From $m\frac{dv}{dt} = mg - kmv$, we obtain $e^{-kt_1} = 1 - \frac{kV}{g}$ and, from $m\frac{dv}{dt} = -mg - kmv$, we obtain $e^{kt_2} = 1 + \frac{k\alpha V}{g}$, leading to $\alpha + 1 = \alpha e^{-kt_1} + e^{kt_2}$ and a similar analysis to that given above.

Reference

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10.1017/mag.2019.30

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Partial fractions and telescoping series

It is a standard question in A-level Mathematics to use partial fractions to sum series such as $\sum_{r=1}^{n} \frac{1}{(r+2)(r+5)}$. Many will have observed that this works because in the partial fractions expansion, here $\frac{\frac{1}{3}}{(r+2)} + \frac{-\frac{1}{3}}{(r+5)}$, the coefficients of the partial fractions sum to zero. We show when this happens, and hence when the method of telescoping can be used.

Preliminary observations

We consider only those partial fractions whose denominators factorise into non-repeated linear factors, and consider $F(r) \equiv \frac{f_l(r)}{(r-a_1)\ldots(r-a_k)}$ with all the a_i distinct and where $f_l(r)$ denotes a polynomial of degree l.

First, without loss of generality we do not need to consider non-monic polynomials and factors. If f_l is non-monic, we simply take out the coefficient of the leading term as a factor. The a_i and the coefficients of the other terms in f_l will be assumed rational.

Second, we assume that $l < k$. If $l \geqslant k$ then we can carry out a standard long division to write

$$F(r) \equiv Q(r) + \sum_{i=1}^{k} \frac{A_i}{(r-a_i)},$$

where $Q(r)$ is a polynomial of degree $l-k$. This case does not in fact interest us in the present context, as plainly $\sum_{r=m}^{\infty}\left[Q(r) + \sum_{i=1}^{k} \frac{A_i}{(r-a_i)}\right]$ does not converge. Hence, without loss of generality, we write the expansion as $F(r) \equiv \sum_{i=1}^{k} \frac{A_i}{(r-a_i)}$.

https://doi.org/10.1017/mag.2019.31 Published online by Cambridge University Press

We now show that we need to consider only $f_l(r) \equiv 1$. We can split any rational expression when f_l is of degree greater than or equal to 1 into separate fractions, so that $F(r) \equiv \sum_{j=1}^{k} \frac{B_j}{(r-a_1)\ldots(r-a_j)}$ where each B_j is constant. The following example shows the algorithm:

$$\begin{aligned}\frac{r^2+2r-7}{(r+2)(r-3)(r-1)} &\equiv \frac{(r-1)(r+3)-4}{(r+2)(r-3)(r-1)}\\ &\equiv \frac{r+3}{(r+2)(r-3)} + \frac{-4}{(r+2)(r-3)(r-1)}\\ &\equiv \frac{r-3+6}{(r+2)(r-3)} + \frac{-4}{(r+2)(r-3)(r-1)}\\ &\equiv \frac{1}{(r+2)} + \frac{6}{(r-3)(r+2)} + \frac{-4}{(r-1)(r-3)(r+2)}.\end{aligned}$$

We will call this method *Algorithm A*. It can be applied for any $F(r)$ with $l < k$ by assumption. Clearly the choice of how to split up $f_l(r)$ is arbitrary; in this example the expression has been written in the form $\sum_{j=1}^{k} \frac{B_j}{(r-a_1)\ldots(r-a_j)}$.

Here is the central result.

Lemma 1

(a) $\sum_{r=m}^{\infty} \frac{f_l(r)}{(r-a_1)\ldots(r-a_k)} \left(m > \max\{a_i\}\right)$ converges

if, and only if,

(b) $l \leqslant k-2$.

Proof:

Intuitively we can argue that the numerator is $O(r^l)$ and the denominator is $O(r^k)$, so that $F(r) = O(r^{l-k})$. The result then follows from the divergence of $\sum_{r=1}^{\infty} \frac{1}{r}$ and the convergence of $\sum_{r=1}^{\infty} \frac{1}{r^k}$ where $k \in \mathbb{Z}, k \geqslant 2$. More formally, we can apply algorithm A to reduce $F(r)$ to $\sum_{j=1}^{k} \frac{B_j}{(r-a_1)\ldots(r-a_j)}$. Then if $B_1 = 0$, all the terms are of order at worst $\frac{1}{r^2}$ and the sum converges, but if $B_1 \neq 0$ we have $\sum_{r=m}^{\infty} \frac{B_1}{(r-a_1)} = \sum_{r=m}^{\infty} O\left(\frac{1}{r}\right)$ which is divergent.

https://doi.org/10.1017/mag.2019.31 Published online by Cambridge University Press

Now consider

(c) $\dfrac{f_l(r)}{(r-a_1)\ldots(r-a_k)}$ can be split into partial fractions with the coefficients summing to 0.

Lemma 2

(b) $\Leftrightarrow$ (c)

Proof:

Write $\dfrac{f_l(r)}{(r-a_1)\ldots(r-a_k)} \equiv \sum_{i=1}^{k} \dfrac{A_i}{(r-a_i)}$ and multiply both sides by $(r-a_1)\ldots(r-a_k)$.

Every term on the right has degree at most $k-1$. The coefficient of r^{k-1} equals $\sum_{i=1}^{k} A_i$, but if $f_l(r)$ has degree less than $k-1$, the coefficient of r^{k-1} on the left is zero. If $f_l(r)$ has degree $k-1$, with the coefficient of r^{k-1} equalling $c \neq 0$, then $\sum_{i=1}^{k} A_i = c \neq 0$.

Example: The series $\sum_{r=1}^{\infty}\left(\dfrac{1}{r}+\dfrac{1}{r+1}+\dfrac{1}{r+2}\right)=\sum_{r=1}^{\infty}\dfrac{3r^2+6r+2}{r(r+1)(r+2)}$ does not converge; the fact that $l = k-1$ corresponds to $A_1 + A_2 + A_3 \neq 0$. In fact, $A_1 + A_2 + A_3 = 3$, the coefficient of r^2 in the numerator.

That (c) $\Rightarrow$ (a) is standard A-level work; the series telescopes, though care may be needed at the beginning and the end of the (finite) series. It is less immediately obvious that (a) $\Rightarrow$ (c); it is easiest to argue that (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

As a final comment, it may be easiest to work out the non-cancelling terms of a convergent series of partial fractions by splitting them up. For instance,

$$\frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2} = \left(\frac{1}{r} - \frac{1}{r+1}\right) - \left(\frac{1}{r+1} - \frac{1}{r+2}\right)$$

so that

$$\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}\right) = \left(1 - \frac{1}{n+1}\right) - \left(\frac{1}{2} - \frac{1}{n+2}\right).$$

10.1017/mag.2019.31

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https://doi.org/10.1017/mag.2019.31 Published online by Cambridge University Press