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The stochastic tetrad magneto-hydrodynamics via functional formalism

Massimo Materassi[1],† **and Giuseppe Consolini**[2]

[1]Istituto dei Sistemi Complessi, CNR, via Madonna del Piano 10, Sesto Fiorentino (Fi), Italy

[2]INAF-Istituto di Astrofisica e Planetologia Spaziali, via del Fosso del Cavaliere 100, 00133 Roma, Italy

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In this work we discuss an application of the Tetrad Dynamics approach, a stochastic dynamical theory already introduced in hydrodynamic turbulence, to incompressible magneto-hydrodynamics. This theoretical framework is capable of taking into account some crucial aspects of turbulent plasmas, namely, (i) its material nature, which is stressed through the adoption of Lagrangian variables, (ii) its probabilistic dynamics, which is fundamental to understand the intermittency and highly irregular nature of turbulence, and (iii) the multi-scale character of interactions, which is approached by promoting the space size of parcels to the role of a dynamical variable. In particular, here, we construct the probabilistic equations of motion for quantities describing the evolution of a turbulent plasma (a matrix $\boldsymbol{\varrho}$ describing the parcel's shape, the plasma velocity and magnetic field coarse-grained gradient tensors, $\boldsymbol{M}$ and $\boldsymbol{W}$), resorting the functional formalism of classical statistical dynamics. Through the introduction of a stochastic action and using a path integral approach, the statistical properties of ($\boldsymbol{\varrho}$, $\boldsymbol{M}$, $\boldsymbol{W}$) can be derived from those of noises appearing in their equations of motion, both at equilibrium and out of equilibrium.

1. Introduction

Plasma is a state of matter and not a state of fields. Indeed, the essence of a plasma system is not that of a 'radiation' described by suitable partial differential equations and boundary conditions, as it is for simple electromagnetic fields, but rather, a 'plasma' is a crowd of particles interacting due to their electric charge via the fields they exchange. In other words, the particle essence of plasmas should never be underestimated, and, not surprisingly, the real complexity of such systems emerges more clearly when the language of 'matter mechanics' is applied instead of 'fields' or 'waves'. Describing plasmas simply in terms of a field theory is equivalent to hiding the real complexity of these systems under a smooth cover. Furthermore, strictly related to the particle nature, there is the emergence of granular, or irregular behaviour, in rather common contexts (Materassi, Consolini & Tassi 2012), i.e. at rather mild gradients or Reynolds numbers, which denotes how a close-up of the apparently continuous matter brings to evidence particles.

The above considerations regarding the necessity of stressing the physical difference between a 'proper' field theory, e.g. that of a Maxwell field (Landau & Lifshitz 1980),

† Email address for correspondence: massimo.materassi@isc.cnr.it

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and the usual Navier–Stokes or magneto-hydrodynamics (MHD) equations (Rai Choudhuri 1998), imply the treatment of fluids and plasmas from the Lagrangian, or material, point of view (Bennett 2006). In the Lagrangian framework, the focus is indeed on the motion of the macroscopic infinitesimal mass of continuum (the parcel) (Materassi 2014) that undergoes the forces applied from outside the fluid (e.g. external gravity and external electric-magnetic fields), the pressure exerted by the nearby parcels and, last but not least, the dissipative processes (Materassi 2015).

Turbulence, and in particular intermittency, is a blatant manifestation of the granularity and complexity of fluids and magneto-fluids (Frisch 1995). Turbulent regimes establish when the interactions among particles and groups of particles, and the systems in the vicinity of the continuum (e.g. obstacles, shocks, solid walls), give rise to fluid velocity distributions that appear richly structured on many space and time scales. When this occurs, it is neither possible to describe the bulk velocity of the turbulent continuum as a smooth field (namely, a function $\boldsymbol{v} \in C^{\infty}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}^3)$), nor to treat the evolution of the emerging structures as deterministic. Conversely, the evolution appears to be so erratic that it is reasonable to describe it in terms of a stochastic process, at least at a certain macroscopic scales (Warhaft 2002). It is still possible to state that the motion of each classical particle of the continuum is deterministic, but at a macroscopic level, the motion results in a stochastic configuration of the velocity field. In the case of magnetized plasmas, we experience a similar situation in many different contexts from laboratory plasmas to space and astrophysical plasmas. In many plasma systems the magnetic field configuration is equally 'turbulent', i.e. irregular (Celani, Mazzino & Pumir 2003; Kallenrode 2004), as a consequence of the matter fluid irregularity.

A crucial feature of turbulence on which the stochastic tetrad dynamics formalism (STD) is based is the presence of inter-scale dynamics. Indeed, in turbulent systems, structures of different scales are supposed to exchange energy with each other, in the form of cascades or coalescence (Frisch 1995; Materassi *et al.* 2012). In order to account of that, a theory of (plasma) turbulence has to consider dynamical variables directly related to the emerging structures at a given scale, so that the exchange of energy among different scales may be transparently represented and calculated. Actually, in a strictly local approach, velocities, densities, fields and their gradients are all quantities defined point-by-point, hence they make sense only 'at zero scale'. Then, purely local quantities cannot describe either processes taking place at finite scales or multiple-scale processes. Consequently, it is necessary to introduce coarse-grained variables. For instance, in Tetrault's papers (Tetrault 1992*a*,*b*), quantities are defined in such a way to represent those processes characterizing the medium at a specific finite scale, and the multi-scale processes are viewed as interactions between quantities pertaining to coarse-graining at different scales.

The full picture emerging from the aforementioned reasoning is thus pointing towards a theory of variables describing the Lagrangian motion of fluid parcels, in which the size of the parcel enters as a dynamical variable. Moreover, a stochastic picture is necessary in order to represent as much as possible the erratic behaviour of turbulent fluids and plasmas.

This work is devoted to approach the plasma turbulent motion following the STD already developed for hydrodynamics (HD). In particular, the aim of this work is to develop an analogous stochastic formalism for the theory of magneto-hydrodynamics (MHD), presenting it in a closed form under some simplifying assumptions. The goal is to take into account the relevant facts of fluids and plasmas. These facts appear underestimated when a formalism is adopted which does not consider how different

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the system may appear at different scales; this happens, for instance, when the system is described by a purely fluid local approach (Kallenrode 2004), i.e. HD or MHD.

The theory we are going to discuss regards the stochastic dynamics of plasma parcels. However, here we merely present the general framework of such a formulation, limiting our discussion to the mathematical tools to perform analytical and numeric studies on turbulent plasmas. Applications will be presented in future works.

Following Chertkov, Pumir & Shraiman (1999) and references therein, the theory with the characteristics invoked is one of stochastic tetrad magneto-hydrodynamics (STMHD). This theory is based on the fundamental 'zero-scale' equations involving the velocity $\boldsymbol{v}$ and the magnetic induction field $\boldsymbol{B}$ as fundamental variables; a set of variables are then constructed to describe the configuration of a fluid parcel of finite size, and the difference in terms of velocity and magnetic field experienced at the different points of the parcel. It is valuable to underline how, as the 'zero-scale' is abandoned, the continuum parcel must be conceived as an extended object (while it is usually understood as a point-particle-plus-thermodynamics, see e.g. Bennett (2006), Materassi (2015)), so one has to assign not the values of fields 'at its position', but the finite differences between the fields 'at its terminal points'.

The need of involving gradients of $\boldsymbol{v}$ and $\boldsymbol{B}$ in the description of a continuum emerges from the necessity to describe the local topology of the field lines (Viellefosse 1984), which determine the streams and currents. Then, in HD, the necessity of passing from zero-scale gradients to finite-scale finite differences emerges as soon as turbulence comes to play (Cantwell 1992; Blackburn, Mansour & Cantwell 1996). Here, this approach is extended to MHD (Consolini *et al.* 2015).

The three-dimensional turbulence scenario is here considered and discussed. The description of continuum parcels in the tetrad formalism is based on the fact that the simplest simplex of $\mathbb{R}^3$ is a tetrahedron. Attaching a tetrahedron of points (the tetrad) to the material motion of the parcel, one may approach the description of the motion itself in the most geometrically essential and exhaustive way. Indeed, from a given initial configuration, the evolution of the parcel is followed through the motion of the tetrad, that travels, stretches and rotates according to the complicated plasma motion. Also, tetrahedra may be defined at different scales, giving rise precisely to the inter-scale dynamics highlighted before.

Tetrad dynamics may be understood as a particular interpretation of the theory of dissipative dynamics of coherent structures naturally arising in astrophysical and planetary plasma (Wu & Chang 2000). The tetrad evolution is the motion of a material portion of fluid, coherently moving, topologically characterized by the invariants that remarkably describe local dissipation processes (Sahoo, Perlekar & Pandit 2011). The STMHD developed here describes the Lagrangian parcel with turbulence inside and around, which may be the building block of a description of extended turbulent plasma regions, resulting in complex systems forced in a condition of self-organized criticality (Chang 1999).

The paper is organized as follows. The dynamical system is described in § 2, where the tetrad coarse-grained variables of MHD are presented together with their stochastic dynamics and their Langevin equations. A particular focus is dedicated to the explanation of the choice of noise statistics made here, which is not the only one possible, and must be thoroughly motivated from a physical point of view. In § 3 we resume the functional formalism of statistical dynamics used to represent STMHD in terms of path integrals. This section is crucial because it presents the hard skeleton of the theory as descending from the many history approach proposed. This formalism

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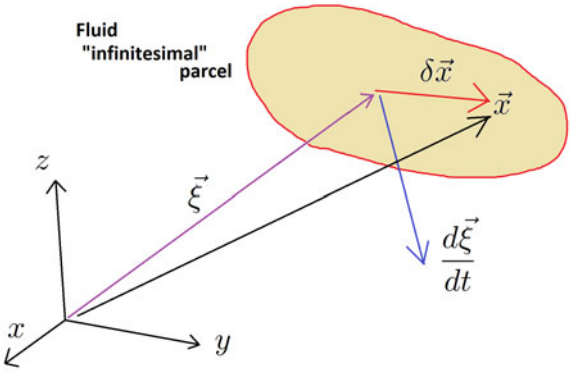


FIGURE 1. The points $\boldsymbol{x}$ in the parcel and the centre-of-mass $\boldsymbol{\xi}$ of the parcel, as in (2.1). The speed $\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t$ represents the collective motion of the parcel, see Materassi (2014, 2015).

applied to the dynamical system considered in § 4, where the physical assumptions of § 4 and the mathematical developments of § 3 are put together. Conclusions and possible development of the findings presented are discussed in § 5 in which a possible roadmap is traced to future investigation based on STMHD.

2. Tetrad magneto-hydrodynamics

The tetrad formalism was developed in the framework of HD. This formalism makes use of Lagrangian variables describing the fluid as a collection of 'macroscopically infinitesimal' parcels (Materassi 2014, 2015). Its introduction was motivated by the need for extending to finite size observation and simulations, the results regarding 'local topology' of the fields in Viellefosse (1984) and Cantwell (1992). As discussed below, the necessity for encoding the effects of scales smaller than the one considered, and of the nearby parcels, yields the introduction of stochastic forces in the equations. This implies the necessity for introducing a stochastic formalism. The physical properties of fluctuations, described in terms of spectrum and correlations, are encoded in the assumptions made about these noise terms.

The parcels are so small that their size may be considered as infinitesimal within a Taylor approximation of fields in a laminar regime, although they are still macroscopic objects to contain enough particles to have proper thermodynamics (Rai Choudhuri 1998; Bennett 2006). Outside laminar regimes, in the presence of turbulence, considerations of the smallness of high-order terms in such expansions must be viewed in terms of ensemble averages, as explained in the following analysis.

Consider a similar small volume of continuum matter. If $\boldsymbol{\xi}$ is its centre-of-mass, a smooth field φ at a point $\boldsymbol{x}$ still within the parcel, so that $\boldsymbol{x} = \boldsymbol{\xi} + \delta\boldsymbol{x}$ (see figure 1) is approximated as $\varphi(\boldsymbol{x}) = \varphi(\boldsymbol{\xi}) + \delta\boldsymbol{x} \cdot \boldsymbol{\partial}\varphi(\boldsymbol{\xi})$. This means the fluid velocity and magnetic field can be approximated as follows:

$$\left.\begin{aligned} v^\alpha(\boldsymbol{x}) &= v^\alpha(\boldsymbol{\xi}) + \partial_\beta v^\alpha(\boldsymbol{\xi})\delta x^\beta + \cdots, \\ B^\alpha(\boldsymbol{x}) &= B^\alpha(\boldsymbol{\xi}) + \partial_\beta B^\alpha(\boldsymbol{\xi})\delta x^\beta + \cdots, \end{aligned}\right\} \tag{2.1}$$

where suspension points mean that higher-order terms are omitted. In (2.1), and in what follows, greek letters are used for the $SO(3)$-vector indices. For the moment,

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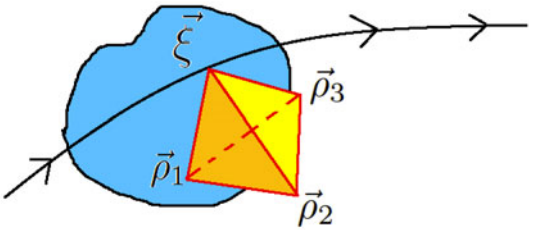


FIGURE 2. The fluid tetrad: the curve passing through $\boldsymbol{\xi}$ represents the trajectory 'of the tetrad', i.e. of the parcel's centre-of-mass.

smoothness of the velocity and magnetic fields is always assumed, however when turbulence is considered, a suitable re-interpretation of (2.1) will be discussed.

The importance of field gradient tensors $\boldsymbol{\partial} \otimes \boldsymbol{v}$ and $\boldsymbol{\partial} \otimes \boldsymbol{B}$ (with $[\partial \otimes X]_{ij} = \partial_i X_j$) in grasping the field configurations within the parcel is immediately understood. The spectra of the two tensors

$$\boldsymbol{A} = \boldsymbol{\partial} \otimes \boldsymbol{v}, \quad \boldsymbol{Z} = \boldsymbol{\partial} \otimes \boldsymbol{B} \tag{2.2a,b}$$

determine the topology of the $\boldsymbol{v}$- and $\boldsymbol{B}$-lines Consolini *et al.* (2015) and a collection of suitable topological invariants may be formulated accordingly.

Following the line of reasoning of the 'tetrad tradition' (see, e.g. Cantwell (1992), Chertkov *et al.* (1999) and references therein), we consider incompressible fluids, $\boldsymbol{\partial} \cdot \boldsymbol{v} = 0$, and since $\boldsymbol{B}$ is naturally solenoidal, one may write:

$$\mathrm{Tr}(\boldsymbol{A}) = A^{\mu}_{\mu} = 0, \quad \mathrm{Tr}(\boldsymbol{Z}) = Z^{\mu}_{\mu} = 0. \tag{2.3a,b}$$

This assumption will be valid also for the coarse-grained versions of these tensors introduced below.

Tetrad variables are a particular way to describe a finite size plasma parcel. The evolution of a parcel's shape is assigned in the simplest way once it is given the topology of the three-dimensional space within which the continuum moves. In particular, three material points $x_{i=1,2,3}$ are chosen within the parcel, and the displacements

$$\rho_i = x_i - \boldsymbol{\xi} \tag{2.4}$$

are considered, $\boldsymbol{\xi}$ being the centre-of-mass of the parcel (or, in practical applications, e.g. direct numerical simulations or space plasma experiments, any reference material point (Chertkov *et al.* 1999; Consolini *et al.* 2015)). The vectors $\rho_{i=1,2,3}$ form a tetrahedron, referred to as the tetrad, advected by the motion of the plasma. It is worth stressing that the term 'material' here, qualifying both $\boldsymbol{\xi}$ and the points x_i, is particularly relevant, because the dynamics of points following the motion of the fluid within the parcel (and moving together with it) is what we are treating. This allows us to work in Lagrangian variables (Materassi 2014, 2015). Figure 2 represents the artist's vision of the fluid tetrad.

The key idea of STD is to consider the velocity of the material point ρ_i as approximated through the (2.1), with the help of tensors in (2.2). If $\delta \boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{\xi})$ is correctly interpreted as the Lagrangian derivative with respect to time of the

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difference between $\boldsymbol{x}$ and $\boldsymbol{\xi}$, this coincides with the Lagrangian time derivative of ρ_i when $\boldsymbol{x} = x_i$ is considered; hence

$$\frac{\mathrm{d}\rho_i^\alpha}{\mathrm{d}t} = \rho_i^\beta A_\beta^\alpha + o(\rho_i), \quad \rho_i = |\delta x_i|. \tag{2.5}$$

In (2.5) the gradient $\boldsymbol{A}$ is the 'point-like' one, $A_\beta^\alpha = \partial_\beta v^\alpha(\boldsymbol{\xi})$, which is calculated for the parcel's centre-of-mass $\boldsymbol{\xi}$. The tetrad variables are inserted instead to describe the motion of the vertices $\rho_{i=1,2,3}$ of the tetrad without making any reference to possibly inaccessible point-like measurements. This is done by generalizing (2.5) to any finite scale by considering it as a linear relationship between $\mathrm{d}\rho_i^\alpha/\mathrm{d}t$ and ρ_i^β, which is perturbed by a 'very small' term $o(\rho_i)$. This perturbation tends to be less and less small as larger and larger tetrad sizes are considered. The linear part is represented by a new tetrad variable, namely a 3×3 matrix $\boldsymbol{M}$, so that

$$u_i^\alpha = \frac{\mathrm{d}\rho_i^\alpha}{\mathrm{d}t} - \rho_i^\beta M_\beta^\alpha. \tag{2.6}$$

What exceeds out of a linear approximation is a 'small term' $o(\rho)$ in the laminar regime, where fluid quantities can be considered to be space- and time-smooth.

Conversely, in the presence of turbulence, u_i^α is not necessarily small, but rather represents an arbitrary fluctuation of the bulk velocity. In this case, the Taylor expansion still holds as the linear approximation of $\mathrm{d}\rho_i^\alpha/\mathrm{d}t$,

$$\frac{\mathrm{d}\rho_i^\alpha}{\mathrm{d}t} = \rho_i^\beta M_\beta^\alpha + u_i^\alpha, \tag{2.7}$$

when the average quantities $\langle \mathrm{d}\rho_i^\alpha/\mathrm{d}t \rangle$, $\langle \rho_i^\beta M_\beta^\alpha \rangle$ and $\langle u_i^\alpha \rangle$ are considered, so that

$$\langle u_i^\alpha \rangle = o(\rho). \tag{2.8}$$

It is convenient to collect the 9 components of the three vectors ρ_i as 9 components of a 3×3 matrix $\boldsymbol{\varrho}$, and the noise terms u_i in the 3×3 matrix $\boldsymbol{U}$, so that, because

$$\varrho^{\beta i} = \rho_i^\beta, \quad U^{\beta i} = u_i^\beta, \tag{2.9a,b}$$

one may collect the three ODEs of (2.7) in matrix form:

$$\dot{\boldsymbol{\varrho}} = \boldsymbol{\varrho}^\dagger \cdot \boldsymbol{M} + \boldsymbol{U}. \tag{2.10}$$

Here, the matrix $\boldsymbol{\varrho}$ describes the size and shape of the tetrad, i.e. of the parcel. From (2.9) two tensors can be introduced:

$$\kappa = \varrho^{-1}, \quad \boldsymbol{\Pi} = \frac{\kappa \cdot \kappa^\dagger}{\mathrm{Tr}(\kappa \cdot \kappa^\dagger)}, \tag{2.11a,b}$$

which define the anisotropy of the tetrad (were the tetrad formed by usual cartesian axes, as the $\hat{i}$, $\hat{j}$ and $\hat{k}$ are orthonormal directions, this $\boldsymbol{\Pi}$ would be proportional the 3×3 identity $\mathbf{1}$, see below).

The tensor $\boldsymbol{M}$ in (2.7) and (2.10), namely the coarse-grained version of the velocity gradient $\boldsymbol{A}$, is a dynamical variable on its own (in the same way as $\boldsymbol{A}$ was in the

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zero-scale treatment in Viellefosse (1984)). In the case of MHD, there is also a tensor that is the tetrad version of the magnetic gradient $\boldsymbol{Z}$, which is named $\boldsymbol{W}$.

In order to obtain the equations of motion for $\boldsymbol{M}$ and $\boldsymbol{W}$, one works on the equations for $\boldsymbol{A}$ and $\boldsymbol{Z}$, derived from the visco-resistive MHD equations (Rai Choudhuri 1998) with the Lagrangian time-derivatives:

$$\left.\begin{aligned}
&\dot{v}^\alpha = -\frac{1}{2\rho}\partial^\alpha B^2 + \frac{1}{\rho}B^\beta \partial_\beta B^\alpha - \frac{1}{\rho}\partial^\alpha p + \frac{1}{\rho}\partial_\kappa \sigma^{\alpha\kappa},\\
&\dot{B}^\alpha = B^\mu \partial_\mu v^\alpha + \zeta\, \partial^2 B^\alpha,\\
&\sigma^{\alpha\kappa} = [\eta(\delta^{\nu\alpha}\delta^{\mu\kappa} + \delta^{\nu\kappa}\delta^{\mu\alpha} - \tfrac{2}{3}\delta^{\alpha\beta}\delta^{\mu\nu}) + \nu\delta^{\alpha\beta}\delta^{\mu\nu}]\partial_\mu v_\nu \overset{def}{=} \Sigma^{\alpha\beta\mu\nu}A_{\mu\nu}.
\end{aligned}\right\} \tag{2.12}$$

The first step is to derive the effect of applying $\partial_t + \boldsymbol{v}\cdot\boldsymbol{\partial}$ to $\boldsymbol{A}$ and $\boldsymbol{Z}$, de facto applying a local gradient to both the Lagrangian equations. The incompressibility hypothesis is assumed valid, so that a space-constant mass density may be assumed, and the continuity equation becomes redundant. Moreover, we introduce an auxiliary tensor $\boldsymbol{\Xi} = \boldsymbol{Z} - \boldsymbol{Z}^\dagger$, such that one has the relationship

$$\Xi^{\mu\nu} = -\epsilon^{\mu\nu\sigma} j_\sigma, \tag{2.13}$$

with electric current density $\boldsymbol{j}$.

As far as the velocity gradient is concerned, in order for the system to be self-consistent, the constraint $\boldsymbol{\partial}\cdot\boldsymbol{v} = 0$ must have null Lagrangian derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\partial}\cdot\boldsymbol{v}) = 0. \tag{2.14}$$

This implies that the fluid pressure must undergo a certain relationship. In particular one has:

$$-\partial^2\left(\frac{p}{\rho} + \frac{\boldsymbol{B}^2}{2\rho}\right) = \mathrm{Tr}(\boldsymbol{A}^2) - \frac{1}{2\rho}\mathrm{Tr}(\boldsymbol{\Xi}^2). \tag{2.15}$$

If the above considerations are taken into account, the ODE for $\boldsymbol{A}$ reads:

$$\left.\begin{aligned}
&\dot{\boldsymbol{A}} + \boldsymbol{A}^2 - \left[\mathrm{Tr}(\boldsymbol{A}^2) - \frac{\mathrm{Tr}(\boldsymbol{\Xi}^2)}{2\rho}\right]\frac{\boldsymbol{1}}{3} - \frac{\boldsymbol{\Xi}\cdot\boldsymbol{Z}}{\rho} = \boldsymbol{H},\\
&\boldsymbol{H} = \frac{\eta}{\rho}\partial^2\boldsymbol{A} - \frac{1}{\rho}\left(\boldsymbol{\partial}\otimes\boldsymbol{\partial} - \frac{\boldsymbol{1}}{3}\partial^2\right)\left(p + \frac{\boldsymbol{B}^2}{2}\right) + \frac{1}{\rho}\boldsymbol{B}\cdot\boldsymbol{\partial}\boldsymbol{Z}.
\end{aligned}\right\} \tag{2.16}$$

The tensor $\boldsymbol{H}$ in (2.16) contains gradients of gradients which are of the same order of the $o(\rho_i)$ terms in (2.5). One can neglect these gradient terms if irregular deformations of the tetrad are disregarded. However, in the presence of turbulence, the negligibility of noises can be assumed only on the average. According to the viewpoint of Chertkov *et al.* (1999), one may treat $\boldsymbol{H}$ as a noise term of assigned statistics, and interpret the ODE in (2.16) as a Langevin equation, in which $\boldsymbol{A}$ and $\boldsymbol{Z}$ (appearing also through its anti-Hermitian part $\boldsymbol{\Xi}$) couple, and are forced by $\boldsymbol{H}$. The noisy nature of $\boldsymbol{H}$ may be justified by considering any derivative operator as 'undressing the singularities'. Thus, if the medium at hand is turbulent, it is natural to expect that the quantity $(\eta/\rho)\partial^2 A^{\mu\nu} - 1/\rho(\boldsymbol{\partial}\otimes\boldsymbol{\partial} - (\delta^{\mu\nu}/3)\partial^2)(p + \boldsymbol{B}^2/2) + (1/\rho)B^\zeta \partial_\zeta Z^{\mu\nu}$ is irregular as a spatio-temporal noise (Frisch 1995).

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A very similar reasoning may be adopted for the $\boldsymbol{Z}$ ODE. Applying the gradient operator to the magnetic equation in (2.12) one may arrive to the matrix form:

$$\dot{\boldsymbol{Z}} + [\boldsymbol{Z}, \boldsymbol{A}] = \zeta \partial^2 \boldsymbol{Z} + \boldsymbol{B} \cdot \partial \boldsymbol{A}. \tag{2.17}$$

Again, the term which renders the ODE 'inhomogeneous', may be singled out:

$$\Theta = \zeta \partial^2 \boldsymbol{Z} + \boldsymbol{B} \cdot \partial \boldsymbol{A}, \tag{2.18}$$

so that the relationships for $\boldsymbol{Z}$ corresponding to (2.16) can be written:

$$\left.\begin{aligned} &\dot{\boldsymbol{Z}} + [\boldsymbol{Z}, \boldsymbol{A}] = \Theta, \\ &\Theta = \zeta \partial^2 \boldsymbol{Z} + \boldsymbol{B} \cdot \partial \boldsymbol{A}. \end{aligned}\right\} \tag{2.19}$$

The two terms in Θ represent the resistivity contribution $\zeta \partial^2 \boldsymbol{Z}$ and the contribution due to inhomogeneous velocity, $\boldsymbol{B} \cdot \partial \boldsymbol{A}$, respectively. Note that (2.19) precisely states that when $\Theta = 0$, the magnetic gradient $\boldsymbol{Z}$ is transported by the operator $[., \boldsymbol{A}]$ along the motion that $\boldsymbol{A}$ generates à la Lie. The dissipation $\zeta \partial^2 \boldsymbol{Z}$ and the deformation $\boldsymbol{B} \cdot \partial \boldsymbol{A}$ are the only things which prevent 'passive transport' from taking place. These foregoing considerations are consistent with what is well-known of MHD (Rai Choudhuri 1998). Θ is recognized to be 'more irregular' than $[\boldsymbol{Z}, \boldsymbol{A}]$, hence in (2.19) it can be assimilated to a noise term in the presence of turbulence. Thus, $\boldsymbol{Z}$ undergoes a proper Langevin equation. Note that, even if we have introduced the possibility of considering turbulence through noise, still no finite size framework has been introduced.

Starting from the zero-size ODEs for $\boldsymbol{A}$ and $\boldsymbol{Z}$, one may introduce the equations of motion of tensors that should represent their finite-size versions, namely the variable $\boldsymbol{M}$ coarse-graining $\boldsymbol{A}$ and appearing in (2.7), and the magnetic tensor $\boldsymbol{W}$ that is the coarse-grained version of $\boldsymbol{Z}$. The ODE system for $(\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W})$ reads:

$$\left.\begin{aligned} &\dot{\boldsymbol{\varrho}} = \varrho^{\dagger} \cdot \boldsymbol{M} + \boldsymbol{U}, \\ &\dot{\boldsymbol{M}} + \boldsymbol{M}^2 - \left[\mathrm{Tr}(\boldsymbol{M}^2) - \frac{\mathrm{Tr}(\boldsymbol{\Xi}^2)}{2\rho}\right] \boldsymbol{\Pi} - \frac{\boldsymbol{\Xi} \cdot \boldsymbol{W}}{\rho} = \boldsymbol{H}, \\ &\dot{\boldsymbol{W}} + [\boldsymbol{W}, \boldsymbol{M}] = \Theta. \end{aligned}\right\} \tag{2.20}$$

In (2.20) we use the tensor $\boldsymbol{\Pi}$ defined in (2.11), this replaces the factor $\mathbf{1}/3$ in the zero-size equations. The tensor quantity $\boldsymbol{\Pi}$ takes into account of the anisotropy of the tetrad (i.e. of the fluid parcel), while $\mathbf{1}/3$ is obviously isotropic, describing the shape of a point-like entity (the zero-size tetrad).

The noise terms forcing the system are $\boldsymbol{U}$, $\boldsymbol{H}$ and Θ. These terms can be due both to the fluctuations emerging at scales smaller than the tetrad scale ℓ, namely defined as

$$\ell = \sqrt{\mathrm{Tr}(\boldsymbol{\varrho}^{\dagger} \boldsymbol{\varrho})}, \tag{2.21}$$

and to the contact forces exerted by the nearby tetrads.

In order to construct a stochastic theory, the distinction in (2.20) between deterministic terms and noise terms is essential. However in this case, looking at (2.16) and (2.19), it may be fairly obvious. Indeed, as it can be understood from the definitions of $\boldsymbol{H}$ and Θ, these 'noises' are reasonably formed by terms which are not expected to all have the same degree of 'irregularity'. Here the choice is to adopt

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a 'minimal model' that generalizes the HD assumptions in Chertkov *et al.* (1999) to MHD. Thus, the two terms $\boldsymbol{H}$ and Θ, are subdivided into pure noise terms h and θ plus deterministic terms proportional to the deterministic part of the ODE already present. Our assumption is:

$$\left.\begin{aligned}\boldsymbol{H} &= h + \alpha_v \left[\boldsymbol{M}^2 - \left(\mathrm{Tr}(\boldsymbol{M}^2) - \frac{\mathrm{Tr}(\boldsymbol{\Xi}^2)}{2\rho}\right)\boldsymbol{\Pi} - \frac{\boldsymbol{\Xi}\cdot\boldsymbol{Z}}{\rho}\right],\\ \Theta &= \theta + \alpha_B[\boldsymbol{Z}, \boldsymbol{A}].\end{aligned}\right\} \tag{2.22}$$

Under these hypotheses, the Langevin ODEs describing the STMHD assume the following form:

$$\left.\begin{aligned}\dot{\boldsymbol{\varrho}} &= \boldsymbol{\varrho}^{\dagger}\cdot\boldsymbol{M} + \boldsymbol{U},\\ \dot{\boldsymbol{M}} &= (\alpha_v - 1)\left\{\boldsymbol{M}^2 - \left[\mathrm{Tr}(\boldsymbol{M}^2) - \frac{\mathrm{Tr}(\boldsymbol{\Xi}^2)}{2\rho}\right]\boldsymbol{\Pi} - \frac{\boldsymbol{\Xi}\cdot\boldsymbol{W}}{\rho}\right\} + h,\\ \dot{\boldsymbol{W}} &= (\alpha_B - 1)[\boldsymbol{W}, \boldsymbol{M}] + \theta.\end{aligned}\right\} \tag{2.23}$$

The system of (2.23) represents the extension of the Lagrangian stochastic tetrad formalism to incompressible MHD. It must be completed with a suitable characterization of the statistics of the noise $\boldsymbol{U}$, h and θ, as borrowed from Chertkov *et al.* (1999) for neutral fluids. This assumption is crucial, because it enables us to complete the construction of STMHD. Furthermore, choosing the magnetic part to be mathematically 'isomorphic' to the velocity components means that the magnetic turbulent fluctuations are assumed to behave like those of velocity, both in terms of energy spectrum and correlations. This choice, defined in (2.29) (please see below), is not the only possible one, but it is the simplest one in order to write a closed form of STMHD.

In the literature there is document much experimental and theoretical evidence that magnetic and velocity fluctuations may follow the same statistical properties (see e.g. Matthaeus & Goldstein 1982; Roberts & Goldstein 1991; Verma *et al.* 1996; Müller & Biskamp 2000).

When the Langevin equations (2.23) are re-written in components, they appear as:

$$\left.\begin{aligned}\dot{\rho}_i^{\alpha} &= \rho_{i\beta} M^{\beta\alpha} + u_i^{\alpha},\\ \dot{M}^{\alpha\beta} &= (\alpha_v - 1)\left[M_{\gamma}^{\alpha} M^{\gamma\beta} - \left(M_{\eta\gamma} M^{\gamma\eta} - \frac{\Xi_{\eta\gamma}\Xi^{\gamma\eta}}{2\rho}\right)\Pi^{\alpha\beta}(\boldsymbol{\rho}) - \frac{\Xi^{\alpha\gamma} W_{\gamma}^{\beta}}{\rho}\right] + h^{\alpha\beta},\\ \dot{W}^{\alpha\beta} &= (\alpha_B - 1)(W^{\alpha\gamma} M_{\gamma}^{\beta} - M^{\alpha\gamma} W_{\gamma}^{\beta}) + \theta^{\alpha\beta},\\ \Xi^{\alpha\beta} &= W^{\alpha\beta} - W^{\beta\alpha}.\end{aligned}\right\} \tag{2.24}$$

Noise u_i^{α}, $h^{\alpha\beta}$ and $\theta^{\alpha\beta}$ are assumed to be time-δ-correlated and Gaussian. As far as the average values of the noise is concerned, this is assumed to vanish:

$$\langle u_i^{\alpha}(\tau)\rangle = 0, \quad \langle h^{\alpha\beta}(\tau)\rangle = 0, \quad \langle \theta^{\alpha\beta}(\tau)\rangle = 0 \tag{2.25a–c}$$

(coherently with the assumption that these averages are $o(\rho)$). In particular, one assumes the correlation functions to be:

$$\left.\begin{aligned}\langle u_i^{\alpha}(\tau) u_j^{\beta}(\tau')\rangle &= \Sigma_{ij}^{(u)\alpha\beta}(\tau)\delta(\tau - \tau'),\\ \langle h^{\alpha\beta}(\tau) h^{\gamma\delta}(\tau')\rangle &= \Sigma^{(h)\alpha\beta\gamma\delta}(\tau)\delta(\tau - \tau'),\\ \langle \theta^{\alpha\beta}(\tau)\theta^{\gamma\delta}(\tau')\rangle &= \Sigma^{(\theta)\alpha\beta\gamma\delta}(\tau)\delta(\tau - \tau'),\end{aligned}\right\} \tag{2.26}$$

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all the other possibile combinations (e.g. $\langle u_i^\alpha(\tau)\theta^{\gamma\delta}(\tau')\rangle$) being null. The tensors $\Sigma_{ij}^{(u)\alpha\beta}(\tau)$, $\Sigma^{(h)\alpha\beta\gamma\delta}(\tau)$ and $\Sigma^{(\theta)\alpha\beta\gamma\delta}(\tau)$ are assumed to be invertible for any τ in the non-trivial sense that, for every τ, there must exist three tensors $\Sigma_{\alpha\beta}^{(u)-1ij}(\tau)$, $\Sigma_{\alpha\beta\gamma\delta}^{(h)-1}(\tau)$ and $\Sigma_{\alpha\beta\gamma\delta}^{(\theta)-1}(\tau)$ so that:

$$\left.\begin{aligned}\Sigma_{ij}^{(u)\alpha\beta}\Sigma_{\beta\gamma}^{(u)-1jk} &= \delta_i^k\delta_\gamma^\alpha,\\ \Sigma^{(h)\alpha\beta\gamma\delta}\Sigma_{\beta\eta\delta\theta}^{(h)-1} &= \delta_\eta^\alpha\delta_\theta^\gamma,\\ \Sigma^{(\theta)\alpha\beta\gamma\delta}\Sigma_{\beta\rho\delta\sigma}^{(\theta)-1} &= \delta_\rho^\alpha\delta_\sigma^\gamma.\end{aligned}\right\} \tag{2.27}$$

Adopting the hypotheses in Chertkov *et al.* (1999), one re-writes (2.26) as:

$$\left.\begin{aligned}\Sigma_{ij}^{(u)\alpha\beta} &= 2\sqrt{\boldsymbol{M}^2}[C_\parallel\rho_i^\alpha\rho_j^\beta + C_\perp(\rho^2\delta^{\alpha\beta}\delta_{ij} - \rho_i^\alpha\rho_j^\beta)],\\ \Sigma^{(h)\alpha\beta\gamma\delta} &= \frac{2C_h}{\rho^2}\left(\delta^{\alpha\delta}\delta^{\beta\gamma} - \frac{1}{3}\delta^{\alpha\beta}\delta^{\gamma\delta}\right)\end{aligned}\right\} \tag{2.28}$$

(here $\|a\|^2 \equiv \boldsymbol{a}^2 = \mathrm{Tr}(a^\dagger a)$ is intended for every matrix $\boldsymbol{a}$).

The form of the correlation matrix $\Sigma^{(\theta)\alpha\beta\gamma\delta}$ for the magnetic noise θ deserves some explanation. In Chertkov *et al.* (1999), the hypotheses of (2.28) appears to follow from the assumption that the velocity fluctuations undergo some K41-like cascade theory. This renders stochastic tetrad hydrodynamics (STHD) suitable for describing fluid turbulence within the inertial range of scales (Frisch 1995). Here, we simply assume for $\boldsymbol{W}$ that grossly the same rules apply as for $\boldsymbol{M}$, both of these being tensors coarse-grained versions of traceless gradient fields. It appears to be reasonable to suppose:

$$\Sigma^{(\theta)\alpha\beta\gamma\delta} = \frac{2C_\theta}{\rho^2}\left(\delta^{\alpha\delta}\delta^{\beta\gamma} - \frac{1}{3}\delta^{\alpha\beta}\delta^{\gamma\delta}\right). \tag{2.29}$$

As anticipated above, this assumption implies that the magnetic fluctuations are statistically equivalent to those of velocity in terms of two time-correlation functions. As in the case of STHD, this limits the validity of our STMHD, under the assumptions of (2.26), (2.28) and (2.29), to the inertial interval of scales, i.e. to a regime of fully developed MHD turbulence. This point is discussed in more detail in §§ 4 and 5.

We now have a system of coupled non-linear Langevin equations, with additive time-δ-correlated Gaussian noise terms. In this case, the evolution of $\boldsymbol{\varrho}$, $\boldsymbol{M}$ and $\boldsymbol{W}$ is probabilistic, and one may wish to calculate the probability to have a given plasma tetrad configuration $(\boldsymbol{\varrho}(t), \boldsymbol{M}(t), \boldsymbol{W}(t))$ at a time t conditioned to the knowledge of the tetrad configuration $(\boldsymbol{\varrho}(t_0), \boldsymbol{M}(t_0), \boldsymbol{W}(t_0))$ at a certain time $t_0 < t$, out of the deterministic and stochastic terms working on the tetrad as in (2.24). The aim of our work is now to introduce a formalism, referred to as functional formalism (FF), rendering this type of calculation possible.

3. The functional formalism of classical statistical dynamics

In 1977, Phythian published the milestone paper (Phythian 1977), in which he constructed a path integral formalism able to produce all the statistical quantities relevant for a classical system undergoing stochastic forces. Such a system, which would undergo a deterministic Cauchy problem in the absence of the random stirring, inherits probabilistic behaviour. The results in Phythian (1977) are the mathematical

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tools to obtain the description of the statistical dynamics of a system from that of the stochastic forces.

A classical system, described by dynamical variables $\psi \in \mathbb{R}^n$ and stirred by stochastic noise $f \in \mathbb{R}^n$, is governed by the Langevin equation

$$\dot{\psi} = \Lambda(\psi) + f. \tag{3.1}$$

In (3.1), $\Lambda \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is the deterministic part of the equation. The presence of noise turns the classical dynamics of ψ into an evolution conditioned time-by-time by the 'probabilistic choice' of f. Because of the noise forcing the system, the system dynamics has infinitely many realizations, even for fixed initial conditions. At each of these different evolutions, a statistical weight is attributed, which is determined by the concurrence of the deterministic part and the noise in the Langevin equation (3.1).

In order to evaluate the effect of the stochasticity of f on ψ, the distribution of this f has to be known. Indeed, if one is interested in properly describing the dynamics of (3.1) between an initial time t_0 and a later time t, it is necessary to assign the probability distribution functional $\mathcal{Q}[f; t_0, t)$ for each of the infinitely many realizations of $f(\tau)$, where $\tau \in (t_0, t]$. Phythian's procedure is a way to obtain the correspondent functional $\mathcal{A}[\psi; t_0, t)$ representing the probability that the particular history $\psi(\tau)$ of the dynamical variables takes place in the time interval from t_0 to t. The map from $\mathcal{Q}[f; t_0, t)$ to $\mathcal{A}[\psi; t_0, t)$ encodes the fact that ψ undergoes the full dynamics of (3.1), in which both chance (the noise f) and necessity (the deterministic part Λ) are considered (Haken 1983): $\mathcal{Q}[f; t_0, t) \overset{f,\Lambda}{\longmapsto} \mathcal{A}[\psi; t_0, t)$.

In order to construct this map, the distribution $\mathcal{Q}[f; t_0, t)$ is assigned through its characteristic functionals, namely its functional Fourier transform in the space of noise realizations:

$$C[\chi; t_0, t) = \int [\mathrm{d}f]\mathcal{Q}[f; t_0, t)\mathrm{e}^{\mathrm{i}\int_{t_0}^{t} \mathrm{d}\tau\, \chi^\dagger(\tau)\cdot f(\tau)}. \tag{3.2}$$

Here, $[\mathrm{d}f]$ is the path integral measure in the ambient of noise realizations. From the last equation, the necessity of defining the variables $\chi \in \mathbb{R}^n$ is evident, nominally Fourier-conjugate to the noises f.

The central result in Phythian (1977) is that, due to the dynamics described in the (3.1), the functional probability distribution that the realizations $\psi(\tau)$ and $\chi(\tau)$ take place in the time interval from t_0 to t, is given by the expression:

$$\boldsymbol{A}[\psi, \chi; t_0, t) = N_0 C[\chi; t_0, t) \exp\left(-\frac{1}{2}\int_{t_0}^{t} \mathrm{d}\tau\, \partial_\psi \cdot \Lambda - \mathrm{i}\int_{t_0}^{t} \mathrm{d}\tau\, \chi^\dagger \cdot (\dot{\psi} - \Lambda)\right), \tag{3.3}$$

where N_0 is a normalization constant (see below). Moreover, in (3.3) the symbol $\partial_\psi \cdot \Lambda$ represents the 'divergence' of the field Λ in the phase space of the system described by ψ, i.e.:

$$\partial_\psi \cdot \Lambda = \frac{\partial \Lambda^1}{\partial \psi^1} + \frac{\partial \Lambda^2}{\partial \psi^2} + \cdots. \tag{3.4}$$

Due to the stochastic ODEs, as in (3.1), Phythian's formalism deals with the so called Ito–Stratonovich dilemma (Van Kampen 1981), which is concerned with the choice made in equation (3) of Phythian (1977), which refines the Hosokawa and Rosen discrete approximations of path integrals to the second-order instead of the first.

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Once the expression of $\boldsymbol{A}[\psi, \chi; t_0, t)$ has been found, the statistical dynamics of the system described by (3.1) is obtained in a rather simple form. This is also true for the transition probabilities among the different configurations (see, for example, Phythian 1977 or Materassi & Consolini 2008). In particular, the probability that the dynamics described in (3.1) gives rise to the transition from an initial configuration $\psi(t_0) = \psi_i$ to a final one $\psi(t) = \psi_f$, may be obtained integrating the functional $\boldsymbol{A}[\psi, \chi; t_0, t)$ over all the realizations $\psi(\tau)$ for $\tau \in (t_0, t)$ while keeping $\psi(t_0)$ fixed as ψ_i, and $\psi(t)$ fixed as ψ_f, and $\chi(\tau)$ for $\tau \in [t_0, t]$:

$$P_{\psi_i \to \psi_f}(t_0, t) = \int [\mathrm{d}\chi] \int_{\psi(t_0)=\psi_i}^{\psi(t)=\psi_f} [\mathrm{d}\psi] \boldsymbol{A}[\psi, \chi; t_0, t). \tag{3.5}$$

In a more compact way, the foregoing expression can be written as

$$P_{\psi_i \to \psi_f}(t_0, t) = \langle 1 \rangle_{\psi(t_0)=\psi_i}^{\psi(t)=\psi_f}, \tag{3.6}$$

where the symbol $\langle F \rangle$ indicates the average value of F over the histories, i.e. $\langle F \rangle = \int [\mathrm{d}\chi] \int [\mathrm{d}\psi] \boldsymbol{A} F$, and $F = 1$ in (3.6).

The explicit calculation of $\boldsymbol{A}[\psi, \chi; t_0, t)$ needs an expression for $C[\chi; t_0, t)$, which is easy to obtain in the particular case when the noise components f^α have a Gaussian distribution q with an assigned correlation function given by:

$$\langle f^\alpha(\tau) f^\beta(\tau') \rangle_{\mathcal{Q}} = \Sigma^{\alpha\beta}(\tau) \delta(\tau - \tau'). \tag{3.7}$$

The matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{n,n}$ in (3.7) is assumed to be invertible, and only its symmetric part play any role because it will be saturated with a symmetric tensor. Thus the distribution of the noise vector at a fixed time $f(\tau)$ reads:

$$q(f(\tau)) = \frac{\mathrm{e}^{-\frac{1}{2}(f(\tau)-f_0(\tau))^\dagger \cdot \sum^{-1}(\tau) \cdot (f(\tau)-f_0(\tau))}}{\sqrt{\det(2\pi \boldsymbol{\Sigma}(\tau))}}. \tag{3.8}$$

In all of these distributions, the functions $f_0^\gamma(\tau)$ represent the expected value of the noise $f^\gamma(\tau)$. The distribution (3.8) is correctly normalized, i.e.

$$\int_{\mathbb{R}^n} \mathrm{d}^n f(\tau) q(f(\tau)) = 1. \tag{3.9}$$

Moreover, the pre-factor $C[\chi; t_0, t)$ can be written as

$$C[\chi; t_0, t) = \mathrm{e}^{(1/2)\int_{t_0}^t \mathrm{d}\tau (2\mathrm{i} f_0^\dagger \cdot \chi - \chi^\dagger \cdot \boldsymbol{\Sigma} \cdot \chi)}. \tag{3.10}$$

This expression enables us to calculate explicitly the kernel as a functional of ψ and χ, by placing (3.10) into (3.3), as done in appendix B. The kernel, expressed as a functional of dynamical variables and stochastic momenta, is given by:

$$\boldsymbol{A}[\psi, \chi; t_0, t) = N_0 \mathrm{e}^{\int_{t_0}^t \mathrm{d}\tau \{-(1/2)\chi^\dagger \cdot \boldsymbol{\Sigma} \cdot \chi + \mathrm{i}\chi^\dagger \cdot [f_0 - (\dot{\psi} - \Lambda)] - (1/2)\partial_\psi \cdot \Lambda(\psi)\}}. \tag{3.11}$$

The expression of (3.11) is, to some extent, not physical, because, while the ψ variables mean 'something physical' for the system, the χ variables are just what

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remains of the noise. If one integrates out the stochastic momenta in $\boldsymbol{A}[\psi, \chi; t_0, t)$, then a more intuitive form of the kernel is obtained,

$$\mathcal{A}[\psi; t_0, t) = \int [\mathrm{d}\chi]\boldsymbol{A}[\psi, \chi; t_0, t). \tag{3.12}$$

In the case of Gaussian time-δ-correlated noises, $\mathcal{A}$ can be expressed (see appendix C for computational details):

$$\mathcal{A}[\psi; t_0, t) = K_0 \mathrm{e}^{-(1/2)\int_{t_0}^{t} \mathrm{d}\tau \{[f_0-(\dot{\psi}^\dagger - \Lambda^\dagger)]\cdot \Sigma^{-1}\cdot[f_0-(\dot{\psi}-\Lambda)]+\partial_\psi \cdot \Lambda(\psi)\}}. \tag{3.13}$$

The kernel $\mathcal{A}$ may be expressed in terms of a functional depending on the evolution of $\psi(\tau)$, namely the stochastic action $S[\psi]$, so that the kernel acquires the form:

$$\mathcal{A}[\psi; t_0, t) = K_0 \mathrm{e}^{-S[\psi; t_0, t)}, \tag{3.14}$$

where the stochastic action $S[\psi; t_0, t)$ is given by the following expression:

$$\begin{aligned} S[\psi; t_0, t) \;=\; & \frac{1}{2}\int_{t_0}^{t} \mathrm{d}\tau\, [(\dot{\psi} - \Lambda)^\dagger \cdot \Sigma^{-1} \cdot (\dot{\psi} - \Lambda) + 2f_0^\dagger \cdot \Sigma^{-1} \cdot (\dot{\psi} - \Lambda) \\ & + f_0^\dagger \cdot \Sigma^{-1} \cdot f_0 + \partial_\psi \cdot \Lambda(\psi)], \end{aligned} \tag{3.15}$$

If the Gaussian noise has zero mean, i.e. if

$$f_0(\tau) = 0 \quad \forall \tau \in [t_0, t], \tag{3.16}$$

then the stochastic action reduces to the expression:

$$S[\psi; t_0, t) = \frac{1}{2}\int_{t_0}^{t} \mathrm{d}\tau [(\dot{\psi} - \Lambda)^\dagger \cdot \Sigma^{-1} \cdot (\dot{\psi} - \Lambda) + \partial_\psi \cdot \Lambda(\psi)]. \tag{3.17}$$

The expressions (3.14) and (3.15), or (3.17) with null average noise, are the crucial ones through which one classical system, with dynamics forced by Gaussian time-δ-correlated noise, is turned into a stochastic system treated via a functional formalism.

4. The stochastic action for tetrad MHD

Now we are in a position to put together the STMHD findings presented in § 2 with Phythian's functional formalism discussed in § 3. To this end, we need to express the action of (3.17) with the assumptions made on the noise statistics by means of (2.26), (2.28) and (2.29). However, it is important to remark that the STMHD presented here is suitable for describing the finite scale statistical dynamics of incompressible MHD turbulence only when the aforementioned assumptions hold for the velocity and magnetic fluctuations. In particular, this is the case of velocity fluctuations that satisfy the K41 results in the inertial range (and the same is expected for magnetic field fluctuations $\boldsymbol{B}$ occurring in turbulent MHD systems) so that a condition on the interval range of scales where the previous framework can be simply set. The other hypotheses on $\langle h^{\alpha\beta}(\tau)h^{\gamma\delta}(\tau')\rangle$ and $\langle \theta^{\alpha\beta}(\tau)\theta^{\gamma\delta}(\tau')\rangle$ could be suitable to describe other regimes. This is a point that will be investigated in future work.

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The inverse tensors of cross correlation matrices in (2.28) and (2.29) are calculated in appendix D. However, the inverse matrix of the first cross-correlation tensor can be written as follows:

$$(\Sigma^{(u)-1})^{ij}_{\alpha\beta} = \frac{1}{2\rho^2\sqrt{\boldsymbol{M}^2}}[C_{\parallel}^{-1}\hat{\rho}_i^{\alpha}\hat{\rho}_j^{\beta} + C_{\perp}^{-1}(\delta^{\alpha\beta}\delta_{ij} - \hat{\rho}_i^{\alpha}\hat{\rho}_j^{\beta})], \tag{4.1}$$

while, as far as the cross-correlation tensor in (2.28) is concerned, the arguments in appendix D imply that, de facto, we are speaking about inverting the identical application of $\mathbb{R}^{3,3}$ onto $\mathbb{R}^{3,3}$. Thus, one may write:

$$(\Sigma^{(h)-1})_{\alpha\beta\gamma\delta} = \frac{\rho^2}{2C_h}\delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{4.2}$$

Moreover, the inversion of the cross correlation tensor for the noise θ leads to the expression:

$$(\Sigma^{(\theta)-1})_{\alpha\beta\gamma\delta} = \frac{\rho^2}{2C_\theta}\delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{4.3}$$

The two conditions in the (2.26) and (2.25) allow to use the relationship in (3.17) to calculate the stochastic action of the Tetrad MHD.

The vector of dynamical variables ψ is:

$$\psi = \begin{pmatrix} \boldsymbol{\varrho} \\ \boldsymbol{M} \\ \boldsymbol{W} \end{pmatrix}, \tag{4.4}$$

where one has to remember that $\boldsymbol{M}$ and $\boldsymbol{W}$ are tensors of components $M^{\alpha\beta}$ and $W^{\alpha\beta}$. Moreover, the deterministic parts of the Langevin equations can be written as:

$$\left.\begin{aligned} &\Lambda_i^{(\rho)\alpha}(\boldsymbol{\rho}, \boldsymbol{M}) = \rho_{i\beta}M^{\beta\alpha}, \\ &\Lambda^{(M)\alpha\beta}(\boldsymbol{\rho}, \boldsymbol{M}, \boldsymbol{W}) \\ &\quad = (\alpha_v - 1)\left[M_\gamma^\alpha M^{\gamma\beta} - \left(M_{\eta\gamma}M^{\gamma\eta} - \frac{\Xi_{\eta\gamma}\Xi^{\gamma\eta}}{2\rho}\right)\Pi^{\alpha\beta}(\boldsymbol{\rho}) - \frac{\Xi^{\alpha\gamma}W_\gamma^\beta}{\rho}\right], \\ &\Lambda^{(W)\alpha\beta}(\boldsymbol{M}, \boldsymbol{W}) = (\alpha_B - 1)(W^{\alpha\gamma}M_\gamma^\beta - M^{\alpha\gamma}W_\gamma^\beta). \end{aligned}\right\} \tag{4.5}$$

We then start by calculating the 'divergence' $\partial_\psi \cdot \Lambda$ of Λ in the phase space of ψ, obtaining:

$$\partial_\psi \cdot \Lambda = \sum_{j,\zeta}\delta_j^i\delta_\alpha^\zeta\frac{\partial\Lambda_i^{(\rho)\alpha}}{\partial\rho_j^\zeta} + \sum_{\xi,\zeta}\delta_\alpha^\xi\delta_\beta^\zeta\frac{\partial\Lambda^{(M)\alpha\beta}}{\partial M^{\xi\zeta}} + \sum_{\xi,\zeta}\delta_\alpha^\xi\delta_\beta^\zeta\frac{\partial\Lambda^{(W)\alpha\beta}}{\partial W^{\xi\zeta}}. \tag{4.6}$$

Detailed calculations can be found in appendix C. Thus, this 'divergence' can be written in the form:

$$\partial_\psi \cdot \Lambda = -2(\alpha_v - 1)\mathrm{Tr}(\boldsymbol{\Pi}(\boldsymbol{\rho}) \cdot \boldsymbol{M}). \tag{4.7}$$

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In order to complete the calculation of the stochastic functional $\mathcal{A}[\psi; t_0, t)$ one has to calculate the expression $(\dot{\psi} - \Lambda)^\dagger \cdot \Sigma^{-1} \cdot (\dot{\psi} - \Lambda)$. This is done, in detail, in the appendix D. The obtained result is:

$$
\begin{aligned}
(\dot{\psi} - \Lambda)^\dagger \cdot \Sigma^{-1} \cdot (\dot{\psi} - \Lambda) = & \frac{\mathrm{Tr}[(\dot{\rho} - \rho^\dagger \cdot \boldsymbol{M})^\dagger \cdot (C_\parallel^{-1} r + C_\perp (I - r)) \cdot (\dot{\rho} - \rho^\dagger \cdot \boldsymbol{M})]}{2\rho^2 \sqrt{M^2}} \\
& + \frac{\rho^2}{2C_h} \left\| \dot{\boldsymbol{M}} - (\alpha_v - 1) \left[M^2 - \left(M^2 - \frac{\Xi^2}{2\rho} \right) \boldsymbol{\Pi}(\boldsymbol{\rho}) - \frac{\boldsymbol{\Xi} \cdot \boldsymbol{W}}{\rho} \right] \right\|^2 \\
& + \frac{\rho^2}{2C_\theta} \| \dot{\boldsymbol{W}} - (\alpha_B - 1)[\boldsymbol{W}, \boldsymbol{M}] \|^2
\end{aligned}
\tag{4.8}
$$

after placing the terms (D 13) in (D 15).

The stochastic action for the tetrad MHD may be calculated considering the relationships of (4.7) and (4.8) in (3.17):

$$
\begin{aligned}
S[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t) = & \frac{1}{2} \int_{t_0}^{t} \mathrm{d}\tau \left\{ \frac{\mathrm{Tr}[(\dot{\boldsymbol{\varrho}} - \varrho^\dagger \cdot \boldsymbol{M})^\dagger \cdot (C_\parallel^{-1} r + C_\perp (I - r)) \cdot (\dot{\boldsymbol{\varrho}} - \varrho^\dagger \cdot \boldsymbol{M})]}{2\rho^2 \sqrt{M^2}} \right. \\
& + \frac{\rho^2}{2C_h} \left\| \dot{\boldsymbol{M}} - (\alpha_v - 1) \left[\boldsymbol{M}^2 - \left(\boldsymbol{M}^2 - \frac{\Xi^2}{2\rho} \right) \boldsymbol{\Pi}(\boldsymbol{\rho}) - \frac{\boldsymbol{\Xi} \cdot \boldsymbol{W}}{\rho} \right] \right\|^2 \\
& \left. + \frac{\rho^2}{2C_\theta} \| \dot{\boldsymbol{W}} - (\alpha_B - 1)[\boldsymbol{W}, \boldsymbol{M}] \|^2 - 2(\alpha_v - 1) \mathrm{Tr}(\boldsymbol{\Pi}(\boldsymbol{\rho}) \cdot \boldsymbol{M}) \right\}.
\end{aligned}
\tag{4.9}
$$

With the aforementioned stochastic action, the transition kernel of the tetrad MHD results in the form:

$$
\mathcal{A}[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t) = \mathcal{N}(t, t_0) \mathrm{e}^{-S[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t)}. \tag{4.10}
$$

The kernel $\mathcal{A}[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t)$ has a paramount role in the construction of any quantity relevant for the statistical dynamics of the STMHD. In § 3 we have already seen how to calculate the transition probability between an initial configuration of the system $\psi(t_0) = \psi_i$ and a final one $\psi(t) = \psi_f$. In this case, the transition would be

$$
\psi(t_0) = (\boldsymbol{\varrho}_i, \boldsymbol{M}_i, \boldsymbol{W}_i) \mapsto \psi(t) = (\boldsymbol{\varrho}_f, \boldsymbol{M}_f, \boldsymbol{W}_f), \tag{4.11}
$$

and the probability for it to take place is:

$$
P_{\psi_i \to \psi_f}(t_0, t) = \int_{(\boldsymbol{\varrho}_i, \boldsymbol{M}_i, \boldsymbol{W}_i)}^{(\boldsymbol{\varrho}_f, \boldsymbol{M}_f, \boldsymbol{W}_f)} [\mathrm{d}\boldsymbol{\varrho}][\mathrm{d}\boldsymbol{M}][\mathrm{d}\boldsymbol{W}] \mathcal{A}[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t), \tag{4.12}
$$

where the kernel of (4.10) and the stochastic action of (4.9) have been used.

If the initial, or final, configuration is not fully observed, e.g. if we want to calculate the transition $\boldsymbol{\varrho}_i \mapsto \boldsymbol{\varrho}_f$ regardless of the configuration of the magnetic and kinetic gradient, one simply avoids fixing $\boldsymbol{M}_f$, $\boldsymbol{W}_f$, $\boldsymbol{M}_i$ and $\boldsymbol{W}_i$ in (4.12). For instance, one has

$$
P_{\boldsymbol{\varrho}_i \to \boldsymbol{\varrho}_f}(t_0, t) = \int_{\boldsymbol{\varrho}_i}^{\boldsymbol{\varrho}_f} [\mathrm{d}\boldsymbol{\varrho}][\mathrm{d}\boldsymbol{M}][\mathrm{d}\boldsymbol{W}] \mathcal{A}[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t). \tag{4.13}
$$

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As explained in Phythian (1977), the functional formalism allows us to calculate the N times correlation functions. In particular, the formula of the N-fold correlation function $\mathcal{G}(\psi^{\alpha_1}(t_1), \ldots, \psi^{\alpha_N}(t_N))$, among N components of ψ taken at N different times, is defined as:

$$\mathcal{G}(\psi^{\alpha_1}(t_1), \ldots, \psi^{\alpha_N}(t_N)) = \left\langle \prod_{k=1}^{N} \psi^{\alpha_k}(t_k) \right\rangle . \tag{4.14}$$

In the STMHD case, the calculation of the correlation between two components of $\boldsymbol{M}$ at two different times, e.g. t_1 and t_2, both within (t_0, t), consists in the computation of:

$$\mathcal{G}(M^{\alpha\beta}(t_1), M^{\gamma\delta}(t_2)) = \int [\mathrm{d}\boldsymbol{\varrho}][\mathrm{d}\boldsymbol{M}][\mathrm{d}\boldsymbol{W}] M^{\alpha\beta}(t_1) M^{\gamma\delta}(t_2) \mathcal{A}[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t). \tag{4.15}$$

Again, this may be used to calculate any correlation function between any components of the STMHD configuration. Furthermore, its analytical form, as derived from (4.10) when it is possible to perform this calculation, would allow us to predict theoretically the time resolved self- or mutual-correlation measurements.

In forthcoming studies, use of the closed form of (4.10) will be made to reproduce what is already known in STHD and to extend the turbulence theory results to STMHD, following the path explained in § 5. The formalism just developed in this paper is represented here in an initial stage, which will be the core of a systematic application wherever incompressible MHD applies. Of course, turbulent solar wind, magnetospheric and boundary layer plasmas are important candidates that will be considered. In particular, in the case of incompressible MHD turbulence, Elsässer variables could be more appropriate than $\boldsymbol{v}$ and $\boldsymbol{B}$ when application to a solar wind context is considered. The description of the above path integral approach in terms of Elsässer variable will be formalized in a successive study. The mathematical core of the present paper is simply aimed at being broader than any specific set of variables.

5. Summary and conclusions

The present paper has focussed on the introduction of the stochastic tetrad in MHD turbulence theory. The main advantages of this approach are:

(1) keeping the method as close as possible to material particle dynamics;
(2) promoting the plasma parcel's size to the role of a dynamical variable;
(3) accepting the stochastic nature of turbulence at the macroscopic level as a result of smaller scale fluctuations and forces exerted by the nearby plasma.

The approach chosen here to meet those requirements is to consider the Lagrangian dynamics of tetrads in fluids, as described in Chertkov *et al.* (1999), and to generalize it to plasmas. This extension includes a new coarse-grained gradient tensor $\boldsymbol{W}$ representing the tetrad version of $\boldsymbol{\partial} \otimes \boldsymbol{B}$ and the noises forcing the new magnetic variable. As in Chertkov's theory (Chertkov *et al.* 1999), here, all the noises are considered time-δ-correlated and Gaussian. This means that the fluctuations pertain to the inertial scale range of the fully developed isotropic turbulence, both for $\boldsymbol{v}$- and $\boldsymbol{B}$-turbulence.

The Langevin equations (2.23) for the tetrad variables $(\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W})$ show clearly how the kinetic and magnetic tensors interact. The $\boldsymbol{M}$ tensor dynamics is fully determined

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by both fluid and Lorentz forces, through terms involving the anti-Hermitian part $\boldsymbol{\Xi} = \boldsymbol{W} - \boldsymbol{W}^{\dagger}$, which is namely the dual of the electric density currents. However, $\boldsymbol{W}$ would be purely dragged by $\boldsymbol{M}$, i.e. by the matter motion, unless of the higher-order derivatives of $\boldsymbol{v}$ and the dissipative effects unfreezing the $\boldsymbol{B}$-flux through the plasma.

The minimal model hypothesis is done in (2.22), which is essentially the same result as in Chertkov *et al.* (1999). Here, the forcing terms are considered to be the sum of the pure noise terms ($\boldsymbol{U}$, h and θ) and the deterministic corrections proportional to the deterministic free parts.

The statistical properties of the tetrad system ($\boldsymbol{\varrho}$, $\boldsymbol{M}$, $\boldsymbol{W}$) may be deduced either through a Fokker–Planck equation, as Chertkov *et al.* did for the HD turbulence, or via a Martin–Siggia–Rose-like mechanism (Martin, Siggia & Rose 1973), namely via a functional formalism inspired by what is presented in Phythian (1977). This is the approach we have chosen here. As it is usually done in quantum mechanics (Feynman & Hibbs 1965), an action functional $S[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t)$ is introduced, whose exponentiation gives the probability for the particular evolution $(\boldsymbol{\varrho}(\tau), \boldsymbol{M}(\tau), \boldsymbol{W}(\tau))$ to occur between t_0 and t. This stochastic action is calculated for full incompressible MHD, and reported in (4.9).

Once the probabilistic weight $\mathcal{A}[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t) = N_0 \exp(-S[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t))$ is calculated, it becomes possible to evaluate any statistically relevant quantity for the stochastic system. For instance, the kernel $\mathcal{A}[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t)$ in (4.12) is used to find the transition probability between a given initial and a given final STMHD configuration, while in (4.15) the use of it is suggested to calculate the mutual correlation between the values of two components of the tensor $\boldsymbol{M}$ at two different times.

The stochastic action S in (4.9) contains three quadratic terms, which are coupled via the inverse cross-correlations of noise Σ^{-1}, and one term $(1 - \alpha_v)\mathrm{Tr}(\boldsymbol{\Pi}(\boldsymbol{\varrho}) \cdot \boldsymbol{M})$ which represents the coupling between the anisotropy of the tetrad and the velocity gradient. We remark that there is a convergence between the STMHD action determined via Phythian's functional formalism and the Green function found in Chertkov *et al.* (1999) from the Fokker–Planck equation, derived using the system (2.23).

The introduction of the kernel $\mathcal{A}[\boldsymbol{\varrho}, \boldsymbol{M}, \boldsymbol{W}; t_0, t)$ is just the starting point of our program of stochastic tetrad MHD, which is intended to cast new light on plasma turbulence. In particular, we believe that this is the right approach to characterise the inherent inhomogeneity, stochasticity and intermittent nature of turbulent plasmas. Indeed, moving from the closed expressions for the statistical weight of histories, we can use the FF for STMHD to make practical predictions of physical quantities, as, for instance, those illustrated in § 4 and many others. The closed forms for quantities of interest can be then compared with the results of direct numerical simulations of how the local topology of a turbulent MHD plasma evolves. An extension to plasmas of the results in Dallas & Alexakis (2013) will be attempted in future work, so that that it will be possible to re-interpret the results via this FF.

Experimental data from space plasmas can be used also in order to validate, criticize and improve the STMHD results, as, for instance, those presented in Volwerk *et al.* (2004) and in Vörös *et al.* (2004). The study of what our multi-scale formalism may teach about space plasma dynamics and topology can finally be of importance to forthcoming and/or future space missions aimed at documenting the role that the inter-scale dynamics takes in turbulent plasmas.

We would like comment with a comparison between the present tetrad dynamics and the use of quaternionic quantities to describe the evolution of a fluid Lagrangian

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parcel (Gibbon & Holm 2007). A tetrad is a 'more flexible' description than a quaternion because the tetrahedron shape may be whichever, and is not purely related to the orthonormality conditions. This aspect of the tetrad approach is crucial to compare theoretical results with multi-probe measurements (e.g. the ESA-CLUSTER mission and the NASA-Magnetospheric Multiscale (MMS) mission in the space plasma context).

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Appendix A. Calculation of $C[\chi]$

Assuming knowledge of the correlation matrix $\boldsymbol{\Sigma}$, let us calculate the expectation value (3.2). To this end, one has to weight the functional

$$\exp\left(\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\,\chi^{\dagger}(\tau)\cdot f(\tau)\right) \tag{A 1}$$

using the joint probability that f^{α} has value $f^{\alpha}(t_0)$ at the initial time t_0, value $f^{\alpha}(t_0+\mathrm{d}\tau)$ at the time $t_0+\mathrm{d}\tau$ immediately after, value $f^{\alpha}(t_0+k\,\mathrm{d}\tau)$ after a delay $k\,\mathrm{d}\tau$ since t_0 and so on, up to the value at $t-\mathrm{d}\tau$, which is some $f^{\alpha}(t-\mathrm{d}\tau)$, and $f^{\alpha}(t)$ at the final time t.

Let the probability distribution of the stochastic variable $f(\tau)$ be the Gaussian expression of (3.8). Then, reasonably, the Gaussian weight to be used in $C[\chi]$ is as follows Feynman & Hibbs (1965):

$$C[\chi]=\left(\prod_{\gamma}\prod_{\tau=t_0}^{t}\int_{\mathbb{R}}\mathrm{d}f^{\gamma}(\tau)\right)\prod_{\tau=t_0}^{t}Q(\boldsymbol{\Sigma},f,\mathrm{d}\tau)\exp\left(\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\,\chi_{\alpha}(\tau)f^{\alpha}(\tau)\right), \tag{A 2}$$

$$Q(\boldsymbol{\Sigma},f,\mathrm{d}\tau)=\boldsymbol{W}(\boldsymbol{\Sigma},\mathrm{d}\tau)\mathrm{e}^{-(1/2)(f(\tau)-f_0(\tau))^{\dagger}\cdot\sum^{-1}(\tau)\cdot(f(\tau)-f_0(\tau))\,\mathrm{d}\tau}. \tag{A 3}$$

The factor $\boldsymbol{W}(\boldsymbol{\Sigma},\mathrm{d}\tau)$ is determined via the normalization:

$$\langle 1\rangle_f=1. \tag{A 4}$$

After some simple algebra, one has

$$\langle 1\rangle_f=\prod_{\tau=t_0}^{t}\boldsymbol{W}(\boldsymbol{\Sigma},\mathrm{d}\tau)\sqrt{\frac{\det(2\pi\sum(\tau))}{\mathrm{d}\tau^{n}}} \tag{A 5}$$

and in order for this to be equal to 1, rendering the condition in (A 4) true, it is sufficient to have:

$$\boldsymbol{W}(\boldsymbol{\Sigma},\mathrm{d}\tau)=\frac{\mathrm{d}\tau^{n/2}}{\sqrt{\det\left(2\pi\sum(\tau)\right)}}. \tag{A 6}$$

For what concerns (A 5) and (A 6), the amount $\mathrm{d}\tau$ should be considered equal to the 'very small' parameter ϵ used in Feynman & Hibbs (1965) for the step-by-step construction of quantum path integrals. In order to achieve mathematical rigour, the quantity $\mathrm{d}\tau$ should be replaced by 'the ϵ that will tend to 0'.

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Thanks to the definition in (A 6) we are ready to calculate $C[\chi; t_0, t)$ explicitly. This goes as follows:

$$C[\chi; t_0, t) = \prod_{\tau=t_0}^{t} \mathrm{e}^{-(1/2)(f_0^\dagger(\tau)\cdot\sum^{-1}(\tau)\cdot f_0(\tau))\,\mathrm{d}\tau} \left(\prod_{\tau=t_0}^{t} \boldsymbol{W}(\boldsymbol{\Sigma}, \mathrm{d}\tau)\right) \prod_{\tau=t_0}^{t} \times \int_{\mathbb{R}^n} \mathrm{d}^n f(\tau) \mathrm{e}^{-(1/2)\,\mathrm{d}\tau f^\dagger(\tau)\cdot\sum^{-1}(\tau)\cdot f(\tau) + \mathrm{d}\tau[\mathrm{i}\chi^\dagger(\tau) + f_0^\dagger(\tau)\cdot\boldsymbol{\Sigma}^{-1}(\tau)]\cdot f(\tau)} \tag{A 7}$$

and since the n-variate Gaussian distributions are integrated as:

$$\int_{\mathbb{R}^n} \mathrm{d}^n f \mathrm{e}^{-(1/2)\,\mathrm{d}\tau f^\dagger\cdot\sum^{-1}\cdot f + \mathrm{d}\tau(\mathrm{i}\chi^\dagger + f_0^\dagger\cdot\sum^{-1})\cdot f(\tau)} = \sqrt{\mathrm{d}\tau^{-n}\det(2\pi\boldsymbol{\Sigma})}\mathrm{e}^{(1/2)\,\mathrm{d}\tau(\mathrm{i}\chi^\dagger + f_0^\dagger\cdot\Sigma^{-1})\cdot\boldsymbol{\Sigma}\cdot(\mathrm{i}\chi + \Sigma^{-1}\cdot f_0)}, \tag{A 8}$$

the calculation of $C[\chi; t_0, t)$ continues as:

$$C[\chi; t_0, t) = \mathrm{e}^{-(1/2)\int_{t_0}^{t}\mathrm{d}\tau(f_0^\dagger\cdot\Sigma^{-1}\cdot f_0)}\mathrm{e}^{(1/2)\int_{t_0}^{t}\mathrm{d}\tau(\mathrm{i}\chi^\dagger + f_0^\dagger\cdot\Sigma^{-1})\cdot\boldsymbol{\Sigma}\cdot(\mathrm{i}\chi + \Sigma^{-1}\cdot f_0)}. \tag{A 9}$$

Now, if the exponent is expanded more explicitly, one has:

$$C[\chi; t_0, t) = \mathrm{e}^{(1/2)\int_{t_0}^{t}\mathrm{d}\tau(-\chi^\dagger\cdot\boldsymbol{\Sigma}\cdot\chi + 2\mathrm{i}f_0^\dagger\cdot\chi)}, \tag{A 10}$$

which is the finding reported in (3.10).

Appendix B. Calculation of $\boldsymbol{A}[\psi, \chi; t_0, t)$ and $\mathcal{A}[\psi; t_0, t)$

The calculation of the functional $\boldsymbol{A}[\psi, \chi; t_0, t)$ is straightforward, if the noise is Gaussian and time-δ-correlated.

Let us start with the expression in (3.11):

$$\boldsymbol{A}[\psi, \chi; t_0, t) = N_0 \mathrm{e}^{\int_{t_0}^{t}\mathrm{d}\tau\{-(1/2)\chi^\dagger\cdot\boldsymbol{\Sigma}\cdot\chi + \mathrm{i}[f_0^\dagger - (\dot{\psi}^\dagger - \Lambda^\dagger)]\cdot\chi\}}\mathrm{e}^{-(1/2)\int_{t_0}^{t}\mathrm{d}\tau\partial_\psi\cdot\Lambda}. \tag{B 1}$$

If the functional $\boldsymbol{A}[\psi, \chi; t_0, t)$ is path-integrated on the histories of stochastic momenta, then a more physical version of it, $\mathcal{A}[\psi; t_0, t)$, which depends on the 'physical' variables ψ only, then:

$$\mathcal{A}[\psi; t_0, t) = \int [\mathrm{d}\chi]\boldsymbol{A}[\psi, \chi; t_0, t) = N_0 \mathrm{e}^{-(1/2)\int_{t_0}^{t}\mathrm{d}\tau\partial_\psi\cdot\Lambda(\psi)} \prod_{\tau=t_0}^{t} \int_{\mathbb{R}^n} \mathrm{d}^n\chi(\tau)\mathrm{e}^{\mathrm{d}\tau\{-(1/2)\chi^\dagger\cdot\boldsymbol{\Sigma}\cdot\chi + \mathrm{i}[f_0^\dagger - (\dot{\psi}^\dagger - \Lambda^\dagger)]\cdot\chi\}}. \tag{B 2}$$

The τth integral,

$$\mathcal{I}(\tau) = \int_{\mathbb{R}^n} \mathrm{d}^n\chi(\tau)\mathrm{e}^{\mathrm{d}\tau\{-(1/2)\chi^\dagger\cdot\boldsymbol{\Sigma}\cdot\chi + \mathrm{i}[f_0^\dagger - (\dot{\psi}^\dagger - \Lambda^\dagger)]\cdot\chi\}} \tag{B 3}$$

in $\mathcal{A}[\psi; t_0, t)$ may be recognized as a Gaussian integral, i.e. the stochastic momenta χ show a Gaussian distribution, if the stirring forces do so. This is not surprising, since the stochastic momenta are related to the stirring forces by some 'Fourier conjugation',

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as anticipated in (3.2), and the Fourier transform of a Gaussian function is still a Gaussian function. The integral $\mathcal{I}(\tau)$ in (B 3) may hence be calculated as a Gaussian multi-variate integral:

$$\mathcal{I}(\tau)=\frac{1}{\sqrt{\det\left(\frac{\boldsymbol{\Sigma}}{2\pi}\,\mathrm{d}\tau\right)}}\exp\left\{-\frac{1}{2}[f_0^\dagger-(\dot{\psi}^\dagger-\Lambda^\dagger)]\cdot\Sigma^{-1}\cdot[f_0-(\dot{\psi}-\Lambda)]\,\mathrm{d}\tau\right\} \tag{B 4}$$

and, after some algebra, the kernel $\mathcal{A}[\psi;t_0,t)$ may be written as:

$$\mathcal{A}[\psi;t_0,t)=N_0\prod_{\tau=t_0}^{t}\frac{1}{\sqrt{\det\left(\frac{\boldsymbol{\Sigma}}{2\pi}\,\mathrm{d}\tau\right)}}\mathrm{e}^{-(1/2)\int_{t_0}^{t}\mathrm{d}\tau\{[f_0^\dagger-(\dot{\psi}^\dagger-\Lambda^\dagger)]\cdot\Sigma^{-1}\cdot[f_0-(\dot{\psi}-\Lambda)]+\partial_\psi\cdot\Lambda(\psi)\}}. \tag{B 5}$$

The constant N_0 times the continuous product $\prod_{\tau=t_0}^{t}(1/\sqrt{\det((\boldsymbol{\Sigma}\,\mathrm{d}\tau)/2\pi)})$ may be indicated with a sole normalization constant K_0 depending on $\boldsymbol{\Sigma}$ and $\mathrm{d}\tau$, so that

$$\int[\mathrm{d}\psi]\mathcal{A}[\psi;t_0,t)=1. \tag{B 6}$$

This is the same result as reported in (3.13).

Appendix C. Calculation of $\partial_\psi\cdot\Lambda$

In this appendix the contribution of the deterministic part of Langevin equations to the stochastic action is calculated as:

$$\begin{aligned}
\partial_\psi\cdot\Lambda &= \sum_{j,\zeta}\delta_j^i\delta_\alpha^\zeta\frac{\partial\Lambda_i^{(\rho)\alpha}}{\partial\rho_j^\zeta}+\sum_{\xi,\zeta}\delta_\alpha^\xi\delta_\beta^\zeta\frac{\partial\Lambda^{(M)\alpha\beta}}{\partial M^{\xi\zeta}}+\sum_{\xi,\zeta}\delta_\alpha^\xi\delta_\beta^\zeta\frac{\partial\Lambda^{(W)\alpha\beta}}{\partial W^{\xi\zeta}}\\
&= \sum_{j,\zeta}\delta_j^i\delta_\alpha^\zeta\frac{\partial}{\partial\rho_j^\zeta}(\rho_{i\beta}M^{\beta\alpha})\\
&\quad+(\alpha_v-1)\sum_{\xi,\zeta}\delta_\alpha^\xi\delta_\beta^\zeta\frac{\partial}{\partial M^{\xi\zeta}}\left[M_\gamma^\alpha M^{\gamma\beta}-\left(M_{\eta\gamma}M^{\gamma\eta}-\frac{\Xi_{\eta\gamma}\Xi^{\gamma\eta}}{2\rho}\right)\right.\\
&\quad\left.\times\ \Pi^{\alpha\beta}(\boldsymbol{\varrho})-\frac{\Xi^{\alpha\gamma}W_\gamma^\beta}{\rho}\right]\\
&\quad+(\alpha_B-1)\sum_{\xi,\zeta}\delta_\alpha^\xi\delta_\beta^\zeta\frac{\partial}{\partial W^{\xi\zeta}}(W^{\alpha\gamma}M_\gamma^\beta-M^{\alpha\gamma}W_\gamma^\beta)\\
&= (\alpha_v-1)\sum_{\xi,\zeta}\delta_\alpha^\xi\delta_\beta^\zeta(\delta_\xi^\alpha\delta_{\gamma\zeta}M^{\gamma\beta}+M_\gamma^\alpha\delta_\xi^\gamma\delta_\zeta^\beta)\\
&\quad-(\alpha_v-1)\sum_{\xi,\zeta}\Pi^{\alpha\beta}(\boldsymbol{\varrho})\delta_\alpha^\xi\delta_\beta^\zeta(\delta_{\eta\xi}\delta_{\gamma\zeta}M^{\gamma\eta}+M_{\eta\gamma}\delta_\xi^\gamma\delta_\zeta^\eta)\\
&= (\alpha_v-1)(3M_\zeta^\zeta+3M_\xi^\xi)-2(\alpha_v-1)\Pi^{\alpha\beta}(\boldsymbol{\varrho})M_{\beta\alpha}\\
&= 6(\alpha_v-1)\mathrm{Tr}(\boldsymbol{M})-2(\alpha_v-1)\Pi^{\alpha\beta}(\boldsymbol{\varrho})M_{\beta\alpha}\\
&= -2(\alpha_v-1)\mathrm{Tr}(\boldsymbol{\Pi}(\boldsymbol{\varrho})\cdot\boldsymbol{M}).
\end{aligned} \tag{C 1}$$

The result is the relationship reported in (4.7).

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Appendix D. Calculation of $(\dot{\psi} - \Lambda)^{\dagger} \cdot \Sigma^{-1} \cdot (\dot{\psi} - \Lambda)$

The calculation of the kinetic part of the stochastic action is done by considering (2.26), (4.4) and (4.5), and expanding the expression $(\dot{\psi} - \Lambda)^{\dagger} \cdot \Sigma^{-1} \cdot (\dot{\psi} - \Lambda)$.

Because noises pertaining to the different variables $\boldsymbol{\varrho}$, $\boldsymbol{M}$ and $\boldsymbol{W}$ are not correlated, $\boldsymbol{\Sigma}(\tau, \tau') = \langle f(\tau) \otimes f(\tau') \rangle$ is a block diagonal matrix

$$\boldsymbol{\Sigma}(\tau, \tau') = \begin{pmatrix} \Sigma^{(u)}(\tau) & 0 & 0 \\ 0 & \Sigma^{(h)}(\tau) & 0 \\ 0 & 0 & \Sigma^{(\theta)}(\tau) \end{pmatrix} \delta(\tau - \tau'), \tag{D 1}$$

whose inverse must have the same form:

$$\Sigma^{-1}(\tau, \tau') = \begin{pmatrix} \Sigma^{(u)-1}(\tau) & 0 & 0 \\ 0 & \Sigma^{(h)-1}(\tau) & 0 \\ 0 & 0 & \Sigma^{(\theta)-1}(\tau) \end{pmatrix} \delta(\tau - \tau'). \tag{D 2}$$

The scalar $(\dot{\psi} - \Lambda)^{\dagger} \cdot \Sigma^{-1} \cdot (\dot{\psi} - \Lambda)$ is given by three addenda:

$$\begin{aligned} (\dot{\psi} - \Lambda)^{\dagger} \cdot \Sigma^{-1} \cdot (\dot{\psi} - \Lambda) \; = \; & \sum_{i=1}^{3} \left(\frac{\mathrm{d}\rho_i}{\mathrm{d}t} - \boldsymbol{\Lambda}_i^{(\rho)} \right)^{\dagger} \cdot (\Sigma^{(u)})^{-1} \cdot \left(\frac{\mathrm{d}\rho_i}{\mathrm{d}t} - \boldsymbol{\Lambda}_i^{(\rho)} \right) \\ & + (\dot{\boldsymbol{M}} - \Lambda^{(M)})^{\dagger} \cdot (\Sigma^{(h)})^{-1} \cdot (\dot{\boldsymbol{M}} - \Lambda^{(M)}) \\ & + (\dot{\boldsymbol{W}} - \Lambda^{(W)})^{\dagger} \cdot (\Sigma^{(\theta)})^{-1} \cdot (\dot{\boldsymbol{W}} - \Lambda^{(W)}). \end{aligned} \tag{D 3}$$

Now, the crucial point is being able to invert the matrices $\Sigma^{(u)}$, $\Sigma^{(h)}$ and $\Sigma^{(\theta)}$ which represent the cross-correlations.

Looking at (2.28) one may re-write $\Sigma_{ij}^{(u)\alpha\beta}$ as:

$$\Sigma_{ij}^{(u)\alpha\beta} = 2\rho^2 \sqrt{\boldsymbol{M}^2} [C_{\parallel} \hat{\rho}_i^{\alpha} \hat{\rho}_j^{\beta} + C_{\perp} (\delta^{\alpha\beta} \delta_{ij} - \hat{\rho}_i^{\alpha} \hat{\rho}_j^{\beta})] \tag{D 4}$$

and, noting that the two addenda $2\rho^2 \sqrt{M^2} C_{\parallel} \hat{\rho}_i^{\alpha} \hat{\rho}_j^{\beta}$ and $2\rho^2 \sqrt{M^2} C_{\perp} (\delta^{\alpha\beta} \delta_{ij} - \hat{\rho}_i^{\alpha} \hat{\rho}_j^{\beta})$ are projectors along orthogonal sub-spaces of $\mathbb{R}^3$, one can conclude that (D 4) is a 'block diagonal expression' itself, whose inverse is simply obtained by inverting the coefficients of the two orthogonal terms as:

$$(\Sigma^{(u)-1})_{\alpha\beta}^{ij} = \frac{1}{2\rho^2 \sqrt{\boldsymbol{M}^2}} [C_{\parallel}^{-1} \hat{\rho}_{\alpha}^{i} \hat{\rho}_{\beta}^{j} + C_{\perp}^{-1} (\delta_{\alpha\beta} \delta^{ij} - \hat{\rho}_{\alpha}^{i} \hat{\rho}_{\beta}^{j})]. \tag{D 5}$$

This is the same expression reported in (4.1).

When the cross-correlation tensor of the h noises is considered

$$\Sigma^{(h)\alpha\beta\gamma\delta}(\tau) = \frac{2C_h}{\rho^2} \left(\delta^{\alpha\delta} \delta^{\beta\gamma} - \frac{1}{3} \delta^{\alpha\beta} \delta^{\gamma\delta} \right), \tag{D 6}$$

one has to keep in mind that this object is proportional to the projector of matrices along their traceless component. Indeed, considering the expression

$$\pi^{\alpha\beta\gamma\delta} = \delta^{\alpha\delta} \delta^{\beta\gamma} - \tfrac{1}{3} \delta^{\alpha\beta} \delta^{\gamma\delta} \tag{D 7}$$

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one may recognize that, for any matrix $\boldsymbol{A} \in \mathbb{R}^{3,3}$, one has:

$$\pi : \boldsymbol{A} = \boldsymbol{A} - \frac{\mathrm{Tr}(\boldsymbol{A})}{3}\mathbf{1}, \tag{D 8}$$

that indeed is traceless:

$$\mathrm{Tr}(\pi : \boldsymbol{A}) = \mathrm{Tr}\left(\boldsymbol{A} - \frac{\mathrm{Tr}(\boldsymbol{A})}{3}\mathbf{1}\right) = 0 \tag{D 9}$$

(here the symbol $\pi : \boldsymbol{A}$ means that the 4-tensor π and the 2-tensor $\boldsymbol{A}$ are contracted on 2 indices). The fact that π is a projector renders it impossible to be inverted along the whole space of $\mathbb{R}^{3,3}$, and it might create a difficulty to apply the recipe in (3.17). The 'disturbing term' in (D 8) is, clearly, the addendum $-(1/3)\delta^{\alpha\beta}\delta^{\gamma\delta}$.

However, this difficulty is easy to circumvent, because we are planning to apply π to traceless matrices, as $\boldsymbol{M}$, $\dot{\boldsymbol{M}}$ or $\Lambda_{\boldsymbol{M}}(\boldsymbol{M})$: thus, we may well state that it is equivalent to the identity along those tensors:

$$\mathrm{Tr}(\boldsymbol{A}) = 0 \Rightarrow \mathrm{Tr}(\pi : \boldsymbol{A}) = \boldsymbol{A}. \tag{D 10}$$

All in all, since we are going to apply $\Sigma^{(h)\alpha\beta\gamma\delta} = ((2C_h)/\rho^2)\pi^{\alpha\beta\gamma\delta}$ to traceless tensors, we could forget about the $-(1/3)\delta^{\alpha\beta}\delta^{\gamma\delta}$ part in π, and simply consider

$$\Sigma^{(h)\alpha\beta\gamma\delta}\boldsymbol{A}_{\gamma\delta} = \boldsymbol{A}^{\alpha\beta} \Leftarrow \mathrm{Tr}(\boldsymbol{A}) = 0. \tag{D 11}$$

The fact that π is restricted to traceless tensors may be assimilated to the identity in $\mathrm{End}(\mathbb{R}^{3,3})$ and helps the computation, because this means that $\Sigma^{(h)\alpha\beta\gamma\delta}$ can be assimilated to $((2C_h)/\rho^2)\delta^{\alpha\delta}\delta^{\beta\gamma}$, and immediately inverted to $(\rho^2/(2C_h))\delta_{\alpha\delta}\delta_{\beta\gamma}$. This leads to the expression in (4.2).

In the same way, one obtains the inversion of the tensor $\Sigma^{(\theta)\alpha\beta\gamma\delta}$, presented in (2.29) and inverted as in (4.3).

Thanks to the results in (4.1)–(4.3) we may construct the term $(\dot{\psi} - \Lambda)^{\dagger} \cdot \Sigma^{-1} \cdot (\dot{\psi} - \Lambda)$ for the tetrad stochastic MHD. The calculation gives:

$$\begin{aligned}
(\dot{\psi} - \Lambda)^{\dagger} \cdot \Sigma^{-1} \cdot (\dot{\psi} - \Lambda) = {} & \frac{(\dot{\rho}_i^{\alpha} - \rho_{i\gamma}M^{\gamma\alpha})(\dot{\rho}_j^{\beta} - \rho_{j\delta}M^{\delta\beta})}{2\rho^2\sqrt{M^2}}\left(\frac{\hat{\rho}_{\alpha}^{i}\hat{\rho}_{\beta}^{j}}{C_{\parallel}} + \frac{\delta_{\alpha\beta}\delta^{ij} - \hat{\rho}_{\alpha}^{i}\hat{\rho}_{\beta}^{j}}{C_{\perp}}\right) \\
& + \frac{\rho^2}{2C_h}\left\{\dot{\boldsymbol{M}}_{\rho\sigma} - (\alpha_v - 1)\left[M_{\rho\gamma}M_{\sigma}^{\gamma} - \left(M_{\eta\gamma}M^{\gamma\eta} - \frac{\Xi_{\eta\gamma}\Xi^{\gamma\eta}}{2\rho}\right)\Pi_{\rho\sigma}(\boldsymbol{\rho}) - \frac{\Xi_{\rho}^{\gamma}W_{\gamma\sigma}}{\rho}\right]\right\} \\
& \times \left\{\dot{\boldsymbol{M}}^{\rho\sigma} - (\alpha_v - 1)\left[M_{\zeta}^{\rho}M^{\zeta\sigma} - \left(M_{\xi\zeta}M^{\xi\zeta} - \frac{\Xi_{\xi\zeta}\Xi^{\xi\zeta}}{2\rho}\right)\Pi^{\rho\sigma}(\boldsymbol{\rho}) - \frac{\Xi^{\rho\xi}W_{\xi}^{\sigma}}{\rho}\right]\right\} \\
& + \frac{\rho^2}{2C_{\theta}}[\dot{W}^{\alpha\beta} - (\alpha_B - 1)(W^{\alpha\xi}M_{\xi}^{\beta} - M^{\alpha\xi}W_{\xi}^{\beta})][\dot{\boldsymbol{W}}_{\alpha\beta} - (\alpha_B - 1)(W_{\alpha}^{\tau}M_{\tau\beta} - M_{\alpha}^{\tau}W_{\tau\beta})].
\end{aligned} \tag{D 12}$$

The foregoing expression in components may be re-written in matrix terms, introducing the matrices

$$\left.\begin{aligned}
& a_i^{\alpha} = \dot{\rho}_i^{\alpha} - \rho_{i\gamma}M^{\gamma\alpha}, \\
& r_{\alpha\beta}^{ij} = \hat{\rho}_{\alpha}^{i}\hat{\rho}_{\beta}^{j}, \quad I_{\alpha\beta}^{ij} = \delta_{\alpha\beta}\delta^{ij}, \\
& b_{\rho\sigma} = \dot{\boldsymbol{M}}_{\rho\sigma} - (\alpha_v - 1)\left[M_{\rho\gamma}M_{\sigma}^{\gamma} - \left(M_{\eta\gamma}M^{\gamma\eta} - \frac{\Xi_{\eta\gamma}\Xi^{\gamma\eta}}{2\rho}\right)\Pi_{\rho\sigma}(\boldsymbol{\rho}) - \frac{\Xi_{\rho}^{\gamma}W_{\gamma\sigma}}{\rho}\right], \\
& c^{\alpha\beta} = \dot{W}^{\alpha\beta} - (\alpha_B - 1)(W^{\alpha\xi}M_{\xi}^{\beta} - M^{\alpha\xi}W_{\xi}^{\beta})
\end{aligned}\right\} \tag{D 13}$$

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and noting that:

$$(\dot{\psi}-\Lambda)^{\dagger}\cdot\Sigma^{-1}\cdot(\dot{\psi}-\Lambda)=\frac{a_i^{\alpha}a_j^{\beta}}{2\rho^2\sqrt{M^2}}\left(\frac{r_{\alpha\beta}^{ij}}{C_{\parallel}}+\frac{I_{\alpha\beta}^{ij}-r_{\alpha\beta}^{ij}}{C_{\perp}}\right)+\frac{\rho^2}{2C_h}b_{\rho\sigma}b^{\rho\sigma}+\frac{\rho^2}{2C_{\theta}}c^{\alpha\beta}c_{\alpha\beta}$$
$$=\frac{\mathrm{Tr}[a^{\dagger}(C_{\parallel}^{-1}r+C_{\perp}(I-r))a]}{2\rho^2\sqrt{M^2}}+\frac{\rho^2}{2C_h}\mathrm{Tr}(b^{\dagger}b)+\frac{\rho^2}{2C_{\theta}}\mathrm{Tr}(c^{\dagger}c). \quad \text{(D 14)}$$

This result

$$(\dot{\psi}-\Lambda)^{\dagger}\cdot\Sigma^{-1}\cdot(\dot{\psi}-\Lambda)=\frac{\mathrm{Tr}[a^{\dagger}(C_{\parallel}^{-1}r+C_{\perp}(I-r))a]}{2\rho^2\sqrt{M^2}}+\frac{\rho^2}{2C_h}\mathrm{Tr}(b^{\dagger}b)+\frac{\rho^2}{2C_{\theta}}\mathrm{Tr}(c^{\dagger}c) \quad \text{(D 15)}$$

is finally the one reported in (4.8).

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