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PARTISAN GERRYMANDERING WITH GEOGRAPHICALLY COMPACT DISTRICTS

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Abstract

Bizarrely shaped voting districts are frequently lambasted as likely instances of gerrymandering. In order to systematically identify such instances, researchers have devised several tests for so-called geographic compactness (i.e. shape niceness). We demonstrate that under certain conditions, a party can gerrymander a competitive state into geographically compact districts to win an average of over 70% of the districts. Our results suggest that geometric features alone may fail to adequately combat partisan gerrymandering.

Keywords: Gerrymandering; Brownian motion

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1. Introduction

Gerrymandering is the manipulation of voting district boundaries to obtain political advantage. The term was coined in 1812 by the Boston Gazette, which likened the contorted shape of a Massachusetts voting district to the profile of a salamander. Ever since, voting districts with bizarre shapes have been criticized as likely instances of gerrymandering (see, for example, [8]). Recently, the plaintiff in *Gill v. Whitford* (see [6]) proposed a nongeometric feature to detect partisan gerrymandering, using recent election data to detect apparent political advantage; the proposal quantified this advantage using the so-called efficiency gap [2], [13]. As a point of comparison, one might use Markov chain Monte Carlo algorithms to uniformly sample the set of compliant maps [7].

In this paper we show that under a certain conditions, a party can gerrymander a competitive state into nicely shaped districts and still manage to win an average of over 70% of the districts; see Figure 1 for a real-world instance of this phenomenon. This suggests that geometric features may be insufficient to adequately diagnose partisan gerrymandering, meaning additional nongeometric features such as an efficiency gap may be necessary to do the job.

To formalize the notion of shape niceness, researchers have devised several methods to quantify so-called geographic compactness, and they can be roughly classified into three different types [5]:

- *Isoperimetry.* Intuitively, a gerrymandered district will spend much of its perimeter selectively including and excluding various portions of a map. One could quantify this waste by simply measuring the perimeter. A scale-invariant alternative is the Polsby–

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Popper score, given by the ratio between the area of the district and the square of its perimeter [9].

- *Convexity.* Congressional districts are confined to state borders, which often exhibit jagged portions due to geographic features such as rivers. These features then induce long district perimeters, thereby incurring false positives with perimeter-based gerrymander detectors. Alternatively, one may compare the area of the district to its convex hull, or to the area of the smallest disk containing the district (as in the Reock score [10]).
- *Dispersion.* Another common feature among gerrymandered districts is *sprawl*. This is quantified by computing the average distance between pairs of points in the district, or by computing the district's moment of inertia. In particular, given a Lebesgue measurable district $D \subseteq \mathbb{R}^2$, the centroid and moment of inertia are given by

$$\mu_D = \frac{1}{|D|} \iint_D x \, \mathrm{d}\,\mathrm{Leb}(x), \qquad I_D = \iint_D \|x - \mu_D\|^2 \, \mathrm{d}\,\mathrm{Leb}(x).$$

In the following section, we introduce a model for voter locations and preferences. In this setting, we show that using a line to split a circular state into two equal populations produces districts with optimal geographic compactness, in the sense of isoperimetry, convexity, and dispersion simultaneously. We then report our main result: under our model of voter locations and preferences, one may split a circular state into two such districts, winning an average of

$$\frac{3}{4} - \frac{1}{2(1+\mathrm{e}^{\pi})} \approx 73\%$$

of the districts (in the limit as the number of voters goes to ∞). The proof of this result involves passing from random walks to Brownian motion, and then manipulating instances of Brownian motion to compute exact probabilities. We discuss possibilities for future work in Section 3. In Section 4 we provide the proof of the main result, and Section 5 contains various technical lemmas.

2. The model and main result

In order to analyze the effectiveness of partisan gerrymandering with geographically compact districts, we need a model for voter locations and preferences. For this, we introduce the (k, n)-*stochastic voter circle model*, which takes kn voters at distinct locations of a circle who cast independent random votes uniformly from $\{\pm 1\}$. (Here, 'circle' refers to the one-dimensional boundary of a disk.) Given this election data, a partisan mapmaker then partitions the circle into k districts so as to maximize the number of majority-positive districts. In this model, we enforce an extreme version of 'one person, one vote' [12] by requiring each district to contain exactly n voters, and we enforce 'geographic compactness' by requiring each district to be a contiguous portion of the circle. We refer to any such partition that maximizes the number of majority-positive districts as an *optimal partisan gerrymander*.

We will devote our attention to the special case where $k = 2$. This case brings a nice interpretation in which voter locations enjoy a more general configuration in the plane: Take n voter locations that, together with their centroid, are in general position, and reflect these locations about the centroid to obtain the other n voter locations. One may project these $2n$ locations onto the unit circle centered at the centroid, and identify permissible partitions of the circle with partitions of the plane into two convex districts that evenly divide the voters.

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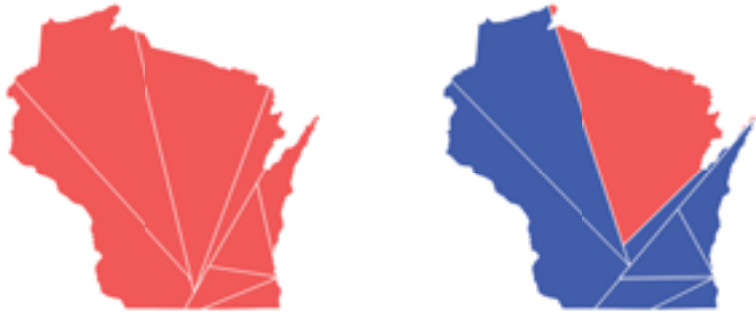


FIGURE 1: Partitions of Wisconsin into eight notional voting districts. Each partition was obtained by selecting lines ('split-lines') to iteratively split regions into two nearly equal populations. For the left-hand panel, these lines were selected so as to maximize the resulting number of majority-Republican districts, whereas majority-Democrat districts (*shaded dark*) were encouraged for the right-hand panel. Here, we applied the 2016 US presidential election returns [14] as a proxy for the spatial distribution of Republicans and Democrats. Wisconsin was particularly competitive in this election, with Trump and Clinton receiving 1,405,284 and 1,382,536 votes, respectively. Under this proxy, Republicans can win all eight of the districts using split-lines (the existence of such a unanimous districting follows from the ham sandwich theorem [11]), while Democrats can achieve seven out of eight (the most possible since they lost overall). Our main result demonstrates that for a certain model of voter locations and preferences, one may use split-lines to gerrymander a competitive state and win an average of over 70% of the allotted districts.

Such partitions of the plane are arguably the most geographically compact possible as we show in the next theorem.

Theorem 1. (Optimal geographic compactness.) *Partition a closed disk D of unit radius into two regions A, B whose closures are homeomorphic to D. Then*

$$\max\{|\partial A|, |\partial B|\} \geq \pi + 2, \qquad \min\left\{\frac{|A|}{|\operatorname{hull}(A)|}, \frac{|B|}{|\operatorname{hull}(B)|}\right\} \leq 1, \qquad I_A + I_B \geq \frac{\pi}{2} - \frac{16}{9\pi}.$$

Equality is simultaneously achieved in all three when A and B are complementary half-disks.

Proof. First, each point in ∂D must lie in $\partial A \cup \partial B$. If $\partial D \subseteq \partial A$ or $\partial D \subseteq \partial B$ then

$$\max\{|\partial A|, |\partial B|\} \geq |\partial D| = 2\pi \geq \pi + 2.$$

Otherwise, $\partial A \cap \partial D$ and $\partial B \cap \partial D$ are both nonempty, and we claim they are both connected.

To see this, suppose to the contrary that there exist $a_1, a_2 \in \partial A \cap \partial D$ and $b_1, b_2 \in \partial B \cap \partial D$, all distinct, arranged in counter-clockwise order as a_1, b_1, a_2, b_2. Then, since the closure $\overline{A}$ is homeomorphic to D, there exists a path P from a_1 to a_2 whose interior points lie in the interior A°. One may also draw a path from a_2 to a_1 by extending radially to the concentric circle of radius 2, and then orbiting along this circle before descending to a_1. Combined, these two paths produce a simple closed curve C that separates b_1 from b_2. Since $\overline{B}$ is homeomorphic to D, there exists a path Q from b_1 to b_2 whose interior points lie in B°. The Jordan curve theorem then gives that an interior point q of Q lies in C, and furthermore, $q \in B^\circ \subseteq D$

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implies that $q \in C \cap D$, i.e. q is also an interior point of P. Overall, $q \in A^\circ \cap B^\circ$, violating the assumption that A and B are disjoint.

At this point, we know that $C_A = \partial A \cap \partial D$ and $C_B = \partial B \cap \partial D$ are nonempty and connected. Let x, y denote the endpoints of C_A (also, of C_B). Since $\overline{A}, \overline{B}$ are homeomorphic to D, there exists a path R from x to y such that $\partial A = C_A \cup R$ and $\partial B = C_B \cup R$. As such,

$$\max\{|\partial A|, |\partial B|\} \geq \max\{|C_A|, |C_B|\} + |R| \geq \max\{|C_A|, 2\pi - |C_A|\} + \|x - y\|.$$

Without loss of generality, we may take $x = (\cos\theta, \sin\theta)$ and $y = (\cos\theta, -\sin\theta)$, and so

$$\max\{|\partial A|, |\partial B|\} \geq \max\{|C_A|, 2\pi - |C_A|\} + \|x - y\| = \max\{2\theta, 2\pi - 2\theta\} + 2\sin\theta.$$

The derivative of the right-hand side is negative for $\theta \in (0, \frac{1}{2}\pi)$ and positive for $\theta \in (\frac{1}{2}\pi, \pi)$, meaning the right-hand side is minimized by $\theta = \frac{1}{2}\pi$, thereby producing the desired bound. In addition, the perimeter of a half-disk is $\pi + 2$, achieving equality in this bound.

For the second bound, we note that every $A \subseteq D$ satisfies $A \subseteq \text{hull}(A)$, and so $|A| \leq |\text{hull}(A)|$ with equality when A is convex, for example, when A is a half-disk.

For the last bound, let μ_A and μ_B denote the centroids of A and B. If $\mu_A = \mu_B$ then

$$I_A + I_B = \iint_D \|x - \mu_A\|^2 \,\mathrm{d}\,\mathrm{Leb}(x) \geq \iint_D \|x\|^2 \,\mathrm{d}\,\mathrm{Leb}(x) = \frac{\pi}{2} \geq \frac{\pi}{2} - \frac{16}{9\pi},$$

where the first inequality follows from the fact that the moment of inertia is minimal about the center of mass. Now suppose that $\mu_A \neq \mu_B$. Then we may define A' and B' as an alternative partition of D obtained from the perpendicular bisector of μ_A and μ_B. (A' and B' are known as the Voronoi regions of μ_A and μ_B, respectively.) Let $\mu_{A'}$ and $\mu_{B'}$ denote the centroids of A' and B'. Then

$$\begin{aligned} I_A + I_B &= \iint_D (\mathbf{1}_A(x)\|x - \mu_A\|^2 + \mathbf{1}_B(x)\|x - \mu_B\|^2) \,\mathrm{d}\,\mathrm{Leb}(x) \\ &\geq \iint_D (\mathbf{1}_{A'}(x)\|x - \mu_A\|^2 + \mathbf{1}_{B'}(x)\|x - \mu_B\|^2) \,\mathrm{d}\,\mathrm{Leb}(x) \\ &\geq \iint_D (\mathbf{1}_{A'}(x)\|x - \mu_{A'}\|^2 + \mathbf{1}_{B'}(x)\|x - \mu_{B'}\|^2) \,\mathrm{d}\,\mathrm{Leb}(x) \\ &= I_{A'} + I_{B'}, \end{aligned}$$

where the inequalities follow from comparing integrands pointwise. As such, we may restrict our attention to partitions of D that arise from a separating line. Without loss of generality, A and B are of the form

$$A = \{(x, y) \in D : x \leq z\}, \qquad B = \{(x, y) \in D : x > z\} \quad \text{for some } z \in (-1, 1).$$

Then $I_A + I_B$ is a function of z that is minimized at $z = 0$ (see Lemma 2 for details), in which case a bit of calculus yields $I_A + I_B = \frac{1}{2}\pi - 16/9\pi$. □

Theorem 2. (Main result.) *For $n \geq 1$, let D_n denote the random number of majority-positive districts in an optimal partisan gerrymander under the $(2, n)$-stochastic voter circle model. Then*

$$\mathbb{P}(D_n = 2) = \frac{1}{2} - \Theta\left(\frac{1}{\sqrt{n}}\right), \qquad \lim_{n\to\infty} \mathbb{P}(D_n = 0) = \frac{1}{1 + \mathrm{e}^{\pi}}.$$

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What follows is a proof of the $D_n = 2$ claim based on a discrete version of the intermediate value theorem. Let P denote the total number of positive votes. We claim that $D_n = 2$ precisely when $P \geq 2\lfloor \frac{1}{2}n + 1 \rfloor$, i.e. $P \geq n + 1$ for n odd and $P \geq n + 2$ for n even. It suffices to prove this claim since

$$\mathbb{P}(P \geq n+1) = \frac{1}{2} - \frac{1}{2^{2n+1}}\binom{2n}{n}, \qquad \mathbb{P}(P \geq n+2) = \frac{1}{2} - \frac{1}{2^{2n+1}}\binom{2n}{n} - \frac{1}{2^{2n}}\binom{2n}{n+1},$$

both of which are equal to $\frac{1}{2} - \Theta(1/\sqrt{n})$ by Stirling's approximation. Since $\lfloor \frac{1}{2}n + 1 \rfloor$ positive votes are required to carry a size-n district, $P < 2\lfloor \frac{1}{2}n + 1 \rfloor$ implies that $D_n < 2$. Now suppose that $P \geq 2\lfloor \frac{1}{2}n + 1 \rfloor$. It is convenient to index the voter locations by $\mathbb{Z}/2n\mathbb{Z}$. Let P_i denote the random number of positive votes in $[i, i + n)$. If $|P_i - P_{i+n}| \leq 1$ for some $i \in \mathbb{Z}/2n\mathbb{Z}$ then

$$\min\{P_i, P_{i+n}\} \geq \tfrac{1}{2}(P_i + P_{i+n}) - \tfrac{1}{2} = \tfrac{1}{2}P - \tfrac{1}{2} \geq \left\lfloor \tfrac{1}{2}n + 1 \right\rfloor - \tfrac{1}{2} > \left\lfloor \tfrac{1}{2}n \right\rfloor,$$

and so $D_n = 2$. As such, we are done if $|P_0 - P_n| \leq 1$. Otherwise, $|P_0 - P_n| \geq 2$, and we may take $P_0 - P_n \geq 2$ without loss of generality. Set $s_i := P_i - P_{i+n}$ for $i \in \{0, \ldots, n\}$. Since $P_i + P_{i+n} = P$, we have $s_i \equiv P \bmod 2$, i.e. s_i has the same parity over $i \in \{0, \ldots, n\}$. We have $s_0 \geq 2$, $s_n \leq -2$, and $s_{i+1} - s_i \in \{-2, 0, 2\}$ for each $i \in \{0, \ldots, n\}$. Let j be the smallest such that $s_j < 2$. Then $2 > s_j = s_{j-1} - 2 \geq 0$, i.e. $|P_j - P_{j+n}| = |s_j| \leq 1$, and so we are done.

Our proof of the $D_n = 0$ claim is more involved, and can be found in Section 4. Note that we analyzed $D_n = 2$ for every fixed n and then applied Stirling's approximation to deduce the reported asymptotic behavior. To analyze our $D_n = 0$ claim, we relate the stochastic voter circle model to a random walk. When suitably scaled, Donsker's invariance principle yields that the walk's distribution converges to that of a standard Brownian motion, and so the desired limit can be written as the probability that standard Brownian motion lies in some event. To compute this probability, we leverage facts about Brownian bridges that have no analog when working with random walks. As such, it is not obvious how to modify our approach to analyze $D_n = 0$ for a fixed n. This is why we report a convergence rate for $\mathbb{P}(D_n = 2)$ but not for $\mathbb{P}(D_n = 0)$.

3. Discussion

In this paper we identified an idealized setting in which a partisan mapmaker can gerrymander to convert an average of 50% of the votes into an average of over 70% of seats, even when constrained to drawing districts that satisfy an extreme notion of geographic compactness. Judging by Figure 1, this phenomenon is not an artifact of our idealized setting, and further research is warranted to generalize and extend these results to more realistic settings. One important direction is the case of more than two districts. For example, one might study the (k, n)-stochastic voter circle model with $k > 2$, or the sequential approach suggested in Figure 1. It would also be nice to consider various fruitful generalizations of how the voter locations and voter preferences are distributed. For example, real-world political geography exhibits dependencies between location and preference. In addition, a real-world mapmaker will not have precise access to voter preferences. Given perfect information about voter preferences, one can always gerrymander with straight lines to obtain 100% of seats from only a majority of votes (see Figure 1 and [11]), but how stable is this behavior to uncertainty in the voter preferences?

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4. Proof of the main result

Draw independent votes $\{x_i\}_{i\in\mathbb{Z}/2n\mathbb{Z}}$ uniformly from $\{\pm 1\}$. We seek the probability that the sum over every interval of length n is nonpositive. Define the random walks

$$A_i = \sum_{j=0}^{i-1} x_j, \qquad B_i = \sum_{j=0}^{i-1} x_{j+n}, \quad i \in \{0, \dots, n\}.$$

Observe that $A_0 = B_0 = 0$. Then the sums of votes over districts $[i, i+n)$ and $[i+n, i)$ are

$$\sum_{j=i}^{i+n-1} x_j = A_n - A_i + B_i, \qquad \sum_{j=i+n}^{i-1} x_j = B_n - B_i + A_i,$$

respectively. As such, both sums are nonpositive for every $i \in \{0, \dots, n\}$ precisely when

$$A_n \le A_i - B_i \le -B_n \quad \text{for all } i \in \{0, \dots, n\}.$$

Define rescaled random walks by

$$W_1^{(n)}(t) = \frac{A_{\lfloor nt \rfloor}}{\sqrt{n}}, \qquad W_2^{(n)}(t) = -\frac{B_{\lfloor nt \rfloor}}{\sqrt{n}}, \quad t \in [0, 1],$$

and let W_1 and W_2 denote independent instances of standard Brownian motion. Then

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}(D_n = 0) \\
&\quad = \lim_{n\to\infty} \mathbb{P}\Big(A_n \le \min_{i\in\{0,\dots,n\}} (A_i - B_i) \le \max_{i\in\{0,\dots,n\}} (A_i - B_i) \le -B_n\Big) \\
&\quad = \lim_{n\to\infty} \mathbb{P}\Big(W_1^{(n)}(1) \le \min_{t\in[0,1]} (W_1^{(n)}(t) + W_2^{(n)}(t)) \le \max_{t\in[0,1]} (W_1^{(n)}(t) + W_2^{(n)}(t)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \le W_2^{(n)}(1)\Big) \\
&\quad = \mathbb{P}\Big(W_1(1) \le \min_{t\in[0,1]} (W_1(t) + W_2(t)) \le \max_{t\in[0,1]} (W_1(t) + W_2(t)) \le W_2(1)\Big),
\end{aligned}$$

where the last step follows from Donsker's invariance principle (see the proof of Lemma 3 for details). Next, consider Brownian bridges $B_k(t) = W_k(t) - tW_k(1)$. Then

$$(W_1(1), B_1(\cdot), W_2(1), B_2(\cdot))$$

are independent, with $W_1(1)$ and $W_2(1)$ exhibiting a standard normal distribution. Set

$$X = \frac{W_1(1) + W_2(1)}{\sqrt{2}}, \qquad Z = \frac{W_1(1) - W_2(1)}{\sqrt{2}}.$$

Then X and Z have a standard normal distribution with $(X, Z, B_1(\cdot), B_2(\cdot))$ independent. As such,

$$W(t) = \frac{B_1(t) + B_2(t)}{\sqrt{2}} + tX, \qquad t \in [0, 1],$$

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is an instance of standard Brownian motion that is independent of Z. Set $L = \min_{t\in[0,1]} W(t)$ and $U = \max_{t\in[0,1]} W(t)$. Then

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(D_n = 0) &= \mathbb{P}\Big(W_1(1) \le \min_{t\in[0,1]} (W_1(t) + W_2(t)) \le \max_{t\in[0,1]} (W_1(t) + W_2(t)) \le W_2(1)\Big)\\ &= \mathbb{P}\bigg(\frac{X+Z}{2} \le L \le U \le \frac{X-Z}{2}\bigg)\\ &= \int_{-\infty}^{\infty} \mathbb{P}\bigg(\frac{X+z}{2} \le L \le U \le \frac{X-z}{2}\bigg) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}\,\mathrm{d}z,\end{aligned}$$

where the last step we condition on Z. Observe the main result in [4], i.e.

$$\mathbb{P}(a \le L \le U \le b \wedge X \in \mathrm{d}x) = f(a,b,x)\,\mathrm{d}x,$$

where

$$f(a,b,x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \displaystyle\sum_{k=-\infty}^{\infty} (\mathrm{e}^{-(x-2k(b-a))^2/2} - \mathrm{e}^{-(x-2b-2k(b-a))^2/2}) & \text{if } a \le 0 \le b,\\ 0 & \text{otherwise.}\end{cases}$$

Since f is continuous (see [4]), Lemma 4 allows us to leverage this expression to obtain

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(D_n = 0) &= \int_{-\infty}^{\infty} \mathbb{P}\bigg(\frac{X+z}{2} \le L \le U \le \frac{X-z}{2}\bigg) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}\,\mathrm{d}z\\ &= \int_{-\infty}^{\infty} \bigg[\int_{-\infty}^{\infty} f\bigg(\frac{x+z}{2}, \frac{x-z}{2}, x\bigg)\,\mathrm{d}x\bigg] \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}\,\mathrm{d}z\\ &= \int_0^{\infty} \int_{-\infty}^{0} f(a,b,a+b) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(a-b)^2/2} 2\,\mathrm{d}a\,\mathrm{d}b.\end{aligned}$$

We wish to convert the integral of the series to a series of integrals. To this end, define

$$\begin{aligned}I_k &= \int_0^{\infty} \int_{-\infty}^{0} \mathrm{e}^{-[(a+b-2k(b-a))^2 + (a-b)^2]/2}\,\mathrm{d}a\,\mathrm{d}b,\\ J_k &= \int_0^{\infty} \int_{-\infty}^{0} \mathrm{e}^{-[(a-b-2k(b-a))^2 + (a-b)^2]/2}\,\mathrm{d}a\,\mathrm{d}b.\end{aligned}$$

Then the triangle inequality yields

$$\begin{aligned}&\sum_{k=-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{0} |\mathrm{e}^{-[(a+b-2k(b-a))^2 + (a-b)^2]/2} - \mathrm{e}^{-[(a-b-2k(b-a))^2 + (a-b)^2]/2}|\,\mathrm{d}a\,\mathrm{d}b\\ &\quad\le \sum_{k=-\infty}^{\infty} I_k + \sum_{k=-\infty}^{\infty} J_k,\end{aligned}$$

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which is finite by Lemmas 5 and 6. As such, an application of the Fubini–Tonelli theorem yields

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}(D_n = 0)\\
&\quad= \int_0^\infty \int_{-\infty}^0 f(a,b,a+b)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(a-b)^2/2} 2\,\mathrm{d}a\,\mathrm{d}b\\
&\quad= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^\infty\int_{-\infty}^0 (\mathrm{e}^{-[(a+b-2k(b-a))^2+(a-b)^2]/2} - \mathrm{e}^{-[(a-b-2k(b-a))^2+(a-b)^2]/2})\,\mathrm{d}a\,\mathrm{d}b\\
&\quad= \frac{1}{2\pi}\Bigg(\sum_{k=-\infty}^{\infty} I_k - \sum_{k=-\infty}^{\infty} J_k\Bigg)\\
&\quad= \frac{1}{1+\mathrm{e}^{\pi}},
\end{aligned}$$

where the last step follows from Lemmas 5 and 6.

5. Lemmata

Lemma 1. *Given a continuous function* $\rho\colon [a,b]\to[0,\infty)$*, denote*

$$M = \int_a^b \rho(x)\,\mathrm{d}x, \qquad \overline{x} = \frac{1}{M}\int_a^b x\rho(x)\,\mathrm{d}x.$$

(i) *If* $\rho\big(\frac{1}{2}(a+b)+t\big) \geq \rho\big(\frac{1}{2}(a+b)-t\big)$ *for every* $t\in\big[0,\frac{1}{2}(b-a)\big]$ *then* $\overline{x}\geq\frac{1}{2}(a+b)$.

(ii) *If* $\rho\big(\frac{1}{2}(a+b)+t\big) \leq \rho\big(\frac{1}{2}(a+b)-t\big)$ *for every* $t\in\big[0,\frac{1}{2}(b-a)\big]$ *then* $\overline{x}\leq\frac{1}{2}(a+b)$.

Proof. We will prove (i), and (ii) follows by negating x. Let ρ_1 denote the symmetric part of ρ, i.e.

$$\rho_1(x) = \begin{cases} \rho(x) & \text{if } x\leq\frac{1}{2}(a+b),\\ \rho(a+b-x) & \text{otherwise,}\end{cases}$$

set $\rho_2(x) = \rho(x) - \rho_1(x)$, and denote

$$M_1 = \int_a^b \rho_1(x)\,\mathrm{d}x, \qquad M_2 = \int_a^b \rho_2(x)\,\mathrm{d}x.$$

Then the symmetry of ρ_1 and the fact that $\rho_2(x)=0$ for $x\leq\frac{1}{2}(a+b)$ together yield

$$\begin{aligned}
\overline{x} &= \frac{1}{M}\int_a^b x\rho(x)\,\mathrm{d}x\\
&= \frac{M_1}{M}\left(\frac{1}{M_1}\int_a^b x\rho_1(x)\,\mathrm{d}x\right) + \frac{M_2}{M}\left(\frac{1}{M_2}\int_a^b x\rho_2(x)\,\mathrm{d}x\right)\\
&= \frac{M_1}{M}\left(\frac{a+b}{2}\right) + \frac{M_2}{M}\left(\frac{1}{M_2}\int_{(a+b)/2}^b x\rho_2(x)\,\mathrm{d}x\right). \qquad (1)
\end{aligned}$$

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Finally,

$$\begin{aligned}\frac{1}{M_2}\int_{(a+b)/2}^{b} x\rho_2(x)\,\mathrm{d}x &\geq \frac{1}{M_2}\int_{(a+b)/2}^{b}\left(\frac{a+b}{2}\right)\rho_2(x)\,\mathrm{d}x\\ &= \frac{a+b}{2}\frac{1}{M_2}\int_{a}^{b}\rho_2(x)\,\mathrm{d}x\\ &= \frac{a+b}{2}.\end{aligned}$$

Combining with (1) and observing $M = M_1 + M_2$ then yields the result. □

Lemma 2. *Consider the following quantities, defined for* $z \in (-1, 1]$*:*

$$M(z) = \int_{-1}^{z} 2\sqrt{1-x^2}\,\mathrm{d}x, \qquad \overline{x}(z) = \frac{1}{M(z)}\int_{-1}^{z} x\cdot 2\sqrt{1-x^2}\,\mathrm{d}x,$$
$$I(z) = \int_{-1}^{z}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (y^2 + (x-\overline{x}(z))^2)\,\mathrm{d}y\,\mathrm{d}x.$$

Then $I(z) + I(-z) \geq 2I(0)$ *for all* $z \in (-1, 1)$.

Proof. Let D denote the unit disk in the xy-plane, and let R_z denote the portion of this disk satisfying $x \leq z$. We start by finding a more convenient expression for the function we are minimizing; thus,

$$\begin{aligned}I(z) + I(-z) &= \iint_{R_z} (y^2 + (x-\overline{x}(z))^2)\,\mathrm{d}x\,\mathrm{d}y + \iint_{R_{-z}} (y^2 + (x-\overline{x}(-z))^2)\,\mathrm{d}x\,\mathrm{d}y\\ &= \iint_{D} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y - 2\overline{x}(z)\iint_{R_z} x\,\mathrm{d}x\,\mathrm{d}y + \overline{x}(z)^2 M(z)\\ &\quad - 2\overline{x}(-z)\iint_{R_{-z}} x\,\mathrm{d}x\,\mathrm{d}y + \overline{x}(-z)^2 M(-z)\\ &= \tfrac{1}{2}\pi - \overline{x}(z)^2 M(z) - \overline{x}(-z)^2 M(-z).\end{aligned}$$

Next, the fundamental theorem of calculus yields

$$\overline{x}(z) = -\frac{2}{3}\frac{(1-z^2)^{3/2}}{M(z)}, \qquad \frac{\mathrm{d}}{\mathrm{d}z}M(z) = 2\sqrt{1-z^2}.$$

These identities allow us to simplify the derivative of our function to obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}(I(z)+I(-z)) = 2\sqrt{1-z^2}(\overline{x}(z)+\overline{x}(-z))(\overline{x}(z)-\overline{x}(-z)-2z). \tag{2}$$

By symmetry, it suffices to show that (2) is nonnegative for $z \in (0, 1)$. To this end, applying Lemma 1(ii) to

$$\rho(x) = \begin{cases} 2\sqrt{1-x^2} & \text{if } x \in [-1, z],\\ 0 & \text{if } x \in (z, 1],\end{cases}$$

yields that $\overline{x}(z) \leq 0$ for every $z \in (-1, 1)$. It remains to show that

$$(\overline{x}(z) - z) - (\overline{x}(-z) - (-z)) = \overline{x}(z) - \overline{x}(-z) - 2z \leq 0 \quad \text{for } z \in (0, 1),$$

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and it suffices to show that $\overline{x}(z) - z$ is decreasing over $z \in (-1, 1)$. To this end, it is straightforward to compute the derivative. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}z}(\overline{x}(z) - z) = 3\frac{\overline{x}(z)^2 - z\overline{x}(z)}{1 - z^2} - 1.$$

To estimate this quantity, we apply Lemma 1(i) to $\rho(x) = 2\sqrt{1 - x^2}$ over $x \in [-1, z]$ to obtain $\overline{x}(z) \geq \frac{1}{2}(z - 1)$. Combining with the more trivial bounds $\overline{x}(z) \leq z \leq 1$ we then obtain

$$-\tfrac{1}{2} \leq \overline{x}(z) - \tfrac{1}{2}z \leq \tfrac{1}{2}z \leq \tfrac{1}{2}.$$

As such, we complete the square to obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}(\overline{x}(z) - z) = 3\frac{(\overline{x} - z/2)^2 - z^2/4}{1 - z^2} - 1 \leq 3\frac{(1/2)^2 - z^2/4}{1 - z^2} - 1 = -\frac{1}{4} < 0,$$

as desired. □

Lemma 3. *Consider the random variables*

$$m^{(n)} = \min_{t\in[0,1]}(W_1^{(n)}(t) + W_2^{(n)}(t)), \qquad m = \min_{t\in[0,1]}(W_1(t) + W_2(t)),$$
$$M^{(n)} = \max_{t\in[0,1]}(W_1^{(n)}(t) + W_2^{(n)}(t)), \qquad M = \max_{t\in[0,1]}(W_1(t) + W_2(t)).$$

Then

$$\lim_{n\to\infty} \mathbb{P}(W_1^{(n)}(1) \leq m^{(n)} \leq M^{(n)} \leq W_2^{(n)}(1)) = \mathbb{P}(W_1(1) \leq m \leq M \leq W_2(1)).$$

Proof. Consider the stochastic processes

$$V^{(n)}(t) = \begin{cases} W_1^{(n)}(t) & \text{if } t \in [0, 1], \\ W_1^{(n)}(1) + W_2^{(n)}(t-1) & \text{if } t \in (1, 2], \end{cases}$$
$$V(t) = \begin{cases} W_1(t) & \text{if } t \in [0, 1], \\ W_1(1) + W_2(t-1) & \text{if } t \in (1, 2]. \end{cases}$$

From Donsker's invariance principle we see that $V^{(n)}(\cdot)$ converges in distribution to the standard Brownian motion $V(\cdot)$. From this, we may conclude that $(W_1^{(n)}(1), m^{(n)}, M^{(n)}, W_2^{(n)}(1))$ converges in distribution to $(W_1(1), m, M, W_2(1))$. To obtain the desired result, it suffices to show that $S = \{(a, b, c, d) \colon a \leq b \leq c \leq d\}$ is a continuity set of $(W_1(1), m, M, W_2(1))$, i.e. that

$$\mathbb{P}((W_1(1), m, M, W_2(1)) \in \partial S) = 0,$$

where ∂S denotes the boundary of S in the standard topology of $\mathbb{R}^4$. Observe that

$$\partial S = \{(a, b, c, d) \colon a = b \vee b = c \vee c = d\},$$

and so the union bound yields

$$\mathbb{P}((W_1(1), m, M, W_2(1)) \in \partial S) \leq \mathbb{P}(W_1(1) = m) + \mathbb{P}(m = M) + \mathbb{P}(M = W_2(1)).$$

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First, $(W_1(\cdot) + W_2(\cdot))/\sqrt{2}$ is standard Brownian motion, which almost surely takes strictly positive values and strictly negative values on $(0, 1)$, and so $m < M$ with probability 1. Below, we demonstrate that $\mathbb{P}(W_1(1) = m) = 0$, and a similar argument yields $\mathbb{P}(M = W_2(1)) = 0$, thereby producing the result.

Consider Brownian bridges $B_k(t) = W_k(t) - tW_k(1)$ and observe that

$$(W_1(1), B_1(\cdot), W_2(1), B_2(\cdot))$$

are independent, with $W_1(1)$ and $W_2(1)$ exhibiting a standard normal distribution. Set

$$X = \frac{W_1(1) + W_2(1)}{\sqrt{2}}, \qquad Z = \frac{W_1(1) - W_2(1)}{\sqrt{2}}.$$

Then X and Z have a standard normal distribution with $(X, Z, B_1(\cdot), B_2(\cdot))$ independent. In particular, Z is independent of $W_1(\cdot) + W_2(\cdot)$. Conditioning on $W_1(\cdot) + W_2(\cdot)$, we have

$$\begin{aligned}\mathbb{P}(W_1(1) = m) &= \mathbb{E}[\mathbb{P}(W_1(1) = m \mid W_1(\cdot) + W_2(\cdot))] \\ &= \mathbb{E}[\mathbb{P}(X + Z = \sqrt{2}m \mid W_1(\cdot) + W_2(\cdot))] \\ &= 0,\end{aligned}$$

where the last step follows from the fact that $W_1(\cdot) + W_2(\cdot)$ determines X and m, while the distribution of Z is continuous. □

Lemma 4. *Let L, U, and X be random variables for which there exists a continuous function f such that*

$$\mathbb{P}(a \le L \le U \le b \wedge X \in [c, d)) = \int_c^d f(a, b, x)\,\mathrm{d}x \quad \text{for all } a, b, c, d \in \mathbb{R},\ a < b,\ c < d.$$

Then for any continuous functions a and b with $a < b$ pointwise, we have

$$\mathbb{P}(a(X) \le L \le U \le b(X) \wedge X \in [c, d)) = \int_c^d f(a(x), b(x), x)\,\mathrm{d}x \quad \text{for all } c, d \in \mathbb{R},\ c < d.$$

Proof. For each h, consider all possible partitions of $[c, d)$ into finitely many half-open intervals $\{I_i\}$ of length less than h. In each interval I_i, consider all possible choices of $x_i^*, x_i^{**} \in \overline{I_i}$. We claim that the following string of equalities hold:

$$\lim_{h\to 0} \sum_i \mathbb{P}(a(x_i^*) \le L \le U \le b(x_i^{**}) \wedge X \in I_i) = \lim_{h\to 0} \sum_i \int_{I_i} f(a(x_i^*), b(x_i^{**}), x)\,\mathrm{d}x \tag{3}$$

$$= \lim_{h\to 0} \sum_i \int_{I_i} f(a(x), b(x), x)\,\mathrm{d}x \tag{4}$$

$$= \int_c^d h(a(x), b(x), x)\,\mathrm{d}x. \tag{5}$$

Indeed, (3) holds by assumption, (4) follows from the uniform continuity of $(s, t, x) \mapsto f(a(s), b(t), x)$ over $s, t, x \in [c, d]$, and (5) follows from the fact that the I_i partition $[c, d)$. Next, define

$$x_i^{*(1)} := \arg\max_{x\in\overline{I_i}} a(x), \qquad x_i^{**(1)} := \arg\min_{x\in\overline{I_i}} b(x),$$

$$x_i^{*(2)} := \arg\min_{x\in\overline{I_i}} a(x), \qquad x_i^{**(2)} := \arg\max_{x\in\overline{I_i}} b(x).$$

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We obtain the following estimates:

$$\mathbb{P}(a(x_i^{*(1)}) \le L \le U \le b(x_i^{**(1)}) \wedge X \in I_i) \le \mathbb{P}(a(X) \le L \le U \le b(X) \wedge X \in I_i) \le \mathbb{P}(a(x_i^{*(2)}) \le L \le U \le b(x_i^{**(2)}) \wedge X \in I_i). \tag{6}$$

At this point, we claim that

$$\begin{aligned}
&\mathbb{P}(a(X) \le L \le U \le b(X) \wedge X \in [c, d)) \\
&\quad = \lim_{h \to 0} \sum_i \mathbb{P}(a(X) \le L \le U \le b(X) \wedge X \in I_i) && (7)\\
&\quad = \lim_{h \to 0} \sum_i \mathbb{P}(a(x_i^*) \le L \le U \le b(x_i^{**}) \wedge X \in I_i) && (8)\\
&\quad = \int_c^d f(a(x), b(x), x)\, \mathrm{d}x. && (9)
\end{aligned}$$

Indeed, (7) follows from equality in the union bound: the I_i partition $[c, d)$ so that the events in the union are disjoint. Finally, (8) applies the squeeze theorem to (6), and (9) comes from (5), completing the proof. □

We note that the remaining lemmata likely enjoy alternative proofs involving probabilistic arguments (see, for example, [3]).

Lemma 5. *It holds that* $\sum_{k=-\infty}^{\infty} I_k = \frac{1}{2}\pi$.

Proof. Recalling that $I_k = \int_0^\infty \int_{-\infty}^0 \exp(-\frac{1}{2}[(a + b - 2k(b - a))^2 + (a - b)^2])\, \mathrm{d}a\, \mathrm{d}b$, and diagonalizing the exponent, we obtain

$$\tfrac{1}{2}[(a + b - 2k(b - a))^2 + (a - b)^2] = (-ka + (k - 1)b)^2 + (-(k + 1)a + kb)^2.$$

This motivates the change of variables $(a, b) \mapsto (x, y)$ given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -k & k-1 \\ -(k+1) & k \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Observe that the above matrix has determinant -1 (as does its inverse). As such,

$$I_k = \iint_{R_k} \mathrm{e}^{-(x^2+y^2)}\, \mathrm{d}x\, \mathrm{d}y,$$

where R_k denotes the cone generated by $(k, k + 1)$ and $(k - 1, k)$. Note that

$$\bigcup_{k=-\infty}^{\infty} R_k = \{(x, y) \colon x < y\} \cup \{(0, 0)\},$$

and, furthermore, $R_i \cap R_j$ has measure zero whenever $i \ne j$. As such, the desired series combines integrals into a computable one, i.e.

$$\sum_{k=-\infty}^{\infty} I_k = \int_{-\infty}^{\infty} \int_{-\infty}^{y} \mathrm{e}^{-(x^2+y^2)}\, \mathrm{d}x\, \mathrm{d}y = \int_{\pi/4}^{5\pi/4} \int_0^\infty \mathrm{e}^{-r^2} r\, \mathrm{d}r\, \mathrm{d}\theta = \frac{\pi}{2}.$$ □

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Lemma 6. *It holds that* $\sum_{k=-\infty}^{\infty} J_k = \frac{1}{2}\pi \tanh(\frac{1}{2}\pi)$.

Proof. Recalling that $J_k = \int_0^\infty \int_{-\infty}^0 \exp\left(-\frac{1}{2}[(a-b-2k(b-a))^2 + (a-b)^2]\right) \mathrm{d}a\, \mathrm{d}b$, and simplifying the exponent yields

$$\tfrac{1}{2}[(a-b-2k(b-a))^2 + (a-b)^2] = (2k^2+2k+1)(-a+b)^2.$$

This motivates the change of variables $x = -a+b$, $y = -a-b$; thus,

$$J_k = \int_0^\infty \int_{-x}^x \mathrm{e}^{-(2k^2+2k+1)x^2} \frac{1}{2}\, \mathrm{d}y\, \mathrm{d}x = \frac{1}{4k^2+2k+2}.$$

At this point, a partial fractions decomposition yields

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} J_k &= \sum_{k=-\infty}^{\infty} \frac{1}{4k^2+2k+2} \\
&= 2\sum_{k=0}^{\infty} \frac{1}{4k^2+2k+2} \\
&= 2\sum_{k=0}^{\infty} \left(\frac{i/4}{k+(1+i)/2} - \frac{i/4}{k+(1-i)/2} \right) \\
&= \frac{i}{2}\left\{ \left[-\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+(1-i)/2} \right) \right] \right. \\
&\qquad \left. - \left[-\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+(1+i)/2} \right) \right] \right\} \\
&= \frac{i}{2}\left(\psi\left(\frac{1-i}{2} \right) - \psi\left(\frac{1+i}{2} \right) \right),
\end{aligned}$$

where γ denotes the Euler–Mascheroni constant, and $\psi(\cdot)$ is the digamma function defined in terms of the gamma function by $\psi(x) = (\mathrm{d}/\mathrm{d}x)\log(\Gamma(x))$. Indeed, the last equality above follows from [1, Section 6.3]. The digamma function satisfies the following reflection formula (see [1, Section 6.3]):

$$\psi(1-z) - \psi(z) = \pi \cot \pi z, \qquad z \in \mathbb{C}.$$

Taking $z = \frac{1}{2}(1+i)$ then yields

$$\sum_{k=-\infty}^{\infty} J_k = \frac{i}{2}\left(\psi\left(\frac{1-i}{2} \right) - \psi\left(\frac{1+i}{2} \right) \right) = \frac{i}{2}\pi\left(\pi \frac{1+i}{2} \right) = \frac{\pi}{2}\tanh\left(\frac{\pi}{2} \right). \qquad \square$$

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