0) - v(6,3) \simeq 0.31863464,$$

compared with the exact value

$$v(0,0) - v(6,3) = \frac{1989\sqrt{3}}{5\pi} - 219 \simeq 0.31863460. \tag{5.20}$$

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6 Conclusion

The question asked in the Introduction about the resistance between different nodes of the hexagonal network can now be answered.

In particular, considering the nodes labelled in Figure 1, the resistances between pairs are typified by the following results

$$AX : \frac{2}{3};\ AO : 1;\ AZ : \frac{2\sqrt{3}}{\pi} \simeq 1.1027;\ AY : \frac{7}{3} - \frac{2\sqrt{3}}{\pi} \simeq 1.2307;$$

$$AC : \frac{6\sqrt{3}}{\pi} - 2 \simeq 1.3080;\ AD : 8 - \frac{12\sqrt{3}}{\pi} \simeq 1.3841.$$

It was shown in § 2 that the solution for the square mesh can be generalized to the case in which the resistance of a link depends on its direction. A similar extension could be made to the cubic lattice problem, and this would lead to the solution for triangular and hexagonal meshes in which the resistance of a link depended on the three possible directions.

Dr David Allwright has pointed out that the case of current flow between different nodes of a finite rectangular network could be derived from the case of the infinite network, by the introduction of a doubly infinite set of image entry and exit points for the current, so as to cancel the current flow in each of the links that join the finite portion to the rest of the infinite network.

The asymptotic approximation shows that, to leading order, the square mesh is equivalent to an electrode of radius $e^{-\gamma}/2\sqrt{2}$ in a conducting sheet of unit specific resistance. The next term corresponds to a relative variation $\frac{1}{2}r^{-2}\cos 4\theta$ in the thickness of the sheet. For the triangular mesh, the specific resistance is $1/\sqrt{3}$ and the electrode radius $e^{-\gamma}/2\sqrt{3}$; the relative thickness variation is $\frac{1}{6}r^{-4}\cos 6\theta$. The higher order of this perturbation may be associated with the greater accuracy shown in the numerical comparison. In the hexagonal mesh case, the equivalent specific resistance is $\sqrt{3}$ and the electrode radius $\frac{1}{2}e^{-\gamma}$.

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