Ergod. Th. & Dynam. Sys. (2019), **39**, 2305–2331 © *Cambridge University Press, 2017*
doi:10.1017/etds.2017.121

Non-rigidity for circle homeomorphisms with several break points

ABDELHAMID ADOUANI and HABIB MARZOUGUI

University of Carthage, Faculty of Science of Bizerte, Department of Mathematics, Jarzouna, 7021, Tunisia

(e-mail: arbi.abdelhamid@gmail.com, hmarzoug@ictp.it, habib.marzougui@fsb.rnu.tn)

(*Received* 14 *April* 2016 *and accepted in revised form* 30 *September* 2017)

Abstract. In this work, we consider two class P-homeomorphisms, f and g, of the circle with break point singularities, that are differentiable maps except at some singular points where the derivative has a jump. Assume that they have the same irrational rotation number of bounded type and that the derivatives Df and Dg are absolutely continuous on every continuity interval of Df and Dg, respectively. We show that if f and g are not break-equivalent, then any topological conjugating h between f and g is a singular function, i.e., it is continuous on the circle, but $\mathrm{Dh}(x) = 0$ almost everywhere (a.e.) with respect to the Lebesgue measure. In particular, this result holds under some combinatorial assumptions on the jumps at break points. It also generalizes previous results obtained for one and two break points and complements that of Cunha–Smania which was established for break equivalence.

1. *Introduction*

Denote the circle by $S^1 = \mathbb{R}/\mathbb{Z}$ and let $p : \mathbb{R} \longrightarrow S^1$ be the canonical projection. Let f be an orientation-preserving homeomorphism of S^1. The homeomorphism f admits a lift $\widehat{f} : \mathbb{R} \longrightarrow \mathbb{R}$ that is an increasing homeomorphism of $\mathbb{R}$ such that $p \circ \widehat{f} = f \circ p$. Conversely, the projection of such a homeomorphism of $\mathbb{R}$ is an orientation-preserving homeomorphism of S^1. Let $x \in S^1$. We call the *orbit* of x by f the subset $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$. Historically, the study of the dynamics of circle homeomorphisms was initiated by Poincaré [**24**, 1886], who introduced the rotation number of a homeomorphism f of S^1 as $\rho(f) = \lim_{n\to+\infty} ((\widehat{f}^n(\widehat{x}) - \widehat{x})/n) \pmod 1$, where $\widehat{x} \in \mathbb{R}$ such that $p(\widehat{x}) = x$. Poincaré shows that this limit exists and does not depend on either x or the lift $\widehat{f}$ of f. We say that f is semi-conjugate to the rotation $R_{\rho(f)}$ if there exists an orientation-preserving surjective continuous map $h : S^1 \longrightarrow S^1$ of degree one such that $h \circ f = R_{\rho(f)} \circ h$.

POINCARÉ'S THEOREM. *Let f be a homeomorphism of S^1 with irrational rotation number $\rho(f)$. Then f is semi-conjugate to the rotation $R_{\rho(f)}$.*

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A natural question is to ask whether the semi-conjugation h could be improved to be a conjugation, i.e., so that h is a homeomorphism. In this case, we say that f is topologically conjugate to the rotation $R_{\rho(f)}$. In this direction, Denjoy [**11**] proved the following theorem.

DENJOY'S THEOREM. [**11**] *Every C^2-diffeomorphism f of S^1 with irrational rotation number $\rho(f)$ is topologically conjugate to the rotation $R_{\rho(f)}$.*

Denjoy asked whether or not C^2-diffeomorphisms f of S^1 are ergodic with respect to the Lebesgue measure m (f is said to be ergodic with respect to m if any f-invariant measurable set A has measure $m(A)$ equal to zero or one). Simultaneously, Herman and Katok gave a positive answer to this question.

HERMAN–KATOK'S THEOREM. [**15**, **18**] *Every C^2-diffeomorphism f with irrational rotation number is ergodic with respect to the Lebesgue measure m.*

It is well known that a homeomorphism f of S^1 with irrational rotation number preserves a unique normalized measure on S^1, denoted by μ_f. If $\widehat{h}$ is the lift of h, by taking $\widehat{h}(0) = 0$, the conjugating homeomorphism h is unique and related to μ_f by $h(x) = p(\mu_f([0, x])) \in S^1$ for $x \in S^1$. Uniqueness of μ_f implies that μ_f is either singular or absolutely continuous with respect to m; in this second case, h is an absolutely continuous function. Recall that μ_f is said to be *singular* with respect to the Lebesgue measure m on S^1 if there exists a measurable subset E of S^1 such that $\mu_f(E) = 1$ and $m(E) = 0$. In fact, if μ_f is absolutely continuous with respect to the Lebesgue measure m and f is a C^2-diffeomorphism, then μ_f is necessarily equivalent to m as a consequence of Herman–Katok's theorem above (i.e., m is absolutely continuous with respect to μ_f and the converse is also true).

In the following, we denote $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.

Definition. A real number $\alpha \in]0, 1[$ is called *Diophantine with exponent* $\delta \geq 0$ if there is a constant $c(\alpha) > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c(\alpha)}{q^{2+\delta}} \quad \text{for any } \frac{p}{q} \in \mathbb{Q}. \tag{1}$$

A number that is neither rational nor Diophantine is called a *Liouville number*.

Every real number $\alpha \in]0, 1[$ has a continued fraction expansion represented by

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} := [a_1, a_2, \ldots, a_n, \ldots],$$

where $a_m \in \mathbb{N}^*$, $m \in \mathbb{N}^*$ are called *partial quotients* of α. When $(a_m)_{m\in\mathbb{N}}$ is bounded, α is said to be of *bounded type*. This is equivalent to the fact that (1) holds with $\delta = 0$.

The problem of smoothness of the conjugacy h of smooth diffeomorphisms to rotations is now very well understood (see, for instance, [**16**, **18**, **20**, **21**, **27**].

We refer the reader to the books [**10**, **23**] for a thorough account of circle homeomorphisms.

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The situation is more complicated for circle homeomorphisms with break points or class P-homeomorphisms, for short (see the definition below). This class are known to satisfy the conclusion of Denjoy's theorem (Corollary 2.6) (see also [**18**]; [**16**, Ch. VI]) and, with additional regularity, of Herman–Katok's theorem (see [**5**]). However, Katznelson–Ornstein's theorem [**18**] cannot be extended, in general, to class P. The study of the regularity of the invariant measures of class P-homeomorphisms then arises naturally.

1.1. *Class P-homeomorphisms.* The following definition is due to Herman.

Definition 1.1. (See [**16**, p. 74]) An orientation-preserving homeomorphism f of S^1 is called a *class P*-homeomorphism if it is differentiable except at finitely many points, the so called *break points* of f, at which left and right derivatives (denoted, respectively, by Df_- and Df_+) exist and such that the derivative $\mathrm{Df} : S^1 \longrightarrow \mathbb{R}_+^*$ has the following properties.

- There exist two constants $0 < a < b < +\infty$ such that $a < \mathrm{Df}(x) < b$, for every x where Df exists.
- $a < \mathrm{Df}_+(c) < b$ and $a < \mathrm{Df}_-(c) < b$ at the break points c.
- log Df has bounded variation on S^1.

We point out that the third condition implies the other two. The ratio $\sigma_f(c) := \mathrm{Df}_-(c)/\mathrm{Df}_+(c)$ is called the *f-jump* at c.

Denote:

- $C(f) = \{c_0, c_1, c_2, \ldots, c_p\}$ to the set of break points of f in S^1;
- $c_{p+1} := c_0$;
- $\pi_{s,O_f(c)}(f) := \prod_{x \in C(f) \cap O_f(c)} \sigma_f(x)$ for every $c \in S^1$;
- $\pi_s(f)$ to be the product of f-jumps at the break points of f, where

$$\pi_s(f) = \prod_{c \in C(f)} \sigma_f(c);$$

- $\mathrm{SO}(f) = \{O_f(c) : c \in C(f) \text{ and } \pi_{s,O_f(c)}(f) \neq 1\}$.

We denote the total variation of log Df by $V = \mathrm{Var}(\log \mathrm{Df})$. We have

$$V := \sum_{j=0}^{p} \mathrm{Var}_{[c_j, c_{j+1}]} \log \mathrm{Df} + |\log(\sigma_f(c_j))| < +\infty.$$

In this case, V is the total variation of log Df, log Df_-, log Df_+.

We notice the following properties.

- If f is a class P-homeomorphism of S^1 which is C^1 on S^1, then f is a C^1-diffeomorphism.
- If f is a class P-homeomorphism of S^1, then $D^2 f \in L^1(S^1)$: indeed, this follows from [**17**, Theorem 18.14, p. 284] since Df has bounded variation on S^1.
- If f is a class P-homeomorphism of S^1 and Df is absolutely continuous on every connected interval of $S^1 \backslash C(f)$, then $f \in C^1(S^1 \backslash C(f))$.

Among the simplest examples of class P-homeomorphisms, there are:

- C^2-diffeomorphisms; and
- piecewise linear (PL)-homeomorphisms.

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An orientation-preserving homeomorphism f of S^1 is called a *PL-homeomorphism* if f is differentiable except at finitely many break points $(c_i)_{1\le i\le p}$ of S^1 such that the derivative Df is constant on each $]c_i, c_{i+1}[$.

Denote:

- $\mathcal{P}(S^1)$ to be the set of class P-homeomorphisms of S^1; and
- $\mathrm{PL}(S^1)$ to be the set of orientation-preserving piecewise linear PL-homeomorphisms of S^1.

We notice that $\mathcal{P}(S^1)$ is a group which contains $\mathrm{PL}(S^1)$ (cf. [**2**]).

Definition 1.2. [**4**] Let $f \in \mathcal{P}(S^1)$. We say that f has the *(D)-property* if the product of f-jumps at the break points of each orbit is trivial, that is, $\pi_{s,O_f(c)}(f) = 1$, for every $c \in C(f)$.

In particular, if f has the (D)-property, then $\pi_s(f) = 1$. Conversely, if all break points belong to the same orbit and $\pi_s(f) = 1$, then f has the (D)-property.

Let $f \in \mathcal{P}(S^1)$. We say that f satisfies the *Katznelson–Ornstein (KO)* if Df is absolutely continuous on every continuity interval of Df.

- Recently, there has been significant progress in the problem of the regularity of the conjugating map between two class P-homeomorphisms with the *same* irrational rotation number. The case of two class P-homeomorphisms with one break point and with the same jump, was studied by Khanin and Khmelev in [**19**] and by Teplinskii and Khanin in [**22**].
- The case of two class P-homeomorphisms with *one* break point and with *distinct* jumps was studied by Dzhalilov, Akin and Temir in [**12**]. Their results show that the conjugation map between two such maps is singular.
- The case of two class P-homeomorphisms with *two* break points was studied by Akhadkulov, Dzhalilov and Mayer in [**6**].
- The case of *several* break points with *distinct total jumps* was studied by Dzhalilov, Mayer and Safarov in [**13**], and independently by the first author in [**1**]. Their results show that the conjugation map between two such maps is singular.

THEOREM. [**1**, **13**] *Let f and g be two class P-homeomorphisms of the circle S^1 with the same irrational rotation number. Assume that f and g satisfy the KO. If $\pi_s(f) \neq \pi_s(g)$, then the homeomorphism map h conjugating f and g is a singular function, i.e., it is continuous on S^1 and $Dh(x) = 0$ a.e. with respect to the Lebesgue measure.*

Definition 1.3. We say that two class P-homeomorphisms, f and g, of the circle S^1 are *break-equivalent* if there exists a topological conjugating h such that:

(1) $h(\mathrm{SO}(f)) = \mathrm{SO}(g)$; and

(2) $\pi_{s,O_g(h(c))}(g) = \pi_{s,O_f(c)}(f)$, for all $c \in C(f)$.

In particular, if all points of $C(f)$ (respectively, $C(g)$) are on pairwise distinct orbits then:

- $\mathrm{card}(\mathrm{SO}(f)) = \mathrm{card}(C(f))$ (respectively, $\mathrm{card}(\mathrm{SO}(g)) = \mathrm{card}(C(g))$); and
- f and g are break-equivalent if there exists a topological conjugation h from f to g such that:

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(1) $h(C(f)) = C(g)$; and
(2) $\sigma_g(h(c)) = \sigma_f(c)$ for all $c \in C(f)$.

This definition coincides with that of [**9**]. It is easy to see that if there is a C^1-conjugation between f and g, then f and g are break-equivalent. The case of break-equivalent $C^{2+\alpha}$ homeomorphisms, f and g, with $\pi_s(f) = \pi_s(g) = 1$ and some combinatorial conditions was studied by Cunha–Smania in [**9**]. It was proved that any two such homeomorphisms are C^1-conjugated. The combinatorial condition in Cunha–Smania [**9**] implies that the circle map has bounded rotation number (but it is likely that there are circle maps with bounded rotation number and many breaks that do not satisfy the combinatorial condition there).

In the present paper, we consider *non-break-equivalent* class P-homeomorphisms having *several break points* with coinciding irrational rotation number of bounded type.

MAIN THEOREM. (Non-rigidity) *Let f and g be two class P-homeomorphisms of the circle S^1 with several break points that satisfy the KO and which have the same irrational rotation number of bounded type. If f and g are not break equivalent, then the homeomorphism h conjugating f to g is a singular function, i.e., it is continuous on S^1 and $Dh(x) = 0$ a.e. with respect to the Lebesgue measure.*

As a consequence we have the following corollary.

COROLLARY 1.4. *Let f and g satisfy the assumptions of the main theorem. If card $(SO(f)) \neq$ card $(SO(g))$, then the map conjugating f to g is a singular function. In particular, this holds if f and g do not have the same number of break points lying on pairwise distinct f-orbits and g-orbits, respectively.*

COROLLARY 1.5. *Let f and g satisfy the assumptions of the Main Theorem. If there is some point $d \in \mathrm{SO}(f)$ such that $\pi_{s,O_g(d)}(g) \notin \{\pi_{s,O_{f(c)}}(f) : c \in C(f)\}$, then the map h conjugating f to g is singular.*

In particular, for *non-break-equivalent* homeomorphisms with two break points, we get, as consequence, the Akhadkulov–Dzhalilov–Noorani theorem (see [**7**]).

COROLLARY 1.6. *Let f and g satisfy the assumptions of the Main Theorem.*
(i) *If g does not have the (D)-property and f has the (D)-property, then the map conjugating f to g is singular.*
(ii) *If f and g have the (D)-property and $D^2 f$, $D^2 g \in L^p(S^1)$ for some $p > 1$, then the map conjugating f to g is absolutely continuous.*

In particular, this leads to the following corollary.

COROLLARY 1.7. [**3**, Theorem B] *Let f satisfy the assumptions of the Main Theorem.*
(i) *If f does not have the (D)-property, then the invariant measure μ_f is singular with respect to the Lebesgue measure m.*
(ii) *If f has the (D)-property and $D^2 f \in L^p(S^1)$ for some $p > 1$, then the measure μ_f is equivalent to the Lebesgue measure m.*

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COROLLARY 1.8. [**3**, Corollary 1.5] *Let $f \in \mathrm{PL}(S^1)$ have irrational rotation number α of bounded type. Then the following are equivalent.*

(i) *f has the (D)-property.*
(ii) *The measure μ_f is equivalent to the Lebesgue measure m.*

Remark.

(1) In [**8**], Akhadkulov *et al* proved, independently and under different regularity assumptions, a similar result to the Main Theorem. Their proof is based on the Khanin–Sinai renormalization. Here, our proof uses ratio distortion estimates.
(2) The Main Theorem and, in particular, Corollary 1.8 cannot be extended to a rotation number that is not of bounded type since, very recently, Teplinsky [**26**] constructed an example of a PL circle homeomorphism with four non-trivial break points lying on different orbits that had invariant measure equivalent to the Lebesgue measure m. The rotation number for such an example can be chosen to be either Diophantine or Liouvillean, but not of bounded type.

1.2. *Strategy of the proof of the Main Theorem.* The strategy can be explained in the following way. First, we show that several breaks lying on the same orbit are, in a sense, equivalent to one single break of a size that is the product of their sizes (see subgraph 2.4); therefore, without loss of generality, one may consider class P-homeomorphisms with breaks that lie on pairwise distinct orbits. After that, we focus our attention on one of the break points only, say, c, and on its associated primary cell, namely, $(y_1(n), y_2(n), y_3(n))$, $n \in \mathbb{N}$, which is contained in some neighbourhood of a point x_0, where $y_2(n) = f^{-i_n(c)}(c)$ belong to the atom $\Delta_0^{(n)}(x_0)$ of the dynamical partition of x_0 (see subgraph 2.1). For any break point $c_1 \notin O_f(c)$, there is an integer $i_n(c_1)$ (respectively, $j_n(c_1)$) such that $f^{-i_n(c_1)}(c_1) \in]y_1(n), y_2(n)[$ (respectively, $f^{-j_n(c_1)}(c_1) \in]y_2(n), y_3(n)[$). When the rotation number $\rho(f)$ of f is of bounded type, we show that if $f^{-i_n(c_1)}(c_1)$ (respectively, $f^{-j_n(c_1)}(c_1)$) is close to $y_2(n)$, then the variation $k_n = i_n(c_1) - i_n(c)$ (respectively, $j_n(c_1) - i_n(c)$) is constant, which leads to c_1 and c being on the same f-orbit, which is a contradiction (see Proposition 3.2). It follows that the point $f^{-i_n(c_1)}(c_1)$ (respectively, $f^{-j_n(c_1)}(c_1)$) is outside a small neighbourhood of $y_2(n)$, for infinitely many n, say, $n \in \mathbb{M}_{c,c_1}$: an infinite subset of $\mathbb{N}$. Using this, we choose a secondary cell (z_1, y_2, z_3) in $(y_1(n), y_2(n), y_3(n))_{n\in\mathbb{M}_{c,c_1}}$ so that $f^{-k_n(c_1)}(c_1) \notin (z_1, y_2, z_3)$ ($k_n(c_1) = i_n(c_1)$ or $j_n(c_1)$) and so are as well for all break points d for which $f^{-k_n(d)}(d)$, ($k_n(d) = i_n(d)$ or $j_n(d)$) are not close to $y_2(n)$ (see Proposition 3.3). The other break points (for which $f^{-k_n(d)}(d)$ is close to $y_2(n)$) form a set $E(c_1, c)$, notice that $c_1 \notin E(c_1, c)$. These allow us, when the conjugacy h between f and g is singular (i.e., $\mathrm{Dh}(x_0) = \omega_0 > 0$), to give a control of the ratio distortion $\mathrm{Dr}_{g^{q_n}}(h(z_1), h(z_2), h(z_3))/\mathrm{Dr}_{f^{q_n}}(z_1, z_2, z_3)$ of the secondary cell (z_1, y_2, z_3) with respect to f^{q_n} and f^{q_n}. We show, on the one hand, that the ratio distortion is close to $\Pi(c_1, c) = \prod_{d\in E(c_1,c)} (\sigma_g(h(d))/\sigma_f(d))$ and, on the other hand, that it is close to one (see §5). This leads to the equality $\Pi(c_1, c) = 1$. As $c_1, c \in B$, $c_1 \neq c$ are arbitrary, and $c_1 \notin E(c_1, c)$, this equality implies that, for every break point c of f, $\sigma_g(h(c)) = \sigma_f(c)$, which contradicts our hypothesis: f and g are not break-equivalent.

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2. *Notation and preliminary results*

2.1. *Dynamical partitions.* Let f be a homeomorphism of S^1 with irrational rotation number $\alpha = \rho(f)$. We identify α to its lift $\widehat{\alpha}$ in $]0, 1[$. Let $(a_n)_{n\in\mathbb{N}^*}$ be the partial quotients of α in the continued fraction expansion. For $n \in \mathbb{N}^*$, the fractions $[a_1, a_2, \ldots, a_n]$ are written in the form of irreducible fractions p_n/q_n. The sequence p_n/q_n converges to α and we say that p_n/q_n are *rational approximations* of α. Their denominators q_n satisfy the recursion relation

$$q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 2, q_0 = 1, q_1 = a_1.$$

Let $x_0 \in S^1$ be fixed. Denote

$$\Delta_0^{(n)}(x_0) = \begin{cases} [x_0, f^{q_n}(x_0)] & \text{if } n \text{ is even,} \\ [f^{q_n}(x_0), x_0] & \text{if } n \text{ is odd,} \end{cases}$$
$$\Delta_i^{(n)}(x_0) := f^i(\Delta_0^{(n)}(x_0)), \quad i \in \mathbb{Z}.$$

In all of the subsequent work, we deal with the case when n is odd (the case when n is even is obtained by reversing the orientation of S^1).

This gives the following lemma.

LEMMA 2.1. (See [**25**]) *The segments* $\Delta_i^{(n-1)}(x_0) = f^i(\Delta_0^{(n-1)}(x_0))$, $0 \leq i < q_n$, *and* $\Delta_j^{(n)}(x_0) = f^j(\Delta_0^{(n)}(x_0))$, $0 \leq j < q_{n-1}$, *cover* S^1 *and their interiors are mutually disjoint.*

The partition denoted by $\xi_n(x_0) := \{\Delta_i^{(n-1)}(x_0); \Delta_j^{(n)}(x_0), 0 \leq i < q_n, 0 \leq j < q_{n-1}\}$ is called the *nth dynamical partition* of the point x_0. It is defined by the order of points $x_0, \ldots, f^{q_{n-1}+q_n-1}(x_0)$. The process of passing from $\xi_n(x_0)$ to $\xi_{n+1}(x_0)$ is described by keeping the elements $\Delta_j^{(n)}(x_0)$ intact while each element $\Delta_i^{(n-1)}(x_0)$, $(0 \leq i < q_n)$ is partitioned into $a_{n+1} + 1$ sub-segments; since $q_{n+1} = a_{n+1}q_n + q_{n-1}$,

$$\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{a_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^{(n)}(x_0).$$

For every $n \geq 1$, $\xi_{n+1}(x_0)$ is finer than $\xi_n(x_0)$ in the following sense: every element $\xi_n(x_0)$ is a union of elements $\xi_{n+1}(x_0)$.

LEMMA 2.2. *We have* $e^{-a_{n+1}V} < (e^{2V}/(e^V - 1))(m(\Delta_0^{(n)}(x_0))/m(\Delta_0^{(n-1)}(x_0)))$. *In particular, if the rotation number* $\rho(f)$ *of* f *is of bounded type, then* $\Delta_0^{(n-1)}(x_0))$ *and* $\Delta_0^{(n)}(x_0))$ *are comparable.*

Proof. Since $\Delta_0^{(n-1)}(x_0) \subset [f^{q_n+q_{n+1}}(x_0), f^{q_{n-1}}(x_0)] = \bigcup_{0\leq j\leq a_{n+1}} f^{q_{n-1}+jq_n}(\Delta_0^{(n)}(x_0))$ (see Figure 1), so, by Denjoy's inequality,

$$\frac{m(\Delta_0^{(n-1)}(x_0))}{m(\Delta_0^{(n)}(x_0)} < \sum_{0\leq j\leq a_{n+1}} e^{(1+j)V} = \frac{e^V}{e^V - 1}(e^{(1+a_{n+1})V} - 1) < \frac{e^{2V}}{e^V - 1} e^{a_{n+1}V}.$$

If $\rho(f)$ is of bounded type, the sequence $(a_n)_{n\in\mathbb{N}^*}$ is bounded and so the conclusion follows from the inequality above. □

Let $x_0 \in S^1$ be fixed and let $n \geq 1$.

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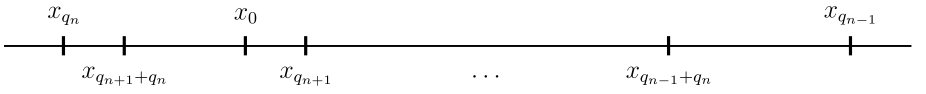


FIGURE 1. $x_i = f^i(x_0)$.

LEMMA 2.3. *Let $-(q_{n+1}+q_n) < k < q_{n+1}+q_{n-1}$ be an integer. Then $\Delta_0^{(n-1)}(x_0) \cap \Delta_k^{(n)}(x_0)$ has non-empty interior if and only if $k \in \{q_{n-1}+jq_n;\ 0 \le j \le a_{n+1}\} \cup \{-jq_n;\ 1 \le j \le a_{n+1}+1\}$.*

Proof. Since

$$\Delta_0^{(n-1)}(x_0) = \Delta_0^{(n+1)}(x_0) \coprod_{0 \le j < a_{n+1}} \Delta_{q_{n-1}+jq_n}^{(n)}(x_0),$$

(where $\coprod$ means pairwise disjoint union), the following hold.

- If $k = q_{n-1} + jq_n$ for some $0 \le j \le a_{n+1}$, then
$$\Delta_0^{(n-1)}(x_0) \cap \Delta_k^{(n)}(x_0) = \begin{cases} \Delta_k^{(n)}(x_0) & \text{if } k = q_{n-1}+jq_n \text{ with } 0 \le j < a_{n+1}, \\ [x_0,\ f^{q_{n+1}}(x_0)] & \text{if } k = q_{n+1}. \end{cases}$$
- If $k = -jq_n$ for some $1 \le j \le a_{n+1}+1$, then
$$\Delta_0^{(n)}(x_0) \cap \Delta_{-k}^{(n-1)}(x_0) = \begin{cases} \Delta_0^{(n)}(x_0) & \text{if } k = -jq_n \text{ with } 1 \le j \le a_{n+1}, \\ [f^{q_n}(x_0),\ f^{q_n+q_{n+1}}(x_0)] & \text{if } k = -(a_{n+1}+1)q_n. \end{cases}$$
In both cases, $\Delta_0^{(n-1)}(x_0) \cap \Delta_k^{(n)}(x_0)$ has non-empty interior.

Conversely, we distinguish the following five cases.

- $0 \le k < q_{n+1}$. If $k \notin \{q_{n-1}+jq_n : 0 \le j < a_{n+1}\}$, then, by Lemma 2.1, $\Delta_0^{(n-1)}(x_0) \cap \Delta_k^{(n)}(x_0)$ has empty interior.
- $q_{n+1} \le k < q_{n+1}+q_{n-1}$. If $k \ne q_{n+1}$, then $k = l + q_{n+1}$ with $1 \le l < q_{n-1}$. Thus
$$\begin{aligned} f^{-q_{n+1}}(\Delta_0^{(n-1)}(x_0) \cap \Delta_k^{(n)}(x_0)) &= [f^{-q_{n+1}}(x_0),\ f^{-q_{n+1}+q_{n-1}}(x_0)] \cap \Delta_l^{(n)}(x_0) \\ &\subset (\Delta_0^{(n)}(x_0) \cup \Delta_0^{(n-1)}(x_0)) \cap \Delta_l^{(n)}(x_0) \end{aligned}$$
and hence $\Delta_0^{(n-1)}(x_0) \cap \Delta_k^{(n)}(x_0)$ has empty interior.
- $-q_n < k \le 0$. In this case, by Lemma 2.1, $\Delta_{-k}^{(n-1)}(x_0) \cap \Delta_0^{(n)}(x_0)$ has empty interior.
- $-(a_{n+1}+1)q_n < k \le -q_n$. If $k \notin \{-jq_n : 1 \le j \le a_{n+1}\}$, then $-k = l + jq_n$ with $1 \le l < q_n$ and $1 \le j \le a_{n+1}$. Thus
$$\begin{aligned} &f^{-jq_n}(\Delta_{-k}^{(n-1)}(x_0) \cap \Delta_0^{(n)}(x_0)) \\ &\quad = \Delta_l^{(n-1)}(x_0) \cap f^{-jq_n}(\Delta_0^{(n)}(x_0)) \subset \Delta_l^{(n-1)}(x_0) \cap \Delta_0^{(n-1)}(x_0), \end{aligned}$$
so $\Delta_{-k}^{(n-1)}(x_0) \cap \Delta_0^{(n)}(x_0)$ has empty interior.
- $-(q_{n+1}+q_n) < k \le -(a_{n+1}+1)q_n$. If $k \ne -(a_{n+1}+1)q_n$, then $-k = l + (a_{n+1}+1)q_n$ with $1 \le l < q_{n-1}$. Thus
$$\begin{aligned} &f^{-(a_{n+1}+1)q_n}(\Delta_{-k}^{(n-1)}(x_0) \cap \Delta_0^{(n)}(x_0)) = \Delta_l^{(n-1)}(x_0) \cap f^{-(a_{n+1}+1)q_n}(\Delta_0^{(n)}(x_0)) \\ &\quad \subset\ \Delta_l^{(n-1)}(x_0) \cap (\Delta_0^{(n-1)}(x_0) \cup f^{q_{n-1}}(\Delta_0^{(n-1)}(x_0))). \end{aligned}$$
Hence $\Delta_{-k}^{(n-1)}(x_0) \cap \Delta_0^{(n)}(x_0)$ has empty interior. □

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LEMMA 2.4.
(1) $\Delta_0^{(n-1)}(x_0) \cap \Delta_k^{(n)}(x_0)$ *has non-empty interior and* $|k| < q_n$ *if and only if* $k = q_{n-1}$.
(2) $\Delta_0^{(n)}(x_0) \cap \Delta_k^{(n)}(x_0)$ *has non-empty interior and* $|k| < q_n + q_{n+1}$ *if and only if* $|k| = 0$ *or* q_{n+1}.

Proof. Assertion (1) follows from Lemma 2.3. For assertion (2), if $|k| = 0$ or q_{n+1}, one has $\Delta_0^{(n)}(x_0) \cap \Delta_k^{(n)}(x_0) = \Delta_0^{(n)}(x_0)$ or $[f^{q_n+q_{n+1}}(x_0), x_0]$, respectively, and $|k| < q_n + q_{n+1}$. Conversely, assume that $\Delta_0^{(n)}(x_0) \cap \Delta_k^{(n)}(x_0)$ has non-empty interior and $|k| < q_n + q_{n+1}$. Assume that $k > 0$. Then $k \geq q_{n-1}$ and so $k = l + q_{n-1}$ with $0 \leq l < q_n + q_{n+1} - q_{n-1}$. Hence $\Delta_0^{(n)}(x_0) \cap \Delta_k^{(n)}(x_0) \subset \Delta_0^{(n)}(x_0) \cap \Delta_l^{(n-1)}(x_0)$. Therefore $\Delta_{-l}^{(n)}(x_0) \cap \Delta_0^{(n-1)}(x_0)$ has non-empty interior. By Lemma 2.3, $-l = -jq_n$ with $1 \leq j \leq a_{n+1} + 1$ and hence $k = q_{n-1} + jq_n$. We conclude that $j = a_{n+1}$ and so $k = q_{n+1}$. Now if $k < 0$, then, by the above, $-k = q_{n+1}$. The proof is complete. □

The following lemmas are easy to check.

LEMMA 2.5. *Let* $n \geq 1$ *be an integer and let* $y_1 \in S^1$. *Set* $y_2 = f^{q_{n-1}}(y_1)$, $y_3 = f^{q_{n-1}}(y_2)$, $\Delta_i^{(n-1)}(y_1) = f^i([y_1, y_2])$ *and* $\Delta_j^{(n-1)}(y_2) = f^j([y_2, y_3])$, $0 \leq i, j < q_n$. *Then* $\Delta_i^{(n-1)}(y_1) \cap \Delta_j^{(n-1)}(y_2)$ *has non-empty interior if and only if one of the following conditions holds.*
(1) $j = i - q_{n-1}$ *and* $q_{n-1} \leq i < q_n$; *in this case,*

$$\Delta_i^{(n-1)}(y_1) = \Delta_j^{(n-1)}(y_2).$$

(2) $j = q_n - q_{n-1} + i$ *and* $0 \leq i < q_{n-1}$; *in this case,*

$$\Delta_j^{(n-1)}(y_2) = f^{q_n}(\Delta_i^{(n-1)}(y_1)).$$

LEMMA 2.6. *Let* $n \geq 1$ *be an integer and let* $y_1 \in S^1$. *Set* $y_2 = f^{q_{n-1}}(y_1)$, $y_3 = f^{q_{n-1}}(y_2)$. *Then, for every* $x \in S^1 \backslash O_f(y_1)$, *there exists a unique* $0 \leq i = i_n(x) < q_n$ *such that* $x \in f^i(]y_1, y_2[) \cup f^{\varphi_n(i)}(]y_2, y_3[)$, *where* $\varphi_n : [0, q_n[\longrightarrow [0, q_n[$ *is a bijective map given by*

$$\varphi_n(i) = \begin{cases} q_n - q_{n-1} + i & \text{if } 0 \leq i < q_{n-1}, \\ i - q_{n-1} & \text{if } q_{n-1} \leq i < q_n. \end{cases}$$

2.2. *Useful inequalities.* The proof of the following results are based on the dynamical partition of S^1.

FINZI DISTORTION LEMMA. [**14**] *Let* $f \in \mathcal{P}(S^1)$ *with irrational rotation number. For all* $z \in S^1$; $x, y \in [z, f^{q_{n-1}}(z)]$ *and for all integer* $0 \leq k \leq q_n$,

$$e^{-V} \leq \frac{Df^k(x)}{Df^k(y)} \leq e^V.$$

Notice that Finzi's distortion lemma is used for the proof of Herman–Katok's theorem.

The following lemma plays a key role in studying metrical properties of the homeomorphism f.

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DENJOY'S INEQUALITY. *Let $f \in \mathcal{P}(S^1)$ with irrational rotation number. For all $x \in S^1$ and all $n \in \mathbb{N}$,*

$$e^{-V} \leq Df^{q_n}(x) \leq e^{V},$$

where $V = \mathrm{Var}(\log Df)$.

Here and further on, by $m([a, b])$ we mean the smallest length of the two arcs $[a, b]$ and $[b, a]$ on S^1. In the following lemma, we have to compare the lengths of different iterates of intervals.

LEMMA 2.7. *Let $f \in \mathcal{P}(S^1)$ with irrational rotation number $\alpha = \rho(f)$. Let $n \in \mathbb{N}^*$ and $z_1 \in S^1$. Set $z_2 = f^{q_{n-1}}(z_1)$, $z_3 = f^{q_{n-1}}(z_2)$. Then, for any segments $K_1, K_2 \subset [z_1, z_3]$,*

$$e^{-2V}\frac{m(K_1)}{m(K_2)} \leq \frac{m(f^j(K_1))}{m(f^j(K_2))} \leq e^{2V}\frac{m(K_1)}{m(K_2)} \quad \textit{for all } |j| \leq q_n.$$

Proof. If $0 \leq j \leq q_n$, this follows from [**1**, Lemma 2.6]. If $-q_n \leq j \leq 0$, we apply the same result to f^{-1} instead of f (since the convergent sequences of the continued fractions of $\varrho(f^{-1}) = 1 - \varrho(f)$ and $\varrho(f)$ have the same denominators). □

2.3. *Some ratio tools.* Let us introduce the notion of ratio distortion with respect to a homeomorphism.

Definition 2.8. Let x_1, x_2, x_3 be real numbers such that $x_1 < x_2 < x_3$.

(i) The *ratio* of the triple (x_1, x_2, x_3) is the real number defined as

$$\mathrm{r}(x_1, x_2, x_3) = \frac{x_2 - x_1}{x_3 - x_2}.$$

(ii) The *ratio distortion* of the triple (x_1, x_2, x_3) with respect to an increasing function $F : \mathbb{R} \longrightarrow \mathbb{R}$ is the real number defined as

$$\mathrm{Dr}_F(x_1, x_2, x_3) = \frac{\mathrm{r}(F(x_1), F(x_2), F(x_3))}{r(x_1, x_2, x_3)} = \frac{F(x_2) - F(x_1)}{x_2 - x_1}\frac{x_3 - x_2}{F(x_3) - F(x_2)}.$$

Notice the following.

- The ratio r is translation invariant: i.e.,

$$\mathrm{r}(x_1 + a, x_2 + a, x_3 + a) = \mathrm{r}(x_1, x_2, x_3), \quad a \in \mathbb{R}.$$

- The ratio distortion Dr is multiplicative with respect to composition: for two functions F and G on $\mathbb{R}$,

$$\mathrm{Dr}_{G\circ F}(x_1, x_2, x_3) = \mathrm{Dr}_G(F(x_1), F(x_2), F(x_3))\mathrm{Dr}_F(x_1, x_2, x_3).$$

Let $z_1, z_2, z_3 \in S^1$ with lifts $\widehat{z_1}, \widehat{z_2}, \widehat{z_3}$ and $z_1 \prec z_2 \prec z_3 \prec z_1$ ordered on S^1. Define

$$\widetilde{z_i} := \begin{cases} \widehat{z_i} & \text{if } \widehat{z_1} < \widehat{z_i} < 1, \\ 1 + \widehat{z_i} & \text{if } 0 \leq \widehat{z_i} < \widehat{z_1}, \end{cases}$$

where $i = 2, 3$. Obviously, $\widetilde{z_1} < \widetilde{z_2} < \widetilde{z_3}$. The vector $(\widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3}) \in \mathbb{R}^3$ is called the *lifted vector* of $(z_1, z_2, z_3) \in (S^1)^3$.

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Now consider a homeomorphism f of S^1 with lift $\widehat{f}$. We define the *ratio distortion* of the triple (z_1, z_2, z_3), $z_1, z_2, z_3 \in S^1$ with respect to f by

$$\mathrm{Dr}_f(z_1, z_2, z_3) := \mathrm{Dr}_{\widehat{f}}(\widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3}),$$

where $(\widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3})$ is the lifted vector of (z_1, z_2, z_3). This definition is independent of the lift of f since the ratio is translation invariant.

Definition 2.9. Let $R > 1$ be a real number and let $x_0 \in S^1$. We say that a triple (z_1, z_2, z_3) of S^1 $(z_1 \prec z_2 \prec z_3)$ satisfies the conditions (a) and (b) for the point x_0 and the constant R if:

(a) $R^{-1} \le m([z_2, z_3])/m([z_1, z_2]) \le R$; and

(b) $\max_{1\le i\le 3} m([x_0, z_i]) \le Rm([z_1, z_2])$.

We call two intervals in S^1 *R-comparable* if the ratio of their lengths is in $[R^{-1}, R]$.

2.4. *Reduction.* In this subsection, we will reduce any homeomorphism $f \in \mathcal{P}(S^1)$ with several break (or non-break) points to one with break points lying on different orbits.

Definition 2.10. Let $c \in C(f)$. A *maximal f-connection* of c is a segment $[f^{-p}(c), \ldots, f^q(c)] := \{f^s(c) : -p \le s \le q\}$ of the orbit $O_f(c)$ which contains all the break points of f contained on $O_f(c)$ and such that $f^{-p}(c)$ (respectively, $f^q(c)$) is the first (respectively, last) break point of f on $O_f(c)$.

Consider the following properties.

- Two break points of f are on the same maximal f-connection if and only if they are on the same orbit.
- Two distinct maximal f-connections are disjoint.

Denote:

- $M_i(f) = [c_i, \ldots, f^{N_i}(c_i)]$, $(N_i \in \mathbb{N})$, to be the maximal f-connections of $c_i \in C(f)$, $(0 \le i \le p)$; and
- $M(f) = \coprod_{i=0}^{p} M_i(f)$.

So we have the decomposition $C(f) = \coprod_{i=0}^{p} C_i(f)$, where $C_i(f) = C(f) \cap M_i(f)$, $0 \le i \le p$. Also,

$$\prod_{d\in C_i(f)} \sigma_f(d) = \prod_{d\in M_i(f)} \sigma_f(d).$$

PROPOSITION 2.11. [**4**, Theorem 2.1] *Let $f \in \mathcal{P}(S^1)$ with irrational rotation number and let $(k_0, \ldots, k_p) \in \mathbb{Z}^{p+1}$. Then there exists a piecewise quadratic homeomorphism $K \in \mathcal{P}(S^1)$ such that $F := K \circ f \circ K^{-1} \in \mathcal{P}(S^1)$ with $C(F) \subset \{K(f^{k_i}(c_i)) = F^{k_i}(K(c_i)); i = 0, 1, \ldots, p\}$ and such that $\sigma_F(F^{k_i}(K(c_i))) = \pi_{s,O_f(c_i)}(f)$, $i = 0, 1, \ldots, p$.*

COROLLARY 2.12. [**4**, Corollary 2.3] *Let $f \in \mathcal{P}(S^1)$ with irrational rotation number. Suppose that f satisfies the KO condition. Then there exists a piecewise quadratic homeomorphism $K \in \mathcal{P}(S^1)$ such that $F = K \circ f \circ K^{-1} \in \mathcal{P}(S^1)$ with $C(F) \subset \{K(c_0), \ldots, K(c_p)\}$, where $c_0, \ldots, c_p \in C(f)$ are on pairwise distinct orbits.*

In particular, this gives the following corollary.

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COROLLARY 2.13. *Let f, K and F as in Corollary 2.12.*

(i) *If f does not satisfy the (D)-property, then there exists $0 \leq i \leq p$ such that $K(c_i)$ is the unique break point of F in its orbit.*

(ii) *If f satisfies the (D)-property then F is a C^1-diffeomorphism with* DF *absolutely continuous on S^1.*

PROPOSITION 2.14. *Let $f, g \in \mathcal{P}(S^1)$ which have the same irrational rotation number such that the break points of f (respectively, g) belong to pairwise distinct f-orbits (respectively, g-orbits). Let h be the conjugating homeomorphism from f to g. Then there exists an integer $q \geq p$, $c_0, c_1, \ldots, c_q$; $d_0, d_1, \ldots, d_q$ belonging to pairwise distinct orbits f-orbits and g-orbits, respectively, such that $C(f) \subset \{c_0, c_1, \ldots, c_q\}$, $C(g) \subset \{d_0, d_1, \ldots, d_q\}$, and a homeomorphism u of S^1 such that $G := u \circ f \circ u^{-1} \in \mathcal{P}(S^1)$ has break points belonging to pairwise distinct G-orbits and satisfies that:*

(i) $C(G) \subset \{u(c_i) : i = 0, 1, \ldots, q\}$;

(ii) $\sigma_G(u(c_i)) = \sigma_g(d_i)$, $i = 0, 1, \ldots, q$;

(iii) $\pi_s(G) = \pi_s(g)$; *and*

(iv) *h is singular if and only if so is u.*

From now on, we denote $B := \{c_0, \ldots, c_q\}$ and $c_0 = c$.

In what follows, we may assume that $f, g \in \mathcal{P}(S^1)$ have the same irrational rotation number α and satisfy the following conditions.

- The points of B belong to pairwise distinct f-orbits.
- The break points of g belong to pairwise distinct g-orbits.
- $C(f) \subset B$.
- The maps f, g satisfy the KO condition.
- The conjugating h is such that $C(g) \subset h(B)$.

3. *Primary cells*

Let $x_0 \in S^1$ and $f \in \mathcal{P}(S^1)$ with irrational rotation number $\alpha = \rho(f)$. Let $c \in B$ and let n be an odd integer. By Lemma 2.1, either $c \in \Delta^{(n-1)}_{i_n(c)}(x_0)$ for some $0 \leq i_n(c) < q_n$ or $c \in \Delta^{(n)}_{i_n(c)}(x_0)$ for some $0 \leq i_n(c) < q_{n-1}$. Set

$$y_2 = f^{-i_n(c)}(c), \quad y_1 = f^{-q_{n-1}}(y_2), \quad y_3 = f^{q_{n-1}}(y_2).$$

Notice that y_1, y_2 and y_3 are defined with respect to f, x_0, c and the number $i_n(c)$ depends on x_0. Let $\delta > 0$ and let $U_\delta(x_0)$ be a δ-neighbourhood of x_0.

PROPOSITION 3.1. (Cf. [**3**, Proposition 3.1]) *Using the above notation, there exists $N = N(x_0, \delta) \in \mathbb{N}$ such that, for all $n \geq N$, there is a triple $(y_1, y_2, y_3) = (y_1(n), y_2(n), y_3(n))_{n \geq N}$ with the following properties.*

(c-0) (y_1, y_2, y_3)(respectively, $(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))_{n \geq N}$) $\subset U_\delta(x_0)$.

(c-1) $y_2 \in \Delta^{(n-1)}_0(x_0)$ *or* $y_2 \in \Delta^{(n)}_0(x_0)$.

(c-2) $m(f^j([y_1, y_3])) \leq K\lambda^n$, *for every $0 \leq j < q_n$, where $\lambda = (1 + e^{-V})^{-1/2}$ and K a constant independent of n.*

(c-3) $\sum_{j=0}^{q_n - 1} m(f^j([y_1, y_3])) \leq 2$.

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(c-4) *The triples* (y_1, y_2, y_3) *and* $(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))$ *satisfy conditions* (*a*) *and* (*b*) *of Definition 2.9 for* x_0 *and the constant* $R = e^{3V} + e^V + 1$.

(c-5)

- $f^i([y_1, y_3])$ *does not contain* c *for every* $0 \le i \ne i_n(c) < q_n$.
- $f^{i_n(c)}([y_1, y_3])$ *does not contain any break point of* f *other than* c.

Definition. (Primary cell) The triple (y_1, y_2, y_3) given in Proposition 3.1 is called a *primary cell associated to* (f, x_0, c, δ, N) for the constant $R = e^{3V} + e^V + 1$.

In the following, set $\mathbb{N}_0 = \{n \in \mathbb{N}, n \ge N\}$.

Let $c \in B$, $d \notin O_f(c)$ and let $(y_1, y_2, y_3) := (y_1(n), y_2(n), y_3(n))_{n\in\mathbb{N}_0}$ be the primary cell associated to (f, x_0, c, δ, N). By Lemma 2.6, for every $n \in \mathbb{N}_0$, denote

$$t_n(d) = \frac{m([f^{-i_n(d)}(d), y_2])}{m([f^{j_n(d)-i_n(d)}(y_2), y_2])},$$

where $j_n(d) = \varphi_n(i_n(d))$ and $i_n(d)$ is the unique integer $0 \le i_n(d) < q_n$ such that $d \in f^{i_n(d)}([y_1, y_2])$ or $d \in f^{j_n(d)}([y_2, y_3])$.

One always has $f^{-i_n(d)}(d) \in [f^{q_n} y_1, y_2]$. Indeed, by Lemma 2.6, there exists a unique $0 \le i_n(d) < q_n$ such that either $d \in f^{i_n(d)}([y_1, y_2])$ or $d \in f^{j_n(d)}([y_2, y_3])$. In the first case, $f^{-i_n(d)}(d) \in [y_1, y_2]$, and in the second case, $f^{-i_n(d)}(d) \in [f^{q_n} y_1, f^{q_n} y_2] \subset [f^{q_n} y_1, y_2]$. One can write

$$t_n(d) = \begin{cases} \dfrac{m([f^{-i_n(d)}(d), y_2])}{m([y_1, y_2])} & \text{if } q_{n-1} \le i_n(d) < q_n, \\ \dfrac{m([f^{-i_n(d)}(d), y_2])}{m([f^{q_n} y_1, y_2])} & \text{if } 0 \le i_n(d) < q_{n-1}. \end{cases}$$

Let $d \in B\backslash\{c\}$ and let A be an infinite subset of $\mathbb{N}_0$.

- Denote by $T(d, A)$ the set of limit points of $(t_n(d))_{n\in A}$. Therefore $T(d, A) \subset [0, 1]$.

We say that d satisfies:

- the $P_1(d, A)$-property if $T(d, A) \cap]0, 1[\ne \emptyset$ and, for every $n \in A$, $0 \le i_n(d) < q_{n-1}$; and
- the $P_2(d, A)$-property if $T(d, A) \cap]0, 1[\ne \emptyset$ and, for every $n \in A$, $q_{n-1} \le i_n(d) < q_n$.

Set:

$$\begin{aligned} Q_{11}(A) &= \{d \in B\backslash\{c\} : \text{d satisfies the } P_1(d, A)\text{-property}\}; \\ Q_{12}(A) &= \{d \in B\backslash\{c\} : \text{d satisfies the } P_2(d, A)\text{-property}\}; \\ Q_1(A) &= Q_{11}(A) \cup Q_{12}(A); \\ Q_2(A) &= \{d \in B\backslash\{c\} : T(d, A) = \{0\}\}; \\ Q_3(A) &= \{d \in B\backslash\{c\} : T(d, A) = \{1\}\}; \textit{ and} \\ Q_4(A) &= \{d \in B\backslash\{c\} : T(d, A) = \{0, 1\}\}. \end{aligned}$$

PROPOSITION 3.2. *Assume that the rotation number* $\rho(f)$ *of* f *is of bounded type. Then, for every point* c, c_1 *in* $B(f)$, $c_1 \ne c$, *there exists an infinite subset* $\mathbb{M}_{c,c_1}$ *of* $\mathbb{N}_0$ *such that* $c_1 \in Q_1(\mathbb{M}_{c,c_1})$.

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Proof. Suppose that there exist two distinct points c, c_1 in $B(f)$ such that, for any infinite subset M of $\mathbb{N}_0$, $T(c_1, M) \subset \{0, 1\}$, in particular, for $M = \mathbb{N}_0$. Then there exist two subsets M_1 and M_2 of $\mathbb{N}_0$ such that $M_2 = \mathbb{N}_0 \setminus M_1$ with

$$c_1 \in \begin{cases} Q_2(M_1) & \text{if } M_2 \text{ is finite,} \\ Q_3(M_2) & \text{if } M_1 \text{ is finite,} \\ Q_2(M_1) \cap Q_3(M_2) & \text{if } M_1 \text{ and } M_2 \text{ are infinite.} \end{cases}$$

Set

$$k_n = \begin{cases} i_n(c_1) - i_n(c) & \text{if } n \in M_1, \\ j_n(c_1) - i_n(c) & \text{if } n \in M_2, \end{cases}$$

and

$$t_n^*(c_1) = \begin{cases} \dfrac{m([c_1,\, f^{k_n}(c)])}{m(\Delta_{k_n - q_{n-1}}^{(n-1)}(c))} & \text{if } n \in M_1, \\ \dfrac{m([c_1,\, f^{k_n}(c)])}{m(\Delta_{k_n}^{(n-1)}(c))} & \text{if } n \in M_2, \end{cases}$$

It is plain that $|k_{n+1} - k_n| < q_{n+1}$ for every $n \in \mathbb{N}_0$.

CLAIM 1. *We have* $\lim_{n\to+\infty} t_n^*(c_1) = 0$.

Proof. We distinguish three cases.

Case 1: M_2 is finite. Then $c_1 \in Q_2(M_1)$, i.e., $\lim_{n\to+\infty, n\in M_1} t_n(c_1) = 0$. We have

$$f^{-i_n(c_1)}(c_1) \in [f^{q_n}(y_1),\, y_2] = \Delta_0^{(n)}(y_1) \cup \Delta_0^{(n-1)}(y_1)$$

and

$$m(\Delta_0^{(n-1)}(y_1)) \le m([f^{q_n}(y_1),\, y_2]) \le (1 + e^V) m(\Delta_0^{(n-1)}(y_1)).$$

As

$$\lim_{n\to+\infty, n\in M_1} \frac{m([f^{-i_n(c_1)}(c_1),\, y_2])}{m([f^{q_n}(y_1),\, y_2])} = 0, \quad \text{then} \quad \lim_{n\to+\infty, n\in M_1} \frac{m([f^{-i_n(c_1)}(c_1),\, y_2])}{m(\Delta_0^{(n-1)}(y_1))} = 0.$$

So there is $n_1 \in \mathbb{N}$ such that, for every $n \in M_1$, $n \ge n_1$, $f^{-i_n(c_1)}(c_1) \in \Delta_0^{(n-1)}(y_1)$. Hence c_1 is an interior point of $\Delta_{k_n - q_{n-1}}^{(n-1)}(c)$ and, by Lemma 2.7, $\lim_{n\to+\infty, n\in M_1} t_n^*(c_1) = 0$.

Case 2: M_1 is finite. Then $c_1 \in Q_3(M_2)$, i.e., $\lim_{n\to+\infty, n\in M_2} t_n(c_1) = 0$. In the same manner as in Case 1, there exists $n_2 \in \mathbb{N}$ such that, for every $n \in M_2$, $n \ge n_2$, $f^{-j_n(c_1)}(c_1) \in \Delta_0^{(n-1)}(y_2)$ and $\lim_{n\to+\infty, n\in M_2} (m([f^{-j_n(c_1)}(c_1),\, y_2])/m(\Delta_0^{(n-1)}(y_2))) = 0$. Hence c_1 is an interior point of $\Delta_{k_n}^{(n-1)}(c)$ and, by Lemma 2.7, $\lim_{n\to+\infty, n\in M_2} t_n^*(c_1) = 0$.

Case 3: M_1 and M_2 are infinite. Then $c_1 \in Q_2(M_1) \cap Q_3(M_2)$, i.e., $\lim_{n\to+\infty, n\in M_1} t_n(c_1) = 0$ and $\lim_{n\to+\infty, n\in M_2} t_n(c_1) = 0$. By Cases 1 and 2, it follows that $\lim_{n\to+\infty} t_n^*(c_1) = 0$. □

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CLAIM 2. *There exists $p \in \mathbb{N}$ such that, for every $n \geq p$, c_1 is an interior point of*

$$\begin{cases} \Delta_{k_n}^{(n+2)}(c) & \text{if } n \in M_1 \text{ and } M_1 \text{ infinite}, \\ \Delta_{k_n}^{(n+3)}(c) & \text{if } n \in M_2 \text{ and } M_2 \text{ infinite}. \end{cases}$$

Proof. If $n \in M_1$ and M_1 is infinite, then c_1 is an interior point of $\Delta_{k_n - q_{n-1}}^{(n-1)}(c)$ and $\lim_{n \to +\infty, n \in M_1} t_n^*(c_1) = 0$. As the rotation number is of bounded type, then, by Lemma 2.2 and Denjoy's inequality, $\Delta_{k_n - q_{n-1}}^{(n-1)}(c)$ and $\Delta_{k_n}^{(n+2)}(c)$ are comparable, and so $\lim_{n \to +\infty, n \in M_1} (m([c_1, f^{k_n}(c)])/m(\Delta_{k_n}^{(n+2)}(c))) = 0$. Since $\Delta_{k_n}^{(n+2)}(c) \subset \Delta_{k_n - q_{n-1}}^{(n-1)}(c)$, the claim follows for this case.

If $n \in M_2$ and if M_2 is infinite, then c_1 is an interior point of $\Delta_{k_n}^{(n-1)}(c)$ and $\lim_{n \to +\infty, n \in M_2} t_n^*(c_1) = 0$. As the rotation number is of bounded type, then, by Lemma 2.2 and Denjoy's inequality, $\Delta_{k_n}^{(n-1)}(c)$ and $\Delta_{k_n}^{(n+3)}(c)$ are comparable, and so $\lim_{n \to +\infty, n \in M_2} (m([c_1, f^{k_n}(c)])/m(\Delta_{k_n}^{(n+3)}(c))) = 0$. Since $\Delta_{k_n}^{(n+3)}(c) \subset \Delta_{k_n}^{(n-1)}(c)$, the claim follows for this case. □

CLAIM 3. *For every $n \geq p$, $k_n = k_{n+1}$.*

Proof. First, we shall prove that $(n, n+1) \notin M_i \times M_i$, $i = 1, 2$. Otherwise, if $(n, n+1) \in M_1 \times M_1$, then, by Claim 2, c_1 is an interior point of $\Delta_{k_n}^{(n+2)}(c) \cap \Delta_{k_{n+1}}^{(n+3)}(c)$. Since $|k_{n+1} - k_n| < q_{n+1} < q_{n+2}$, this contradicts Lemma 2.4. If $(n, n+1) \in M_2 \times M_2$, then, by Claim 2, c_1 is an interior point of $\Delta_{k_n}^{(n+3)}(c) \cap \Delta_{k_{n+1}}^{(n+4)}(c)$. Since $|k_{n+1} - k_n| < q_{n+1} < q_{n+3}$, this contradicts Lemma 2.4. Now, if $(n, n+1) \in M_1 \times M_2$, then, by Claim 2, c_1 is an interior point of $\Delta_{k_n}^{(n+2)}(c) \cap \Delta_{k_{n+1}}^{(n+4)}(c) \subset \Delta_{k_n}^{(n+2)}(c) \cap \Delta_{k_{n+1}}^{(n+2)}(c)$. Since $|k_{n+1} - k_n| < q_{n+1} < q_{n+3}$, by Lemma 2.4, $k_n = k_{n+1}$. If $(n, n+1) \in M_2 \times M_1$, then, by Claim 2, c_1 is an interior point of $\Delta_{k_n}^{(n+3)}(c) \cap \Delta_{k_{n+1}}^{(n+3)}(c)$. Since $|k_{n+1} - k_n| < q_{n+1} < q_{n+3}$, by Lemma 2.4, $k_n = k_{n+1}$. □

We end the proof of Proposition 3.2 as follows. By Claim 1, $\lim_{n \to +\infty} t_n^*(c_1) = 0$, and by Claim 3, $k_n = k_p$ for every $n \geq p$. As $\lim_{n \to +\infty} m(\Delta_{k_n}^{(n-1)}(c)) = 0$ and $\lim_{n \to +\infty} m(\Delta_{k_n - q_{n-1}}^{(n-1)}(c)) = 0$, then $c_1 = f^p(c) \in O_f(c)$. This implies that $c_1 = c$, which is a contradiction. □

Following [**1**, Proposition 3.2], there exists an infinite subset M of $\mathbb{M}_{c,c_1}$ such that $Q_1(\mathbb{M}_{c,c_1})$, $Q_2(\mathbb{M}_{c,c_1})$ and $Q_3(\mathbb{M}_{c,c_1})$ form a partition of $B \backslash \{c\}$. In the following, we write $\mathbb{N}_0$ instead of M for simplicity, and so $B \backslash \{c\} = Q_1(\mathbb{N}_0) \sqcup Q_2(\mathbb{N}_0) \sqcup Q_3(\mathbb{N}_0)$.

- Denote $E(c_1, c) := \{c\} \cup Q_2(\mathbb{N}_0) \cup Q_3(\mathbb{N}_0)$.

For $a \in S^1$, $n \in \mathbb{N}_0$ and $\gamma > 0$, set:

- $l_n := m(\Delta^{(n)}(y_1))$;
- $V_{n,\gamma}^{(f)}(a) := [a - \gamma l_{n-1}, a + \gamma l_{n-1}]$; and
- $k_n(d) = \begin{cases} i_n(d) & \text{if } d \in \{c\} \cup Q_2(\mathbb{N}_0), \\ j_n(d) & \text{if } d \in Q_3(\mathbb{N}_0). \end{cases}$

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PROPOSITION 3.3. *There exists a positive constant $\gamma_0 > 0$ such that:*

(1) *for every $d \in E(c_1, c)$ and every $0 < \gamma < \gamma_0$, there exists $n_\gamma \in \mathbb{N}_0$ such that, for any $n \in \mathbb{N}_0$, $n \geq n_\gamma$, there exists a unique integer $0 \leq k_n(d) < q_n$ such that*

$$f^{-k_n(d)}(d) \in V_{n,\gamma}^{(f)}(f^{-i_n(c)}(c)) \subset]y_1, y_3[;$$

(2) *for every $n \in \mathbb{N}_0$, $\bigcup_{i=0}^{q_n-1} f^{-i}(B \backslash E(c_1, c)) \subset S^1 \backslash V_{n,\gamma_0}^{(f)}(f^{-i_n(c)}(c))$;*

(3) *for every $d \in Q_{11}(\mathbb{N}_0)$ and $n \in \mathbb{N}_0$, $f^{-i_n(d)}(d) \in]f^{q_n}(y_1) + \gamma_0 l_{n-1}, y_2 - \gamma_0 l_{n-1}[$; and*

(4) *for every $d \in Q_{12}(\mathbb{N}_0)$ and $n \in \mathbb{N}_0$, $f^{-j_n(d)}(d) \in]y_2 + \gamma_0 l_{n-1}, y_3 - \gamma_0 l_{n-1}[$.*

Proof. Let $d \in B \backslash \{c\}$.

Assertion (1). The intervals $f^{i_n(d)}([y_1, y_2])$ and $f^{j_n(d)}([y_2, y_3])$ are comparable (that is, there is a positive constant $0 < K < 1$ such that $K \leq m(f^{i_n(d)}([y_1, y_2]))/m(f^{j_n(d)}([y_2, y_3])) \leq K^{-1}$ for every $n \in \mathbb{N}_0$) and are comparable to their union $[f^{j_n(d)}(y_2), f^{i_n(d)}(y_2)]$ which implies, by Lemma 2.7 and Denjoy's inequality, that the four intervals $[y_1, y_2]$, $[y_2, y_3]$, $[f^{j_n(d)-i_n(d)}(y_2), y_2]$, $[y_2, f^{i_n(d)-j_n(d)}(y_2)]$ are comparable. In particular, the ratios $m([f^{-i_n(d)}(d), y_2])/m([y_1, y_2])$ and $m([f^{-i_n(d)}(d), y_2])/m([f^{j_n(d)-i_n(d)}(y_2), y_2]) = t_n(d)$ are comparable. So, for $d \in Q_2(\mathbb{N}_0)$, $\lim_{n\to+\infty} t_n(d) = 0$, which is equivalent to the assertion that

$$\begin{aligned}&\text{for all } \gamma > 0, \text{ there exists } n_\gamma \in \mathbb{N}_0\\ &\quad\text{such that, for every } n \geq n_\gamma,\ n \in \mathbb{N}_0 : f^{-i_n(d)}(d) \in V_{n,\gamma}^{(f)}(y_2).\end{aligned}$$

Similarly, the ratios $m([y_2, f^{-j_n(d)}(d)])/m([y_1, y_2])$ and $m([y_2, f^{-j_n(d)}(d)])/m([y_2, f^{i_n(d)-j_n(d)}(y_2)])$ are comparable, which is comparable to the ratio $m([f^{j_n(d)-i_n(d)}(y_2), f^{-i_n(d)}(d)])/m([f^{j_n(d)-i_n(d)}(y_2), y_2]) = 1 - t_n(d)$ (by Lemma 2.7). So, for $d \in Q_3(\mathbb{N}_0)$, $\lim_{n\to+\infty} t_n(d) = 1$, which is equivalent to the assertion that

$$\begin{aligned}&\text{for all } \gamma > 0, \text{ there exists } n_\gamma \in \mathbb{N}_0\\ &\quad\text{such that, for every } n \geq n_\gamma,\ n \in \mathbb{N}_0 : f^{-j_n(d)}(d) \in V_{n,\gamma}^{(f)}(y_2).\end{aligned}$$

Assertion (2). Let $d \in Q_1(\mathbb{N}_0)$. There exists $0 < \gamma(d) < 1$ such that, for every $n \in \mathbb{N}_0$, $t_n(d) \geq \gamma(d)$ and $1 - t_n(d) \geq \gamma(d)$. Since $Q_1(\mathbb{N}_0)$ is finite, there exists $0 < \gamma_0 < 1$ and an infinite subset M_2 of $\mathbb{N}_0$ ($\mathbb{N}_0 \backslash M_2$ is finite) such that the points $f^{-i_n(d)}(d)$ and $f^{-j_n(d)}(d)$ are contained in the set $S^1 \backslash V_{n,\gamma_0}^{(f)}(y_2)$ for all $d \in Q_1(M_2)$ and $n \in M_2$. By Lemma 2.6, $d \notin f^i([y_1, y_2]) \cup f^j([y_2, y_3])$ for every $0 \leq i < q_n$, $i \neq i_n(d)$, $j = \varphi_n(i)$. Hence, for every $0 \leq i < q_n$, $i \neq i_n(d) : f^{-i}(d) \notin [y_1, y_2]$, and for every $0 \leq j < q_n$, $j \neq j_n(d) = \varphi_n(i_n(d)) : f^{-j}(d) \notin [y_2, y_3]$. In particular, for every $0 \leq i < q_n$, $i \neq i_n(d)$, $i \neq j_n(d) : f^{-i}(d) \notin [y_1, y_3]$. So $f^{-i}(d) \notin V_{n,\gamma_0}^{(f)}(y_2)$ since $V_{n,\gamma_0}^{(f)}(y_2) \subset [y_1, y_3]$. Then $\bigcup_{0\leq i<q_n} f^{-i}(B \backslash E(c_1, c)) \subset S^1 \backslash V_{n,\gamma_0}^{(f)}(y_2)$.

Write $\mathbb{N}_0$ instead of M_2.

Assertion (3). Let $d \in Q_{11}(\mathbb{N}_0)$. Then there exists an integer $n(d) \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0 (n \geq n(d))$, $0 \leq i_n(d) < q_{n-1}$,

$$\begin{aligned}f^{-i_n(d)}(d) &\in [y_1, y_2] \cup f^{j_n(d)-i_n(d)}([y_2, y_3])\\ &= [y_1, y_2] \cup [f^{q_n}(y_1), f^{q_n}(y_2)] = [f^{q_n}(y_1), y_2]\end{aligned}$$

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and

$$\gamma_0 \le t_n(d) = \frac{m([f^{-i_n(d)}(d),\, y_2])}{m([f^{q_n}(y_1),\, y_2])} \le 1 - \gamma_0.$$

Since the intervals $[f^{q_n}(y_1),\, y_2]$ and $[y_1,\, y_2]$ are comparable, there exists $0 < \gamma_1 < \gamma_0$ and an integer $n'(d) \in \mathbb{N}_0$ such that, for all $d \in \mathbb{N}_0$ and $n \ge n'(d)$, $f^{-i_n(d)}(d) \in [f^{q_n}(y_1) + \gamma_1 l_{n-1},\, y_2 - \gamma_1 l_{n-1}]$. Since $Q_{11}(\mathbb{N}_0)$ is finite, there exists an integer $n_1 \in \mathbb{N}_0$ such that, for all $d \in Q_{11}(\mathbb{N}_0)$ and $n \in \mathbb{N}_0$, $n \ge n_1$,

$$f^{-i_n(d)}(d) \in [f^{q_n}(y_1) + \gamma_1 l_{n-1},\, y_2 - \gamma_1 l_{n-1}].$$

Assertion (4). Let $d \in Q_{12}(\mathbb{N}_0)$. Then there exists an integer $n(d) \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0$, $n \ge n(d)$, $q_{n-1} \le i_n(d) < q_n$, $f^{-j_n(d)}(d) \in [y_2,\, y_3]$ and

$$\gamma_0 \le t_n(d) = \frac{m([f^{-i_n(d)}(d),\, y_2])}{m([y_1,\, y_2])} \le 1 - \gamma_0.$$

Since the ratios

$$\frac{m([f^{-j_n(d)}(d),\, y_3])}{m([y_2,\, y_3])} \quad \text{and} \quad \frac{m([f^{-i_n(d)}(d),\, y_2])}{m([y_1,\, y_2])}$$

are comparable, there exists $0 < \gamma_2 < \gamma_0$ such that $f^{-j_n(d)}(d) \in [y_2 + \gamma_2 l_{n-1},\, y_3 - \gamma_2 l_{n-1}]$ for all $d \in Q_{12}(\mathbb{N}_0)$. Since $Q_{12}(\mathbb{N}_0)$ is finite, there exists an integer $n_2 \in \mathbb{N}_0$ such that, for all $d \in Q_{12}(\mathbb{N}_0)$ and $n \in \mathbb{N}_0$, $n \ge n_2$,

$$f^{-j_n(d)}(d) \in [y_2 + \gamma_2 l_{n-1},\, y_3 - \gamma_2 l_{n-1}]. \qquad \square$$

COROLLARY 3.4. *For every $d \in E(c_1, c)$,*

$$\lim_{n \to +\infty, n \in \mathbb{N}_0} \frac{m([f^{-k_n(d)}(d),\, y_2])}{m([y_1,\, y_2])} = 0.$$

Proof. It is a consequence of Proposition 3.3. $\square$

4. *Secondary cells*

In this section, we introduce a new triple (z_1, z_2, z_3) in $U_\delta(x_0)$ depending on two real parameters β and γ. Let $\beta, \gamma \in]0, \gamma_0[(\beta < \gamma)$. For $n \in \mathbb{N}_0$ $(n \ge n_\beta)$, we define the points $z_1, z_2, z_3 \in S^1$ $(z_1 \prec z_2 \prec z_3)$ as $z_2 = y_2$ and $z_1, z_3 \in V^{(f)}_{n,\gamma}(z_2) \setminus V^{(f)}_{n,\beta}(z_2)$, i.e.,

$$\beta \le \frac{m([z_1, z_2])}{m([y_1, y_2])} \le \gamma \quad \text{and} \quad \beta \le \frac{m([z_2, z_3])}{m([y_1, y_2])} \le \gamma.$$

Definition. (Secondary cell) The triple (z_1, z_2, z_3) is called a (β, γ)-derived cell associated to $(f, x_0, c, \mathbb{N}_0, \delta)$.

PROPOSITION 4.1. *Using the above notation, for every $n \in \mathbb{N}_0$, $n \ge n_\beta$:*

(0) $[z_1, z_3] \subset V^{(f)}_{n,\gamma}(y_2) \subset [y_1, y_3]$;

(1) *for any $d \in E(c_1, c)$, there exists a unique $0 \le k_n(d) < q_n$ such that $f^{-k_n(d)}(d) \in V^{(f)}_{n,(1/4)\beta}(y_2)$;*

(2) *$m(f^j([z_1, z_3])) \le K\lambda^n$, for every $0 \le j < q_n$, where $\lambda = (1 + e^{-V})^{-1/2}$ and K is a constant independent of n, $V = \mathrm{Var}(\log(\mathrm{Df}))$;*

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(3) $\sum_{j=0}^{q_n-1} m(f^j([z_1, z_3])) \le 2$;
(4) *the triples* (z_1, z_2, z_3) *and* $(f^{q_n}(z_1), f^{q_n}(z_2), f^{q_n}(z_3))$ *satisfy conditions* (a) *and* (b) *of Definition 2.9 for* x_0 *and the constant* $R(\beta) = Re^{2V}/\beta$, *where* $R = e^{3V} + e^V + 1$;
(5) $f^i([z_1, z_3])$ *does not contain any point of* B *for every* $0 \le i < q_n$ *and* $i \ne k_n(d)$, $d \in E(c_1, c)$; *and*
(6) *for every* $d \in E(c_1, c)$, $d \in f^{k_n(d)}(]z_1, z_3[)$.

Proof. Assertion (0). This results from the definition of (z_1, z_2, z_3), for small γ.

Assertion (1). This follows from assertion (1) in Proposition 3.3.

Assertions (2) and (3) hold since $[z_1, z_3] \subset [y_1, y_3]$ and by applying the same proof of the properties (c-2) and (c-3) of Proposition 3.1 to (z_1, z_2, z_3) instead of (y_1, y_2, y_3) .

Assertion (4). We have

$$\frac{\beta}{\gamma} = \frac{\beta m([y_1, y_2])}{\gamma m([y_1, y_2])} \le \frac{m([z_2, z_3])}{m([z_1, z_2])} \le \frac{\gamma m([y_1, y_2])}{\beta m([y_1, y_2])} = \frac{\gamma}{\beta}.$$

Then, by Lemma 2.7,

$$\begin{aligned} e^{-2V}\frac{\beta}{\gamma} = e^{-2V}\frac{\beta m([y_1, y_2])}{\gamma m([y_1, y_2])} \le e^{-2V}\frac{m([z_2, z_3])}{m([z_1, z_2])} \le \frac{m(f^{q_n}([z_2, z_3]))}{m(f^{q_n}([z_1, z_2]))} \\ \le e^{2V}\frac{m([z_2, z_3])}{m([z_1, z_2])} \le e^{2V}\frac{\gamma m([y_1, y_2])}{\beta m([y_1, y_2])} = e^{2V}\frac{\gamma}{\beta}. \end{aligned}$$

Hence (z_1, z_2, z_3) satisfies conditions (a) and (b) of Definition 2.9 for the constant $R(\beta, \gamma) = (R/\beta)e^{2V}$. On the other hand, since $[z_1, z_3] \subset [f^{q_n}(y_1), y_3]$, $x_0 \in [f^{q_n}(y_1), y_3]$ and $\max(m([f^{q_n}(y_1), x_0]); m([x_0, y_3])) \le Rm([y_1, y_2])$, it follows that

$$\begin{aligned} \max_{1\le i\le 3}(m([x_0, z_i]) &\le \max(m([f^{q_n}(y_1), x_0]); m([x_0, y_3])) \\ &\le Rm([y_1, y_2]) \le \frac{R}{\beta}m([z_1, z_2]). \end{aligned}$$

Furthermore, as $[z_1, z_3] \subset]y_1, y_3[$, $f^{q_n}([z_1, z_3]) \subset f^{q_n}(]y_1, y_3[)$. Since $x_0 \in [f^{q_n}(y_1), y_3]$ and $f^{q_n}(]y_1, y_3[) \cup f^{q_n-1}(]f^{q_n}(y_1), y_2[) \cup f^{q_n-q_{n-1}}(]y_2, y_3[) \subset [f^{q_n}(y_1), y_3]$,

$$\begin{aligned} \max_{1\le i\le 3} m([f^{q_n}(z_i), x_0]) &\le \max(m([f^{q_n}(y_1), x_0]); m([x_0, y_3])) \le \frac{R}{\beta}m([z_1, z_2]) \\ &\le \frac{R}{\beta}e^{2V}m(f^{q_n}([z_1, z_2])) \text{(by Denjoy's inequality)}. \end{aligned}$$

Therefore $(f^{q_n}(z_1), f^{q_n}(z_2), f^{q_n}(z_3))$ satisfies conditions (a) and (b) of Definition 2.9 for the constant $R(\beta, \gamma) = (R/\beta)e^{2V}$.

Assertion (5). If $d \in B$ and $0 \le i < q_n$ is an integer such that $d \in f^i([z_1, z_3])$, then $f^{-i}(d) \in [z_1, z_3] \subset V^{(f)}_{n,\gamma}(y_2) \subset V^{(f)}_{n,\gamma_0}(y_2)$. According to assertion (2) of Proposition 3.3, we obtain $f^{-i}(d) \notin \bigcup_{j=0}^{q_n-1} f^{-j}(B\backslash E(c_1, c))$, $0 \le i < q_n$. In particular, $f^{-i}(d) \notin f^{-i}(B\backslash E(c_1, c))$. Hence $d \in E(c_1, c)$ such that $0 \le i < q_n$ and $f^{-i}(d) \in V^{(f)}_{n,\gamma}(y_2)$. By assertion (1), $i = k_n(d)$.

Assertion (6) is a consequence of assertions (0) and (1). □

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PROPOSITION 4.2. [**1**, Proposition 4.2] *Let* $(y_1, y_2, y_3) \subset U_\delta(x_0)$ *be the primary cell and let* (z_1, z_2, z_3) *be a* (β, γ) *secondary cell associated to* $(f, x_0, c, \mathbb{N}_0)$. *Let* h *be the homeomorphism conjugating* f *to* g *and assume that it admits in* x_0 *a positive derivative* $Dh(x_0) > 0$. *Then the following properties hold.*

(1) $(h(y_1), h(y_2), h(y_3))$ *is a primary cell associated to* $(g, h(x_0), h(c), \mathbb{N}_0)$.

(2) $(h(z_1), h(z_2), h(z_3))$ *is a* $(\beta/2, 2\gamma)$*-derived cell associated to* $(g, h(x_0), h(c), \mathbb{N}_0)$.

(3) *There is an integer* $n_\gamma \in \mathbb{N}_0$ *such that, for every* $n \in \mathbb{N}_0$, $n \geq n_\gamma$,

$$h(V^{(f)}_{n,\gamma}(y_2)) \subset V^{(g)}_{n,2\gamma}(h(y_2)) \subset h(V^{(f)}_{n,4\gamma}(y_2)).$$

PROPOSITION 4.3. *Using the notation of Proposition 4.2, for every* $n \in \mathbb{N}_0$, $n \geq n_\beta$:

(h-0) $[h(z_1), h(z_3)] \subset V^{(g)}_{n,2\gamma}(h(y_2)) \subset [h(y_1), h(y_3)]$;

(h-1) *for any* $d \in E(c_1, c)$, $k_n(d)$ *is the unique integer in* $[0, q_n[$ *such that* $g^{-k_n(d)}(h(d)) \in V^{(g)}_{n,(1/2)\beta}(h(y_2))$;

(h-2) $m(g^j([h(z_1), h(z_3)])) \leq K'(\lambda')^n$, *for every* $0 \leq j < q_n$, *where* $\lambda' = (1 + e^{-V'})^{-1/2}$ *and* K' *is a constant independent of* n, $V' = \mathrm{Var}(\log(Dg))$;

(h-3) $\sum_{j=0}^{q_n-1} m(g^j([h(z_1), h(z_3)])) \leq 2$;

(h-4) $g^i([h(z_1), h(z_3)])$ *does not contain any break point of* $h(B)$ *for every* $0 \leq i < q_n$ *and* $i \neq k_n(d)$, $d \in E(c_1, c)$;

(h-5) *for every* $d \in E(c_1, c)$,

$$\lim_{n \to +\infty} \frac{m([g^{-k_n(d)}(h(d)), h(y_2)])}{m([h(y_1), h(y_2)])} = 0;$$

(h-6) *for every* $d \in E(c_1, c)$, $h(d) \in g^{k_n(d)}(]h(z_1), h(z_3)[)$.

Proof. Assertion (h-0) is a consequence of Proposition 4.2(3).

(h-1). Let $d \in E(c_1, c)$. By Proposition 4.1(1), there exists $0 \leq k_n(d) < q_n$ such that $f^{-k_n(d)}(d) \in V^{(f)}_{n,(1/4)\beta}(z_2)$. Then $g^{-k_n(d)}(h(d)) = h(f^{-k_n(d)}(d)) \in h(V^{(f)}_{n,(1/4)\beta}(z_2)) \subset V^{(g)}_{n,(1/2)\beta}(h(z_2))$ for every $n \geq n''_\beta$, $n \in \mathbb{N}_0$. Uniqueness: if $0 \leq i, j < q_n$ satisfy $g^{-i}(h(d))$, $g^{-j}(h(d)) \in V^{(g)}_{n,2\gamma}(h(z_2))$, then $f^{-i}(d)$, $f^{-j}(d) \in h^{-1}(V^{(g)}_{n,\gamma}(h(z_2)) \subset V^{(f)}_{n,2\gamma}(z_2)$. Therefore, by Proposition 4.1(1), $i = j$.

(h-2) and (h-3). By Proposition 4.2(2), $(h(z_1), h(z_2), h(z_3))$ is a $(\beta/2, 2\gamma)$-derived cell associated to $(g, h(x_0), h(c), \mathbb{N}_0)$. Therefore the proof is similar to that of Proposition 4.1(2) and (3), with f replaced by g.

(h-4). Let $0 \leq i < q_n$ and assume that $g^i([h(z_1), h(z_3)])$ contains a break point $d' = h(d)$ of g for a break point d of f. Then $f^i([z_1, z_3])$ contains the break point d of f. By Proposition 4.1(1) and (5), $d \in E(c_1, c)$ and $i = k_n(d)$.

(h-5). Let $d \in E(c_1, c)$ and $\gamma > 0$. By Proposition 3.1(6), there is an integer $n'_\gamma \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0$, $n \geq n'_\gamma$, $f^{-k_n(d)}(d) \in V^{(f)}(y_2)$. On the other hand, by Proposition 4.2(3), there is an integer $n_\gamma \in \mathbb{N}_0$, $n_\gamma \geq n'_\gamma$ such that, for every $n \in \mathbb{N}_0$, $n \geq n_\gamma$, $h(V^{(f)}_{n,(\gamma/2)}(y_2)) \subset V^{(g)}_{n,\gamma}(h(y_2))$. As $g^{-k_n(d)}(h(d)) = h(f^{-k_n(d)}(d))$, it follows that $g^{-k_n(d)}(h(d)) \in V^{(g)}_{n,\gamma}(h(y_2))$ for every $n \geq n_\gamma$, $n \in \mathbb{N}_0$. Since $\gamma > 0$ is arbitrary,

$$\lim_{n \to +\infty} \frac{m([g^{-k_n(d)}(h(d)), h(y_2)])}{m([h(y_1), h(y_2)])} = 0.$$

(h-6) is a consequence of Assertions (h-0) and (h-1). □

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5. *Control of distortions*

PROPOSITION 5.1. [**1**, Proposition 5.1] *Let $\mathbb{N}_0$ and γ_0 be as in Proposition 3.3. Let $\beta, \gamma \in]0, \gamma_0[(\beta < \gamma)$. Assume that the conjugation homeomorphism h from f to g admits at x_0 a positive derivative $\mathrm{Dh}(x_0) = \omega_0 > 0$. Then there exists an integer $n_\beta \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0$, $n \geq n_\beta$,*

$$\left| \frac{\mathrm{Dr}_{g^{q_n}}(h(z_1), h(z_2), h(z_3))}{\mathrm{Dr}_{f^{q_n}}(z_1, z_2, z_3)} - 1 \right| \leq \frac{2}{1-\gamma_0}\beta.$$

The next proposition is another distortion control which is opposite to the one of Proposition 5.1. This allows us to prove that the conjugation from f to g is singular with respect to the Lebesgue measure.

PROPOSITION 5.2. *Assume that the irrational rotation number of f is of bounded type. Let c, c_1 in $B(f)$, $c \neq c_1$, $\mathbb{N}_0$ and let γ_0 be as in Proposition 3.3. Then, for any $\varepsilon > 0$, there exists $0 < \gamma < \gamma_0$ such that, for any $\beta \in]0, \gamma[$, there exists $n_\beta \in \mathbb{N}_0$ such that, for all $n \geq n_\beta$, $n \in \mathbb{N}_0$, the (β, γ) secondary cell (z_1, z_2, z_3) associated to $(f, x_0, c, \mathbb{N}_0, \delta)$ satisfies the inequality*

$$\left| \frac{Dr_{g^{q_n}}(h(z_1), h(z_2), h(z_3))}{Dr_{f^{q_n}}(z_1, z_2, z_3)} - \Pi(c_1, c) \right| \leq A\varepsilon \quad \text{for some constant } A > 0,$$

where

$$\Pi(c_1, c) = \prod_{d \in E(c_1, c)} \frac{\sigma_g(h(d))}{\sigma_f(d)}.$$

The proof of Proposition 5.2 is an elaboration of the proof of [**1**, Proposition 5.3] and so we only describe the changes that are necessary.

LEMMA 5.3. *Assume that the rotation number α of f is of bounded type. Let $c_1 \in C(f)\backslash\{c\}$, u_0, γ_{c,c_1} and let $\mathbb{N}_0$ be as in Proposition 3.3. Let h be the conjugation from f to R_α: $h \circ f = R_\alpha \circ h$. Assume that $Dh(x_0) > 0$. Then, for any $\varepsilon > 0$, there exists $\eta > 0$ such that, for any $u \in]0, \eta[$, there exists $n_{u,r} \in \mathbb{N}_{c,c_1}$ such that for, any $n \in \mathbb{N}_{c,c_1}$, $n \geq n_{u,r}$, the (r, u)-derived cell (z_1, z_2, z_3) associated to (f, x_0, c) satisfies*

$$|Dr_{f^{q_n}}(z_1, z_2, z_3) - \Pi_f(c, c_1)| \leq C_1\varepsilon,$$

where

$$\Pi_f(c, c_1) = \prod_{b \in E(c_1, c)} \sigma_f(b)$$

and C_1 is a positive constant.

Proof. Let $\varepsilon > 0$. Since Df is absolutely continuous on every interval $[c_i, c_{i+1}](0 \leq i \leq p)$, there exists a real $0 < \eta_0 < e^{2V}\gamma_0$ such that, for every disjoint interval $([a_j, b_j])_{j \in J}$ of continuity intervals of Df (these later are the $[c_i, c_{i+1}](0 \leq i \leq p)$) such that if $\sum_{j \in J} |a_j - b_j| < \eta_0$, then $\sum_{j \in J} |\mathrm{Df}(a_j) - \mathrm{Df}(b_j)| < \varepsilon$. Let $0 < \beta < \gamma < \frac{1}{5}e^{-2V}\eta_0$ and let (z_1, z_2, z_3) be the (β, γ)-derived cell associated to $(f, x_0, c, \mathbb{N}_0)$. For $n \in \mathbb{N}_0$, denote the following.

- $I := \{0, 1, \ldots, q_n - 1\}\backslash\{k_n(d) : d \in E(c_1, c)\}$.
- $D_i(f) := \mathrm{Dr}_f(f^i(z_1), f^i(z_2), f^i(z_3)), 0 \leq i < q_n$.

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- $P := \prod_{i\in I} D_i(f)$; $P_1 := \prod_{d\in E(c_1,c)} D_{k_n(d)}(f)$.
- $D_n := \mathrm{Dr}_{f^{q_n}}(z_1, z_2, z_3)$.

One has $D_n = P P_1$.

Step 1. We consider $D_i(f)$, $i \in I$. By Proposition 4.1(5), Df is continuous on the interval $[f^i(z_1),\ f^i(z_3)]$. Then, by the mean value theorem, there exist $a_i \in]f^i(z_1),\ f^i(z_2)[$ and $b_i \in]f^i(z_2),\ f^i(z_3)[$ such that $D_i(f) = \mathrm{Df}(a_i)/\mathrm{Df}(b_i)$. Since $([a_i,\ f^i(z_2)])_{i\in I}$ are disjoint intervals of continuity of Df, by Lemma 2.7, they satisfy the inequalities

$$\begin{aligned}\sum_{i\in I} m([a_i,\ f^i(z_2)]) &\le \sum_{i\in I} m([f^i(z_1),\ f^i(z_2)])\\ &\le \sum_{i\in I} e^{2V}\frac{m([z_1, z_2])}{m([y_1, y_2])} m([f^i(y_1),\ f^i(y_2)])\\ &\le e^{2V}\gamma \sum_{i\in I} m([f^i(y_1),\ f^i(y_2)]) \le e^{2V}\gamma \le \eta_0.\end{aligned}$$

Consequently,

$$\sum_{i\in I} |\mathrm{Df}(f^i(z_2)) - \mathrm{Df}(a_i)| \le \varepsilon. \tag{5.1}$$

Also $([f^i(z_2),\ b_i])_{i\in I}$ are disjoint intervals of continuity of Df, so, by Lemma 2.7, they satisfy

$$\begin{aligned}\sum_{i\in I} m([f^i(z_2), b_i]) &\le \sum_{i\in I} m([f^i(z_2),\ f^i(z_3)])\\ &\le \sum_{i\in I} e^{2V}\frac{m([z_2, z_3])}{m([y_1, y_2])} m([f^i(y_1),\ f^i(y_2)])\\ &\le e^{2V}\gamma \sum_{i\in I} m([f^i(y_1),\ f^i(y_2)]) \le \eta_0.\end{aligned}$$

Consequently,

$$\sum_{i\in I} |\mathrm{Df}(f^i(z_2)) - \mathrm{Df}(b_i)| \le \varepsilon. \tag{5.2}$$

Combining (5.1) and (5.2),

$$\sum_{i\in I} |\mathrm{Df}(b_i) - \mathrm{Df}(a_i)| \le 2\varepsilon. \tag{5.3}$$

Set $D_i^*(f) := \max(D_i(f), (D_i(f))^{-1})$ and $P^* := \prod_{i\in I} D_i^*(f)$.

Since Df takes its values in $[m_1, M_1]$ $(m_1 > 0)$, the following properties hold.

- $0 \le D_i^*(f) - 1 \le (1/m_1)|\mathrm{Df}(b_i) - \mathrm{Df}(a_i)|$.
- $\sum_{i\in I} |D_i^*(f) - 1| \le (2/m_1)\varepsilon$.
- $|\log D_i(f)| = \log D_i^*(f) \le D_i^*(f) - 1$.
- $1/P^* \le P \le P^*$.

It follows that

$$e^{-C_0\varepsilon} \le P \le e^{C_0\varepsilon} \quad \text{where } C_0 = \frac{2}{m_1}. \tag{5.4}$$

Step 2. We consider $D_{k_n(d)}(f)$, $d \in E(c_1, c)$. We know that $d \in]f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_3)[$.

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Case 2a. $d \in]f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)[$ with $k_n(d) = i_n(d)$. By Proposition 4.1(5), Df is continuous on the intervals $[f^{k_n(d)}(z_1),\ d]$, $[d,\ f^{k_n(d)}(z_2)]$ and $[f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)]$. By the mean value theorem, there exist $t_1 \in]f^{k_n(d)}(z_1),\ d[$, $t_2 \in]d,\ f^{k_n(d)}(z_2)[$ and $t_3 \in]f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)[$ such that

$$D_{k_n(d)}(f) = (1 - r_n(d))\frac{\mathrm{Df}(t_1)}{\mathrm{Df}(t_3)} + r_n(d)\frac{\mathrm{Df}(t_2)}{\mathrm{Df}(t_3)},$$

where

$$r_n(d) = \frac{m([d,\ f^{k_n(d)}(z_2)])}{m([f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)])}.$$

We have

$$|D_{k_n(d)}(f) - \sigma_f(d)| \le \left|\frac{\mathrm{Df}(t_1)}{\mathrm{Df}(t_3)} - \frac{D_-f(d)}{D_+f(d)}\right| + r_n(d)\frac{|\mathrm{Df}(t_1) - \mathrm{Df}(t_2)|}{\mathrm{Df}(t_3)}.$$

Since

$$\left|\frac{\mathrm{Df}(t_1)}{\mathrm{Df}(t_3)} - \frac{D_-f(d)}{D_+f(d)}\right| \le \frac{M_1}{m_1^2}(|\mathrm{Df}(t_1) - D_-f(d)| + |\mathrm{Df}(t_3) - D_+f(d)|),$$

by Lemma 2.7,

$$\begin{aligned} r_n(d) &= \frac{m(f^{k_n(d)}([f^{-k_n(d)}(d),\ z_2]))}{m(f^{k_n(d)}([z_1,\ z_2])} \le e^{2V}\frac{m([f^{-k_n(d)}(d),\ z_2])}{m([z_1,\ z_2])} \\ &\le e^{2V}\frac{m([y_1,\ y_2])}{m([z_1,\ z_2])} \times \frac{m([f^{-k_n(d)}(d),\ z_2])}{m([y_1,\ y_2])} \le \frac{e^{2V}}{\beta}s_n(d) \end{aligned}$$

with

$$s_n(d) := \frac{m([f^{-k_n(d)}(d),\ z_2])}{l([y_1,\ y_2])}.$$

It follows that

$$|D_{k_n(d)}(f) - \sigma_f(d)| \le K_0\left(\frac{s_n(d)}{\beta} + |\mathrm{Df}(t_1) - D_-f(d)| + |\mathrm{Df}(t_3) - D_+f(d)|\right),$$

where $K_0 = \max((M_1/m_1)e^{2V},\ M_1/m_1^2)$.

Since $\lim_{n\to+\infty} s_n(d) = 0$ (Corollary 3.4), there exists $n_{d,\beta} \in \mathbb{N}_0$ such that, for any $n \in \mathbb{N}_0$, $n \ge n_{d,\beta}$, $s_n(d) < \beta\varepsilon$. The intervals $(]t_1,\ d[;\ ]d,\ t_3[)_{d\in E(c_1,c)}$ are disjoint intervals of continuity of Df; they satisfy, by Lemma 2.7,

$$\begin{aligned} m([t_1,\ d]) + m([d,\ t_3]) &\le 2m([f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)]) + m([f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)]) \\ &\le 3e^{2V}\gamma m([f^{k_n(d)}(y_1),\ f^{k_n(d)}(y_2)]) \le 3e^{2V}\gamma < \eta_0. \end{aligned}$$

So

$$|\mathrm{Df}(t_1) - D_-f(d)| + |\mathrm{Df}(t_3) - D_+f(d)| < \varepsilon.$$

Hence there exists $n_{d,\beta} \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0$, $n \ge n_{d,\beta}$,

$$|D_{k_n(d)}(f) - \sigma_f(d)| \le 2K_0\varepsilon. \tag{5.5}$$

Case 2b. $d \in]f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)[$ with $k_n(d) = j_n(d)$. By Proposition 4.1(b-5), Df is continuous on the intervals $[f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)]$, $[f^{k_n(d)}(z_2),\ d]$ and $[d,\ f^{k_n(d)}(z_3)]$.

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By the mean value theorem, there exist $s_1 \in]d,\ f^{k_n(d)}(z_3)[$, $s_2 \in]f^{k_n(d)}(z_2),\ d[$ and $s_3 \in]f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)[$ such that

$$D_{k_n(d)}^{-1}(f) = (1 - r_n'(d))\frac{\mathrm{Df}(s_1)}{\mathrm{Df}(s_3)} + r_n'(d)\frac{\mathrm{Df}(s_2)}{\mathrm{Df}(s_3)},$$

where

$$r_n'(d) = \frac{m([f^{k_n(d)}(z_2),\ d])}{m([f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)])}.$$

As in Case 2a,

$$|D_{k_n(d)}(f)^{-1} - \sigma_f(d)^{-1}| \le \left|\frac{\mathrm{Df}(s_1)}{\mathrm{Df}(s_3)} - \frac{D_+f(d)}{D_-f(d)}\right| + r_n'(d)\frac{|\mathrm{Df}(s_1) - \mathrm{Df}(s_2)|}{\mathrm{Df}(s_3)}.$$

Since

$$\left|\frac{\mathrm{Df}(s_1)}{\mathrm{Df}(s_3)} - \frac{D_+f(d)}{D_-f(d)}\right| \le \frac{M_1}{m_1^2}(|\mathrm{Df}(s_1) - D_+f(d)| + |\mathrm{Df}(s_3) - D_-f(d)|),$$

by Lemma 2.7,

$$\begin{aligned} r_n'(d) &= \frac{m(f^{k_n(d)}([z_2,\ f^{-k_n(d)}(d)]))}{m(f^{k_n(d)}([z_2,\ z_3])} \le e^{2V}\frac{m([z_2,\ f^{-k_n(d)}(d)])}{m([z_2,\ z_3])} \\ &\le e^{2V}\frac{m([y_1,\ y_2])}{m([z_2,\ z_3])} \times \frac{m([z_2,\ f^{-k_n(d)}(d)])}{m([y_1,\ y_2])} \le \frac{e^{2V}}{\beta}s_n'(d), \end{aligned}$$

where

$$s_n'(d) := \frac{m([z_2,\ f^{-k_n(d)}(d)])}{m([y_1,\ y_2])}.$$

It follows that

$$|D_{k_n(d)}(f)^{-1} - \sigma_f(d)^{-1}| \le K_0\left(\frac{s_n'(d)}{\beta} + |\mathrm{Df}(s_1) - D_+f(d)| + |\mathrm{Df}(s_3) - D_-f(d)|\right),$$

where $K_0 = \max((M_1/m_1)e^{2V},\ M_1/m_1^2)$. By Corollary 3.4, $\lim_{n\to+\infty} s_n'(d) = 0$ and hence there exists $n_{d,\beta} \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0$, $n \ge n_{d,\beta}$, $s_n'(d) < \beta\varepsilon$. The intervals $(]d,\ s_1[;\ ]s_3,\ d[)_{d\in E(c_1,c)}$ are disjoint intervals of continuity of Df. They satisfy, by Lemma 2.7,

$$\begin{aligned} m([d,\ s_1]) + m([s_3,\ d]) &\le m([f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)]) + 2m([f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)]) \\ &\le 3e^{2V}\gamma m([f^{k_n(d)}(y_1),\ f^{k_n(d)}(y_2)]) \le 3e^{2V}\gamma < \eta_0. \end{aligned}$$

So

$$|\mathrm{Df}(s_1) - D_+f(d)| + |\mathrm{Df}(s_3) - D_-f(d)| < \varepsilon.$$

Hence there exists $n_{d,\beta} \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0$, $n \ge n_{d,\beta}$,

$$|D_{k_n(d)}(f)^{-1} - \sigma_f(d)^{-1}| \le 2K_0\varepsilon. \tag{5.6}$$

Therefore, from Cases 2a and 2b, we conclude that there exists $n_{d,\beta} \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0$, $n \ge n_{d,\beta}$,

$$|D_{k_n(d)}(f) - \sigma_f(d)| \le K_0\varepsilon. \tag{5.7}$$

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Since $E(c_1, c)$ is finite, there exists $n_\beta \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0$, $n \geq n_\beta$,

$$\left| \prod_{d \in E(c_1,c)} D_{k_n(d)}(f) - \prod_{d \in E(c_1,c)} \sigma_f(d) \right| \leq \varepsilon. \tag{5.8}$$

Hence, (5.4) and (5.8) imply that there exist $n_\beta \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0$, $n \geq n_\beta$,

$$|\mathrm{Dr}_{f^{q_n}}(z_1, z_2, z_3) - \nu(c)| \leq C_1 \varepsilon, \tag{5.9}$$

where $\nu(c) = \prod_{d \in E(c_1,c)} \sigma_f(d)$ and C_1 is a positive constant. □

LEMMA 5.4. *Under the hypothesis of Proposition 5.2, for any $\varepsilon > 0$, there exists $0 < \gamma < \gamma_0$ such that, for any $\beta \in]0, \gamma[$, there exist $n_\beta \in \mathbb{M}_0$ such that, for any $n \in \mathbb{M}_0$, $n \geq n_\beta$, the (β, γ)-secondary cell (z_1, z_2, z_3) associated to $(f, x_0, b, \mathbb{M}_0, \delta)$ satisfies*

$$|\mathrm{Dr}_{g^{q_n(b)}}(h(z_1), h(z_2), h(z_3)) - \Pi_g(h(c), h(c_1))| \leq C_2 \varepsilon,$$

where $\Pi_g(h(c), h(c_1)) = \prod_{d \in E(c_1,c)} \sigma_g(h(d))$ and C_2 is a positive constant.

Proof. The proof is a consequence of Lemma 5.3 and (Proposition 4.3(h-5)) applied to the $(\beta/2, \gamma)$-derived cell $(h(z_1), h(z_2), h(z_3))$ associated to $(g, h(x_0), h(c), \mathbb{N}_0)$ instead of the (β, γ)-derived cell (z_1, z_2, z_3) associated to $(f, x_0, c, \mathbb{N}_0)$. □

Proof of Proposition 5.2. The proof results from Lemmas 5.3 and 5.4. □

6. *Proof of Main Theorem*

Let $f, g \in \mathcal{P}(S^1)$ with the same irrational rotation number α of bounded type. Suppose that f and g satisfy the KO condition. By Corollary 2.12, there exist two piecewise quadratic homeomorphisms $K, L \in \mathcal{P}(S^1)$ such that $F = L \circ f \circ L^{-1}$ and $G = K \circ g \circ K^{-1}$ have the following properties.

(1) $F, G \in \mathcal{P}(S^1)$ and have the same irrational rotation number α.
(2) The break points of F (respectively, G) are on *pairwise distinct* F-orbits (respectively, G-orbits).
(3) F and G satisfy the KO condition.
(4) Let h be the conjugating map between f and g, i.e., $h \circ f = g \circ h$, and set $v = K \circ h \circ L^{-1}$. Then:
   (a) v is an absolutely continuous (respectively, a singular) function if and only if h also is; and
   (b) $v \circ F = G \circ v$.

Therefore we may assume that all break points of f (respectively, g) are on *pairwise distinct* f-orbits (respectively, g-orbits). Now, by Proposition 2.14, there exists a homeomorphism u of S^1 such that $G = u \circ f \circ u^{-1} \in \mathcal{P}(S^1)$ with the following properties.

- $C(f) \subset B := \{c_i : i = 0, 1, \ldots, q\}, (q \geq p)$.
- $C(G) \subset u(B) := \{u(c_i) : i = 0, 1, \ldots, q\}$.
- $\sigma_G(u(c_i)) = \sigma_g(d_i)$, $i = 0, 1, \ldots, q$.
- $\pi_s(G) = \pi_s(g)$.
- h is singular if and only if u is also.

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So we may assume that $u = h$ and $G = g$.

To show that the conjugation homeomorphism h from f to g is singular with respect to the Lebesgue measure m, it suffices to prove that its derivative Dh is zero on a set of Lebesgue total measure. Assume, on the contrary, that h admits at a point x_0 a positive derivative $Dh(x_0) > 0$.

Let $\varepsilon > 0$, $c \in B$. For $0 < \beta < \gamma < \gamma_0$, write

$$D_n(\beta, \gamma) := \frac{\mathrm{Dr}_{g^{qn}}(h(z_1), h(z_2), h(z_3))}{\mathrm{Dr}_{f^{qn}}(z_1, z_2, z_3)}.$$

By Proposition 5.1, there exists $n_\beta \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0$, $n \geq n_\beta$,

$$|D_n(\beta, \gamma) - 1| \leq \frac{2}{1 - \gamma_0}\beta. \tag{6.1}$$

By Proposition 5.2, there exists $\gamma_\varepsilon < \gamma_0$ such that, for every $0 < \beta < \gamma_\varepsilon$, there exists $n_\beta(\varepsilon) \in \mathbb{N}_0$ such that, for every $n \in \mathbb{N}_0$, $n \geq n_\beta(\varepsilon)$, the (β, γ) secondary cell $(z_1, z_2, z_3)(\gamma < \gamma_\varepsilon)$- associated to $(f, x_0, c, \mathbb{N}_0, \delta)$ satisfies

$$|D_n(\beta, \gamma) - \Pi(c_1, c)| \leq A\varepsilon, \tag{6.2}$$

where

$$\Pi(c_1, c) := \prod_{d \in E(c_1, c)} \frac{\sigma_g(h(d))}{\sigma_f(d)}$$

and A is a positive constant.

Let $\beta < \gamma < \gamma_\varepsilon$ and let $n \in \mathbb{N}_0$, $n \geq n_\beta(\varepsilon)$. Then (6.1) and (6.2) imply that

$$|\Pi(c_1, c) - 1| \leq |D_n(\beta, \gamma) - 1| + |D_n(\beta, \gamma) - \Pi(c_1, c)| \leq \frac{2\beta}{1 - \gamma_0} + A\varepsilon.$$

Since ε and β are arbitrary, it follows that $\Pi(c_1, c) = 1$, i.e.,

$$\prod_{d \in E(c_1, c)} \frac{\sigma_g(h(d))}{\sigma_f(d)} = 1. \tag{6.3}$$

Since $c_1, c \in B$,$c_1 \neq c$ are arbitrary, for every $(i, k) \in \{0, 1, \ldots, q\}^2 (i \neq k)$,

$$\prod_{d \in E(c_k, c_i)} \frac{\sigma_g(h(d))}{\sigma_f(d)} = 1.$$

Now, by Proposition 3.2, $c_k \notin E(c_k, c_i)$. Therefore

$$\prod_{0 \leq j \leq q} \left(\frac{\sigma_g(h(c_j))}{\sigma_f(c_j)} \right)^{e(i,k,j)} = 1,$$

where $e(i, k, j) \in \{0, 1\}$ with $\varepsilon(i, k, i) = 1$ and $\varepsilon(i, k, k) = 0$.

It follows that, for every $(i, k) \in \{0, 1, \ldots, q\}^2$, $i \neq k$,

$$\sum_{0 \leq j \leq q} e(i, k, j) \log \left(\frac{\sigma_g(h(c_j))}{\sigma_f(c_j)} \right) = 0.$$

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For $(i, j, k) \in \{0, \ldots, q\}^3$, set

$$\varepsilon_{k+i(q+1),j} = \begin{cases} e(i,k,j) & \text{if } i \neq k,\ j \neq i \text{ and } j \neq k, \\ 1 & \text{if } i \neq k \text{ and } j = i, \\ 0 & \text{if } i \neq k \text{ and } j = k, \\ 0 & \text{if } i = k. \end{cases}$$

It follows that, for every $(i, k) \in \{0, \ldots, q\}^2$,

$$\sum_{j=0}^{q} \varepsilon_{k+i(q+1),j} \log\left(\frac{\sigma_g(h(c_j))}{\sigma_f(c_j)}\right) = 0.$$

Set $A_{q+1} = (a_{l,j})_{0 \leq l < (q+1)^2, 0 \leq j < q} \in \mathcal{M}_{(q+1)^2, q+1}(\{0, 1\})$, the matrix defined by

$$a_{l,j} = \varepsilon_{k+i(q+1),j} \quad \text{where } l = k + i(q+1) \text{ with } 0 \leq i, k \leq q.$$

The matrix A_{q+1} has a rank $q+1$ (see [**3**, Lemma 6.1]). Therefore, for every $j \in \{0, 1, \ldots, q\}$, $\sigma_g(h(c_j)) = \sigma_f(c_j)$. Hence f and g are break-equivalent, which is a contradiction. The proof is complete. □

Proof of Corollary 1.4. This follows from the fact that if f and g do not have the same number of singular orbits, then f and g are not break-equivalent. □

Proof of Corollary 1.5. Suppose that f and g are break-equivalent. After reduction, one can suppose that $\mathrm{SO}(f) = C(f)$ and $\mathrm{SO}(g) = C(g)$. Then, for every $d \in C(g)$, there is $e \in C(f)$ such that $d = h(e)$ and $\sigma_g(d) = \sigma_f(e)$, i.e., $\sigma_g(d) \in \{\sigma_f(c) : c \in C(f)\}$. This contradicts the hypothesis of the corollary. □

Proof of Corollary 1.6. Proof of (i). Since g does not have the (D)-property and f has the (D)-property, so there exists a point $d \in C(g)$ such that $\pi_{s,O_g(d)}(d) \neq 1$ and $\pi_{s,O_f(c)}(f) = 1$ for every $c \in C(f)$. We conclude by Corollary 1.5.

Proof of (ii). By Corollary 2.12, f is conjugate to a class P-homeomorphism $F = K \circ f \circ K^{-1}$ through a piecewise quadratic class P homeomorphism K of S^1. Since f satisfies the (D)-property, $\pi_{s,O_f(c)}(f) = 1 = \sigma_F(K(c))$ for every $c \in C(f)$. Thus F is a C^1- diffeomorphism. Moreover, DF is absolutely continuous on S^1 and $D^2F \in L^p(S^1)$ for some $p > 1$. By Katznelson–Ornstein's theorem, μ_F is equivalent to the Lebesgue measure m and so is μ_f. Hence there exists an absolutely continuous map φ such that $\varphi \circ f = R_{\rho(f)} \circ \varphi$. Similarly, there exists an absolutely continuous map ψ such that $\psi \circ g = R_{\rho(f)} \circ \psi$. In addition, ψ^{-1} is absolutely continuous. It follows that $(\psi \circ h \circ \varphi^{-1}) \circ R_{\rho(f)} = R_{\rho(f)} \circ (\psi \circ h \circ \varphi^{-1})$. As $\rho(f)$ is irrational, $\psi \circ h \circ \varphi^{-1} = R_\beta$, a rotation, for some $\beta \in S^1$. Therefore $h = \psi^{-1} \circ R_\beta \circ \varphi$, which is an absolutely continuous map. This completes the proof. □

Proof of Corollary 1.7. The proof follows easily from Corollary 1.6 by taking g the rotation $R_{\rho(f)}$. □

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Acknowledgements. The authors would like to thank the referee for valuable comments and suggestions. This work was supported by the research unit, 'Dynamical systems and their applications', UR17ES21, Ministry of Higher Education and Scientific Research, Tunisia.

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