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A NOTE ON GROUPS WITH MANY NORMAL-BY-FINITE SUBGROUPS

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Abstract

A subgroup X of a group G is called normal-by-finite if it is finite over its core X_G. We investigate the behaviour of uncountable groups G of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal-by-finite. Moreover, generalised soluble groups whose subgroups of infinite rank are either normal-by-finite or contranormal are characterised, and the analogous problem is considered for uncountable groups whose subgroups of large cardinality satisfy the same condition.

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1. Introduction

A subgroup X of a group G is called *normal-by-finite* if the core X_G of X in G has finite index in X. Clearly, finite subgroups and normal subgroups of an arbitrary group are examples of normal-by-finite subgroups. A group G is said to be *core-finite* (or a *CF-group*) if all its subgroups are normal-by-finite. Moreover, G is called *boundedly core-finite* (or a *BCF-group*) if there exists a positive integer m such that $|X : X_G| \leq m$ for every subgroup X of G. In general, such groups appear to be highly complicated, as shown by Tarski groups (that is, infinite simple groups whose proper nontrivial subgroups have prime order), which provide examples of CF-groups. It is clear that Tarski groups are far from being abelian-by-finite; however, Buckley *et al.* [1] showed that any locally finite CF-group is abelian-by-finite and BCF. A generalisation of this result, due to Smith and Wiegold, can be found in [21]. Over the years, there has been considerable interest in studying groups where the normal-by-finite property is required only for certain families of subgroups. In [15], Franciosi *et al.* considered groups whose subnormal subgroups are normal-by-finite, while in [11], de Giovanni and Saccomanno investigated generalised soluble groups by first imposing the CF-property on proper subgroups and later restricting it to proper subgroups of infinite rank.

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Recall that a group G is said to have *finite rank* if there exists a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements; if such an r does not exist, we say that the group G has *infinite rank*. The influence of the behaviour of subgroups of infinite rank on the structure of a (generalised) soluble group has been investigated by many authors (see [4], where a full reference list on this subject can be found). In many relevant cases, it turns out that the imposition of an embedding property $\mathcal{X}$ on all subgroups of infinite rank of a group G forces either G to have finite rank or all its subgroups to satisfy $\mathcal{X}$. For instance, Evans and Kim [14] proved that if G is a locally soluble group of infinite rank in which all subgroups of infinite rank are normal, then G is a Dedekind group, that is, all its subgroups are normal. Similar results have been obtained by imposing generalised normality conditions to subgroups of infinite rank (see [5, 8]). In particular, De Falco, de Giovanni and Musella showed that if G is a generalised radical group of infinite rank whose subgroups of infinite rank are normal-by-finite, then G is an abelian-by-finite CF-group (see [5, Theorem C]). Here, a group G is called *generalised radical* if it has an ascending series whose factors are either locally finite or locally nilpotent. Results of this type suggest that the behaviour of *small* subgroups in a *large* group is negligible, at least for a right choice of the definition of largeness and within a suitable universe. This point of view is adopted also in the papers [2, 3, 6, 7, 10], by considering uncountable groups G whose subgroups of the same cardinality as G are normal or satisfy other relevant embedding properties that generalise normality.

In Section 2, we provide a further contribution to this topic, by investigating the behaviour of uncountable groups G in which all subgroups equipotent to G are normal-by-finite. The main obstacle in the study of groups of large cardinality $\aleph$ is the existence of infinite groups in which all proper subgroups have cardinality strictly smaller than $\aleph$ (the so-called *Jónsson groups*). Clearly, a countably infinite group is a Jónsson group if and only if all its proper subgroups are finite, so that Prüfer groups and Tarski groups have the Jónsson property. However, it is known that if G is an uncountable Jónsson group, then $G/Z(G)$ is simple and $G = G'$ (see [3, Lemma 2.7]). Relevant examples of Jónsson groups of cardinality $\aleph_1$ have been constructed by Shelah [20] and Obraztsov [18].

Section 3 is devoted to the study of groups in which every subgroup is either normal-by-finite or contranormal. Recall that a subgroup X of a group G is said to be *contranormal* if it is not contained in any proper normal subgroup of G, that is, if $X^G = G$. Clearly, contranormal subgroups may be viewed as *antipodes* of normal subgroups, being diametrically opposed to them. Over the years, a significant focus has been placed on characterising groups with only two types of subgroups that are antipodal to each other. Groups with only normal or contranormal subgroups were studied by Subbotin [22]; subsequently, the normality condition was weakened in several directions. De Falco *et al.* [9] investigated groups in which every subgroup is either subnormal or contranormal. Later, Kurdachenko *et al.* [17] considered groups whose subgroups are either almost normal or contranormal, where a subgroup X of a group G is *almost normal* in G if the normaliser $N_G(X)$ has finite index in G.

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More recently, Zaccardo [24] studied groups with only nearly normal or contranormal subgroups; here, a subgroup X of a group G is said to be *nearly normal* in G if X has finite index in its normal closure X^G. Further results concerning groups in which every subgroup satisfies either a normality condition or an antipodal embedding condition can be found in [9, 16, 22].

In light of these considerations, it is natural to study the analogous problem for groups of infinite rank. Groups whose subgroups of infinite rank are either normal or contranormal were investigated by De Luca and Di Grazia [13], while the case involving nearly normal and contranormal subgroups has been analysed by Zaccardo [24]. In the present paper, we consider groups whose subgroups of infinite rank are either normal-by-finite or contranormal. In particular, we prove that, within a suitable class of generalised soluble groups, such groups either have finite rank or all their subgroups are either normal-by-finite or contranormal.

In the final part of the section, we investigate the corresponding problem for uncountable groups, focusing on groups in which all *large* subgroups are either normal-by-finite or contranormal. Throughout this paper, $\aleph$ will denote an uncountable regular cardinal number and if G is a group of cardinality $\aleph$, a subgroup X of G will be called *large* if it likewise has cardinality $\aleph$, and *small* otherwise.

Most of our notation is standard and can be found in [19].

2. Uncountable groups whose large subgroups are normal-by-finite

In several recent papers, it has been established that many normality conditions in uncountable groups containing a large abelian subgroup pass from large subgroups to small ones (see, for instance, [2, 3, 6]). The following lemma shows that this phenomenon occurs for the normal-by-finite property.

LEMMA 2.1. *Let G be an uncountable group of regular cardinality $\aleph$ whose large subgroups are normal-by-finite. If G contains a large abelian subgroup, then all subgroups of G are normal-by-finite. In particular, if G is a generalised radical group, then G is abelian-by-finite.*

PROOF. Let X be any small subgroup of G and let A be an abelian subgroup of G having cardinality $\aleph$. It is known (see for instance [12, Lemma 3.2]) that A contains large subgroups U and V such that

$$\langle U, V\rangle = U \times V$$

and

$$X \cap \langle U, V\rangle = \{1\}.$$

As the indices $|U : U_G|$ and $|V : V_G|$ are finite, it follows that U_G and V_G have cardinality $\aleph$. Therefore, the large subgroups XU_G and XV_G are normal-by-finite, and hence also

$$X = XU_G \cap XV_G$$

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is finite over its core X_G. Finally, if G is a generalised radical group, then all periodic homomorphic images of G are locally finite and the statement follows from [21, Theorem 2]. □

COROLLARY 2.2. *Let G be an uncountable locally finite group of regular cardinality $\aleph$ whose large subgroups are normal-by-finite. If G contains a large abelian subgroup, then G is an abelian-by-finite BCF-group.*

Let G be an uncountable group of regular cardinality $\aleph$ in which all large subgroups are normal-by-finite. It is easy to see that this property is inherited by large subgroups and homomorphic images; of course, every quotient of G by a large normal subgroup is a CF-group. As a consequence of Lemma 2.1, we can find a bigger CF-quotient arising from the consideration of a large normal subgroup.

COROLLARY 2.3. *Let G be an uncountable group of regular cardinality $\aleph$ whose large subgroups are normal-by-finite. If N is a normal subgroup of cardinality $\aleph$ of G, then G/N' is a CF-group.*

PROOF. Clearly, N' is a normal subgroup of G and the statement is obvious if N' has cardinality $\aleph$. Suppose now that N' has cardinality strictly smaller than $\aleph$. Then, N/N' is a large abelian subgroup of G/N', so that G/N' is a CF-group by Lemma 2.1. □

Recall that the *Baer radical* of a group G is the subgroup generated by all abelian subnormal subgroups of G, and that G is a *Baer group* if it coincides with its Baer radical or, equivalently, if every cyclic subgroup of G is subnormal. It is known that any Baer group having all (subnormal) subgroups normal-by-finite is nilpotent (see [15, Theorem 3.2]). This gives the following result.

PROPOSITION 2.4. *Let G be an uncountable group of regular cardinality $\aleph$ whose large subgroups are normal-by-finite. If G contains a large abelian subgroup, then the Baer radical of G is a large nilpotent abelian-by-finite BCF-group.*

PROOF. Let A be a large abelian subgroup of G. Then, the core A_G of A in G is a large abelian normal subgroup of G contained in the Baer radical B of G, and hence, B has cardinality $\aleph$. However, by Lemma 2.1, G is a CF-group and so is B. Therefore, B is nilpotent and the statement follows from [21, Theorem 3]. □

COROLLARY 2.5. *Let G be an uncountable Baer group of regular cardinality $\aleph$ whose large subgroups are normal-by-finite.*

(1) *If G contains a large nilpotent subgroup, then G is nilpotent and there exists a small characteristic subgroup N of G such that G/N is an abelian-by-finite BCF-group.*
(2) *If G contains a large abelian subgroup, then G is a nilpotent abelian-by-finite BCF-group.*

PROOF. (1) Let A be a large nilpotent subgroup of G; clearly, as every large subgroup of G is normal-by-finite, we may assume that A is normal in G. The Baer group G/A'

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is a CF-group by Corollary 2.3 and, hence, G/A' is nilpotent. Now, by a well-known nilpotency criterion of P. Hall (see [19, 5.2.16]), it follows that G itself is nilpotent. Thus, there exists a positive integer s such that $\gamma_s(G)$ has cardinality $\aleph$ and $\gamma_{s+1}(G)$ has cardinality strictly smaller than $\aleph$. Therefore, the factor group $G/\gamma_{s+1}(G)$ contains the large abelian subgroup $\gamma_s(G)/\gamma_{s+1}(G)$, so that $G/\gamma_{s+1}(G)$ is a CF-group by Lemma 2.1 and, hence, it is an abelian-by-finite BCF-group (see [21, Theorem 3]).

(2) The statement follows immediately from Proposition 2.4. □

The imposition of generalised normality conditions on large subgroups of an uncountable group often leads to strong restrictions on its commutator subgroup. For instance, if G is an uncountable group whose large subgroups are nearly normal, it turns out that G has a finite commutator subgroup G', provided that G contains a large FC-subgroup (see [6, Theorem 3.6]). Recall that a group G is an FC-group if all its elements admit only finitely many conjugates, that is, if the centraliser $C_G(g)$ has finite index in G for each element g of G. In our situation, we obtain information on the second commutator subgroup.

PROPOSITION 2.6. *Let G be an uncountable generalised radical group of regular cardinality $\aleph$ whose large subgroups are normal-by-finite. Then, the factor group $G''/G'' \cap H_G$ is locally finite, for every large subgroup H of G.*

PROOF. Let H be any large subgroup of G; then the core H_G of H in G has finite index in H and, in particular, H_G is a large subgroup of G. It follows that all subgroups of the factor group G/H_G are normal-by-finite. As every periodic homomorphic image of G is locally finite, the second commutator subgroup $G''H_G/H_G$ of G/H_G is locally finite (see [11, Lemma 1]). □

COROLLARY 2.7. *Let G be an uncountable generalised radical group whose large subgroups are normal-by-finite. If the commutator subgroup G' of G is small, then $G''Z(G)/Z(G)$ is residually (locally) finite.*

PROOF. Let $g(G'' \cap Z(G)) \neq G'' \cap Z(G)$ be a nontrivial element of $G''/(G'' \cap Z(G))$. Then, $g \notin Z(G) = \bigcap_{x\in G} C_G(x)$, so there exists an element $h \in G$ such that $g \notin C_G(h)$. As the normal closure $\langle h\rangle^G$ is contained in $\langle h, G'\rangle$, it is a small subgroup of G. Therefore, the index $|G : C_G(h)|$ is strictly smaller than $\aleph$ and, hence, the centraliser $C_G(h)$ is a large subgroup of G. Put $H = C_G(h)$. Clearly, the element $g \notin H_G$, and thus, $g(G'' \cap Z(G)) \notin (G'' \cap H_G)/(G'' \cap Z(G))$. Moreover, it follows from Proposition 2.6 that the factor group,

$$\frac{G''/(G'' \cap Z(G))}{(G'' \cap H_G)/(G'' \cap Z(G))} \simeq \frac{G''}{(G'' \cap H_G)},$$

is locally finite and the statement is proved. □

LEMMA 2.8. *Let G be an uncountable generalised radical group of regular cardinality $\aleph$ whose large subgroups are normal-by-finite. If G contains a large FC-subgroup, then the second commutator subgroup G'' of G is locally finite.*

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PROOF. As G contains a large FC-subgroup, there exists a large subgroup K of G whose commutator subgroup K' is finite (see [23, page 155]). Therefore, K/K' contains an abelian subgroup of the form $U/K' \times V/K'$ where both U and V have cardinality $\aleph$ (see [6, Corollary 3.2]), so that U_G and V_G are large subgroups of G. Moreover, the intersection $U_G \cap V_G \leq U \cap V = K'$ is finite. As all subgroups of the factor groups G/U_G and G/V_G are normal-by-finite, and every periodic homomorphic image of G is locally finite, it follows that the second commutator subgroups $G''U_G/U_G$ of G/U_G and $G''V_G/V_G$ of G/V_G are both locally finite (see [11, Lemma 1]). However, the factor group $G''(U_G \cap V_G)/(U_G \cap V_G)$ is isomorphic to a subgroup of the locally finite group

$$\frac{G''}{G'' \cap U_G} \times \frac{G''}{G'' \cap V_G}$$

and, hence, G'' is likewise locally finite. □

COROLLARY 2.9. *Let G be an uncountable generalised radical group of regular cardinality $\aleph$ whose large subgroups are normal-by-finite. If G contains a large abelian subgroup, then G'' is an abelian-by-finite BCF-group.*

3. Groups with only normal-by-finite or contranormal subgroups

We start this section by providing a complete description of nonperfect groups with no nontrivial perfect quotients in which all subgroups are either normal-by-finite or contranormal. For our purposes, we note that if H is a contranormal subgroup of a group G, then $G = G'H$. Conversely, if G has no nontrivial perfect quotients and $G = G'H$, then H is contranormal in G. Indeed, in this case, $G = G'H^G$, so that the quotient G/H^G is perfect and, hence, trivial.

PROPOSITION 3.1. *Let G be a generalised radical group whose subgroups are either normal-by-finite or contranormal. If N is a proper normal subgroup of G, then every subgroup of N is normal-by-finite in G and N is abelian-by-finite.*

PROOF. Let X be any subgroup of N. Clearly, as N is a proper normal subgroup of G, X cannot be contranormal in G. Thus, every subgroup of N is normal-by-finite in G and the statement follows from [21, Theorem 2]. □

THEOREM 3.2. *Let G be a nonperfect group with no nontrivial perfect quotients. Then, every subgroup of G is either normal-by-finite or contranormal if and only if one of the following conditions holds:*

(i) *G is a CF-group;*
(ii) *all cyclic subgroups of G are normal-by-finite, G/G' is a group of type p^∞ for some prime p and all subgroups of G' are normal-by-finite in G;*
(iii) *$G = G'\langle x\rangle$ for some $x \in G$ and there exists a prime number p such that all subgroups of $G'\langle x^p\rangle$ are normal-by-finite in G and $\langle x^n\rangle$ is contranormal in G for every integer n which is not divisible by p.*

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PROOF. Assume first that every subgroup of G is either normal-by-finite or contranormal. If G is not of type (i), then there exists a proper subgroup B which is not normal-by-finite and, hence, it is contranormal; in particular, $G = G'B$. Put $\overline{B} = B/(B \cap G')$, and suppose that $\overline{B}$ is generated by two proper subgroups $\overline{X}$ and $\overline{Y}$. Then, applying Dedekind's modular law, it follows that the normal subgroups XG' and YG' are properly contained in G; thus, X and Y cannot be contranormal in G. Therefore, X and Y are both normal-by-finite in G, and hence, $B = XY$ is likewise a normal-by-finite subgroup of G, which is a contradiction. Thus, $\overline{B}$ cannot be generated by two proper subgroups, so it is either a group of type p^∞ or a cyclic p-group for some prime p.

If $\overline{B}$ is a group of type p^∞, then also G/G' is a group of type p^∞. However, since G' is a proper normal subgroup of G, all its subgroups are normal-by-finite in G. Moreover, if $\langle x \rangle$ is a cyclic subgroup of G which is not contained in G', it follows that $\langle x \rangle G'$ is a proper normal subgroup of G and, hence, all its subgroups are normal-by-finite in G. In particular, $\langle x \rangle$ is normal-by-finite in G, so G is a group of type (ii). If $\overline{B}$ is a cyclic p-group, then also G/G' is a cyclic p-group and G is a group of type (iii).

Conversely, suppose that G satisfies one of the conditions (i)–(iii). Clearly, if condition (i) holds, every subgroup of G is either normal-by-finite or contranormal. Assume now that G satisfies condition (ii) and let H be a subgroup of G. Since every perfect quotient of G is trivial, if $G = G'H$, then H is contranormal in G. However, if $G \neq G'H$, it follows that $G'H/G'$ is a proper subgroup of G/G' and, hence, $G'H/G'$ is a cyclic group. Therefore, $H/(H \cap G')$ is likewise cyclic and $H = (H \cap G')\langle y \rangle$ for some $y \in G$. As all subgroups of G' and all cyclic subgroups of G are normal-by-finite in G, also H is normal-by-finite in G, so that G has the required property. Finally, assume that G satisfies condition (iii) and let H be a subgroup of G. Since every perfect quotient of G is trivial, if $G = G'H$, then H is contranormal in G. However, if $G \neq G'H$, then $G'H$ is contained in $G'\langle x^p \rangle$, so that H is normal-by-finite in G and the statement is proved. □

We now turn to the study of groups whose subgroups of infinite rank are either normal-by-finite or contranormal. As in many problems concerning groups of infinite rank, the presence of a proper normal subgroup of infinite rank plays a crucial role. Note that the existence of such a subgroup is guaranteed at least within the class of locally soluble groups G of infinite rank. Indeed, assuming by contradiction that every proper normal subgroup of G has finite rank, the subgroup R generated by all proper normal subgroups of G likewise has finite rank (see [13, Lemma 2.2]). Therefore, the factor group G/R is a simple locally soluble group of infinite rank, which is impossible.

LEMMA 3.3. *Let G be a generalised radical group whose subgroups of infinite rank are either normal-by-finite or contranormal, and let N be a proper normal subgroup of infinite rank of G. Then, every subgroup of N is normal-by-finite in G.*

PROOF. Clearly, as N is a proper normal subgroup of G, any subgroup of infinite rank of N cannot be contranormal in G and, hence, is normal-by-finite in G. Therefore,

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[5, Theorem C] ensures that every subgroup of N is normal-by-finite and N is abelian-by-finite. In particular, N contains an abelian characteristic subgroup of finite index, so without loss of generality we may assume that N is abelian. Let X be any subgroup of finite rank of N. Since N has infinite rank, it contains subgroups U and V of infinite rank such that

$$\langle U, V\rangle = U \times V$$

and

$$X \cap \langle U, V\rangle = \{1\}.$$

As the indices $|U : U_G|$ and $|V : V_G|$ are finite, it follows that U_G and V_G have infinite rank. Therefore, the products XU_G and XV_G are normal-by-finite in G, and hence, also

$$X = XU_G \cap XV_G$$

is finite over its core X_G. □

LEMMA 3.4. *Let G be a generalised radical group whose subgroups of infinite rank are either normal-by-finite or contranormal. If G contains a proper normal subgroup of infinite rank, then every subgroup of G is either normal-by-finite or contranormal.*

PROOF. Let X be any noncontranormal subgroup of finite rank of G. If X^G has infinite rank, by Lemma 3.3, every subgroup of X^G is normal-by-finite in G and so is X. Assume now that X^G has finite rank. Let N be a proper normal subgroup of infinite rank of G; then N contains an abelian subgroup A of infinite rank (see [4, Lemma 5]). Thus, there exists a subgroup B of A such that B has infinite rank and X^GB is a proper subgroup of G (see [13, Lemma 2.3]). Clearly, B cannot be contranormal in G and, hence, it is normal-by-finite in G. It follows that the product X^GB is finite over its core $(X^GB)_G$, so that $(X^GB)_G$ has infinite rank and Lemma 3.3 ensures that all its subgroups are normal-by-finite in G. In particular, X is normal-by-finite in G and the statement is proved. □

THEOREM 3.5. *Let G be a locally soluble generalised radical group whose subgroups of infinite rank are either normal-by-finite or contranormal. Then, either G has finite rank or every subgroup of G is either normal-by-finite or contranormal.*

PROOF. We may assume that the locally soluble group G has infinite rank, so that it contains a proper normal subgroup of infinite rank. Therefore, it follows from Lemma 3.4 that every subgroup of G is either normal-by-finite or contranormal. □

We now consider the corresponding problem for uncountable groups.

LEMMA 3.6. *Let G be an uncountable group of regular cardinality $\aleph$ whose large subgroups are either normal-by-finite or contranormal, and let N be a proper normal subgroup of cardinality $\aleph$ of G. If N contains a large abelian subgroup, then every subgroup of N is normal-by-finite in G.*

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PROOF. Let H be any large subgroup of N. Clearly, as N is a proper normal subgroup of G, H cannot be contranormal in G. Thus, every large subgroup of N is normal-by-finite in G and the statement follows from Lemma 2.1. □

THEOREM 3.7. *Let G be a nonperfect uncountable group of regular cardinality $\aleph$ whose large subgroups are either normal-by-finite or contranormal. If G contains a large abelian subgroup, then every subgroup of G is either normal-by-finite or contranormal in G.*

PROOF. Let A be a large abelian subgroup of G and assume first that A is not contranormal in G. Therefore, A is normal-by-finite in G and, hence, its core A_G is a large abelian normal subgroup of G. Suppose now that $A^G = G$. Since large subgroups of G' cannot be contranormal in G, if $A \cap G'$ has cardinality $\aleph$, then it is normal-by-finite in G. Thus, the core $(A \cap G')_G$ is a large abelian normal subgroup of G. However, if $A \cap G'$ has cardinality strictly smaller than $\aleph$, there exists a proper large subgroup B of A such that $A \cap G'$ is properly contained in B. Suppose that B is contranormal in G. It follows that $G = G'B$ and, applying Dedekind's modular law,

$$B = (A \cap G')B = A \cap G'B = A,$$

which is a contradiction. Therefore, the large subgroup B is normal-by-finite in G and its core B_G is a large abelian normal subgroup of G. This argument shows that, without loss of generality, we can assume that the large abelian subgroup A is normal in G.

Let H be any small noncontranormal subgroup of G. If $A \cap H^G$ has cardinality $\aleph$, then H^G is a proper large normal subgroup of G and, hence, Lemma 3.6 ensures that H is normal-by-finite in G. Thus, we may assume that $A \cap H^G$ has cardinality strictly smaller than $\aleph$, so there exists a proper large subgroup C of A such that $A \cap H^G$ is properly contained in C. In particular, C is a large normal-by-finite subgroup of G. Suppose that $H^G C = G$; applying Dedekind's modular law,

$$C = (A \cap H^G)C = A \cap H^G C = A,$$

which is a contradiction. It follows that the product $H^G C$ is a proper large normal-by-finite subgroup of G and, hence, its core $(H^G C)_G$ is a proper large normal subgroup of G. As the index $|C : A \cap (H^G C)_G|$ is finite, the intersection $A \cap (H^G C)_G$ is a large abelian subgroup of $(H^G C)_G$ and, by Lemma 3.6, every subgroup of $(H^G C)_G$ is normal-by-finite in G. In particular, H is normal-by-finite in G and the statement is proved. □

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