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A Pair of Forbidden Subgraphs and 2-Factors

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Dedicated to Richard H. Schelp

In this paper, we consider pairs of forbidden subgraphs that imply the existence of a 2-factor in a graph. For $d \geqslant 2$, let $\mathcal{G}_d$ be the set of connected graphs of minimum degree at least d. Let F_1 and F_2 be connected graphs and let $\mathcal{H}$ be a set of connected graphs. Then $\{F_1, F_2\}$ is said to be a forbidden pair for $\mathcal{H}$ if every $\{F_1, F_2\}$-free graph in $\mathcal{H}$ of sufficiently large order has a 2-factor. Faudree, Faudree and Ryjáček have characterized all the forbidden pairs for the set of 2-connected graphs. We first characterize the forbidden pairs for $\mathcal{G}_2$, which is a larger set than the set of 2-connected graphs, and observe a sharp difference between the characterized pairs and those obtained by Faudree, Faudree and Ryjáček. We then consider the forbidden pairs for connected graphs of large minimum degree. We prove that if $\{F_1, F_2\}$ is a forbidden pair for $\mathcal{G}_d$, then either F_1 or F_2 is a star of order at most $d+2$. Ota and Tokuda have proved that every $K_{1,\lfloor \frac{d+2}{2} \rfloor}$-free graph of minimum degree at least d has a 2-factor. These results imply that for $k \geqslant d+2$, no connected graphs F except for stars of order at most $d+2$ make $\{K_{1,k}, F\}$ a forbidden pair for $\mathcal{G}_d$, while for $k \leqslant \lfloor \frac{d+2}{2} \rfloor$ every connected graph F makes $\{K_{1,k}, F\}$ a forbidden pair for $\mathcal{G}_d$. We consider the remaining range of $\lfloor \frac{d+2}{2} \rfloor < k < d+2$, and prove that only a finite number of connected graphs F make $\{K_{1,k}, F\}$ a forbidden pair for $\mathcal{G}_d$.

1. Introduction

In this paper, we consider only finite graphs. For a connected graph H, a graph G is said to be H-free if G does not contain an induced subgraph which is isomorphic to H.

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More generally, for a set of connected graphs $\mathcal{H}$, G is said to be $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. The set $\mathcal{H}$ is often referred to as forbidden subgraphs. Duffus, Gould and Jacobson [4] have proved that every $\{K_{1,3}, \mathrm{Net}\}$-free 2-connected graph is Hamiltonian, where Net is the unique connected graph having degree sequence $(3,3,3,1,1,1)$ (Figure 1). This result stimulated the research on the relationship between forbidden subgraphs and Hamiltonicity. After several sets of forbidden subgraphs implying the existence of a Hamiltonian cycle were found, Faudree and Gould [6] characterized all the pairs of forbidden subgraphs that force the existence of a Hamiltonian cycle in a 2-connected graph of sufficiently large order.

In order to state results on forbidden subgraphs clearly, we introduce several notations and definitions. Let $\mathcal{G}$ be the set of all the connected graphs of order at least three. For $\mathcal{H} \subset \mathcal{G}$, we denote by $\mathcal{F}(\mathcal{H})$ the set of all the $\mathcal{H}$-free graphs in $\mathcal{G}$. Then for $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{G}$, $\mathcal{H}_1 \subset \mathcal{H}_2$ implies $\mathcal{F}(\mathcal{H}_2) \subset \mathcal{F}(\mathcal{H}_1)$. This means that when we try to characterize the sets of forbidden subgraphs that force a certain graph property, we have only to determine their inclusion-minimal elements.

For $G_1, G_2 \in \mathcal{G}$, we write $G_1 \prec G_2$ if G_2 has an induced subgraph isomorphic to G_1, and for $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{G}$, we write $\mathcal{H}_1 \leqslant \mathcal{H}_2$ if, for every $H_2 \in \mathcal{H}_2$, there exists $H_1 \in \mathcal{H}_1$ with $H_1 \prec H_2$. Fujita and co-workers [7] observed that for $\boldsymbol{H} \subset 2^{\mathcal{G}}$, if no pair of elements in $\boldsymbol{H}$ are comparable with respect to inclusion, then $(\boldsymbol{H}, \leqslant)$ is a partially ordered set. They also remarked that $\mathcal{H}_1 \prec \mathcal{H}_2$ implies $\mathcal{F}(\mathcal{H}_1) \subset \mathcal{F}(\mathcal{H}_2)$. Therefore, when we try to determine the sets of forbidden subgraphs for a graph property, we have only to characterize their maximal elements with respect to $\leqslant$.

A Hamiltonian cycle is interpreted as a connected 2-factor. In this sense, all the pairs of forbidden subgraphs that appear in [6] also guarantee the existence of a 2-factor in a 2-connected graph of sufficiently large order. However, if we only require the existence of a 2-factor, connected or disconnected, we may expect more pairs. In fact, Faudree, Faudree and Ryjáček [5] have characterized such pairs. The graphs $N(3,1,1)$ and $B(4,1)$ are the ones shown in Figure 1. We denote by P_l the path of order l.

Theorem A ([5]). *Let F_1 and F_2 be connected graphs of order at least three. Then there exists a positive integer n_0 such that every 2-connected $\{F_1, F_2\}$-free graph of order at least n_0 has a 2-factor if and only if $\{F_1, F_2\} \leqslant \{K_{1,3}, B(4,1)\}$, $\{F_1, F_2\} \leqslant \{K_{1,3}, N(3,1,1)\}$ or $\{F_1, F_2\} = \{K_{1,4}, P_4\}$.*

Now we focus on the assumption of connectivity. Since every graph with a Hamiltonian cycle is 2-connected, we may naturally assume that the graphs in consideration are 2-connected when we discuss the existence of a Hamiltonian cycle. On the other hand, not every graph with a 2-factor is 2-connected. Therefore, the assumption of 2-connectedness seems to be too strong when we discuss 2-factors. Since every graph with a 2-factor has minimum degree at least two, we consider that a natural domain of our research is the set of connected graphs of minimum degree at least two. (We do not have to worry about disconnected graphs since a graph has a 2-factor if and only if all of its components have a 2-factor.) For a positive integer d, let $\mathcal{G}_d$ be the set of connected graphs of minimum degree at least d. By the above observation, we start our research in $\mathcal{G}_2$.

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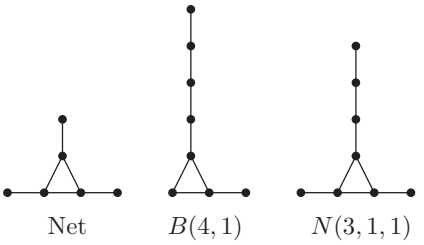


Figure 1. Net, $B(4,1)$ and $N(3,1,1)$.

Since $\mathcal{G}_2$ is larger than the set of 2-connected graphs, not all the pairs that appear in [5] may imply the existence of a 2-factor in $\mathcal{G}_2$ any longer. This actually happens, as shown in [2].

Theorem B ([2]). *Let F_1 and F_2 be connected graphs of order at least three. If there exists a positive integer n_0 such that every connected $\{F_1, F_2\}$-free graph of order at least n_0 and minimum degree at least two has a 2-factor, then either $F_1 \prec K_{1,3}$ or $F_2 \prec K_{1,3}$.*

For $\mathcal{H} \subset \mathcal{G}_2$ and $H_1, H_2 \in \mathcal{G}$, we say that $\{H_1, H_2\}$ is a *forbidden pair for* $\mathcal{H}$ if there exists a positive constant n_0 such that every $\{H_1, H_2\}$-free graph of order at least n_0 in $\mathcal{H}$ has a 2-factor. Theorems A and B say that $\{K_{1,4}, P_4\}$ is a forbidden pair for the set of 2-connected graphs, but not for $\mathcal{G}_2$.

In this paper, we first continue this line of research and determine all the forbidden pairs for $\mathcal{G}_2$. It turns out that the characterized pairs are much fewer than those determined in [5] for 2-connected graphs.

We then investigate the effect of minimum degree. For an integer d with $d \geqslant 2$, we consider the same problem in $\mathcal{G}_d$. As d grows larger, the set $\mathcal{G}_d$ becomes smaller. Then a weaker condition may guarantee the existence of a 2-factor. In terms of forbidden pairs, those for $\mathcal{G}_d$ may increase as d grows larger. However, we prove that we always need a star of bounded order.

Theorem 1.1. *Let d be an integer with $d \geqslant 2$, and let F_1 and F_2 be connected graphs of order at least three. If there exists a positive integer n_0 such that every connected $\{F_1, F_2\}$-free graph of order at least n_0 and minimum degree at least d has a 2-factor, then either $F_1 \prec K_{1,d+1}$ or $F_2 \prec K_{1,d+1}$.*

We also investigate the number of pairs that imply the existence of a 2-factor. Let $F \in \mathcal{G}$ and suppose F is not a star. Also let k be a positive integer. By Theorem 1.1, if $k \geqslant d+2$, then $\{K_{1,k}, F\}$ cannot be a forbidden pair for $\mathcal{G}_d$. On the other hand, Ota and Tokuda [9] have proved the following theorem.

Theorem C ([9]). *Let d be an integer with $d \geqslant 2$. Then*

(1) *every $K_{1,\lfloor \frac{d+2}{2} \rfloor}$-free graph of minimum degree at least d has a 2-factor, and*

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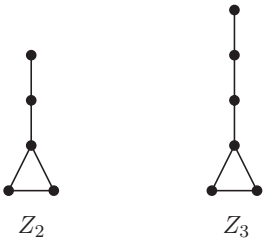


Figure 2. Z_2 and Z_3.

(2) *there exist infinitely many* $K_{1,\lceil \frac{d+3}{2} \rceil}$*-free graphs of minimum degree* d *which do not have a* 2*-factor.*

This theorem says that if we forbid a star with $\lfloor \frac{d+2}{2} \rfloor$ or fewer edges in $\mathcal{G}_d$, then we can guarantee the existence of a 2-factor without forbidding another graph. In other words, if $k \leqslant \lfloor \frac{d+2}{2} \rfloor$, then $\{K_{1,k}, F\}$ is a forbidden pair for $\mathcal{G}_d$ for any connected graph F.

Now a question arises: How many graphs F make $\{K_{1,k}, F\}$ a forbidden pair for $\mathcal{G}_d$ in the range $\lceil \frac{d+3}{2} \rceil \leqslant k \leqslant d+1$? We give the following partial answer to this question.

Theorem 1.2. *Let* d *and* k *be integers with* $d \geqslant 3$ *and* $\lceil \frac{d+3}{2} \rceil \leqslant k \leqslant d+1$. *Then there are only finitely many connected graphs* F *such that* $\{K_{1,k}, F\}$ *is a forbidden pair for the set of connected graphs of minimum degree at least* d.

In the next section, we characterize the forbidden pairs for $\mathcal{G}_2$. In Section 3, we study the effect of minimum degree, and we make some concluding remarks in Section 4.

For standard graph-theoretic notation not explained in this paper, we refer the reader to [3]. Let G be a graph and let x be a vertex in G. Then we denote by $N_G(x)$ and $\deg_G x$ the neighbourhood and the degree of x in G, respectively. We also denote by $\Delta(G)$, $\delta(G)$ and $\alpha(G)$ the maximum degree, the minimum degree and the independence number of G, respectively. For $X \subset V(G)$, $G[X]$ denotes the subgraph of G induced by X. Given another graph H, $G \cup H$ and $G + H$ denote the union and the join of G and H, receptively, and for a positive integer n, let nG denote the union n copies of G. A block B of G is called an end-block if B contains exactly one cut-vertex of G.

A complete subgraph of a graph G is called a clique. A clique H is said to be maximal if G does not have a clique that contains H as a proper subgraph, and H is called maximum if H is a clique in G of the largest order.

2. Forbidden subgraphs which imply the existence of a 2-factor

In this section, we characterize the forbidden pairs for the set of connected graphs of minimum degree at least two. The graph Z_k is a graph obtained from a triangle C_3 and P_k by joining one vertex in C_3 and one of the end-vertices of P_k by an edge (Figure 2).

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Theorem 2.1. *Let F_1 and F_2 be connected graphs of order at least three. Then there exists a positive integer n_0 such that every connected $\{F_1, F_2\}$-free graph of order at least n_0 and minimum degree at least two has a 2-factor if and only if $\{F_1, F_2\} \leqslant \{K_{1,3}, Z_2\}$.*

As we mentioned in the Introduction, this theorem says that the forbidden pairs for $\mathcal{G}_2$ are much fewer than those for the set of 2-connected graphs determined in [5]. Moreover, unlike the case of 2-connected graphs, where the pairs have three maximal elements with respect to $\leqslant$, those for $\mathcal{G}_2$ have a maximum element.

Now we give a proof of Theorem 2.1. We first prove the 'only if' part.

Lemma 2.2. *Let F_1 and F_2 be connected graphs of order at least three. If there exists a positive integer n_0 such that every connected $\{F_1, F_2\}$-free graph of order at least n_0 and minimum degree at least two has a 2-factor, then $\{F_1, F_2\} \leqslant \{K_{1,3}, Z_2\}$.*

Proof. If $F_1 = K_{1,2}$ or $F_2 = K_{1,2}$, then the conclusion follows since $K_{1,2} \prec K_{1,3}$ and $K_{1,2} \prec Z_2$. Therefore, we may assume $F_1 \neq K_{1,2}$ and $F_2 \neq K_{1,2}$. By Theorem B, we may assume $F_1 = K_{1,3}$. Since every 2-connected graph belongs to $\mathcal{G}_2$ and $\{F_1, F_2\} \neq \{K_{1,4}, P_4\}$, we also have $\{F_1, F_2\} \leqslant \{K_{1,3}, B(4,1)\}$ or $\{F_1, F_2\} \leqslant \{K_{1,3}, N(3,1,1)\}$ by Theorem A. Since neither $B(4,1)$ nor $N(3,1,1)$ contains $K_{1,3}$ as an induced subgraph, we have $F_2 \prec B(4,1)$ or $F_2 \prec N(3,1,1)$.

Let $m_0 = \max\{n_0, 3\}$. Prepare two copies H_1 and H_2 of K_{m_0} and choose one vertex x_i from H_i $(i = 1, 2)$. Introduce a new vertex y and add two edges x_1y and x_2y. Let G_1 be the resulting graph. Then $|V(G_1)| \geqslant n_0$ and $\delta(G_1) = 2$, but G_1 does not have a 2-factor. Since G_1 is claw-free, this implies $F_2 \prec G_1$. Since F_2 is also an induced subgraph of $B(4,1)$ or $N(3,1,1)$, we have $F_2 \prec Z_3$.

Next prepare a copy of K_{m_0} and take one of its vertices a. Introduce four new vertices b, c, d, e and add six edges ab, bc, ac, cd, de, ce. Let G_2 be the resulting graph. Then $|G_2| \geqslant n_0$ and $\delta(G_2) = 2$, but G_2 does not have a 2-factor. Since G_2 is claw-free, G_2 contains F_2 as an induced subgraph. Moreover, since G_2 is also P_5-free, F_2 is P_5-free. Since F_2 is connected and $F_2 \prec Z_3$, this implies $F_2 \prec Z_2$. □

Next, we prove the 'if' part. Actually, it is a corollary of the following theorem.

Theorem D (Gould and Jacobson [8]). *Every 2-connected $\{K_{1,3}, Z_2\}$-free graph is Hamiltonian.*

We now prove the following statement.

Lemma 2.3. *A connected $\{K_{1,3}, Z_2\}$-free graph of minimum degree at least two is either Hamiltonian or isomorphic to $K_1 + (K_l \cup K_m)$ for some integers l and m with $l, m \geqslant 2$.*

This lemma immediately yields that every connected $\{K_{1,3}, Z_2\}$-free graph of order at least six and minimum degree at least two has a 2-factor with at most two components.

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Proof of Lemma 2.3. Let G be a connected $\{K_{1,3}, Z_2\}$-free graph of minimum degree at least two. If G is 2-connected, then G is Hamiltonian by Theorem D and hence it has a 2-factor. Thus, we may assume that G is not 2-connected. Let H be an end-block of G and let x be the unique cut-vertex of G contained in H. Let $G' = G - V(H)$. Since $\delta(G) \geqslant 2$, $|V(H)| \geqslant 3$ and $\deg_H(x) \geqslant 2$. Let x_1 and x_2 be distinct neighbours of x in H. Let $A = N_G(x) \cap V(H)$ and $B = N_G(x) \cap V(G')$. Since G is $K_{1,3}$-free, both A and B induce complete graphs in G.

If $B \neq V(G')$, then since G is connected, there exists an edge yz in G' with $y \in B$ and $z \notin B$. Then $\{x_1, x_2, x, y, z\}$ induces Z_2. This contradicts the assumption. Therefore, $B = V(G')$ and $G[B \cup \{x\}]$ is an end-block of G. This implies $|B| \geqslant 2$. Then, by symmetry, $|A| \geqslant 2$ and $A = V(H)$. Therefore, G is isomorphic to $K_1 + (K_l \cup K_m)$ for some l, $m \geqslant 2$. □

3. Minimum degree and forbidden pairs

In this section, we consider forbidden pairs for a 2-factor in $\mathcal{G}_d$, where $d \geqslant 2$. We first prove Theorem 1.1, which claims that each pair contains a star of order at most $d+2$.

In [1], we proved that for $k \geqslant 2$, every forbidden pair for the set of k-connected graphs contains a star.

Theorem E ([1]). *Let k be an integer with $k \geqslant 2$, and let F_1 and F_2 be connected graphs of order at least three. If every k-connected $\{F_1, F_2\}$-graph has a 2-factor, then either F_1 or F_2 is a star.*

We use this theorem as a starting point for the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $m_0 = n_0 + d$. Since every m_0-connected $\{F_1, F_2\}$-free graph is a connected $\{F_1, F_2\}$-free graph of minimum degree at least d and order at least n_0, either F_1 or F_2 is a star by Theorem E. We may assume $F_1 = K_{1,m}$ for some $m \geqslant 2$.

By Theorem C (2), if both F_1 and F_2 are stars, then one of them is a star with at most $\lfloor \frac{d+2}{2} \rfloor$ end-vertices. Since $\lfloor \frac{d+2}{2} \rfloor \leqslant d+1$, the theorem follows in this case. Therefore, we may assume that F_2 is not a star.

We now claim $m \leqslant d+1$. Assume, to the contrary, $m \geqslant d+2$. Let $G_1 = K_{d-1} + (dK_2 \cup K_{m_0})$. Then G_1 is a connected graph of order at least n_0 and minimum degree d, but it does not have a 2-factor. Since every connected induced subgraph of G_1 either is a star with at most $d+1$ end-vertices or contains a triangle, G_1 is $K_{1,m}$-free and hence $F_2 \prec G_1$. Moreover, since F_2 is not a star, F_2 contains a triangle.

Introduce $2m_0$ vertices $x_0, x_1, \ldots, x_{m_0-1}, y_0, y_1, \ldots, y_{m_0-1}$ and join x_i and y_{i+j} by an edge for each i, j with $0 \leqslant i \leqslant m_0 - 1$ and $0 \leqslant j \leqslant d$, where each suffix is counted modulo m_0. Let G_2' be the resulting graph. Then G_2' is a $(d+1)$-regular bipartite graph of order $2m_0$. Let $G_2 = G_2' - x_0$. Then G_2 is a connected graph of order at least n_0 and minimum degree d. Since $\Delta(G_2) \leqslant d+1$, G_2 is $K_{1,m}$-free. Moreover, since G_2 is a bipartite graph and F_3 contains a triangle, G_2 is F_2-free. However, since G_2 is an unbalanced bipartite graph, it does not have a 2-factor. This is a contradiction, and the theorem follows. □

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By Theorem 1.1, when we deal with a forbidden pair for $\mathcal{G}_d$, we may assume that it is written as $\{K_{1,k}, F\}$, where F is a connected graph of order at least three. Furthermore, if we forbid two stars, one of them is always redundant. Therefore, we may assume that F is not a star. Again by Theorem 1.1, for $k \geqslant d+2$, no F makes $\{K_{1,k}, F\}$ a forbidden pair. On the other hand, Theorem C says that for $k \leqslant \lfloor \frac{d+2}{2} \rfloor$, every F makes $\{K_{1,k}, F\}$ a forbidden pair because F is redundant. In other words, there exist infinitely many forbidden pairs of the form $\{K_{1,k}, F\}$. In the remaining range of $\lceil \frac{d+3}{2} \rceil \leqslant k \leqslant d+1$, we prove the following. Note that since the case $d = 2$ is settled by Theorem 2.1, we may assume $d \geqslant 3$.

Theorem 3.1. *Let F be a connected graph of order at least three, and let k and d be positive integers with $d \geqslant 3$ and $\lceil \frac{d+3}{2} \rceil \leqslant k \leqslant d+1$. If there exists a positive integer n_0 such that every connected $\{K_{1,k}, F\}$-free graph of order at least d and minimum degree at least d has a 2-factor, then $|F| \leqslant \frac{1}{2}(4d^2 + 3d + 3)$.*

This theorem says that there are only finitely many connected graphs F which make $\{K_{1,k}, F\}$ a forbidden pair for $\mathcal{G}_d$ if $\lceil \frac{d+3}{2} \rceil \leqslant k \leqslant d+1$.

In the proof of Theorem 3.1, we use the following theorem, which is a restricted version of Tutte's f-factor theorem. For a graph G and disjoint sets of vertices S and T in G, we define $\delta_G(S, T)$ by

$$\delta_G(S, T) = 2|S| + \sum_{x \in T} \deg_{G-S}(x) - 2|T| - h_G(S, T),$$

where $h_G(S, T)$ is the number of components C of $G - (S \cup T)$ with $e_G(C, T) \equiv 1 \pmod 2$.

Theorem F (Tutte [10]). *A graph G has a 2-factor if and only if $\delta_G(S, T) \geqslant 0$ for every S, $T \subset V(G)$ with $S \cap T = \emptyset$.*

Proof of Theorem 3.1. Let $t = \lceil \frac{d+3}{2} \rceil$. Let H_0 be a copy of K_{t-2} and let $H_1, H_2, \ldots, H_{t-1}$ be $t-1$ copies of K_2 with $V(H_i) = \{x_{i,1}, x_{i,2}\}$ $(1 \leqslant i \leqslant t-1)$. Let $G_0 = H_0 + (H_1 \cup \cdots \cup H_{t-1})$.

Given another connected graph Γ and a vertex y in Γ, we take $d-t+1$ copies $\Gamma^1_{i,j}, \Gamma^2_{i,j}, \ldots, \Gamma^{d-t+1}_{i,j}$ of Γ for each $x_{i,j}$ $(1 \leqslant i \leqslant t-1$ and $1 \leqslant j \leqslant 2)$. Let $y^p_{i,j}$ be the vertex in $\Gamma^p_{i,j}$ that corresponds to y in Γ $(1 \leqslant p \leqslant d-t+1)$. Join $x_{i,j}$ with $y^p_{i,j}$ by an edge for each i, j and p with $1 \leqslant i \leqslant t-1$, $1 \leqslant j \leqslant 2$ and $1 \leqslant p \leqslant d-t+1$. Let $G(\Gamma, y)$ be the resulting graph (Figure 3). Note that since $d \geqslant 3$, we have $d-t+1 \geqslant 1$, and hence $G(\Gamma, y)$ contains at least one copy of Γ.

The graph $G(\Gamma, y)$ is connected, and if $|\Gamma| \geqslant n_0$, then $|G(\Gamma, y)| \geqslant n_0$. For $v \in V(G(\Gamma, y))$, let B_v be the subgraph of $G(\Gamma, y)$ induced by $N_{G(\Gamma,y)}(v)$. If $v \in V(H_0)$, then

$$\deg_{G(\Gamma,y)} v = 3t - 5 = 3\left\lceil \frac{d+3}{2} \right\rceil - 5 > d$$

and $\alpha(B_v) = t - 1 < \lceil \frac{d+3}{2} \rceil$. If $v \in V(H_1) \cup \cdots \cup V(H_{t-1})$, then $\deg_{G(\Gamma,y)}(v) = d$ and

$$\alpha(B_v) = d - t + 2 = \left\lfloor \frac{d+1}{2} \right\rfloor < t.$$

Since $k \geqslant t$, no vertex in G_0 can be the centre of $K_{1,k}$ in $G(\Gamma, y)$.

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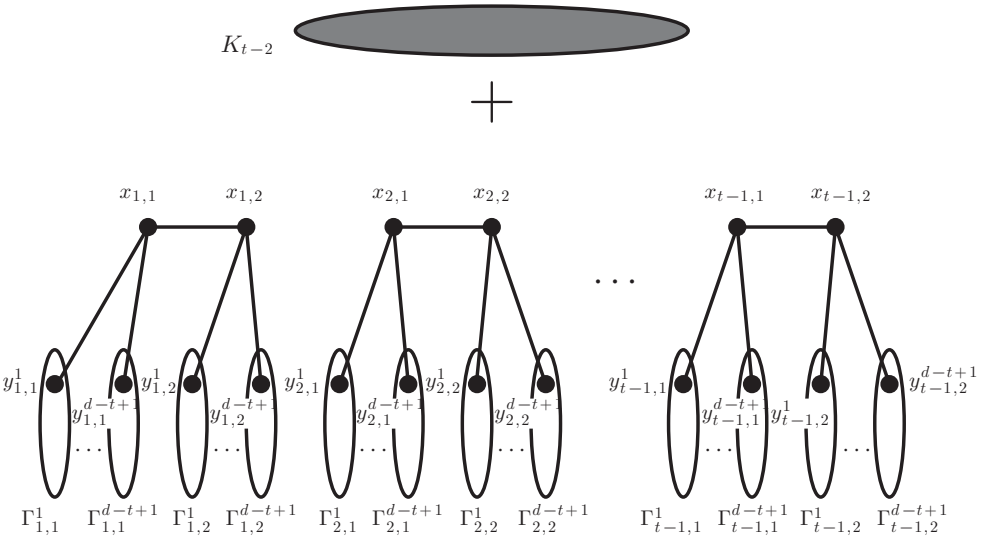


Figure 3. $G(\Gamma, y)$.

Let $S = V(H_0)$ and let $T = V(H_1) \cup \cdots \cup V(H_{t-1})$. Then each $\Gamma^p_{i,j}$ is a component of $G - (S \cup T)$ with $e_{G(\Gamma,y)}(V(\Gamma^p_{i,j}), T) = 1$, and hence

$$\begin{aligned}\delta_{G(\Gamma,y)}(S,T) &= 2|S| + \sum_{t\in T} \deg_{G(\Gamma,y)-S}(t) - 2|T| - h_{G(\Gamma,y)}(S,T)\\ &= 2(t-2) + (d-t+2)\cdot 2(t-1) - 2\cdot 2(t-1) - (d-t+1)\cdot 2(t-1) = -2.\end{aligned}$$

Therefore, by Theorem F, $G(\Gamma, y)$ does not have a 2-factor, regardless of the structure of Γ.

Let $m_0 = n_0 + d + 1$. Prepare K_{m_0}, take one vertex y in K_{m_0} and let $G_1 = G(K_{m_0}, y)$. Since $m_0 \geqslant n_0$, $|G_1| \geqslant n_0$. For $v \in V(G_1) - V(G_0)$, $\deg_{G_1}(v) \geqslant m_0 - 1 \geqslant d$ and $\alpha(B_v) \leqslant 2 < t \leqslant k$. Therefore, G_1 is $K_{1,k}$-free and $\delta(G_1) \geqslant d$. Since G_1 does not contain a 2-factor, G_1 contains F as an induced subgraph.

Assume $|F| > \frac{1}{2}(4d^2 + 3d + 3)$. Let $k_0 = |V(F) \cap V(H_0)|$, $k_1 = |V(F) \cap V(\Gamma^1_{1,1})|$, $k_2 = |V(F) \cap V(\Gamma^1_{1,2})|, \ldots, k_{2(t-1)(d-t+1)} = |V(F) \cap V(\Gamma^{d-t+1}_{t-1,2})|$ and let l_1 (resp. l_2) be the largest (resp. the second-largest) number among $k_0, \ldots, k_{2(t-1)(d-t+1)}$. We claim that either $l_1 \geqslant t+2$ or $l_2 \geqslant 5$. If $l_1 \leqslant t+1$ and $l_2 \leqslant 4$, then since $|G_0 - H_0| = 2t - 2$,

$$|F| \leqslant 2t - 2 + t + 1 + 2(t-1)(d-t+1)\cdot 4 = \begin{cases} \frac{1}{2}(4d^2+3d+3) & \text{if } d \text{ is odd,}\\ \frac{1}{2}(4d^2+3d-6) & \text{if } d \text{ is even,}\end{cases}$$

a contradiction. Thus either $l_1 \geqslant t+2$ or $l_2 \geqslant 5$. This implies that F contains either a maximum clique of order at least $t+2$ or two disjoint maximal cliques of order at least five.

Let Γ_0 be a connected $(t-2)$-regular graph of girth at least 4 and order at least m_0. For each vertex v in Γ_0, we prepare a graph K_v which is a complete graph of order two. Then, for each edge uv in Γ_0, we add four edges which join two vertices in K_u and two

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Table 1.

Range of k	Number of graphs F
$k \leqslant \lfloor \frac{d+2}{2} \rfloor$	infinite (F is redundant)
$\lfloor \frac{d+2}{2} \rfloor < k < d+2$	finite
$k \geqslant d+2$	none

vertices in K_v. Let Γ_1 be the resulting graph. Note that a maximum clique in Γ_1 has order four and $N_{\Gamma_1}(y)$ induces $(t-2)K_2 + K_1$ for each vertex y in Γ_1.

Take one vertex y_1 of Γ_1 and let $G_2 = G(\Gamma_1, y_1)$. Since $|\Gamma_1| \geqslant |\Gamma_0| \geqslant m_0$, $|G_2| \geqslant m_0$. Let $u \in V(G_2) - V(G_0)$. If u is adjacent to a vertex in G_0, $\deg_{G_2}(u) = 2(t-2)+1+1 \geqslant d+1$, otherwise $\deg_{G_2}(u) = 2(t-2)+1 \geqslant d$. Thus, $\delta(G_2) = d$. Also, by the construction, $\alpha(B_u) \leqslant (t-2)+1 < t \leqslant k$. Therefore, G_2 is $K_{1,k}$-free. Note that a maximum clique in G_0 has order t and a maximum clique in Γ_1 has order four. Since $t+2 = \lceil \frac{d+3}{2} \rceil + 2 \geqslant 5$, G_2 is K_{t+2}-free. Since F contains either a maximal clique of order at least $t+2$ or two disjoint maximal cliques of order at least five, G_2 is F-free. Now we see that G_2 is a connected $\{K_{1,k}, F\}$-free graph of order at least n_0 and minimum degree at least d but it does not have a 2-factor. This contradicts the assumption, and the theorem follows. □

4. Concluding remarks

In this paper, we have considered pairs of forbidden subgraphs which imply the existence of a 2-factor in a connected graph of sufficiently large order. Our research has focused on minimum degree. Recalling that $\mathcal{G}_d$ is the set of connected graphs of minimum degree at least d, we summarize the results in this paper as follows.

(1) The forbidden pairs for $\mathcal{G}_2$ are characterized.
(2) For $d \geqslant 3$, every forbidden pair for $\mathcal{G}_d$ contains a star of order at most $d+2$.
(3) The number of connected graphs F other than stars that make $\{K_{1,k}, F\}$ a forbidden pair is given in Table 1.

Let $\{K_{1,k}, F\}$ be a forbidden pair for $\mathcal{G}_d$, where k is in the range of $\lfloor \frac{d+2}{2} \rfloor < k < d+2$. In order to give an upper bound of the order of F in the proof of Theorem 3.1, we have introduced a graph $G(\Gamma, y)$. The key idea of the proof is that $G(\Gamma, y)$ does not have a 2-factor, regardless of the structure of Γ. It strongly suggests that the large part of F lies in $G_0 = K_{t-2} + (t-1)K_2$, which appeared in the proof, where $t = \lceil \frac{d+3}{2} \rceil$. Although we have drawn only the information on $|V(F)|$ in this paper, this fact seems to give more structural information on F, which may lead us to the characterization of the forbidden pairs for $\mathcal{G}_d$.

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