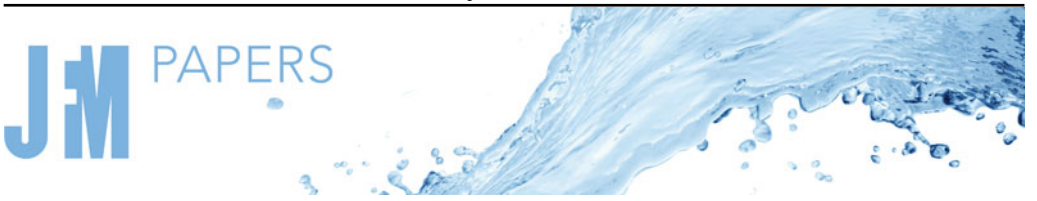
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On exponential decay with depth for irrotational water waves

Evgeniy Lokharu

Centre for Mathematical Sciences, Lund University, SE-22100 Lund, Sweden
Corresponding author: Evgeniy Lokharu, evgeniy.lokharu@math.lu.se

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We consider two-dimensional irrotational steady gravity waves and derive new explicit analytical bounds for the velocity field and the slope of the free surface. Using an auxiliary function tailored to the streamfunction formulation, we obtain an explicit exponential decay estimate which is optimal for linear waves. The same method yields a new slope estimate that improves existing bounds in the moderate-amplitude regime.

Key words: solitary waves, surface gravity waves

1. Introduction

The mathematical theory of two-dimensional irrotational steady gravity water waves is a classical problem in fluid dynamics. The model describes a steady, periodic (or solitary) travelling wave propagating with constant speed over an inviscid, incompressible fluid under the influence of gravity. Despite the apparent simplicity of the governing equations – the incompressible Euler equations – the combination of harmonicity, nonlinear boundary conditions and the unknown geometry of the free surface makes the problem both subtle and rich in structure.

The existence of small-amplitude periodic waves was first established via perturbation and bifurcation techniques by Levi-Civita (1925) and Nekrasov (1951). Global branches of large-amplitude Stokes waves, including the limiting wave with a stagnation point at the crest, were obtained through global bifurcation theory by Krasovskiĭ (1960), Amick & Toland (1981*a*) and Keady & Norbury (1978*a*). Subsequent analytical advances have yielded results on symmetry, monotonicity and regularity (Craig & Sternberg 1988, 1992), as well as the existence of solitary and generalised solitary waves via variational and topological methods (Amick & Toland 1981*b*; Buffoni, Groves & Toland 1996). The existence and properties of extreme or highest waves have also been the focus of sustained

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research; see Toland (1978), Amick, Fraenkel & Toland (1982), Amick & Fraenkel (1987) and McLeod (1997).

A number of further contributions concern structural properties of Stokes and solitary waves. The flow-force constant, a key invariant, was introduced by Benjamin & Lighthill (1954). In their study, Keady & Norbury (1975) introduced conjugate streams and derived fundamental inequalities involving fluid depth, velocities, the flow-force constant S and the total head. In particular, they established the inequality $S \geqslant S_-$, where S_- corresponds to the supercritical conjugate flow. Later, Benjamin (1995) confirmed the conjecture $S_- \leqslant S \leqslant S_+$ for arbitrary Stokes waves; see also Kozlov & Kuznetsov (2010), Kozlov, Kuznetsov & Lokharu (2017) and Keady & Norbury (1978*b*). The conjecture was established in full generality in Lokharu (2024), and its role in the irrotational setting was further emphasised in Lokharu (2021).

In this paper, we obtain a new decay estimate for the vertical component of the relative velocity. In contrast to previous results by Longuet-Higgins (1953), Aleman & Constantin (2020) and Roberti (2020), our estimate is explicit and is optimal in the small-amplitude regime. It follows from applying the maximum principle to a suitably chosen auxiliary function, which additionally yields a new slope bound.

2. Mathematical formulation

2.1. *Euler equations for steady irrotational flow*

We consider a two-dimensional fluid domain

$$\Omega_\eta = \{(x, y) \in \mathbb{R}^2 : 0 < y < \eta(x)\}, \tag{2.1}$$

bounded above by the free surface $y = \eta(x)$ and below by a flat bed $y = 0$. Note that according to our notations the flat bed is at level zero, so that for solitary waves we have $\eta(x) \to d > 0$ as $|x| \to \infty$. In a frame moving with constant wave speed $c > 0$, the steady incompressible Euler equations for the velocity field $u = (u, v)$, pressure P and gravitational acceleration $g > 0$ read

$$\begin{cases} (u - c)u_x + vu_y = -P_x, \\ (u - c)v_x + vv_y = -P_y - g, \\ u_x + v_y = 0, \end{cases} \qquad (x, y) \in \Omega_\eta. \tag{2.2}$$

The boundary conditions are

$$\begin{cases} v = (u - c)\eta'(x), & \text{on } y = \eta(x), \\ P = P_{atm}, & \text{on } y = \eta(x), \\ v = 0, & \text{on } y = 0. \end{cases} \tag{2.3}$$

The first expresses that the free surface is a material boundary, the second fixes the pressure at the free surface to be atmospheric and the third enforces impermeability at the bottom.

2.2. *Streamfunction formulation*

It is convenient to introduce a streamfunction ψ defined in the moving frame by

$$\psi_x = v, \quad \psi_y = c - u. \tag{2.4}$$

With this convention, ψ decreases with height, and its level sets represent streamlines of the flow. Since the flow is incompressible, ψ is harmonic

$$\Delta\psi = 0, \quad (x, y) \in \Omega_\eta. \tag{2.5}$$

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We may normalise

$$\psi = m \quad \text{on} \quad y = \eta(x), \qquad \psi = 0 \quad \text{on} \quad y = 0, \tag{2.6}$$

where $m > 0$ denotes the (dimensional) relative mass flux

$$m = \int_0^{\eta(x)} (c - u)\, \mathrm{d}y. \tag{2.7}$$

The dynamic boundary condition in terms of ψ becomes

$$\frac{1}{2}|\nabla\psi|^2 + gy = Q, \quad y = \eta(x), \tag{2.8}$$

for some constant Q (Bernoulli constant). Equation (2.8) and the harmonicity of ψ together with (2.6) constitute the classical free-boundary formulation of the steady wave problem.

2.3. *Non-dimensionalisation*

To simplify the equations, we introduce non-dimensional variables in which the gravitational acceleration and the mass flux are scaled to unity. Let $m > 0$ be the dimensional mass flux and $g > 0$ the gravitational constant. We define

$$\tilde{x} = \frac{x}{m^{2/3}g^{-1/3}}, \quad \tilde{y} = \frac{y}{m^{2/3}g^{-1/3}}, \quad \tilde{\psi} = \frac{\psi}{m}, \quad \tilde{\eta}(\tilde{x}) = \frac{\eta(x)}{m^{2/3}g^{-1/3}}. \tag{2.9}$$

In these variables, the governing equations preserve their form

$$\tilde{\psi}_{\tilde{x}\tilde{x}} + \tilde{\psi}_{\tilde{y}\tilde{y}} = 0, \quad \text{in} \quad \tilde{\Omega}_{\tilde{\eta}}, \tag{2.10}$$

with boundary conditions

$$\tilde{\psi} = 1 \quad \text{on} \quad \tilde{y} = \tilde{\eta}(\tilde{x}), \qquad \tilde{\psi} = 0 \quad \text{on} \quad \tilde{y} = -\tilde{h}, \tag{2.11}$$

and the Bernoulli condition

$$\frac{1}{2}|\tilde{\nabla}\tilde{\psi}|^2 + \tilde{y} = \tilde{Q}, \quad \tilde{y} = \tilde{\eta}(\tilde{x}). \tag{2.12}$$

Henceforth, dropping tildes for notational simplicity, we assume $g = 1$ and $m = 1$, obtaining the dimensionless system

$$\Delta\psi = 0, \quad \text{in} \quad \Omega_\eta, \tag{2.13}$$

$$\psi = 1, \quad \text{on} \quad y = \eta(x), \tag{2.14}$$

$$\psi = 0, \quad \text{on} \quad y = 0, \tag{2.15}$$

$$\frac{1}{2}|\nabla\psi|^2 + y = r, \quad \text{on} \quad y = \eta(x). \tag{2.16}$$

Note that, here, we replaced Q by r in order to avoid any issues when interpreting dimensionless results in physical variables. This normalised formulation will be useful for deriving our main results.

In what follows, we will consider Stokes and solitary wave solutions. Any Stokes wave is assumed to be even in x and symmetric around each crest and trough lines. In particular, we have $\psi_x = 0$ below troughs and crests. For a solitary wave, we will also assume that $\eta(x) \to d > 0$, $\psi_y \to y/d$ and $\psi_x \to 0$ as $|x| \to \infty$.

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3. Conjugate laminar flows

In the study of two-dimensional steady water waves, laminar flows play a fundamental role as exact solutions of the governing Euler equations. Laminar flows are characterised by a velocity field that depends solely on the vertical coordinate and corresponds to a parallel shear flow. Specifically, for a flow over a flat bed at $y = 0$, a laminar flow is described by

$$\boldsymbol{u} = (u(y), 0), \tag{3.1}$$

with $u(y) = 1/d$, where d stands for the flow depth.

A conjugate laminar flow is defined as a laminar flow that shares the same mass flux and Bernoulli constant as a given laminar flow but possesses a different free-surface height; see Keady & Norbury (1975). In non-dimensional variables, given a fixed Bernoulli constant r the conjugate flows are defined by the depths which are roots of the equation

$$\frac{1}{2d^2} + d = r. \tag{3.2}$$

This equation can be derived from (2.16) by inserting in $\psi_y = 1/d$ and $\eta = d$.

For any $r > 1.5$ there are exactly two roots $d_-(r) < 1 < d_+(r)$. The corresponding flows are called super- and sub-critical, respectively.

Let us define

$$\hat{\eta} = \max \eta, \quad \check{\eta} = \inf \eta \tag{3.3}$$

to be the fluid height (depth) at the trough and crest, respectively. Then, as shown by Keady & Norbury (1975), we have

$$d_-(r) < \check{\eta} < d_+(r) < \hat{\eta}. \tag{3.4}$$

These inequalities hold true for all Stokes waves; see also Kozlov & Kuznetsov (2009) and Kozlov, Kuznetsov & Lokharu (2015) for a rotational analogue. A similar fact is valid for solitary waves with the equality $d_-(r) = \check{\eta}$ instead of the inequality.

4. Main results

Below we formulate our main results for Stokes waves. However, we shall emphasise that all inequalities, such as (4.1), (4.3) and slope bounds, are also valid for solitary waves. Below we will consider an evolution of Stokes waves starting from a laminar flow. From mathematical point of view we will consider a connected family (as elements in an infinite dimensional space) of Stokes waves. Such family can be constructed using various methods, for instance, by an application of the global bifurcation theory; see Toland (1978), Constantin & Strauss (2004) or Kozlov & Lokharu (2023).

THEOREM. *Let $\mathcal{C}$ be any connected smooth family of Stokes waves containing a laminar flow. Then the following inequality:*

$$\psi_{xy} - \check{k}\psi_x \leqslant 0 \quad \textit{in} \quad \Omega \tag{4.1}$$

holds true along $\mathcal{C}$, where $\check{k} = (2(r - \check{\eta}))^{-1} = \check{\psi}_y^{-2}$. Here, $\check{\psi}_y$ and $\check{\eta}$ stand for the relative horizontal velocity and the fluid height, respectively, at a trough.

Note that

$$d_-^2(r) < \check{k} < d_+^2(r), \tag{4.2}$$

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where $d_-(r)$ and $d_+(r)$ are the depths of the corresponding conjugate flows. For solitary waves $\check{k} = d_-^2(r)$, while for small-amplitude Stokes wave solutions $\check{k} \approx d_+^2(r)$. We shall see below that inequality (4.1) becomes an equality at the linearised level.

We also point out that $\check{k}$ is separated from zero by an absolute constant. Indeed, for near-critical values of r we have $\check{k} \approx 1$. For large r the amplitude of every solution is of order r^{-2}, as recently shown in Lokharu (2021). Thus, the trough height is approximately r, while $\check{\psi}_y$ is then small. Then $\check{k}$ is large for $r >> 1$. For intermediate values of r we can use the estimate $\check{k} > d_-^2(r)$ and the continuity of $d_-(r)$. Together that gives and absolute lower bound for $\check{k}$. This observation is essential for the next result, which is an immediate consequence of the theorem.

COROLLARY 1. *The following estimate:*

$$|v(x,\, y)| \leqslant |v(x,\, \eta(x)|e^{-\check{k}(\eta(x)-y)} \tag{4.3}$$

is true for any solution in $\mathcal{C}$ everywhere inside of the fluid domain. Here, v is the vertical component of the velocity field. The constant $\check{k}$ is the same as in the theorem and is separated from zero by an absolute constant. This estimate is optimal for small-amplitude and solitary waves in the sense that one obtains equality at the linear level.

COROLLARY 2. *The slope bound*

$$\max \eta'^2 = \eta'^2(x_0) \leqslant \frac{\eta(x_0) - \check{\eta}}{2r - \eta(x_0) - \check{\eta}} \tag{4.4}$$

is valid along $\mathcal{C}$.

The claim follows directly from the inequality $\psi_{xy} - \check{k}\psi_x \leqslant 0$ evaluated at the surface for $\eta'' = 0$. When $r - \eta$ is small the bound reduces to $\eta' \leqslant 1$ so the estimate is most informative for waves of moderate amplitudes (far from stagnation). In practice, the wave height is typically much smaller than its maximum admissible value r (more than ten times), so let us assume that $r - \hat{\eta} \geqslant 9/10(r - d_+(r))$ (a quantitative form for the term moderate amplitude). Then (4.4) provides us with the bound

$$\eta'^2 + \frac{9d_-^2(r)}{10d_+^2(r)}(1 + \eta'^2) \leqslant 1, \tag{4.5}$$

which for near-critical waves with $d_- \approx d_+ \approx 1$ leads to $\eta'^2 \leqslant 1/19$, resulting in the maximal slope of approximately 12.92°.

4.1. *Proof of the theorem*

Proof. For an arbitrary $k > 0$ we define

$$f = \psi_{xy} - k\psi_x. \tag{4.6}$$

First, we need to ensure that k is chosen so that $f < 0$ for all small-amplitude solutions in $\mathcal{C}$. Here small-amplitude waves are used only to fixed the sign of f, while the claim of the theorem is true for arbitrary large waves.

For small-amplitude solitary waves one obtains $f < 0$ by the approximation argument. Indeed, every small-amplitude solitary wave can be obtained as a limit of Stokes waves, also of small amplitudes, along a connected family of solutions. Thus, if the claim of the theorem is true for Stokes waves it is also valid for small-amplitude solitary waves. The remaining part of the argument is just the same and is based on the maximum principle.

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At the linear level we have

$$\psi = 1 - \epsilon \cos(\tau x)\varphi(y) + O(\epsilon^2), \tag{4.7}$$

where $\varphi'' = \tau^2\varphi$ on $[0, d_+(r)]$, $\varphi(0) = 0$ and $\varphi'(d_+(r)) = d_+^2(r)\varphi(d_+(r))$. Thus, we compute

$$f = \psi_{xy} - k\psi_x = \epsilon\tau \sin(\tau x)(\varphi' - k\varphi) + O(\epsilon^2). \tag{4.8}$$

Let us show that any $k < d_+^2(r)$ will work for small-amplitude waves. Indeed, the function $\varphi' - k\varphi$ for $k = d_+(r)$ is strictly positive on $[0, d_+(r))$ because otherwise the equation

$$\tau = d^2 \sinh(\tau d) \tag{4.9}$$

would have two distinct roots $y_1 < y_2 = d_+(r)$ for the same fixed value of τ, which is impossible. Now if we choose $k < d_+^2(r)$ we would only increase the function.

Next, we need to prove that f will be strictly negative inside Ω along $\mathcal{C}$. We will prove the claim for $k = (1 - \delta)\hat{k}$ with an arbitrary small $\delta > 0$. Then the main result will follow by passing to the limit $\delta \to 0$, because $\mathcal{C}$ is independent of δ. Note also that $\hat{k}$ is separated from zero and then $k > 0$ is well defined.

Recall that f is zero on the vertical boundaries and is negative along the bottom boundary of Ω. This is due to the symmetry of Stokes and solitary waves around troughs and crests. Thus, because the function f is harmonic, a positive maximum can appear only on the surface. For small-amplitude waves we already know that $f < 0$ and then by continuity we have two potential options of forming a positive maximum: (i) for some solution in $\mathcal{C}$ the zero maximum of f occurs somewhere on the surface of Ω strictly in between trough and crest; or (ii) when a small positive maximum appears near the trough or crest. We will show that both scenarios are not possible, and so f has to stay negative along the continuum. □

4.2. *The case (i), when the zero maximum of f occurs somewhere on the surface of Ω strictly in between trough and crest*

Assume there exists a solution in $\mathcal{C}$ with such a property. We will show below that the normal derivative at the point of maximum is negative, leading to a contradiction. In order to compute the normal derivative, we need to express all partial derivatives of ψ up to the third order at the boundary in terms of η. For the second-order derivatives of ψ one needs to solve a 3 to 3 system of linear equations including

$$\begin{aligned} &\psi_{xx} + \psi_{yy} = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}x}(\psi_x(x, \eta(x)) + \eta'\psi_y(x, \eta(x))) = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\psi_x^2(x, \eta(x)) + \frac{1}{2}\psi_y^2(x, \eta(x)) + \eta(x)\right) = 0. \end{aligned} \tag{4.10}$$

Solving the system we find

$$\psi_{xx}(x, \eta(x)) = \frac{\eta'(x)^2\left(\eta''(x)\,[\psi_y(x, \eta(x))]^2 + 2\right) - \eta''(x)\,[\psi_y(x, \eta(x))]^2}{\left(\eta'(x)^2 + 1\right)^2 \psi_y(x, \eta(x))}, \tag{4.11}$$

$$\psi_{yy}(x, \eta(x)) = \frac{\eta''(x)\,[\psi_y(x, \eta(x))]^2 - \eta'(x)^2\left(\eta''(x)\,[\psi_y(x, \eta(x))]^2 + 2\right)}{\left(\eta'(x)^2 + 1\right)^2 \psi_y(x, \eta(x))}, \tag{4.12}$$

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$$\psi_{xy}(x,\eta(x)) = \frac{\eta'(x)\left(-2\eta''(x)\left[\psi_y(x,\eta(x))\right]^2+\eta'(x)^2-1\right)}{\left(\eta'(x)^2+1\right)^2\psi_y(x,\eta(x))}. \tag{4.13}$$

Next, we need to compute $D^3\psi$ (all third-order partials) in a similar way. But now we have four unknowns and four equations

$$\begin{aligned}
&\psi_{xxx}+\psi_{xyy}=0,\\
&\psi_{xxy}+\psi_{yyy}=0,\\
&\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\psi_x(x,\eta(x))+\eta'\psi_y(x,\eta(x)))=0,\\
&\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{1}{2}\psi_x^2(x,\eta(x))+\frac{1}{2}\psi_y^2(x,\eta(x))+\eta(x)\right)=0.
\end{aligned} \tag{4.14}$$

The exact formulas for the solution are too complicated to be presented here, but one can easily compute it by using any mathematical software.

Finally, after a long computation, we obtain

$$-\eta' f_x + f_y = \frac{M(4kr-2)-2k(2M+\eta'(x)^2)\,\eta(x)+\eta'(x)^2(2kr-1)-8M^2}{4\sqrt{2}\,(\eta'(x)^3+\eta'(x))\left(\dfrac{r-\eta(x)}{\eta'(x)^2+1}\right)^{3/2}}. \tag{4.15}$$

Here,

$$M=\frac{\eta''(x)(r-\eta(x))}{1+\eta'^2(x)}. \tag{4.16}$$

The expression for the normal derivative does not contain $\eta^{(3)}$ because we can express it in terms of lower derivatives from the relation $f_x+\eta' f_y=0$, which gives

$$\begin{aligned}
\eta^{(3)}(x) = \frac{1}{8(r-\eta(x))^2\left(\eta'(x)^3+\eta'(x)\right)}\big[&-(\eta'(x)^3+\eta'(x))^2\left(\eta'(x)^2(2kr-2k\eta(x)-1)\right.\\
&\left.+2kr-2k\eta(x)+1\right)+2(r-\eta(x))(\eta'(x)^2+1)\eta''(x)\left(\eta'(x)^2(2kr-2k\eta(x)+7)\right.\\
&\left.+2kr-2k\eta(x)-1\right)+8(r-\eta(x))^2(4\eta'(x)^2-1)\eta''(x)^2\big].
\end{aligned} \tag{4.17}$$

Next, we need to use the fact that $f=0$ at the point of maximum, which allows us to express η'' (or M) in terms of lower-order derivatives

$$M = -\frac{\eta'(x)-2kr\,\eta'(x)-(1+2kr)\,\eta'(x)^3+2k\,\eta'(x)\left(1+\eta'(x)^2\right)\eta(x)}{4\,\eta'(x)}. \tag{4.18}$$

Using this formula for M inside (4.15), we conclude

$$\begin{aligned}
&-\eta' f_x + f_y =\\
&-\frac{t\left[4k^2(t^2+1)(r-\eta(x))^2+4k(t^2-1)(r-\eta(x))+t^2+1\right]}{8\sqrt{2}\,(t^2+1)\left(\dfrac{r-\eta(x)}{t^2+1}\right)^{3/2}} =: A(k).
\end{aligned} \tag{4.19}$$

Here, $t=\eta'(x)$. The numerator here is a quadratic function of k and the corresponding determinant equals

$$-t^2(r-\eta)^2<0, \tag{4.20}$$

and so the normal derivative is strictly negative, leading to a contradiction.

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4.3. *The case (ii), when a small positive maximum of f occurs near the trough or crest*

We will argue in a similar fashion, but instead computing M from the relation $f = 0$ we express M in terms of f directly

$$M = -\frac{-t^3(2kr+1) - 2kr\,t + 2k(t^2+1)\,t\,\eta(x) + f\psi_y(1+t^2)^2 + t}{4t}. \tag{4.21}$$

Using this formula we compute

$$-\eta' f_x + f_y = A(k) + \frac{f\,(t^2+1)\Big[-2f(r-\eta(x)(1+t^2) + t\psi_y(-1+2t^2+2k(r-\eta(x))(1+2t^2)\Big]}{8\sqrt{2}\,t^3\left(\dfrac{r-\eta(x)}{t^2+1}\right)^{3/2}}, \tag{4.22}$$

where $A(k) < 0$ is the expression from (4.19). Now it is clear that the expression in square brackets is negative for small t, provided

$$2k(r - \check{\eta}) < 1, \tag{4.23}$$

which is granted by the choice of k. Then the normal derivative is also negative, leading to a contradiction. This finishes the proof of the theorem.

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