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Black-hole magnetostatics

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Abstract. The general multipole solutions obeying the appropriate boundary conditions are presented for stationary non-axisymmetric electromagnetic fields in the Kerr background geometry.

1. *Introduction and equations.* In the search for effects of astrophysical importance connected with black holes, a central position is occupied by the study of perturbing electromagnetic and gravitational fields in the black-hole background geometry. Because of a chain of theorems associated chiefly with the names of Israel, Carter, Hawking and Robinson, one now knows that once the geometry settles down to a stationary state this background must be a member of the Kerr–Newman family of metrics. Moreover, in most conceivable astrophysical situations any charge on the hole is small enough to be itself treated as a perturbation on the (uncharged) Kerr black-hole background. In this paper, I shall give the full stationary non-axisymmetric multipole solutions of the Maxwell equations on the Kerr background by making use of the work of Teukolsky (11). I take the Kerr metric in the Boyer–Lindquist form

$$ds^2 = (1-2Mr/\Sigma)\,dt^2 + 4Mar\,\Sigma^{-1}\sin^2\theta\,dt\,d\phi - \Sigma/\Delta\,dr^2 - \Sigma\,d\theta^2 - \sin^2\theta(r^2+a^2+2Ma^2r\,\Sigma^{-1}\sin^2\theta)\,d\phi^2, \quad (1)$$

where

$$\Sigma = r^2+a^2\cos^2\theta, \quad \Delta = r^2-2Mr+a^2 = (r-r_+)(r-r_-), \quad r_\pm = M\pm\sqrt{(M^2-a^2)}.$$

(The units are such that $G = c = 1$.) Using the Kinnersley (6) null tetrad, whose $[t, r, \theta, \phi]$ components are

$$\left.\begin{aligned} l^\mu &= [(r^2+a^2)/\Delta, 1, 0, a/\Delta], \quad n^\mu = [r^2+a^2, -\Delta, 0, a]/2\Sigma \\ m^\mu &= [ia\sin\theta, 0, 1, i/\sin\theta]/[\sqrt{2}(r+ia\cos\theta)], \end{aligned}\right\} \quad (2)$$

one can describe the electromagnetic field $F_{\mu\nu}$ by the three complex quantities

$$\phi_0 = F_{\mu\nu}l^\mu m^\nu, \quad \phi_1 = \tfrac{1}{2}F_{\mu\nu}(l^\mu n^\nu + m^{*\mu}m^\nu), \quad \phi_2 = F_{\mu\nu}m^{*\mu}n^\nu. \quad (3)$$

One can in turn express $F_{\mu\nu}$ in terms of the ϕ's, but a more intuitively appealing representation of the field is to give the components $\mathscr{E}_{(a)}$ and $\mathscr{B}_{(a)}$ of electric and magnetic fields as seen in the orthonormal ZAMO frame of Bardeen (1). This frame has $[t, r, \theta, \phi]$ components

$$\left.\begin{aligned} e^\mu_{(t)} &= [A^{\frac{1}{2}}/(\Sigma\Delta)^{\frac{1}{2}}, 0, 0, 2Mar/(A\Sigma\Delta)^{\frac{1}{2}}], \\ e^\mu_{(r)} &= [0, \Delta^{\frac{1}{2}}/\Sigma^{\frac{1}{2}}, 0, 0], \quad e^\mu_{(\theta)} = [0, 0, \Sigma^{-\frac{1}{2}}, 0], \\ e^\mu_{(\phi)} &= [0, 0, 0, \Sigma^{\frac{1}{2}}/(A^{\frac{1}{2}}\sin\theta)], \end{aligned}\right\} \quad (4)$$

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where $$A = (r^2+a^2)^2 - a^2\Delta\sin^2\theta.$$

Defining $\mathscr{E}_{(a)}$ and $\mathscr{B}_{(a)}$ by

$$\mathscr{E}_{(a)} = F_{\mu\nu}e^{\mu}_{(a)}e^{\nu}_{(t)}, \quad \mathscr{B}_{(a)} = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\tau}F^{\lambda\tau}e^{\mu}_{(a)}e^{\nu}_{(t)},$$

one finds the components

$$\begin{aligned}
\mathscr{E}_{(r)} &= \left\{\frac{ia\sin\theta}{\sqrt{2}\rho A^{\frac{1}{2}}}(\phi_2 - \tfrac{1}{2}\Delta\rho^2\phi_0) + \text{c.c.}\right\} + 2\frac{r^2+a^2}{A^{\frac{1}{2}}}\operatorname{Re}\phi_1,\\
\mathscr{E}_{(\theta)} &= \left\{\frac{\Delta^{\frac{1}{2}}(r^2+a^2)}{\sqrt{2}\rho^* A^{\frac{1}{2}}\Sigma}\left(\tfrac{1}{2}\phi_0 - \frac{\phi_2}{\rho^2\Delta}\right) + \text{c.c.}\right\} - \frac{2a\Delta^{\frac{1}{2}}}{A^{\frac{1}{2}}}\operatorname{Im}\phi_1\sin\theta,\\
\mathscr{E}_{(\phi)} &= -i\Delta^{\frac{1}{2}}\rho\left(\frac{\phi_0}{2\sqrt{2}} + \frac{\phi_2}{\sqrt{2}\rho^2\Delta}\right) + \text{c.c.},\\
\mathscr{B}_{(r)} &= \left\{\frac{a\sin\theta}{\sqrt{2}\rho A^{\frac{1}{2}}}(\phi_2 - \tfrac{1}{2}\Delta\rho^2\phi_0) + \text{c.c.}\right\} + 2\frac{r^2+a^2}{A^{\frac{1}{2}}}\operatorname{Im}\phi_1,\\
\mathscr{B}_{(\theta)} &= -\left\{\frac{i\Delta^{\frac{1}{2}}(r^2+a^2)}{\sqrt{2}\rho^* A^{\frac{1}{2}}\Sigma}\left(\tfrac{1}{2}\phi_0 - \frac{\phi_2}{\rho^2\Delta}\right) + \text{c.c.}\right\} + \frac{2a\Delta^{\frac{1}{2}}}{A^{\frac{1}{2}}}\operatorname{Re}\phi_1\sin\theta,\\
\mathscr{B}_{(\phi)} &= -\Delta^{\frac{1}{2}}\rho\left(\frac{\phi_0}{2\sqrt{2}} + \frac{\phi_2}{\sqrt{2}\rho^2\Delta}\right) + \text{c.c.},
\end{aligned} \tag{5}$$

where $\rho = -(r - ia\cos\theta)^{-1}$ and 'c.c.' denotes the complex conjugate of the preceding terms. Using spinors (5) or by direct computation from (5) one gets the field invariants $\mathscr{E}.\mathscr{B}$, $\mathscr{E}^2 - \mathscr{B}^2$ expressed in terms of the ϕ's:

$$2\mathscr{E}.\mathscr{B} = -4\operatorname{Im}(\phi_0\phi_2 + \phi_1^2), \quad \mathscr{E}^2 - \mathscr{B}^2 = -4\operatorname{Re}(\phi_0\phi_2 - \phi_1^2). \tag{6}$$

As shown by Fackerell and Ipser (3), Maxwell's equations take a slightly simpler form when expressed in terms of the variables

$$\Phi_0 = \Delta\rho\sin\theta\phi_0, \quad \Phi_1 = \phi_1\rho^{-2}, \quad \Phi_2 = \rho^{-1}\sin\theta\phi_2.$$

In the stationary case ($\partial_t\phi_A = 0, A = 0, 1, 2$), they become

$$(\Delta\partial_r + a\partial_\phi)\Phi_1 = 2^{-\frac{1}{2}}\rho^{-2}(\sin\theta)^{-1}\eth'_{(0)}\Phi_0, \tag{7}$$

$$(-\Delta\partial_r + a\partial_\phi)\Phi_1 = 2^{\frac{1}{2}}\rho^{-2}(\sin\theta)^{-1}\eth_{(0)}\Phi_2, \tag{8}$$

$$(\Delta\partial_r + a\partial_\phi)\Phi_2 = 2^{-\frac{1}{2}}\Delta\rho^2\sin\theta\eth'_{(0)}\Phi_1, \tag{9}$$

$$(-\Delta\partial_r + a\partial_\phi)\Phi_0 = 2^{\frac{1}{2}}\Delta\rho^2\sin\theta\eth_{(0)}\Phi_1, \tag{10}$$

where $$\eth_{(0)} = -(\partial_\theta + i\operatorname{cosec}\theta\partial_\phi), \quad \eth'_{(0)} = -(\partial_\theta - i\operatorname{cosec}\theta\partial_\phi).$$

$\eth_{(0)}, \eth'_{(0)}$ are in fact the spin-raising and lowering operators $\eth, \eth'$ of Newman and Penrose (7) when applied to quantities of zero spin-weight.

It was Teukolsky's achievement (11) to derive (in the full non-stationary case) separable equations for quantities characterizing perturbations by fields of various spins in the Kerr background. For electromagnetism (spin one), the relevant quantities are ϕ_0 and $\phi_2\rho^{-2}$, Teukolsky's equations being integrability conditions for the Maxwell equations (7)–(10). In the stationary case, the separation of variables may be expressed in the form

$$\phi_0^{(l,m)} = R_{lm}(r)\eth Y_{lm}, \quad \phi_2^{(l,m)}\rho^{-2} = S_{lm}(r)\eth' Y_{lm} \tag{11}$$

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for an (l, m) multipole, where Y_{lm} is the usual spherical harmonic. (l is an integer $\geqslant 1$ and m an integer with $|m| \leqslant l$; $\eth Y_{lm}$, $\eth' Y_{lm}$ are related to the 'spin-one spherical harmonics' (4) ${}_{\pm 1}Y_{lm}$ by $\eth Y_{lm} = [l(l+1)]^{\frac{1}{2}}\,{}_1Y_{lm}$, $\eth' Y_{lm} = -[l(l+1)]^{\frac{1}{2}}\,{}_{-1}Y_{lm}$.) $R_{lm}(r), S_{lm}(r)$ obey the equations

$$\frac{d}{dr}\left(\Delta^2 \frac{dR_{lm}}{dr}\right) + [a^2m^2 + 2iam(r-M) - (l-1)(l+2)\Delta] R_{lm} = 0, \tag{12}$$

$$\Delta^2 \frac{d^2S_{lm}}{dr^2} + [a^2m^2 - 2iam(r-M) - l(l+1)\Delta] S_{lm} = 0. \tag{13}$$

One can easily verify the following symmetry property of equations (12) and (13): if R_{lm} is a solution of (12) for particular values of l and m, then $S_{lm} = \Delta R_{l,-m}$ is a solution of (13) for the same l and m. Thus a knowledge of the solutions of (12) is sufficient to supply those of (13). Further, since the coefficients of (12) are real except for the term $2iam(r-M)$, one has $R^*_{lm} = R_{l,-m}$.

2. *Boundary conditions.* To complete the specification of solutions of the Maxwell equations (7)–(10), one needs appropriate boundary conditions. These will take the form of requiring the solutions to decay correctly at $r = \infty$ and demanding 'regularity' at the future event horizon $r = r_+$. This 'regularity' amounts to ensuring that no physically well-behaved observer sees infinite fields near the hole. From the electromagnetic 'no-hair' theorems (3), (11) one knows that the only solution source-free for $r_+ \leqslant r \leqslant \infty$ satisfying both boundary conditions is that corresponding to putting a charge (or magnetic monopole) on the hole, namely $\phi_0 = \phi_2 = 0$, $\phi_1 = \text{constant} \times \rho^2$. Thus one will get two kinds of multipole solutions (11): those satisfying regularity at the horizon but violating it at infinity and vice versa. To produce the full solution to a given physical problem appropriate combinations of the two kinds of solution must be matched across sources in the way familiar from flat-space electrodynamics.

From equation (12), one finds for large r the asymptotic solutions $R_{lm} \sim r^{l-1}, r^{-l-2}$. Thus the boundary condition at infinity can be taken as

$$R^{(\infty)}_{lm} = r^{-l-2} + O(r^{-l-3}), \tag{14}$$

where we have normalized $R^{(\infty)}_{lm}$ to unity in its leading term.

The formulation of the boundary condition at $r = r_+$ is rather more complicated: I shall follow here essentially the discussions of Teukolsky (11) and Stewart (10). The first point is that both the Boyer–Lindquist coordinates of (1) and the tetrad (2) become singular at the horizon. This is overcome by transforming to Kerr coordinates

$$dv = dt + dr^*, \quad d\tilde{\phi} = d\phi + a(r^2+a^2)^{-1}\,dr^*$$

with $dr^*/dr = (r^2+a^2)/\Delta$, and by applying an appropriate null rotation (11) to the tetrad (2). Kerr coordinates cover the *future* event horizon (which is all we are interested in) in a regular way. Transforming the radial equations (12) and (13) to these coordinates yields near the future horizon ($r^* \to -\infty$) the asymptotic solutions

$${}^{(1)}R_{lm} \sim e^{ikr^*}, \quad {}^{(2)}R_{lm} \sim \Delta^{-1} e^{-ikr^*},$$

$${}^{(1)}S_{lm} \sim e^{ikr^*}, \quad {}^{(2)}S_{lm} \sim \Delta\, e^{-ikr^*},$$

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with $k = -am/2Mr_+$. Because of the null rotation of the tetrad the ϕ's are also transformed, the two asymptotic solutions above corresponding to

$$^{(1)}\phi_0^{\text{New}} \sim e^{im\phi}e^{ikr^*}\Delta, \quad ^{(2)}\phi_0^{\text{New}} \sim e^{im\phi}e^{-ikr^*},$$
$$\rho^{-2}\,^{(1)}\phi_2^{\text{New}} \sim e^{im\phi}e^{ikr^*}\Delta^{-1}, \quad \rho^{-2}\,^{(2)}\phi_2^{\text{New}} \sim e^{im\phi}e^{-ikr^*},$$

respectively. It turns out (see below) that the Maxwell equations require the solutions $^{(1)}R_{lm}$, $^{(1)}S_{lm}$ and $^{(2)}R_{lm}$, $^{(2)}S_{lm}$ to occur in pairs, i.e. R_{lm} has the form $^{(1)}R_{lm}$ near $r = r_+$ if and only if S_{lm} has the form $^{(1)}S_{lm}$ there, with a similar statement for $^{(2)}R_{lm}$, $^{(2)}S_{lm}$. Correspondingly the $^{(1)}\phi_0^{\text{New}}$ and $\rho^{-2}\,^{(1)}\phi_2^{\text{New}}$ etc. occur in pairs. We can now give the boundary condition at $r = r_+$: since the future horizon is a perfectly regular region of spacetime (as demonstrated by the existence of Kerr coordinates there) and the ϕ^{New} are simply the components of the electromagnetic field relative to a regular null tetrad there, it follows that for a regular electromagnetic field the ϕ^{New} should be finite at the future horizon. This is easily seen to be the case for $^{(2)}\phi_0^{\text{New}}$ and $^{(2)}\phi_2^{\text{New}}$, as they may both be rewritten in the form $e^{im\tilde{\phi}}$. The other pair of solutions must however be rejected for $^{(1)}\phi_2^{\text{New}}$ is infinite on the future horizon. Thus the boundary condition at $r = r_+$ can be taken as $R_{lm}^{(H)} \sim \Delta^{-1}e^{-ikr^*}$ (which will imply $S_{lm}^{(H)} \sim \Delta e^{ikr^*}$). Adopting the notation (9) $\delta = 2\sqrt{(M^2-a^2)}$ we have for $r \sim r_+$

$$r^* \sim 2Mr_+\delta^{-1}\log|(r-r_+)/r_+|,$$

so that the boundary condition at $r = r_+$ becomes

$$R_{lm}^{(H)} \sim (r-r_+)^{-1+iam/\delta}. \tag{15}$$

One expects this solution to behave like r^{l-1} for large r, so the normalization will be chosen to make the coefficient of this leading term unity.

Two remarks are perhaps in order here. The first is that it can be shown (11) that (15) is equivalent to requiring that a physically well-behaved observer who is not himself falling into the hole sees all wave packets as having a negative radial group velocity. (See however the remarks at the end of the next section.) Secondly one should note that, in the full non-stationary case, it is not in general sufficient simply to demand (as is sometimes done) that the invariants (6) of the electromagnetic field remain finite at the future horizon in order to derive the boundary condition there. To see this, consider a wave with a positive radial group velocity. By the first remark, this does not obey (15) and hence leads to divergent field components in any regular frame on the future horizon. However, since the invariants for a wave are identically zero, the solution will nevertheless obey the condition of finite invariants at $r = r_+$.

3. *Stationary solutions of Maxwell's equations.* It is easily seen that equation (12) has just three regular singular points: at $r = r_-, r_+$ and ∞. It may therefore be transformed into the hypergeometric equation (11). As a first step we move the singular points to the standard positions 0, 1 and ∞ by the substitution

$$x = (r-r_+)/(r_--r_+) = (r-r_+)/(-\delta) \tag{16}$$

(13) then reduces to

$$\frac{d^2R_{lm}}{dx^2}+\left[\frac{2}{x}+\frac{2}{x-1}\right]\frac{dR_{lm}}{dx}+\left[-\frac{a^2m^2+iam\delta}{x\delta^2}+\frac{a^2m^2-iam\delta}{(x-1)\delta^2}-(l-1)(l+2)\right]\frac{R_{lm}}{x(x-1)} = 0$$

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which assumes the standard form of Papperitz's equation (8)

$$\frac{d^2R_{lm}}{dx^2}+\left[\frac{1-\lambda-\lambda'}{x}+\frac{1-\mu-\mu'}{x-1}\right]\frac{dR_{lm}}{dx}$$
$$+\left[-\frac{\lambda\lambda'}{x}+\frac{\mu\mu'}{x-1}+\nu(1-\lambda-\lambda'-\mu-\mu'-\nu)\right]\frac{R_{lm}}{x(x-1)}=0 \qquad (17)$$

provided

$$\left.\begin{aligned}&\lambda+\lambda'=-1, \quad \mu+\mu'=-1,\\ &\lambda\lambda'=(a^2m^2+iam\delta)/\delta^2, \quad \mu\mu'=(a^2m^2-iam\delta)/\delta^2,\\ &\nu(1-\lambda-\lambda'-\mu-\mu'-\nu)=-(l-1)(l+2) \quad (\Rightarrow \nu=-(l-1), l+2).\end{aligned}\right\} \qquad (18)$$

A solution of (17) is, (8),

$$R_{lm}=x^{\lambda}(x-1)^{\mu}F(\alpha,\beta,\gamma,x)$$

with $F(\alpha,\beta,\gamma,x)$ the standard hypergeometric function, and

$$\gamma=\lambda-\lambda'+1, \quad \alpha+\beta=\lambda+\mu-\lambda'-\mu'+1,$$
$$\alpha\beta=[\nu+\lambda+\mu][1-\nu+\lambda'+\mu'].$$

Suitably choosing a solution of (18) and arranging the normalization gives in a straightforward manner (note that F is a polynomial here)

$$R_{lm}^{(H)}=\delta^{l+1}\frac{2q(2q+1)\dots(2q+l)}{l(l+1)\dots(2l)}(r-r_+)^{-1+q}(r-r_-)^{-1-q}F[-l-1,l,2q,(r-r_+)/(-\delta)] \qquad (19)$$

with
$$q=iam/\delta \qquad (20)$$

as the solution obeying the boundary condition (15) and also $R_{lm}^{(H)}=r^{l-1}+O(r^{l-2})$ as $r\to\infty$. The fact that (12) can be solved in terms of hypergeometric functions was pointed out by Teukolsky (11): that in the case of $R_{lm}^{(H)}$ the hypergeometric functions reduce to polynomials (called *ipso facto* Jacobi polynomials) because the parameter α in F is a negative integer was noted by Stewart (10). (Note that this is not true for $R_{lm}^{(\infty)}, S_{lm}^{(\infty)}$ derived below.) What is done in this paper is to derive full sets of consistent solutions for the ϕ's; i.e. starting with ϕ_0 I derive the corresponding ϕ_1, ϕ_2. To get $R_{lm}^{(\infty)}$ we take the solution of the hypergeometric equation about the regular singular point $x=\infty$, finding

$$R_{lm}^{(\infty)}=(r-r_+)^{-l-1+q}(r-r_-)^{-1-q}F[l,l+1-2q,2l+2,-\delta/(r-r_+)] \qquad (21)$$

which obeys (14). (We assume for the time being $q\neq 0$; solutions with $am=0$ are given by taking the limit $q\to 0$. This procedure is discussed further at the end of this section.)

In a completely analogous way one finds for $\phi_2\rho^{-2}$ the solutions

$$S_{lm}^{(H)}=C_{lm}^{(H)}(r-r_+)^{1+q}(r-r_-)^{1-q}F[-l+1,l+2,2+2q,(r-r_+)/(-\delta)], \qquad (22)$$

$$S_{lm}^{(\infty)}=C_{lm}^{(\infty)}(r-r_+)^{-l-1+q}(r-r_-)^{1-q}F[l+2,1+l-2q,2l+2,-\delta/(r-r_+)]. \qquad (23)$$

Here $S_{lm}^{(H)}, S_{lm}^{(\infty)}$ are regular at the horizon and at infinity respectively, so one expects that in the full solution of the Maxwell equations $R_{lm}^{(H)}, S_{lm}^{(H)}$ and $R_{lm}^{(\infty)}, S_{lm}^{(\infty)}$ will occur

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in pairs. The coefficients $C_{lm}^{(H)}, C_{lm}^{(\infty)}$ must of course be determined by solving the full set of equations (7)–(10), to which task we now turn. From (7) we find on substituting (11) the relation

$$(\Delta\partial_r + a\partial_\phi)\,\Phi_1 = -ia2^{-\frac{1}{2}}\Delta R_{lm}\sin\theta\eth Y_{lm} - l(l+1)\,2^{-\frac{1}{2}}\rho^{-1}\Delta R_{lm}Y_{lm},$$

where use has been made of the identity

$$(\sin\theta)^{-1}\eth'_{(0)}\sin\theta\eth_{(0)}Y_{lm} = \eth'\eth Y_{lm} = -l(l+1)\,Y_{lm}.$$

Similarly (8) gives

$$(\Delta\partial_r - a\partial_\phi)\,\Phi_1 = ia2^{\frac{1}{2}}S_{lm}\sin\theta\eth' Y_{lm} + 2^{\frac{1}{2}}l(l+1)\,\rho^{-1}S_{lm}Y_{lm}.$$

It is clear that Φ_1 must have $e^{im\phi}$ dependence, so for the axisymmetric $m = 0$ multipoles the left-hand sides of these equations are the same. Further, $\eth Y_{l0} = \eth' Y_{l0}$, so we conclude that $S_{l0} = -\frac{1}{2}\Delta R_{l0}$, a well-known result. Adding the two equations above and integrating with respect to r gives

$$\begin{aligned}\Phi_1 = {} & -ia2^{-\frac{3}{2}}\sin\theta\eth Y_{lm}\int^r R_{lm}\,dr + ia2^{-\frac{1}{2}}\sin\theta\eth' Y_{lm}\int^r S_{lm}/\Delta\,dr \\ & - 2^{-\frac{3}{2}}l(l+1)\,Y_{lm}\int^r \rho^{-1}(R_{lm} - 2S_{lm}/\Delta)\,dr + f_{lm}(\theta,\phi),\end{aligned} \tag{24}$$

where f_{lm} is an arbitrary function of θ, ϕ. Substituting this form into (9) and (10) and using identities like

$$\begin{aligned}\eth_{(0)}(\sin\theta\eth_{(0)}Y_{lm}) &= (\eth'_{(0)} - 2i\operatorname{cosec}\theta\,\partial_\phi)(\sin\theta\eth_{(0)}Y_{lm}) \\ &= -l(l+1)\sin\theta Y_{lm} + 2m\eth_{(0)}Y_{lm}\end{aligned}$$

produces the two equations:

$$\left.\begin{aligned} iam\rho^{-1}R_{lm} - \Delta R_{lm} - \rho^{-1}\partial_r(\Delta R_{lm}) &= -iam\int^r R_{lm}\,dr - \tfrac{1}{2}l(l+1)\int^r \rho^{-1}(R_{lm} - 2S_{lm}/\Delta)\,dr \\ &\quad + 2^{\frac{1}{2}}\eth_{(0)}f_{lm}/\eth_{(0)}Y_{lm}, \\ iam\Delta^{-1}\rho^{-1}S_{lm} + S_{lm} + \rho^{-1}\partial_r S_{lm} &= -iam\int^r \Delta^{-1}S_{lm}\,dr - \tfrac{1}{4}l(l+1)\int^r \rho^{-1}(R_{lm} - 2S_{lm}/\Delta)\,dr \\ &\quad + 2^{-\frac{1}{2}}\eth'_{(0)}f_{lm}/\eth'_{(0)}Y_{lm},\end{aligned}\right\} \tag{25}$$

where one clearly sees the symmetry between ΔR_{lm} and $-2S_{l,-m}$.

As yet we have not determined the lower limits in the integrals occurring above and in the expression for Φ_1. These will depend on whether we ask for solutions regular at the horizon or at infinity. Considering the former for the moment, we take the asymptotic form of equations (25) near $r = r_+$ and for large r. This shows first that S_{lm} must be the $S_{lm}^{(H)}$ of equation (22), and moreover with the coefficient

$$C_{lm}^{(H)} = -\delta^{l-1}\frac{(2q+2)\ldots(2q+l)}{2(l+2)\ldots(2l)} \tag{26}$$

so that $S_{lm}^{(H)} \sim -(r^2/2)\,R_{lm'}^{(H)}$, for large r and any m, m'. Secondly, by choosing the lower limits in the integrals to be r_+ one can take $\eth_{(0)}f_{lm} = \eth'_{(0)}f_{lm} = 0$, which implies

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f_{lm} = constant. The value of this constant is fixed by requiring the (l, m) multipole to represent a state of zero electric and magnetic monopole charge on the black hole. (If necessary we can always add charge to the hole by adding a constant to Φ_1.) From Gauss's law, this is equivalent to demanding $\int_{S_\infty} \mathscr{E}.dS = \int_{S_\infty} \mathscr{B}.dS = 0$, where S_∞ is the sphere at infinity. Using equations (5) for $\mathscr{E}_{(r)}$ and $\mathscr{B}_{(r)}$ and standard properties of spherical harmonics, we get $f_{lm} = 0$. Almost identical reasoning shows that to get solutions regular at infinity one takes $R_{lm} = R_{lm}^{(\infty)}$ (equation (21)), $S_{lm} = S_{lm}^{(\infty)}$ (equation (23)) with $C_{lm}^{(\infty)} = -\frac{1}{2}$. Φ_1 is given by (24) with $f_{lm} = 0$ and the lower limits of integration are now $r = \infty$. Collecting our results we have the general (l, m) multipole solution of Maxwell's equations in the Kerr background:

$$\left.\begin{aligned} \phi_0^{(l,m)} &= R_{lm}(r)\,\eth Y_{lm}, \quad \phi_2^{(l,m)}\rho^{-2} = S_{lm}(r)\,\eth' Y_{lm} \\ \phi_1^{(l,m)}\rho^{-2} &= -ia2^{-\frac{1}{2}}\sin\theta\,\eth Y_{lm}\int_{r_0}^{r} R_{lm}\,dr + ia2^{-\frac{1}{2}}\sin\theta\,\eth' Y_{lm}\int_{r_0}^{r} S_{lm}/\Delta\,dr \\ &\quad - 2^{-\frac{3}{2}}\,l(l+1)\,Y_{lm}\int_{r_0}^{r}\rho^{-1}(R_{lm} - 2S_{lm}/\Delta)\,dr, \end{aligned}\right\}$$

where for solutions regular at the horizon one has $r_0 = r_+$ in ϕ_1 and

$$R_{lm} = R_{lm}^{(H)} = \delta^{l+1}\frac{2q(2q+1)\ldots(2q+l)}{l(l+1)\ldots(2l)}(r-r_+)^{-1+q}(r-r_-)^{-1-q} \times F[-l-1, l, 2q, (r-r_+)/(-\delta)]$$

$$S_{lm} = S_{lm}^{(H)} = -\delta^{l-1}\frac{(2q+2)\ldots(2q+l)}{2(l+2)\ldots(2l)}(r-r_+)^{1+q}(r-r_-)^{1-q} \times F[-l+1, l+2, 2+2q, (r-r_+)/(-\delta)]$$

($q = iam/\delta$). For solutions regular at infinity one takes $r_0 = \infty$ and

$$R_{lm} = R_{lm}^{(\infty)} = (r-r_+)^{-l-1+q}(r-r_-)^{-1-q}F[l, l+1-2q, 2l+2, -\delta/(r-r_+)],$$

$$S_{lm} = S_{lm}^{(\infty)} = -\tfrac{1}{2}(r-r_+)^{-l-1+q}(r-r_-)^{1-q}F[l+2, l+1-2q, 2l+2, -\delta/(r-r_+)].$$

A curious feature of these solutions emerges in the case $am = 0$: taking the limit $q \to 0$, $R_{lm}^{(H)}$ is seen not to obey the boundary condition (15) at $r = r_+$; rather we have $R_{lm}^{(H)} \to$ constant there. $S_{lm}^{(H)}$ does however obey the boundary condition analogous to (15), namely $S_{lm}^{(H)} \sim (r - r_+)$. That these solutions are the correct ones is seen by verifying that the ϕ^{New} of Sect. 2 are finite. There does indeed exist a solution obeying (15) even for $am = 0$: for $l = 1$ it is easily found from (12) to be $\int \Delta^{-2} dr$. Since this dies away at infinity one expects trouble at $r = r_+$: the invariants (6) turn out to diverge there. Thus Teukolsky's 'wave-packet' argument breaks down in this special case: this is not very surprising since for $am = 0$ stationary solutions are identical with standing waves. For $am \neq 0$ this is not so because of the existence of the ergosphere for $a \neq 0$. There the non-axisymmetry of an $m \neq 0$ mode means that even 'stationary' (i.e. t-independent) modes appear as time-varying to all physical observers, since these are forced to move in the ϕ-direction because of dragging of inertial frames. By

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manipulation of the hypergeometric functions one can verify that the R_{lm} reduce to the solutions given previously by Cohen *et al.* (2) for the case $am = 0$:

$$R^{(H)}_{lm} = \frac{l!(l-1)!}{(2l)!}\,\delta^l \frac{d}{dr} P_l\left(\frac{2(r-M)}{\delta}\right),$$

$$R^{(\infty)}_{lm} = -\frac{\sqrt{2(2l+1)!}}{l!(l+1)!}\,\delta^{-l-1} \frac{d}{dr} Q_l\left(\frac{2(r-M)}{\delta}\right)$$

with $S_{lm} = -\frac{1}{2}\Delta R_{lm}$ and P_l, Q_l Legendre functions of the first and second kinds. (Note that for $a = 0$ the solutions are independent of m.)

In subsequent papers the solutions found here will be applied, for example to the problem of a rotating black hole immersed in an asymptotically uniform magnetic field whose axis of symmetry does not coincide with that of the hole.

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