Ergod. Th. & Dynam. Sys. (2013), **33**, 1813–1863 © *Cambridge University Press, 2012*
doi:10.1017/etds.2012.114

Lamination languages

LUIS-MIGUEL LOPEZ† and PHILIPPE NARBEL‡

† *Tokyo University of Social Welfare, 2020-1 Sanno-cho, Isesaki, 372-0831 Gunma, Japan*
(e-mail: lopez@ed.tokyo-fukushi.ac.jp)

‡ *University of Bordeaux 1, LaBRI - UFR Math-Info, 33405 Talence, France*
(e-mail: narbel@labri.fr)

(*Received* 22 *November* 2010 *and accepted in revised form* 7 *July* 2012)

Abstract. Leaves of laminations can be symbolically represented by deforming them into paths of labeled embedded carrier graphs, including train tracks. Here, we describe and characterize the languages of two-way infinite words coming from this kind of coding, called lamination languages, first, by using carrier graph sequences, and second, by using word combinatorics. These characterizations generalize those existing for interval exchange transformations. We also show that lamination languages have ultimately affine factor complexity, and we present effective techniques to build these languages.

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https://doi.org/10.1017/etds.2012.114 Published online by Cambridge University Press

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1. *Introduction*

The idea of representing curves and geodesics on surfaces by infinite words has a long history [**21, 35–37**]. In this paper, we study the symbolic representation which relies on coding curves by deforming them into paths of embedded graphs like *train tracks* [**41, 49**], so that these graphs become carrier graphs for these curves. Using ordinary directed labeled graphs as carrier graphs, we present a general approach that includes all these graph-based symbolic representations, and investigate their properties. This study is done on oriented topological surfaces admitting hyperbolic metrics with finite area, and on the sets of curves forming *laminations* [**49**]. We define *lamination languages* as the sets of words arising from this general coding setting. These languages are *shift spaces*, and include the classic codings of the *interval exchange transformation* dynamics as their simplest cases.

We first characterize lamination languages by using their relationships with specific infinite sequences of carrier graphs, which happen to be usual representations of the combinatorics of the word *factors* (or *subblocks*) of these languages. These graphs are commonly called *Rauzy graphs* [**4, 44**] (or *edge graphs* [**28**]). We show that these graph sequences can be built step by step through geometric moves, called *slittings*, which are just specific kinds of compositions of usual *splittings* (or *unzippings*) in train track theory [**41**]. The main result we prove in this respect, generalizing a similar result obtained for interval exchange transformations in [**5, 6**], is the following (see §3.5).

THEOREM A.1. *A closed shift-invariant language L of two-way infinite words is a lamination language coding a lamination $\mathcal{L}$ in a surface Σ if and only if there is a sequence of graphs $\{\Gamma_n\}_{n\in\mathbb{N}}$ embedded in Σ, obtained by successive slitting steps, whose positive terms are the Rauzy graphs of L.*

By exploiting the local planarity property of the slittings applied to the embedded Rauzy graphs, we derive another characterization of lamination languages, which also generalizes results obtained for interval exchange transformations [**18**]. This characterization relies on order relation constraints over the *bispecial factors* of languages L, i.e. factors having at least two distinct letter prolongations in both left and right directions which are still factors of L (see §4.3).

THEOREM A.2. *A closed shift-invariant language L of two-way infinite words is a lamination language coding a lamination $\mathcal{L}$ in a surface Σ if and only if there exists a pair of partial orders over the coding alphabet that applies to the letter prolongations of the bispecial factors of L.*

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As an extension of these characterization results, we also investigate the form of the *complexity functions* [**36, 37**] of lamination words and languages, i.e. functions over $\mathbb{N}^*$ defined for each n as the number $p(n)$ of distinct length-n factors occurring in them. These functions determine the *topological entropy* of infinite words and languages [**1**]. Families of low complexity objects, i.e. zero-entropy ones, have been much investigated [**2, 7, 13, 17**], in particular those having *linear complexity*, i.e. when $p(n)$ is $O(n)$, and more specifically those having *(ultimate) affine complexity*, i.e. when $p(n) = an + b$, for all $n > n_0$, $a \in \mathbb{N}$, $b \in \mathbb{Z}$ for some $n_0 \geq 0$. For instance, the languages of symbolic orbits of minimal interval exchange transformations have ultimate affine complexity. We prove here that lamination languages are also such zero-entropy languages (see §4.1).

THEOREM B. *A lamination language has ultimate affine complexity.*

This low and regular complexity is a consequence of the fact that lamination languages represent simple and pairwise non-intersecting curves, which induces strong constraints on the combinatorics of their codings. Incidentally, these constraints explain why, although they resemble *edge shifts* [**28**], lamination languages are generally not *shifts of finite type*. Note also that there exist affine complexity languages which are not lamination languages. In particular, *Arnoux–Rauzy words* [**4**] (or *strict episturmian words* [**20**]) are minimal words with affine complexity whose associated shift invariant languages are generally not lamination languages (see Corollary 4.2.4).

The last section of the paper presents techniques for obtaining lamination languages in an effective way, and gives examples illustrating the above results. A first main construction (see §5.2) exploits one of the classic techniques to describe measured laminations from weighted carrier graphs [**23, 41**], closely related to slittings. Using the connection between slittings and Rauzy graphs, factors of lamination languages are then recovered by explicitly building their associated Rauzy graph sequences. Also, by taking advantage of the relationships between minimal laminations and interval exchange transformations (see §4.4), we show how to apply *Rauzy induction* [**43**], so as to obtain factors of lamination words by composing substitutions [**19, 46**]. Second, we present a generalization of a technique described in [**31**] (see §5.3), based on building lamination languages after a Thurston's construction of pseudo-Anosov surface maps [**40, 50**]; in this context, minimal lamination words are obtained as *purely substitutive words* [**3, 19**]. Third, by using specific *graph moves*, namely *edge subdivisions, identifications* and *contractions*, we show how to modify carrier graphs so as to transform and manipulate lamination languages accordingly (see §5.4). All these constructions illustrate the fact that the lamination coding framework we present here, not only gives ways of obtaining explicit symbolic descriptions of surface laminations, but also ways of constructing low complexity words and languages.

2. *Basic definitions*

2.1. *Laminations and their carrier graphs.* A closed surface with finitely many punctures is called a *surface of finite type*. In this paper, Σ always denotes an oriented hyperbolic surface of finite type, with the Poincaré disk $\mathbb{H}^2$ as universal covering space.

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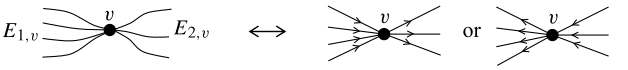
Also here, a *geodesic* in a surface Σ is always the image of a complete geodesic in $\mathbb{H}^2$ (as in [**14**]).

Definition 1. [**14, 49**] A *geodesic lamination* $\mathcal{L}$ on a surface Σ is a non-empty closed subset of Σ forming a union of simple and pairwise disjoint geodesics. These geodesics are called the *leaves* of $\mathcal{L}$.

Because of the assumptions on Σ, a lamination is a union of leaves in only one way. A *curve* γ in a surface Σ is a continuous map, either from a closed connected subset $J \subseteq \mathbb{R}$ to Σ, or from S^1 to Σ. In the latter case, γ is said to be *closed*; if the map defining the curve is injective, γ is said to be *simple*; if $J = \mathbb{R}$, γ is said to be *two-way infinite*; if J is bounded and γ is simple, γ is called an *arc*. A *realization* of a leaf ℓ of a lamination is a parametrization of ℓ to give it a curve structure.

A finite set of pairwise non-homotopic compact geodesics in Σ is a lamination, but laminations are generally made of uncountably many leaves. A lamination $\mathcal{L}$ is said to be *minimal* if it does not contain a lamination as a proper subset. If a minimal lamination $\mathcal{L}$ is not reduced to a single compact leaf, $\mathcal{L}$ is said to be *aperiodic minimal*. A lamination can be covered with finitely many product charts U_i of the form $[0, 1] \times [0, 1]$, each intersected by the leaves along sets of the form $(F_i \subseteq [0, 1]) \times [0, 1]$, where F_i is closed. A *transverse measure* is then a positive Borel measure on the leaf space in each product chart, compatible on the overlap of distinct charts. Such a measure assigns a non-negative number to each *transversal*, i.e. to each arc everywhere transverse to the leaves, and is invariant under leaf-preserving homotopies between transversals [**9, 10, 41**]. A lamination endowed with a transverse measure whose support is the whole lamination, is called a *measured lamination*.

Geodesic laminations have also been described in more combinatorial terms via *train tracks*, i.e. finite embedded C^1 graphs in Σ with a well-defined tangent space at each vertex [**41, 49**]. When the tangent space is oriented at each vertex of such a graph, it locally determines incoming and outgoing edges. In this respect, to take an even more combinatorial point of view, we say that a tamely embedded graph $\Gamma = (V, E)$ in Σ, with V as the set of vertices and E as the set of edges, without isolated vertex, is *train track-like* if its vertices of degree one (if any) correspond to punctures of Σ, and if for each vertex $v \in V$ in Σ, its set of incident edges has been non-trivially partitioned into $E_{1,v} \sqcup E_{2,v}$, where $E_{1,v}$, $E_{2,v}$ are each formed of consecutive edges around v, using either cyclic order. These local partitions induce then two possible local orientations in a small neighborhood of each vertex with positive degree: the edges in $E_{1,v}$ are either considered incoming at v, and the ones of $E_{2,v}$ outgoing from v, or the other way around.



Note that these local orientations need not extend to a global orientation of the graph. An *admissible (edge) path* (or *trainpath*) in a train track-like graph Γ is a sequence of consecutive edges in Γ, such that at each vertex v crossed, the path enters v by an incoming edge and leaves it by an outgoing edge, using one of the two possible local orientations at v.

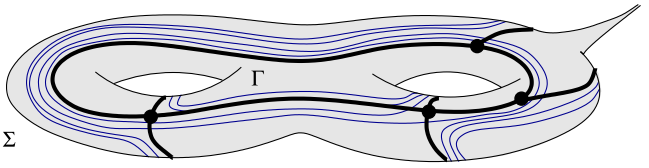
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Definition 2. A curve γ is said to be *carried* by a train track-like graph Γ if:

(i) when γ is closed, it is freely homotopic to a closed admissible path of Γ;

(ii) when γ is two-way infinite, it is uniformly homotopic (i.e. the homotopy is uniformly continuous) to an admissible path of Γ;

(iii) when γ is an arc with its endpoints on Γ's vertices in Σ, it is homotopic rel endpoints to a finite admissible path of Γ.



A uniform homotopy between two curves γ_1 and γ_2 means they have preimage components $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ by the universal covering map at bounded distance from each other in $\mathbb{H}^2$. In particular, if $\tilde{\gamma_1}$ has two endpoints on the boundary of $\mathbb{H}^2$, then $\tilde{\gamma_2}$ has the same endpoints, and both correspond to the same geodesic in $\mathbb{H}^2$. Here, homotopies between curves, including isotopies between simple ones, are always understood to be uniformly continuous maps, thus always preserving these endpoints. Now, to ensure that every closed or two-way infinite curve carried by a graph Γ has one of its preimage components (hence all) with two distinct endpoints on the boundary of $\mathbb{H}^2$, the embeddings of Γ are restricted by forbidding two types of regions in $\Sigma \backslash \Gamma$: disks bounded by a graph cycle, and disks with one puncture bounded by a graph cycle. When a train track-like graph Γ satisfies these constraints we call it simply a *(lamination) carrier graph*. It is then known that for any curve γ carried by a carrier graph, $\tilde{\gamma}$ is always uniformly homotopic in $\mathbb{H}^2$ to its corresponding geodesic [**52**]. Definition 2 can then be extended as follows: a leaf of a lamination is *carried* by a carrier graph Γ, if one of its curve realizations is carried by Γ. A lamination is *carried* by Γ if all its leaves are carried by Γ (this is similar to the notion of *weakly carried* in [**52**]).

THEOREM 2.1.1. [**49**] *Every geodesic lamination $\mathcal{L}$ in an orientable hyperbolic surface of finite type Σ is carried by some carrier graph Γ embedded in Σ.*

Now, here is an alternative, metric independent definition of the notion of lamination (also contained in Thurston's work [**49**]).

Definition 3. A *lamination* $\mathcal{L}$ on a surface Σ is a non-empty subset of Σ forming a union of pairwise disjoint simple curves, all closed or two-way infinite, such that: the curves are pairwise non-homotopic, they are all carried by a common carrier graph Γ and contained in a regular neighborhood of it, and $\mathcal{L}$ is maximal with respect to inclusion for this Γ and remains so after any isotopies applied to the curves.

A *leaf* with respect to a lamination $\mathcal{L}$ as in Definition 3 is a carried curve of $\mathcal{L}$ up to parametrization. All the other notions defined for geodesic laminations as in Definition 1 are defined in the same way for laminations as in Definition 3. Moreover, these two definitions give rise to the same laminations up to isotopy. Indeed, given a carrier graph Γ, every set of curves it carries can be deformed into a set of geodesics. Also, any geodesic γ

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in the closure of such a set is necessarily carried by Γ [**52**], thus if maximality is assumed, γ is already in this set. Conversely, given a geodesic lamination $\mathcal{L}$, to prove Theorem 2.1.1 Thurston builds a family of carrier graphs Γ_ε, all carrying $\mathcal{L}$, obtained from the open ε-neighborhoods of $\mathcal{L}$ in Σ [**49**, §8.9]. At most finitely many geodesics can be carried by Γ_ε while missing $\mathcal{L}$, since the lifts of these geodesics must share their endpoints on the boundary of $\mathbb{H}^2$ with the lifts of $\mathcal{L}$'s boundary leaves. But then, since $\mathcal{L}$ is closed, there is an $\varepsilon_0 > 0$ such that on each of these geodesics (if any) there is a point at distance at least ε_0 from $\mathcal{L}$, so according to [**49**, Corollary 8.9.3], as soon as $\varepsilon < \varepsilon_0$ no such geodesic can be carried by Γ_ε. Thus, $\mathcal{L}$ is maximal relative to Γ_ε if ε is small enough.

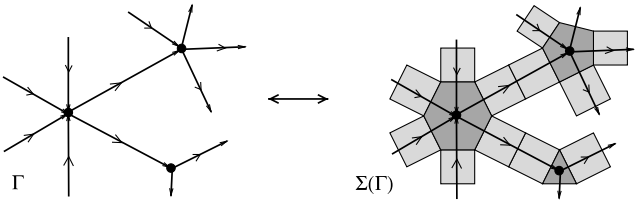
Now, this paper focuses on coding laminations into shifts by using labeled carrier graphs. This particular point of view upon lamination theory where carrying is central allows one to consider general carrier graphs, making use of the properties below.

- *Directedness*. For a carrier graph Γ, if one local orientation—out of the possible two—can be fixed at each vertex, so that each edge of Γ has an outgoing and an incoming end, then Γ becomes a usual *directed graph*. A path is *admissible* in a directed graph if it complies with these fixed local orientations. In this paper, a carrier graph is always assumed to be directed, without loss of generality: if Γ is not directed, we consider its directed double cover, to which admissible paths lift, reflecting the fact that an admissible path may visit an edge in both directions [**49**, §9.5]. The initial embedding surface Σ is then not preserved, but for the symbolic descriptions based on carrying as studied here, this is not constraining (we state this point more precisely further below).
- *Coherence*. A directed graph Γ is said to be *coherently embedded* (*coherent* for short) in Σ if all the incoming edges incident with any vertex are consecutive around it, hence the outgoing edges are too. By the above definitions, directed carrier graphs are coherent graphs. However, carrying a lamination $\mathcal{L}$ mainly relies on the notion of admissible path, and thus, can be readily extended to directed non-coherent graphs. The symbolic descriptions studied here will thus include the use of non-coherent graphs.
- *Maximality*. A lamination $\mathcal{L}$ can be carried by a graph Γ without the maximality condition of Definition 3, but when this condition holds, $\mathcal{L}$ is said to be *maximal* relative *to* Γ. Definition 3 just says that a lamination is maximal relative to at least one carrier graph.
- *Freeness*. A carrier graph Γ is said to be *free* if $\Sigma \backslash \Gamma$ contains no annulus bounded by a pair of closed admissible paths, and no disk bounded by a pair of admissible paths (if Γ is coherent, these additional conditions over $\Sigma \backslash \Gamma$ make Γ similar to a usual train track [**41, 52**]). Freeness means that each curve carried by Γ is homotopic to a unique admissible path of Γ. In this paper, carrier graphs are always free.
- *Fullness*. A carrier graph Γ is said to *fully carry* a lamination $\mathcal{L}$ if each edge of Γ is visited by some of the admissible paths associated with the carrying of $\mathcal{L}$. In this paper, carrying is always assumed to be full.
- *No punctures at vertices*. A carrier graph Γ is here always assumed to have no vertices at punctures, so that every vertex of Γ has admissible paths going through it,

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and has degree > 1; allowing vertices at punctures does not add new interesting combinatorial behaviors, while inducing technical intricacies.

Non-uniqueness of the embedding surfaces. As just said above, the embedding surfaces are not crucial in this study. Only the lamination structure captured by the carrier graphs is. From this point of view, if $\mathcal{L}$ is a lamination in Σ carried by a graph Γ, it is possible e.g. to add handles and punctures to Σ in the complement of any regular neighborhood N of Γ containing $\mathcal{L}$, without altering the way $\mathcal{L}$ is carried. In order to deal with this flexibility, instead of carrier graphs embedded in specific surfaces, we will use *ribbon graphs* (or *fat graphs*), i.e. graphs endowed with cyclic orderings on the sets of edges incident with each of their vertices. More precisely, a ribbon graph Γ is defined by (V, H, h, i, ξ), where V is a set of vertices; H is a set of half-edges; $h : H \to H$ is an involution without fixed points, which exchanges the pairs of half-edges, thus inducing a set E of full edges for Γ; $i : H \to V$ is an incidence map, which indicates the vertex of Γ each half-edge is incident with; ξ is a permutation on H defined as the product of the cyclic orderings defined on each subset $i^{-1}(v)$, with $v \in V$. A ribbon graph is *directed* if its full edges are oriented, and *coherent* if the cyclic permutation of each $i^{-1}(v)$ makes all the incoming (equivalently outgoing) edges consecutive around v. To a ribbon graph $\Gamma = (V, H, h, i, \xi)$, one associates a compact oriented surface with boundary, unique up to isometry, called the *ribbon graph surface* $\Sigma(\Gamma)$, by replacing vertices with polygons, and edges with oriented rectangles (the 'ribbons'), gluing them together according to the ribbon graph structure. More precisely, to each vertex in V with degree $d > 2$, one associates an oriented Euclidean regular polygon with d sides of, say, length 1; to the other vertices in V and also to each half-edge in H, one associates an oriented Euclidean square of side length 1. Next, all these polygons are glued together according to the patterns given by h, i and ξ.



A tame embedding of Γ in $\Sigma(\Gamma)$ is obtained by putting each vertex $v \in V$ in its corresponding polygon, joining these in the obvious way; $\Sigma(\Gamma)$ itself becomes a regular neighborhood of Γ.

Construction 2.1.2. (Hyperbolic surfaces from carrier ribbon graphs) Consider a ribbon graph $\Gamma = (V, H, h, i, \xi)$ and its ribbon graph surface $\Sigma(\Gamma)$. Assuming that Γ is used to carry laminations, we define the following associated hyperbolic surface: first, we cap off the boundary components of $\Sigma(\Gamma)$ with disks to form a surface Σ without boundary, and if necessary, these disks are once or twice punctured to ensure that Γ is a free carrier graph in Σ. This surface Σ has Euler–Poincaré characteristic $\chi(\Sigma) = f - |H|/2 + |V| - m$, where $|\cdot|$ denotes the cardinality of a set, f denotes the number of boundary components of $\Sigma(\Gamma)$, and $m \geq 0$ is the number of punctures. If $\chi(\Sigma) \geq 0$, we add more punctures

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until $\chi(\Sigma)$ becomes negative, so that Σ admits a hyperbolic structure—the classification of closed orientable surfaces tells us that no more than three additional punctures are needed here. The resulting surface is denoted by $\Sigma_0(\Gamma)$, and we call it the *standard surface of* Γ.

Thus, to define or speak about a carried lamination $\mathcal{L}$ we only need a carrier ribbon graph Γ: the lamination $\mathcal{L}$ exists then in $\Sigma(\Gamma)$, itself embedded in $\Sigma_0(\Gamma)$. Moreover, note that if in Construction 2.1.2 any disk used to cap off $\Sigma(\Gamma)$ is replaced by a connected sum of this disk with any kind of surface of finite type, the lamination $\mathcal{L}$ also exists in $\Sigma(\Gamma)$. From now on, if Γ is a carrier graph, we implicitly consider it as a ribbon graph when necessary.

Non-uniqueness of carrier graphs. No univocal relationship exists between laminations and their carrier graphs. In this respect, we will define several graph moves in the paper to transform a graph while preserving the carrying of its carried laminations. One significant instance of this non-uniqueness is that every minimal lamination can be carried by a graph made of a single vertex and $m \geq 1$ edges, called a *bouquet of circles*. There are indeed relationships between laminations and coherent bouquets of circles, which are described via *interval exchange transformations* (*iets* for short) [**26, 33, 34**]; an iet is an orientation-preserving and piecewise isometric map $T : I \to I$, where $I = [0, 1)$, whose effect is to permute a finite number of semi-open subintervals $I_1, \ldots, I_m$ partitioning I. More precisely, T is defined by (λ, π), where $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a positive length vector, whose entries are the I_i's lengths, and π is a permutation of $\{1, \ldots, m\}$, and the effect of T is to concatenate in its image the I_i in such a way that the length vector becomes $(\lambda_{\pi^{-1}(1)}, \ldots, \lambda_{\pi^{-1}(m)})$.

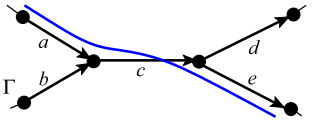
- *Iets* $\to$ *Laminations carried by coherent bouquets*. Let T be an iet over m intervals defined by (λ, π). Let $\mathcal{F}$ be the *suspension flow* of T [**33**]. Let Γ be a coherent bouquet of m circles, such that π induces the ribbon structure of Γ, where ξ is the cyclic permutation on the $2m$ half-edges obtained by concatenating the outgoing ones in the order $(1, \ldots, n)$ to the incoming ones in the order $(\pi^{-1}(n), \ldots, \pi^{-1}(1))$. Then $\mathcal{F}$ can be embedded in $\Sigma(\Gamma)$, so that each trajectory of $\mathcal{F}$ is carried by Γ. Fixing an embedding hyperbolic surface, like the standard surface $\Sigma_0(\Gamma)$, each of these trajectories has one geodesic homotopic to it. The closure of this union of carried geodesics is a geodesic lamination $\mathcal{L}$ associated with T (see the details in [**9**]), and $\mathcal{L}$ is carried by Γ. Moreover, the suspension flow leaves invariant a measure coming from λ on the underlying surface of $\mathcal{F}$, inducing in turn a transverse measure for $\mathcal{L}$, with $\mathcal{L}$ as support.
- *Laminations carried by coherent bouquets* $\to$ *Iets*. A measured lamination $\mathcal{L}$ carried by a coherent ribbon bouquet of circles Γ can be associated with an area-preserving flow by replacing the edges of Γ with foliated rectangles [**10, 23, 41**] (see also Construction 5.2.1 below). Then this flow can be seen as the suspension $\mathcal{F}$ of the iet (λ, π), where λ is given by the invariant measure on $\mathcal{F}$, and π is determined by the permutation ξ of Γ [**9**].
- *Minimal laminations* $\to$ *Laminations carried by coherent bouquets*. Every oriented measured lamination can be associated with a measure preserving flow $\mathcal{F}$, and the

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first-return map, or *Poincaré map*, on any transversal to $\mathcal{F}$ essentially gives an iet [**33, 34**]. Since every minimal lamination is measurable, this is how, given such a lamination $\mathcal{L}$, we can find coherent bouquets of circles carrying $\mathcal{L}$ [**27**] (see also §4.4 below).

2.2. *Coding laminations.* Let A be a finite alphabet. Let A^* denote the set of finite words over A. Let A^ω (respectively ${}^\omega A$) denote the set of right (respectively left) one-way infinite words, and let ${}^\omega A^\omega$ denote the set of two-way infinite words. A *language* is a subset of $A^* \cup A^\omega \cup {}^\omega A \cup {}^\omega A^\omega$. A *factor* (or *sub-block*) of a word w is a finite word u such that $w = w'uw''$, with w', w'' possibly empty, and where $w'uw''$ denotes the concatenation of the words w', u and w. The set of all the distinct factors of a word w is denoted by Fact_w, and if L is a language, $\mathrm{Fact}_L = \bigcup_{w\in L} \mathrm{Fact}_w$.

Following Definition 3, a lamination $\mathcal{L}$ carried by Γ can be symbolically represented by a language of two-way infinite words. First, here, a carrier graph Γ is said to be *labeled* by an alphabet A if it is endowed with a bijection from its set of edges to A. Every graph in this paper is assumed to be labeled. Next, the *label* of an admissible path η of Γ is the word obtained by concatenating the letters in A labeling the edges whose sequence forms η. Then, if γ is a curve carried by Γ, and if its homotopic path η in Γ is unique up to re-parametrization, Γ is said to *code* γ by the label of η. The label of η is the *coding* of γ by Γ. For instance in the following figure, the coding of the drawn piece of curve is 'ace'.



By convention, the coding of a closed curve γ is the periodic word ${}^\omega u^\omega$, where u is the label of the closed path in Γ freely homotopic to γ. A coding of a leaf is the coding of any of its curve realizations. If Γ codes every leaf of a lamination $\mathcal{L}$, then Γ is said to *code* $\mathcal{L}$.

Definition 4. A *lamination language* is a set of two-way infinite words in ${}^\omega A^\omega$ which corresponds to the codings of all the curve realizations of the leaves in a lamination $\mathcal{L}$, when $\mathcal{L}$ is coded by a directed carrier graph Γ labeled by A. A *lamination word* is a word in a lamination language.

When the carrying is full, every letter of A is used to code $\mathcal{L}$. If Γ is free, then Γ codes all the possible individual leaves it carries, hence also every lamination it carries. When Γ is a ribbon graph, Γ is always free in its standard surface $\Sigma_0(\Gamma)$ (see Construction 2.1.2).

A topology is given to ${}^\omega A^\omega$ by the *Cantor metric*: let $\ldots w_{-1}w_0w_1\ldots$ and $\ldots w'_{-1}w'_0w'_1\ldots$ be two words in ${}^\omega A^\omega$, then their distance is 0 if they are equal, and 2^{-k} if they are not, where $k \geq 0$ is the smallest integer for which $w_k \neq w'_k$ or $w_{-k} \neq w'_{-k}$. The *shift map* σ on ${}^\omega A^\omega$ sends $\ldots w_{-1}w_0w_1\ldots$ to $\ldots w'_{-1}w'_0w'_1\ldots$ where $w'_i = w_{i+1}$ for $i \in \mathbb{Z}$. A language L in ${}^\omega A^\omega$ is *shift-invariant* if $\sigma(L) = L$, and a closed shift-invariant language is called a *shift*. Lamination languages are shifts. Indeed, by Definition 4, a lamination language L includes the coding of every realization of each leaf of the lamination $\mathcal{L}$ it codes, so L is shift-invariant. Moreover L is closed, since first, up to

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deformation, $\mathcal{L}$ can be assumed to be a geodesic lamination (see §2.1); second, for any sequence of words $\{w^{(i)}\}$ in L converging to w in $\overline{L}$, where $\overline{L}$ denotes the topological closure of L in ${}^{\omega}A^{\omega}$, there is a geodesic γ obtained as the limit of the geodesics coded by the $w^{(i)}$. Then, w is the coding of γ, and since $\mathcal{L}$ is closed, $\gamma \in \mathcal{L}$, hence $w \in L$. We can also sometimes construct laminations from shifts.

LEMMA 2.2.1. [**31**] *Let Γ be a coherent carrier graph labeled by A, embedded in a surface Σ. Let L be a shift in ${}^{\omega}A^{\omega}$, and let $\mathcal{C}_L$ be a set of finite simple curves in Σ carried by Γ such that:*

(i) *each factor in* Fact_L *is the coding of a curve in $\mathcal{C}_L$, and vice-versa;*

(ii) *up to moving their extremities in an arbitrarily small neighborhood of Γ's vertices, all the curves in $\mathcal{C}_L$ can be made simultaneously pairwise disjoint in Σ.*

Then, L is the coding of a closed set $\mathcal{C}$ of simple curves, pairwise disjoint, pairwise non-homotopic, such that $\mathcal{C}$ is a lamination, and thus L is a lamination language.

A consequence of Lemma 2.2.1 is that, if w is a lamination word, then the shift $\overline{\{\sigma^k(w)\}}_{k\in\mathbb{Z}}$ is a lamination language. Another consequence is that every lamination in the sense of Definition 3 can be assumed to form a closed set in any regular neighborhood it lies in, up to moving its curves by isotopy. A shift-invariant language L is said to be *minimal* if the orbit set $\{\sigma^k(w)\}_{k\in\mathbb{Z}}$ of each word $w \in L$ is dense in L, i.e. $\overline{\{\sigma^k(w)\}}_{k\in\mathbb{Z}} = L$. Minimal laminations are coded by minimal languages. Moreover, every word w of a minimal language L is such that $\mathrm{Fact}_w = \mathrm{Fact}_L$, and w itself is said to be *minimal* (or *uniformly recurrent*) [**36**], i.e. every $u \in \mathrm{Fact}_w$ occurs infinitely often with bounded gaps in w. Also, let w' be any one-way right infinite half-word in L, and pad it to the left with a dummy letter $z \notin A$ to transform it into ${}^{\omega}zw'$. Then, by minimality, the shift made of all the words in $\overline{\{\sigma^k({}^{\omega}zw')\}}_{k\in\mathbb{Z}}$ where z does not occur, i.e. the *two-way infinite boundary language* of $\mathrm{Fact}_{w'}$ [**38**], is L too. Here are two additional facts.

- *Lamination languages are not finite type shifts*. Shifts can be defined by languages of infinite words whose factors do not fall into a prescribed set. When this set is finite, shifts are said to be of *finite type*. A way of describing shifts of finite type is to consider the sets of all the infinite admissible path labels of directed labeled graphs, i.e., *edge shifts* [**28**]. Lamination languages look like edge shifts, being also shifts based on admissible path codings, but they are generally shifts of non-finite type. Words in a lamination language represent simple and pairwise disjoint curves on a surface, and these geometric constraints imply that lamination languages are generally far from being all the possible admissible path labels of a graph. As a matter of fact, all non-trivial minimal shifts of finite type have positive entropy [**28**], whereas all lamination languages have zero entropy (see Theorem B below). In this respect, lamination languages should also be contrasted with *symbolic geodesic flows*, whose associated shifts are of finite type [**47, 48**], while their coding scheme (the *Morse coding*) can be interpreted through carrier graphs too.
- *Classic iet codings are lamination languages*. With the discussion at the end of §2.1 in mind, we define an *iet language* as the coding of a lamination carried by a coherent bouquet of circles. Moreover, iet languages correspond to the standard way of coding iet iterations [**26**]. Indeed, let $T : I \to I$ be an iet, and let $\mathrm{cod} : I \to A$

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assign a distinct letter of A to each subinterval I_i of the partition of I. Then for every $x \in I$, the *orbit* of x is $\{T^k(x)\}_{k\in\mathbb{Z}}$, and its *symbolic orbit* is the word $\ldots \mathrm{cod}(T^{-1}(x))\mathrm{cod}(x)\mathrm{cod}(T(x))\mathrm{cod}(T^2(x))\ldots$. In the corresponding bouquet of circles labeled by A, each edge is associated with one subinterval I_i of T, so that the corresponding iet language is equal to the closure of all the symbolic orbits of T. Thus, lamination languages include the usual way of coding symbolic orbits of iets.

2.3. *Factor combinatorics: complexity and Rauzy graphs.* The set of all the distinct length-n factors of a word w is denoted by $\mathrm{Fact}_w(n)$, and if L is a language, $\mathrm{Fact}_L(n) = \bigcup_{w\in L} \mathrm{Fact}_w(n)$. The *(factor) complexity* of an infinite word w [**1, 13, 36, 37**] is the function $p_w : \mathbb{N}^* \to \mathbb{N}^*$, where $p_w(n) = |\mathrm{Fact}_w(n)|$, i.e. the cardinality of $\mathrm{Fact}_w(n)$. The complexity of a language L of infinite words is $p_L(n) = |\mathrm{Fact}_L(n)|$. For a minimal language L, since all the words $w \in L$ have the same sets of factors, then $p_w \equiv p_L$. A language of finite words is said to be *prolongable* if its words can be prolongated at least in one way to the right and to the left, such that these prolongations remain in the language. If L is a language made of two-way infinite words, Fact_L is prolongable. If w is a one-way minimal infinite word, Fact_w is also prolongable. A length-n factor $u \in \mathrm{Fact}_L(n)$ is said to be *right special* (respectively *left special*) if u has at least two right (respectively left) letter prolongations in Fact_L, that is, u is the prefix (respectively suffix) of at least two distinct length-$(n+1)$ factors in $\mathrm{Fact}_L(n+1)$. The *right* (respectively *left*) *prolongation order* $m_r(u)$ (respectively $m_l(u)$) of a factor u is the number of its right (respectively left) prolongations minus 1. For every word u in a prolongable language, we then have $m_r(u) \geq 0$, $m_l(u) \geq 0$.

Example 2.3.1. Let L be such that $\mathrm{Fact}_L(5) = \{aabaa, abaab, ababa, baaba, babaa, aabab\}$. Since $\mathrm{Fact}_{\mathrm{Fact}_L(n)}(n-k) = \mathrm{Fact}_L(n-k)$, then $\mathrm{Fact}_L(4) = \{aaba, abaa, abab, baab, baba\}$, $\mathrm{Fact}_L(3) = \{aab, aba, bab, baa\}$, $\mathrm{Fact}_L(2) = \{aa, ab, ba\}$. Then, the words ba, aba, $aaba$ are right special with $m_r(\cdot) = 1$, and ab, aba, $abaa$ are left special with $m_l(\cdot) = 1$.

When Fact_L is prolongable, the orders of either all the left or all the right special factors of each length determine the first difference complexity function [**11, 13**]:

$$p'_L(n) = p_L(n+1) - p_L(n) = \sum_{u\in\mathrm{Fact}_L(n)} m_r(u) = \sum_{u\in\mathrm{Fact}_L(n)} m_l(u).$$

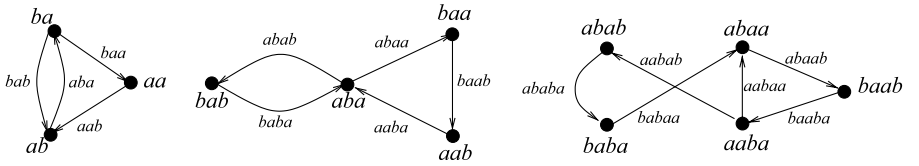
A factor which is both right and left special is said to be *bispecial*. In Example 2.3.1, aba is the only length-3 bispecial factor among the listed special factors. The *prolongation order* of a factor u is defined by $m(u) = |\mathrm{Fact}_L \cap AuA| - (m_l(u) + m_r(u) + 1)$, where AuA denotes all the prolongations of u by one letter to the left and one letter to the right. For instance, in Example 2.3.1, $m(aba) = 0$. The orders of all the bispecial factors determine the second difference complexity function (the other factors contribute zero to the sum):

$$p''_L(n) = p'_L(n+1) - p'_L(n) = \sum_{u\in\mathrm{Fact}_L(n)} m(u).$$

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In order to study the sets of factors, there exists a classic graph-based representation defined as follows: for a language L (similarly for a word w) and $n \geq 0$, the *nth Rauzy graph* [**4, 44**] (or *nth higher edge graph* [**28**]) is the directed labeled graph $\Gamma_{L,n} = (V_n, E_n)$, where the set of vertices is $V_n = \mathrm{Fact}_L(n)$, and where there is an edge in E_n between two vertices $au, ub \in \mathrm{Fact}_L(n)$ with $a, b \in A$, if $aub \in \mathrm{Fact}_L(n+1)$, and aub is the label of this edge. Note that for $n > 0$, we have $|V_n| = p_L(n)$ and $|E_n| = p_L(n+1)$. For $n = 0$, $\Gamma_{L,0}$ is a bouquet of m circles where $m = |\mathrm{Fact}_L(1)| = |A|$. The *Rauzy graph sequence* of a language L is defined as the sequence $\{\Gamma_{L,n}\}_{n\geq 0}$.

Example 2.3.2. Consider the factor sets $\mathrm{Fact}_L(2)$, $\mathrm{Fact}_L(3)$ and $\mathrm{Fact}_L(4)$ in Example 2.3.1. Here are the three corresponding Rauzy graphs $\Gamma_{L,2}$, $\Gamma_{L,3}$ and $\Gamma_{L,4}$:



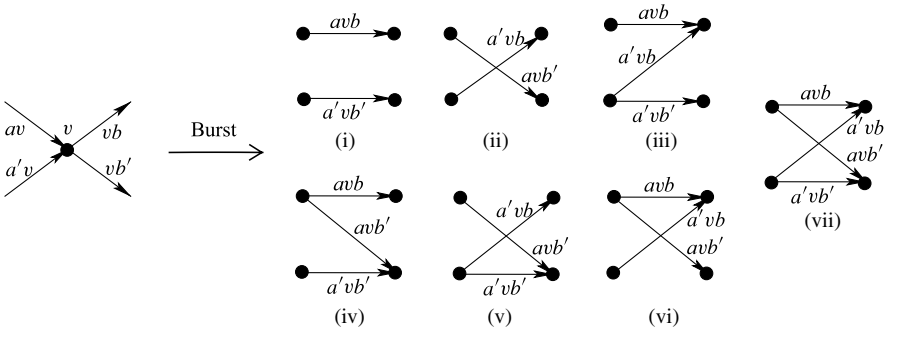
Construction 2.3.3. (Rauzy graph sequences) In $\{\Gamma_{L,n}\}_{n\geq 0}$, the $(n+1)$th Rauzy graph $\Gamma_{L,n+1} = (V_{n+1}, E_{n+1})$ is obtained from $\Gamma_{L,n} = (V_n, E_n)$ as follows: first, we build up the *line graph* $LG(\Gamma_{L,n}) = (V_{LG}, E_{LG})$ where $V_{LG} = E_n$, and where E_{LG} is such that there is an edge between e_1 and e_2, if e_1e_2 corresponds to a length-2 admissible path in $\Gamma_{L,n}$. Next, we put $V_{n+1} = V_{LG}$, and E_{n+1} is obtained by pruning off E_{LG} so that only the edges corresponding to $\mathrm{Fact}_L(n+2)$ remain.

For every $n \geq 0$, a vertex v in $\Gamma_{L,n}$ whose indegree is $\partial^-(v) \geq 2$ (respectively, whose outdegree is $\partial^+(v) \geq 2$) corresponds to a length-n left (respectively right) special factor, and in terms of the prolongation orders, $\partial^-(v) = m_l(v) + 1$ and $\partial^+(v) = m_r(v) + 1$ (where v also stands for the factor it corresponds to). A vertex v such that $\partial^-(v) \geq 2$ and $\partial^+(v) \geq 2$ corresponds to a bispecial factor, and we call it a *bispecial vertex*. Now, according to Construction 2.3.3, each vertex v of $\Gamma_{L,n}$ gives rise in $LG(\Gamma_{L,n})$ to a small graph, whose vertices correspond to the edges of $\Gamma_{L,n}$ incident with v, and whose edges correspond to all the length-2 admissible paths going through v. This graph is pruned off by keeping only the edges corresponding to $\mathrm{Fact}_L \cap AvA$, and the resulting subgraph in $\Gamma_{L,n+1}$ is called the *burst* of v. Notice then that if Fact_L is prolongable, and if v is a vertex of $\Gamma_{L,n}$ such that $\partial^-(v) = 1$ or $\partial^+(v) = 1$, the burst depends only on the definition of $LG(\cdot)$, whereas if v is bispecial, the pruning off by $\mathrm{Fact}_L \cap AvA$ determines a specific burst. Indeed, the inclusion of $\Gamma_{L,n+1}$ in $LG(\Gamma_{L,n})$, and more generally the evolution of a Rauzy graph sequence $\{\Gamma_{L,n}\}_{n\geq 0}$ is determined by the bursts of the bispecial vertices [**4, 44, 45**]. In order to represent a burst of a vertex v in a convenient way, we associate a bipartite graph $\Gamma_v = (V_{l,v} \sqcup V_{r,v}, E_v)$ to it, where $V_{l,v}$ (respectively $V_{r,v}$) corresponds to the incoming (respectively outgoing) half-edges incident with v, labeled as their corresponding edges, and where E_v corresponds to the factors in $\mathrm{Fact}_L \cap AvA$, the edge orientations being from $V_{L,v}$ to $V_{r,v}$. Compared to the burst, the graph Γ_v thus relies on the incident half-edges instead of the incident edges at v. Each Γ_v is then isomorphic to its burst in $\Gamma_{L,n+1}$, except when loops are incident with v; for each such loop, an identification of the two distinct vertices of Γ_v corresponding to its half-edges

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yields an isomorphic graph. In any case, Γ_v has the same edge cardinality as the burst it corresponds to.

Example 2.3.4. Let v be a length-n bispecial factor with $m_l(v) = m_r(v) = 1$, i.e. $\partial^-(v) = \partial^+(v) = 2$ as a vertex in $\Gamma_{L,n}$. In a prolongable context, there are seven possible distinct bursts for v when going from $\Gamma_{L,n}$ to $\Gamma_{L,n+1}$, i.e. seven corresponding bipartite graphs Γ_v since there are at most four possible factors, avb, $a'vb$, avb', $a'vb'$ in $\mathrm{Fact}_L(n+2)$, with $a, a', b, b' \in A$, not necessarily distinct (cases (i), (ii) correspond to $m(v) = -1$, cases (iii)–(vi) to $m(v) = 0$, case (vii) to $m(v) = 1$):



The bursts are closely related to the prolongation orders of the factors producing them, hence also to the second difference complexity function p''. The number of edges generated by the burst of a vertex v in $\Gamma_{L,n}$, i.e. $|\mathrm{Fact}_L \cap AvA|$, is at least $k = \max(m_l(v) + 1, m_r(v) + 1)$ since Fact_L is prolongable, and is at most $k' = (m_l(v) + 1)(m_r(v) + 1)$, i.e. the number of edges of the associated bipartite graph Γ_v at v, when Γ_v is complete. The number $m_l(v) + m_r(v) + 1$ in $[k, k']$, implying $m(v) = 0$, corresponds to the number of edges for Γ_v such that the sum of $m_r(\cdot)$ over the left vertices of Γ_v and the sum of $m_l(\cdot)$ over the right vertices of Γ_v are preserved, respectively equal to $m_r(v)$ and $m_l(v)$, so that $p''(n)$ remains unchanged.

A *recoding by a length-n sliding window* of a word $w = w_1 \dots w_h$, $h \geq n$, is based on a map $\kappa : \mathrm{Fact}_w(n) \to B$, where B is an alphabet, and it consists in rewriting w into $w'_1 \dots w'_{h-n+1}$ where each $w'_i = \kappa(w_i \dots w_{i+n-1})$, $1 \leq i \leq h - n + 1$. This recoding scheme is extended in a straightforward way to infinite words and languages. It is a continuous map on ${}^{\omega}A^{\omega}$. In the context of Rauzy graphs, we may assume that $\mathrm{Fact}_L(n)$ stands itself for the recoding alphabet B. As a consequence, every admissible path label of $\Gamma_{L,n}$ can be seen as a recoding by a length-$(n+1)$ sliding window over $\mathrm{Fact}_L(n+1)$ of a word over A. This kind of recoding for two-way and right (respectively left) one-way infinite words has also an inverse which is given by projecting each occurrence of $\mathrm{Fact}_L(n+1)$ to its first (respectively last) letter.

3. *Lamination languages and Rauzy graphs*

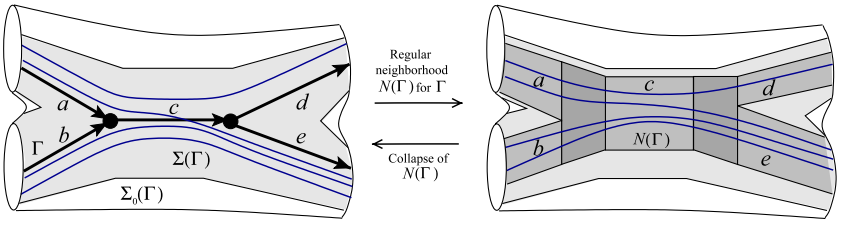
3.1. *Lamination-based slittings.*

In order to establish relationships between lamination languages and Rauzy graphs, we now introduce classic notions of train track theory in our context.

Construction 3.1.1. (Regular neighborhoods as Euclidean complexes) Let Γ be a coherent ribbon graph. To each edge of Γ we associate a closed Euclidean rectangle, called

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an *edge-rectangle*, and to each vertex of Γ we associate a closed trapezium, called a *vertex-trapezium*. Edge-rectangle heights are arbitrarily chosen, and the lengths of vertex-trapezium bases are respectively set to the sum of their corresponding incoming edge-rectangle heights and to the sum of their outgoing ones, where edge-rectangles are matched along these bases according to the pattern induced by the ribbon structure of Γ. We thus obtain a Euclidean complex $N(\Gamma)$ embeddable into the ribbon graph surface $\Sigma(\Gamma)$, where each vertex-trapezium of $N(\Gamma)$ is isometrically embedded in its corresponding polygon of $\Sigma(\Gamma)$, the latter being embeddable into the standard hyperbolic surface $\Sigma_0(\Gamma)$. Hence, $\Gamma \subset N(\Gamma) \subset \Sigma(\Gamma) \subset \Sigma_0(\Gamma)$. The edge-rectangles of $N(\Gamma)$ are labeled after the labels of Γ.



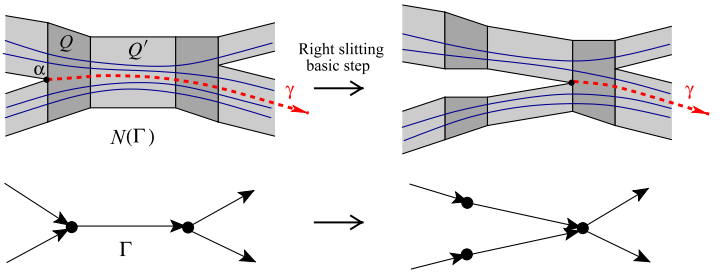
Since Γ is directed, we can define in a consistent manner right, left, top and bottom sides of each quadrilateral making $N(\Gamma)$. These quadrilaterals can also be fibered with fibers parallel to their right and left sides, defining a fibration in all of $N(\Gamma)$. We can then recover Γ from $N(\Gamma)$: first by collapsing the fibers, next by contracting to one vertex each arc resulting from the collapsing of a vertex-trapezium. The resulting graph is embeddable in $N(\Gamma)$ in such a way that $N(\Gamma)$ becomes a regular neighborhood of it. This graph is called a *core graph* of $N(\Gamma)$. Any such graph is isomorphic to Γ (as a ribbon graph too), and any two core graphs are isotopic in $N(\Gamma)$. Thus, by abuse of language we refer to any such graph as *the* core graph of $N(\Gamma)$, and call it Γ too.

When Γ carries a lamination $\mathcal{L}$, given a regular neighborhood $N(\Gamma)$, the leaves of $\mathcal{L}$ can all be moved so as to be contained in $N(\Gamma)$. Moreover, using the Euclidean metric and the fibers, $\mathcal{L}$ can be further moved so that all its leaves become everywhere transverse to the fibers. For every $N(\Gamma)$ and every $\mathcal{L}$ carried by Γ, we will assume that $\mathcal{L} \subset N(\Gamma)$ with the preceding property. We say that a curve in $N(\Gamma)$ is *$\mathcal{L}$-avoiding* if it has no intersection with $\mathcal{L}$ and if it is everywhere transverse to all the fibers of $N(\Gamma)$. A *slitting curve* is an $\mathcal{L}$-avoiding curve starting at an intersection point between two edge-rectangles of $N(\Gamma)$, occurring in some side of a vertex-trapezium. If a slitting curve γ can reach another edge-rectangle intersection of $N(\Gamma)$ while preserving $\mathcal{L}$-avoidance, we force it to do so as soon as it can, and call γ a *finite slitting curve*. Because of this definition and because of the $\mathcal{L}$-avoidance, the sequence of quadrilaterals that are visited by a slitting curve γ is determined by $\mathcal{L}$. Note that there is no finite slitting curve if and only if $\mathcal{L}$ is maximal relative to Γ (see Proposition 3.3.3 below). Now, based on slitting curves, the following transformations of $N(\Gamma)$ are defined, and can be understood to take place in the ribbon graph surface $\Sigma(\Gamma)$, thus in a Euclidean context. A *right* (respectively *left*) *slitting basic step* $\mathrm{slit}_{\alpha,\mathcal{L}}$ of $N(\Gamma)$ along a slitting curve γ starting at an edge-rectangle intersection α occurring at the left (respectively right) side of a vertex-trapezium Q consists of:

(i) dividing Q along $\gamma \cap Q$ into two trapezia placed one above the other in $\Sigma(\Gamma)$ at distance $\epsilon > 0$;

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(ii) if γ does not end at an edge-rectangle intersection in the other side of Q, dividing the next edge-rectangle Q' that γ enters along $\gamma \cap Q'$ into two edge-rectangles, placed one above the other at a distance which monotonically decreases from ϵ to 0.



The result of $\mathrm{slit}_{\alpha,\mathcal{L}}(N(\Gamma))$ is again a Euclidean complex as in Construction 3.1.1, which does not depend on the choice of a specific slitting curve γ starting from α, or on the choice of ϵ, up to isotopy. A slitting basic step is thus also interpretable as a transformation of Γ (see the above figure): $\mathrm{slit}_{\alpha,\mathcal{L}}(\Gamma)$ is defined as the core graph of $\mathrm{slit}_{\alpha,\mathcal{L}}(N(\Gamma))$ in $\Sigma(\Gamma)$. A slitting basic step is similar to a usual *train track splitting* (also called *unzipping*) [**39, 41, 49**]. When step (ii) of the definition of a slitting basic step cannot be applied, that is, when it applies only to Q, then $\mathrm{slit}_{\alpha,\mathcal{L}}(\Gamma)$ is a *vertex cutting*. Note that if $\mathcal{L}$ is a measured lamination carried by a coherent bouquet of circles Γ, i.e. $\mathcal{L}$ corresponds to an iet T (see the end of §2.1), to have no finite slitting curve in $N(\Gamma)$ is equivalent to the *infinite distinct orbit condition* (*idoc* for short), i.e. the orbits of the separation points between the subintervals I_i of the partition of I by T are infinite and disjoint [**26**].

All the main properties of carrier graphs are preserved through a slitting basic step: let $\mathrm{slit}_{\alpha,\mathcal{L}}(\Gamma)$ be denoted by Γ'. According to the definition of a slitting curve, every leaf of $\mathcal{L}$ is still carried by Γ'. Conversely, every curve carried by Γ' is also carried by Γ, since the difference from $N(\Gamma')$ to $N(\Gamma)$ is made up of at most a rectangle and an adjacent triangle, both fibered. Hence, a slitting basic step also preserves maximality. According to the definition of a finite slitting curve, and because of the carrying being full, near its endpoints a slitting curve γ has at least one leaf above it and one leaf under it, and this remains true all along γ. Thus, a full carrying of $\mathcal{L}$ remains full after a slitting basic step. Coherence is also preserved since after a slitting basic step, all the leaves entering a vertex-trapezium of $N(\Gamma')$ by one side must exit by the opposite one. Finally, freeness is preserved in $\Sigma(\Gamma)$ and, in particular $\mathcal{L}$ is still coded by Γ'. Indeed, if two curves carried by Γ' are homotopic in $\Sigma(\Gamma)$, they are also carried by Γ and homotopic within $N(\Gamma)$, since by freeness they are homotopic to a unique path in Γ; but then, using again the difference from $N(\Gamma')$ to $N(\Gamma)$, they are also homotopic within $N(\Gamma')$, hence homotopic to a unique path in Γ'.

Thus, slitting basic steps can be iterated along slitting curves without losing any carrying property. An iteration of slitting basic steps which goes all along a finite slitting curve is called a *finite slitting* (it is similar to a *collision* in train track theory [**41**]). Slitting basic steps along distinct slitting curves can also be composed, since these curves are always separated by leaves of $\mathcal{L}$ when the carrying is full, thus being well-defined. Here, we will use the following specific compositions of slitting basic steps.

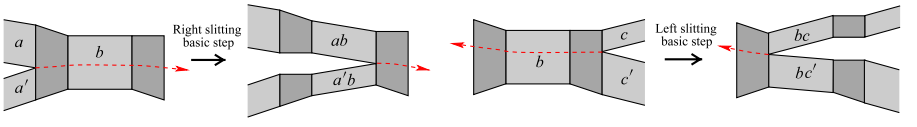
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Definition 5. Let Γ be a coherent graph with a regular neighborhood $N(\Gamma)$, and let $\mathcal{L}$ be a lamination carried by Γ. A *right* (respectively *left*) *slitting global step* $\mathrm{Slit}_{r,\mathcal{L}}$ (respectively $\mathrm{Slit}_{l,\mathcal{L}}$) of $N(\Gamma)$ is the result of applying all the possible right (respectively left) slitting basic steps to $N(\Gamma)$, i.e. $\mathrm{slit}_{\alpha_1,\mathcal{L}} \circ \cdots \circ \mathrm{slit}_{\alpha_n,\mathcal{L}}$, where the α_i are all the edge-rectangle intersections occurring to the left (respectively right) sides of the vertex-trapezia of $N(\Gamma)$.

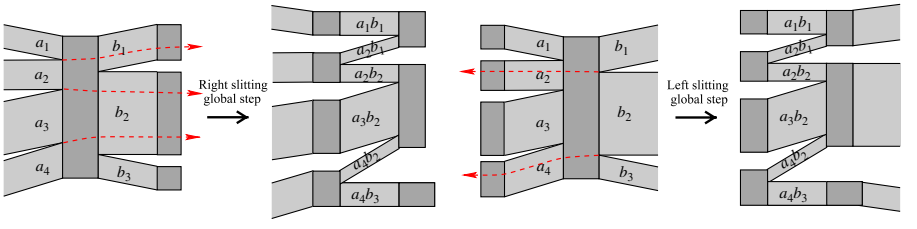
Accordingly, the result of $\mathrm{Slit}_{\cdot,\mathcal{L}}(\Gamma)$ is the core graph of $\mathrm{Slit}_{\cdot,\mathcal{L}}(N(\Gamma))$, still embedded in the ribbon graph surface $\Sigma(\Gamma)$.

3.2. *Slittings and embedded Rauzy graphs.* The rationale of Definition 5 is that applying slitting global steps to a carrier graph Γ is closely related to Construction 2.3.3 of Rauzy graphs. Slitting curves of $N(\Gamma)$ are indeed dividing lines for the way the leaves of a carried lamination $\mathcal{L}$ enter or exit quadrilaterals of $N(\Gamma)$, and as such, locally determine the carrying length-2 admissible paths. To capture this fact, we label $\mathrm{Slit}_{\cdot,\mathcal{L}}(N(\Gamma))$ as follows: we first assume that each edge-rectangle of $\mathrm{Slit}_{\cdot,\mathcal{L}}(N(\Gamma))$ keeps the label of the edge-rectangle of $N(\Gamma)$ it comes from before the slitting. Next, when a, b, c label consecutive edge-rectangles in $\mathrm{Slit}_{r,\mathcal{L}}(N(\Gamma))$ (respectively $\mathrm{Slit}_{l,\mathcal{L}}(N(\Gamma))$), the one labeled by b is relabeled by ab (respectively bc); this relabeling is unique since, after a global slitting step, the edge-rectangle labeled by b, either has a unique preceding (respectively succeeding) edge-rectangle labeled by a, or more than one but all labeled by a (respectively c). The core graph $\mathrm{Slit}_{\cdot,\mathcal{L}}(\Gamma)$ inherits its edge labeling from $\mathrm{Slit}_{\cdot,\mathcal{L}}(N(\Gamma))$'s edge-rectangles.

Example 3.2.1. The effect of left and right slitting basic steps on a trivalent configuration.



The local effect of slitting global steps on a more involved configuration.



LEMMA 3.2.2. *Let Γ be a coherent graph which codes a lamination $\mathcal{L}$ by a language L. Let Γ' denote* $\mathrm{Slit}_{r,\mathcal{L}}(\Gamma)$ *(or* $\mathrm{Slit}_{l,\mathcal{L}}(\Gamma)$*). Then $ab \in \mathrm{Fact}_L(2)$ if and only if ab occurs as an edge label in Γ'. Also, two edges ab and $a'b'$, in this order, make a length-2 path in Γ' if and only if $b = a'$.*

Proof. The carrying of $\mathcal{L}$ remains full after the slitting, so if ab occurs as a label of an edge of Γ', there are leaves of $\mathcal{L}$ carried along the corresponding length-2 path ab in Γ, hence $ab \in \mathrm{Fact}_L(2)$. Conversely, if $ab \in \mathrm{Fact}_L(2)$, then a and b are two consecutive edges in Γ used to carry $\mathcal{L}$. According to the labeling definition of Γ', ab must occur as an edge label of Γ'. The form of this labeling also gives the label constraint on consecutive edges. □

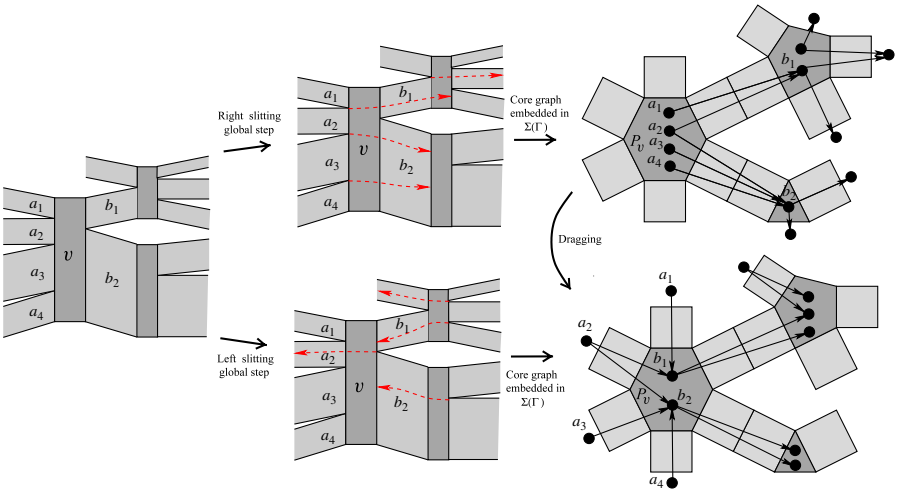
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Now, when slitting global steps are iterated, the labeling definition of the obtained graphs inductively applies using longer labels as follows: assuming first that each edge-rectangle of $\mathrm{Slit}^n_{\cdot,\mathcal{L}}(N(\Gamma))$ keeps the label of the edge-rectangle of $\mathrm{Slit}^{n-1}_{\cdot,\mathcal{L}}(N(\Gamma))$ it comes from, then when $u_a = a_0 \dots a_{n-1}$, $u_b = b_0 \dots b_{n-1}$, $u_c = c_0 \dots c_{n-1}$, with $a_i, b_i, c_i \in A$, label consecutive edge-rectangles of $\mathrm{Slit}^n_{r,\mathcal{L}}(N(\Gamma))$ (respectively $\mathrm{Slit}^n_{l,\mathcal{L}}(N(\Gamma))$), the edge-rectangle u_b is uniquely relabeled by $a_0 u_b$ (respectively $u_b c_{n-1}$). We can then generalize Lemma 3.2.2 (its proof is similar).

LEMMA 3.2.3. *Let Γ be a coherent graph which codes a lamination $\mathcal{L}$ by a language L. Let Γ' denote $\mathrm{Slit}^n_{r,\mathcal{L}}(\Gamma)$ (or $\mathrm{Slit}^n_{l,\mathcal{L}}(\Gamma)$) with $n > 0$. Then $u \in \mathrm{Fact}_L(n+1)$ if and only if u occurs as an edge label in Γ'. Also, two edges $u = a_0 \dots a_n$ and $u' = b_0 \dots b_n$, in this order, make a length-2 path in Γ' if and only if $a_1 \dots a_n = b_0 \dots b_{n-1}$.*

LEMMA 3.2.4. *Let Γ be a coherent graph which codes a lamination $\mathcal{L}$ by a language L. Then $\mathrm{Slit}_{r,\mathcal{L}}(\Gamma)$ is isotopic to $\mathrm{Slit}_{l,\mathcal{L}}(\Gamma)$, with the same labeling.*

Proof. We prove the property with respect to Lemma 3.2.2 only (it generalizes straightforwardly with Lemma 3.2.3). By this lemma, both graphs have the same set of edges labeled by $\mathrm{Fact}_L(2)$, linking the same set of vertices, i.e. the single letters of L. Moreover, the fact that ab, $a'b'$ are consecutive if and only if $b = a'$ does not depend on the direction of the global slitting steps, so the vertex adjacencies are the same. Thus, the graphs are isomorphic. They are also isotopic: let us consider the ribbon structure of Γ and its surface $\Sigma(\Gamma)$. For each vertex v of Γ, let P_v denote its corresponding polygon of $\Sigma(\Gamma)$. We then embed $\mathrm{Slit}_{r,\mathcal{L}}(\Gamma)$ in $\Sigma(\Gamma)$ with respect to Γ as follows. For each vertex v of Γ: (i) each edge a_i of Γ incoming at v has its corresponding vertex a_i of $\mathrm{Slit}_{r,\mathcal{L}}(\Gamma)$ placed in P_v; (ii) each edge b_j of Γ outgoing from v gives a set of edges $\{a_i b_j\}$ in $\mathrm{Slit}_{r,\mathcal{L}}(\Gamma)$, corresponding to the factors in $\mathrm{Fact}_L(2)$ of this form, each one going out of P_v, and linking a_i to b_j. We similarly embed $\mathrm{Slit}_{l,\mathcal{L}}(\Gamma)$ in $\Sigma(\Gamma)$. For each vertex v in Γ: (i) each edge b_j of Γ outgoing from v has its corresponding vertex b_j of $\mathrm{Slit}_{l,\mathcal{L}}(\Gamma)$ placed in P_v; (ii) each edge a_i of Γ incoming in v gives a set of edges $\{a_i b_j\}$ in $\mathrm{Slit}_{l,\mathcal{L}}(\Gamma)$, each one coming in to P_v, and linking a_i to b_j.



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According to the definition of a right slitting global step, for a vertex labeled b_j in $\mathrm{Slit}_{r,\mathcal{L}}(\Gamma)$, all the edges $a_i b_j$ incoming at b_j are such that the vertices a_i at their other end are necessarily in the same polygon P_v of $\Sigma(\Gamma)$. Thus, we can drag b_j along these $a_i b_j$ until b_j becomes a vertex in P_v (see the above figure). We drag in the same way all the vertices of $\mathrm{Slit}_{r,\mathcal{L}}(\Gamma)$. As a result, in each polygon of $\Sigma(\Gamma)$, each vertex b_j is now as prescribed by the embedding of $\mathrm{Slit}_{l,\mathcal{L}}(\Gamma)$. Moreover, the polygons of $\Sigma(\Gamma)$ contain all the vertices we expect them to contain. Indeed, if $a_i b_j$ is an edge in $\mathrm{Slit}_{r,\mathcal{L}}(\Gamma)$, it is also one in $\mathrm{Slit}_{l,\mathcal{L}}(\Gamma)$, since by Lemma 3.2.2, both sets of edges exactly correspond to $\mathrm{Fact}_L(2)$, therefore all the b_j corresponding to the edges of Γ outgoing from a vertex v are now in P_v. Also, there are no vertices other than the b_j since all the a_i originally in a polygon P_v have been dragged out of P_v. Thus all the $a_i b_j$ are incoming into P_v as prescribed for the embedding of $\mathrm{Slit}_{l,\mathcal{L}}(\Gamma)$. Now, dragging is an isotopy, hence the result. □

As a consequence, a slitting global step can be simply denoted by $\mathrm{Slit}_{\mathcal{L}}$ whenever carrier graphs are considered up to isotopy.

PROPOSITION 3.2.5. (Slitting global steps and Rauzy graphs) *Let Γ be a coherent graph which codes a lamination $\mathcal{L}$ by a language L. Then, for all $n > 0$, $\mathrm{Slit}^n_{\mathcal{L}}(\Gamma)$ is isomorphic to the nth order Rauzy graph $\Gamma_{L,n}$. Moreover, $\mathrm{Slit}^n_{\mathcal{L}}(\Gamma)$ can be coherently embedded into $\Sigma(\Gamma)$, and preserves the carrying properties of Γ with respect to $\mathcal{L}$.*

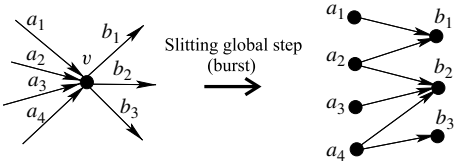
Proof. By Lemma 3.2.2, $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ has edges labeled by $\mathrm{Fact}_L(2)$, in such a way that ab and $a'b'$ are consecutive if and only if $b = a'$. These constraints are the same as the ones defining the first-order Rauzy graph $\Gamma_{L,1}$. According to Lemma 3.2.3, the same arguments hold inductively for all $\Gamma_{L,n}$, for all $n > 1$. Slitting global steps preserve carrying, maximality, fullness, coherence, freeness, and the fact that $\mathcal{L}$ is coded in $\Sigma(\Gamma)$ (see §3.1), hence the same is true for $\mathrm{Slit}^n_{\mathcal{L}}(\Gamma)$, for all $n > 0$, with respect to Γ. □

In the following, we will consider carrier ribbon graph sequences of the form $\mathcal{S} = \{\mathrm{Slit}^n_{\mathcal{L}}(\Gamma)\}_{n\geq 0}$. Note however that, in terms of Rauzy graphs, Proposition 3.2.5 only ensures that $\mathrm{Slit}^n_{\mathcal{L}}(\Gamma) = \Gamma_{L,n}$ for $n > 0$ (from now on, when there is no ambiguity we use the equality sign for graphs to denote either isomorphism or isotopy). As a matter of fact, a carrier graph Γ is the zeroth-order Rauzy graph $\Gamma_{L,0}$ if Γ is a bouquet of circles, where the single vertex corresponds to the empty word, and the edges correspond to $\mathrm{Fact}_L(1)$. Otherwise, Γ is of different nature from a Rauzy graph (similarly to what happens for *higher edge graph sequences* of *edge shifts* [**28**]); it is a general graph, describing more than $\mathrm{Fact}_L(1)$ in terms of constraints over L, but less than $\mathrm{Fact}_L(2)$. This exception in $\mathcal{S}$ gives some of the extended possibilities compared to the iet case which requires Γ to be a coherent bouquet of circles.

3.3. *Bursts, bipartite graphs and maximality.* We now analyze the constraints on the sequences of the form $\{\mathrm{Slit}^n_{\mathcal{L}}(\Gamma)\}_{n\geq 0}$. When building $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ from Γ, each vertex $v \in \Gamma$ gives rise in $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ to a small graph, to which one can associate a bipartite graph $(V_{l,v} \sqcup V_{r,v}, E_v)$, where $V_{l,v}$ (respectively $V_{r,v}$) corresponds to the incoming (respectively outgoing) half-edges incident with v, and where E_v corresponds to the length-2 admissible paths going through v and used to carry $\mathcal{L}$. Applying Proposition 3.2.5, if Γ is a Rauzy graph, this bipartite graph is isomorphic to the graph Γ_v, as introduced in §2.3. But

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slitting applies also when Γ is not a Rauzy graph, thus by extension, we also denote $(V_{l,v} \sqcup V_{r,v}, E_v)$ by Γ_v for a vertex v in any slitted Γ. The graph Γ_v represents the local graph transformation at v induced by $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ which we also call a *burst*. Moreover, since Γ is coherent, we can place $V_{l,v}$ and $V_{r,v}$ in two parallel lines in a neighborhood of v in $\Sigma(\Gamma)$, using the order over the half-edges given by the ribbon structure of Γ. Then E_v is realized by arcs connecting $V_{l,v}$ to $V_{r,v}$, making Γ_v an embedded graph in $\Sigma(\Gamma)$ since it comes from a slitting. The following figure shows an example (see the second case in Example 3.2.1).

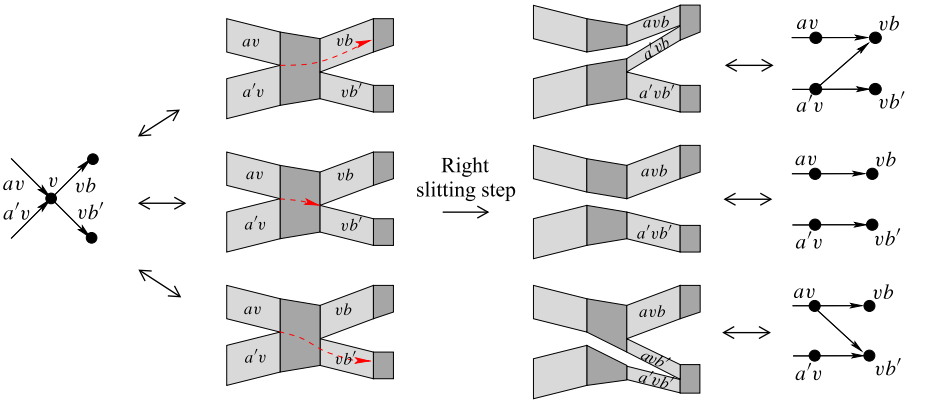


Such an embedding of a bipartite graph is classically called a *biplanar drawing*. From now on, Γ_v denotes the bipartite graph $(V_{l,v} \sqcup V_{r,v}, E_v)$ with its biplanar drawing in $\Sigma(\Gamma)$, both induced by $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$, and representing the burst it corresponds to.

Remark 3.3.1. Let Γ be a coherent graph carrying a lamination $\mathcal{L}$. Then every vertex v in Γ is such that its corresponding Γ_v induced by $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ is topologically connected if and only if there is no vertex cutting at v.

Proof. Let $N(\Gamma)$ be a neighborhood of Γ. A vertex cutting at v means a slitting step $\mathrm{slit}_{\alpha,\mathcal{L}}$ restricted to slit the vertex-trapezium Q_v in $N(\Gamma)$ corresponding to v (see §3.1). In the core graph, this vertex cutting results in a local disconnection, hence by construction, into a disconnection of Γ_v since the carrying $\mathcal{L}$ is assumed to be full. The converse also holds since a disconnection of Γ_v means there is a finite slitting within Q_v. □

Example 3.3.2. Let v be a bispecial vertex of a coherent graph with $\partial^-(v) = \partial^+(v) = 2$. There are three possibilities for a slitting step at v, reflected into three local graph transformations: these are exactly the three forms of bursts for v for which their corresponding Γ_v are biplanar among all the possible ones (see Example 2.3.4 (i), (iii) and (iv)). The middle case corresponds to a vertex cutting. These graphs are similar to the usual three forms of *splittings* in train track theory [**41**].

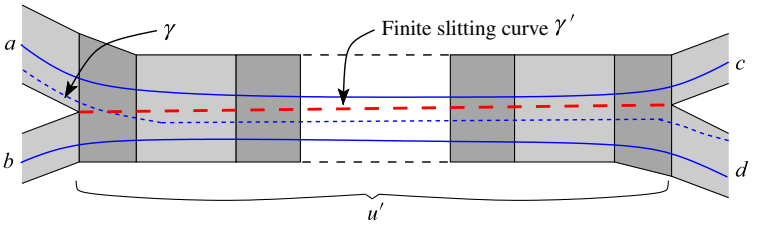


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If a bipartite graph drawing is biplanar then it is a disjoint union of *caterpillar trees*, where a caterpillar tree is such that deleting all its leaves yields a linear path graph [**22**]. This property gives us a criterion to prove the following proposition.

PROPOSITION 3.3.3. (Maximality and finite slittings) *Let $\mathcal{L}$ be a lamination carried by a coherent graph Γ. Then $\mathcal{L}$ is maximal relative to Γ if and only if there is no finite slitting of Γ with respect to $\mathcal{L}$.*

Proof. ($\Leftarrow$): Assume there is a curve γ carried by Γ but disjoint from $\mathcal{L}$ and not homotopic to any curve of $\mathcal{L}$. Consider a neighborhood $N(\Gamma)$ containing $\mathcal{L} \cup \{\gamma\}$. Let Γ be labeled by A, and let L be the lamination language which codes $\mathcal{L}$ by Γ. Let w_γ denote the coding of γ by Γ. Since $w_\gamma \notin L$, there must be factors which occur in w_γ but not in L (L is a shift over A). Let u be a shortest one, having length n. Since Γ is assumed to carry $\mathcal{L}$ in a full way, all the letters in A occur in L, therefore $n > 1$. Let Γ' denote $\mathrm{Slit}_{\mathcal{L}}^{n-2}(\Gamma)$, with edges labeled with length-$(n-1)$ factors. By construction, every factor of length at most $n-1$ in w_γ is also a factor of L. Thus, the slittings to get Γ' can be performed on $N(\Gamma)$ using slitting curves disjoint from γ and coded by factors of length $n-1$, so that γ is still carried by Γ'. We write $u = au'd$ with $a, d \in A$, $u' \in A^*$, that is, u corresponds to a length-2 path in Γ' made of the edges au' and $u'd$, linked by the vertex u'. Now, au' and $u'd$ are factors of L, but by assumption u is not. Hence, since Fact_L is prolongable, u' must be bispecial in L. Indeed, au' is prolongated to the right by at least another letter $c \in A$, giving rise to $au'c \in \mathrm{Fact}_L$, and $u'd$ is prolongated to the left by at least another letter $b \in A$, giving rise to $bu'd \in \mathrm{Fact}_L$. Therefore, when constructing $\mathrm{Slit}_{\mathcal{L}}(\Gamma')$, there is a burst at u', with its corresponding embedded bipartite graph $\Gamma'_{u'}$. For $\mathrm{Slit}_{\mathcal{L}\cup\{\gamma\}}(\Gamma')$, the existence of $u = au'd$ means to add an edge to $\Gamma'_{u'}$, and the result must remain biplanar, i.e. it must be a caterpillar forest. Thus, without u, the graph $\Gamma'_{u'}$ could not be topologically connected. According to Remark 3.3.1 there is a vertex cutting on Γ' at u', that is, there is a finite slitting curve coded by u' on Γ.



($\Rightarrow$): Assume there is a finite slitting curve γ' linking two edge-rectangle intersections in $N(\Gamma)$, say between the edge-rectangles a, b at one end, and c, d at the other (see the above figure). Let u' be the coding of γ', and let γ_1, γ_2 be two curves of $\mathcal{L}$, with respective codings $w_l au'c w_r$ and $w'_l bu'd w'_r$, where $w_l, w'_l \in {}^{\omega}A$ and $w_r, w'_r \in A^{\omega}$, such that γ_1 is the lowermost curve going through a, and γ_2 is the uppermost one going through d. The curves γ_1, γ_2 exist since the carrying is assumed full and $\mathcal{L}$ can be assumed to be closed. But then γ' can be prolongated from each of its endpoints into a two-way infinite curve γ coded by $w_l au'd w'_r$ (see the above figure). By construction, γ is carried by Γ, and cannot be homotopic to any other curve of $\mathcal{L}$, since $au'd \notin \mathrm{Fact}_L$. Hence, $\mathcal{L}$ is not maximal relative to Γ. □

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COROLLARY 3.3.4. *Let $\mathcal{L}$ be a lamination carried by a coherent graph Γ, but not maximal relative to Γ. Then, there exists $k > 0$ such that $\mathcal{L}$ is maximal relative to* $\mathrm{Slit}^k_{\mathcal{L}}(\Gamma)$.

Proof. According to Proposition 3.3.3, if $\mathcal{L}$ is not maximal relative to Γ, there are finite slitting curves associated with Γ. But a carrier graph has only a finite number of associated slitting curves, since these start from the edge-rectangle intersections of $N(\Gamma)$. Hence, there is a $k > 0$ such that k slitting global steps exhaust all the finite slittings. □

We can now generalize Remark 3.3.1.

PROPOSITION 3.3.5. (Maximality and vertex bursts) *Let $\mathcal{L}$ be a lamination carried by a coherent graph Γ. Then $\mathcal{L}$ is maximal relative to Γ if and only if every vertex v in the graphs in $\{\mathrm{Slit}^n_{\mathcal{L}}(\Gamma)\}_{n\geq 0}$ is such that its burst representation Γ_v induced by $\mathrm{Slit}_{\mathcal{L}}$ is topologically connected, hence maximally biplanar (no edge can be added without impairing biplanarity), and with $\partial^-(v) + \partial^+(v) - 1$ edges.*

Proof. If $\mathcal{L}$ is maximal relative to Γ, every vertex burst in $\mathrm{Slit}^n_{\mathcal{L}}(\Gamma)$, for every $n \geq 0$, is associated with a topologically connected $\Gamma_v = (V_{l,v} \sqcup V_{r,v}, E_v)$. Otherwise by Remark 3.3.1, for some index $n_0 \geq 0$, a vertex cutting would occur when obtaining $\mathrm{Slit}^{n_0+1}_{\mathcal{L}}(\Gamma)$ from $\mathrm{Slit}^{n_0}_{\mathcal{L}}(\Gamma)$, meaning a finite slitting in Γ, and contradicting Proposition 3.3.3. Since Remark 3.3.1 and Proposition 3.3.3 give necessary and sufficient conditions, the converse holds too. Now, by definition of a slitting global step, Γ_v is such that $|V_{l,v}| = \partial^-(v)$ and $|V_{r,v}| = \partial^+(v)$. Thus, if Γ_v is topologically connected, it contains a tree, and it has at least $\partial^-(v) + \partial^+(v) - 1$ edges. Moreover, being induced by slittings, Γ_v is biplanar, and as such it must be a caterpillar tree, that is, $\partial^-(v) + \partial^+(v) - 1$ is the maximal number of edges. □

3.4. *Non-coherent graphs and language-based slittings.* Coherent embeddings for carrier graphs are the natural geometric context whereto slittings apply, since slitting curves are well-defined for them, univocally starting from the edge-rectangle intersections of $N(\cdot)$, i.e. crotches of these graphs. The preceding sections exploited this fact, and the same is true for classic train track theory. However, using the relationships between slittings, Rauzy graphs and vertex bursts (see §3.3), we can extend the slitting operations to non-coherent graphs by applying in a general way the main ideas of Construction 2.3.3.

Construction 3.4.1. (Slittings from line graph immersions) Let Γ be a ribbon graph labeled by A, and let L be a language in ${}^{\omega}A^{\omega}$ of labels of admissible paths of Γ, which is equivalent to ask that $\mathrm{Fact}_L(2)$ is a set of labels of length-2 admissible paths of Γ. Using Construction 2.3.3, we first consider the line graph $LG(\Gamma)$, and we immerse it in $\Sigma(\Gamma)$ as follows:

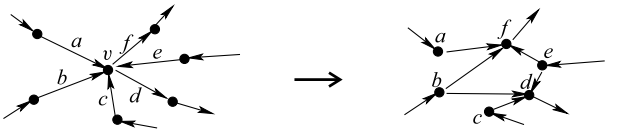
(i) one vertex is placed in the interior of each edge of Γ;
(ii) the vertices of each pair of vertices lying in a length-2 admissible path of Γ are linked by an arc in Γ which is contained in this length-2 path.

Next, we prune $LG(\Gamma)$ of its edges not corresponding to length-2 factors in $\mathrm{Fact}_L(2)$, and we put them in general and minimal intersection position (with vertices fixed). The resulting graph Γ' is an immersion of the first-order Rauzy graph $\Gamma_{L,1}$ in $\Sigma(\Gamma)$. If the

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language L involves all the edge labels of Γ, and if Γ' is embedded in $\Sigma(\Gamma)$ (not just immersed), we say that Γ is *L-slittable*, and we denote Γ' by $\mathrm{Slit}_L(\Gamma)$. If L is a lamination language coding a lamination $\mathcal{L}$ carried by Γ, then Γ is L-slittable; moreover, if Γ is coherent then by Proposition 3.2.5, $\mathrm{Slit}_L(\Gamma)$ is just $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ (up to isotopy), so if Γ is non-coherent, by extension, we also denote $\mathrm{Slit}_L(\Gamma)$ by $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$.

Example 3.4.2. Here is the effect of Construction 3.4.1 (a burst in this context) at a non-coherent vertex v, where the involved factors of L are $\{af, bf, bd, cd, ed, ef\} \subset \mathrm{Fact}_L(2)$.



The properties of carrier graphs are preserved by Slit_L, as they are by $\mathrm{Slit}_{\mathcal{L}}$, leading firstly to the following lemma.

LEMMA 3.4.3. *Let Γ be a (not necessarily coherent) graph. Let L be a language such that Γ is L-slittable. Then if γ is a curve carried by Γ with a path whose label is in L, γ is also carried by* $\mathrm{Slit}_L(\Gamma)$. *Conversely, if γ is any curve carried by* $\mathrm{Slit}_L(\Gamma)$, *γ is also carried by Γ.*

Proof. According to Construction 3.4.1, each edge e of $\mathrm{Slit}_L(\Gamma)$ corresponds to a length-2 admissible path e_1e_2 of Γ, which can be homotopically deformed into an arc embedded in Γ, with start and end fixed in e_1 and e_2, respectively. Now, if γ is carried by $\mathrm{Slit}_L(\Gamma)$, it has a homotopic path η' in $\mathrm{Slit}_L(\Gamma)$. This path η' can be deformed into a path η in Γ described by the sequence of overlapping length-2 paths given by the sequence of the deformed edges of η'. Hence γ is also carried by Γ. Conversely, let γ be carried by Γ with a path η whose label is in L. The path η can be decomposed into a sequence of length-2 overlapping subpaths. By definition of $\mathrm{Slit}_L(\Gamma)$, each of these subpaths contains one of the arcs described above, with endpoints at the vertices of $\mathrm{Slit}_L(\Gamma)$ and homotopic to one edge of this graph, defining a path η' in $\mathrm{Slit}_L(\Gamma)$ homotopic to γ. □

A set of curves fully carried by Γ with paths whose labels are in L is still fully carried by $\mathrm{Slit}_L(\Gamma)$. Freeness is also preserved by $\mathrm{Slit}_L(\Gamma)$ in $\Sigma(\Gamma)$. Indeed, by Lemma 3.4.3, any two homotopic curves in $\Sigma(\Gamma)$ carried by $\mathrm{Slit}_L(\Gamma)$ are also carried by Γ, and with a unique path η if Γ is free; then, η is transformed into a unique path η' in $\mathrm{Slit}_L(\Gamma)$ following the length-2 overlapping subpaths of η.

COROLLARY 3.4.4. *Let Γ be a (not necessarily coherent) graph, and let $\mathcal{L}$ be a lamination coded by Γ. Then $\mathcal{L}$ is coded by* $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$. *Conversely, if γ is any curve coded by* $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$, *then γ is also coded by Γ.*

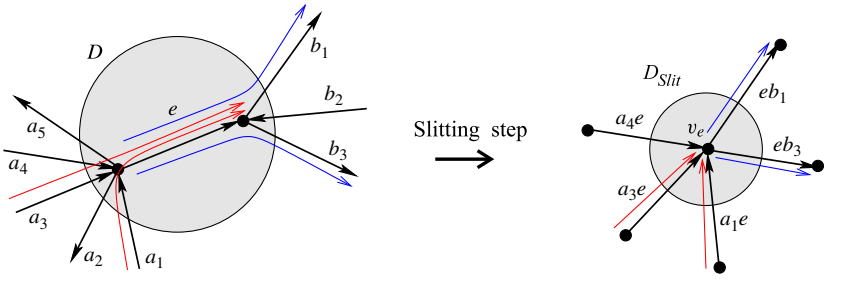
Proof. If L is the coding of $\mathcal{L}$ by Γ, then Γ is L-slittable, and Lemma 3.4.3 applies. □

Moreover, if $\mathcal{L}$ is maximal relative to Γ, it is also relative to $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$. Finally, dealing with non-coherence is simplified by the following property.

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PROPOSITION 3.4.5. (Coherence in one slitting global step) *Let* Γ *be a (not necessarily coherent) graph, and let* L *be a language such that* Γ *is* L*-slittable. Then* $\mathrm{Slit}_L(\Gamma)$ *is a coherent graph.*

Proof. Let e be an edge of Γ, and let v_e be its corresponding vertex in $\mathrm{Slit}_L(\Gamma)$. First, assume that e is not a loop. Let $D \subset \Sigma(\Gamma)$ be a disk containing e, such that e is the only edge of Γ completely included in D. We assume its boundary ∂D is in minimal intersection position with each edge a_i (respectively b_j) incident with the initial vertex (respectively the terminal vertex) of e. Similarly, let $D_{\mathrm{Slit}} \subset \Sigma(\Gamma)$ be a disk containing v_e, such that v_e is the only vertex contained by D_{Slit}.



According to Construction 3.4.1, the edges of $\mathrm{Slit}_L(\Gamma)$ incoming at v_e (respectively outgoing from v_e) intersecting $\partial D_{\mathrm{Slit}}$ correspond to length-2 paths $a_i e$ (respectively eb_j) of Γ used to describe factors in $\mathrm{Fact}_L(2)$. Now, any incident edge a_i with the initial vertex of e, which has another orientation than the ones determined by the above length-2 paths, is such that $a_i e$ is not an admissible path, so that there is no possible corresponding edge incident with v_e in $\mathrm{Slit}_L(\Gamma)$. The same holds for the incident edges at the terminal vertex of e. Hence, the $a_i e$'s intersections with $\partial D_{\mathrm{Slit}}$ are all consecutive, and the eb_j's intersections too, that is, $\mathrm{Slit}_L(\Gamma)$ is coherent at v_e. The same reasoning applies to every edge of Γ.

If there are loops in Γ, we subdivide each of them into two edges by adding a new vertex on it, and call Γ' the resulting graph. We then use the above argument to obtain a coherent graph $\mathrm{Slit}_L(\Gamma')$. Next, we consider any of these loops in Γ', subdivided into e_1, e_2, and for which v_{e_1} and v_{e_2} are the corresponding vertices in $\mathrm{Slit}_L(\Gamma')$. There is a unique edge e' from v_{e_1} to v_{e_2}, and $\partial^+(v_{e_1}) = \partial^-(v_{e_2}) = 1$. Hence, contracting e' preserves coherence. Applying contraction to every such edge of $\mathrm{Slit}_L(\Gamma')$ yields a coherent graph which is $\mathrm{Slit}_L(\Gamma)$. □

COROLLARY 3.4.6. *Let* Γ *be a (not necessarily coherent) graph, and let* $\mathcal{L}$ *be a lamination carried by* Γ*. Then* $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ *is a coherent graph.*

A consequence of the above Corollary 3.4.6 is that, given a carried lamination $\mathcal{L}$, only Γ may be not coherent in a sequence $\mathcal{S} = \{\mathrm{Slit}^n_{\mathcal{L}}(\Gamma)\}_{n\geq 0}$. Thus, using Construction 3.4.1 and Corollary 3.4.4, the close relationship between sequences like $\mathcal{S}$ and Rauzy graph sequences, i.e. Proposition 3.2.5, also holds when Γ is non-coherent. Moreover by Lemma 3.4.3, given a lamination or a language to obtain $\mathrm{Slit}_{(\cdot)}(\Gamma)$, one can work with curves carried by this $\mathrm{Slit}_{(\cdot)}(\Gamma)$, that is, in the context of coherent graphs, and then go back to Γ, by which these curves are carried too.

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3.5. *A characterization of lamination languages.* In order to describe a sequence of slitted graphs with respect to a language L, we inductively extend Construction 3.4.1 to every Rauzy graph $\Gamma_{L,n}$, $n > 0$: we first define an immersion of $\Gamma_{L,n+1}$ in $\Sigma(\Gamma)$ from $\Gamma_{L,n}$ by using $\mathrm{Fact}_L(n+1)$; next, if $\Gamma_{L,n+1}$ is embedded in $\Sigma(\Gamma)$, we also say that $\Gamma_{L,n}$ is *L-slittable*. This extension leads to the following definition.

Definition 6. An infinite sequence $\{\Gamma_n\}_{n\geq 0}$ of ribbon graphs is said to be an *L-slittable graph sequence* if L is a language in ${}^{\omega}A^{\omega}$, and if Γ_n is L-slittable, for all $n \geq 0$, so that $\Gamma_{n+1} = \mathrm{Slit}_L(\Gamma_n)$, i.e. $\{\Gamma_n\}_{n\geq 0} = \{\mathrm{Slit}_L^n(\Gamma_0)\}_{n\geq 0}$.

Thus, in an L-slittable graph sequence, $\mathrm{Slit}_L^n(\Gamma_0)$ is the nth order Rauzy graph $\Gamma_{L,n}$, for all $n > 0$, and the starting graph Γ_0 is not necessarily a Rauzy graph, like in a sequence of the form $\{\mathrm{Slit}_{\mathcal{L}}^n(\Gamma)\}_{n\geq 0}$ based on a lamination $\mathcal{L}$ carried by Γ (see the end of §3.2). Also, an L-slittable graph sequence induces a ribbon structure for each of the Γ_n, all embeddable in $\Sigma(\Gamma_0)$, that is, in the same hyperbolic surface, like the standard surface $\Sigma_0(\Gamma_0)$. Therefore, Definition 6 is stronger than just asking that each slitted Γ_n be independently embeddable.

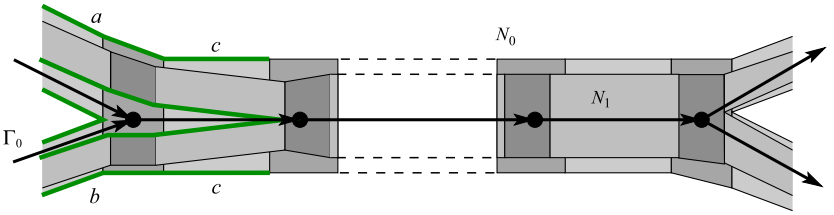
We now characterize lamination languages by using the above graph sequences. This result is an extension of the iet language case [**5, 6**].

THEOREM A.1. *Let L be a shift in ${}^{\omega}A^{\omega}$. Then L is a lamination language if and only if there exists a corresponding L-slittable graph sequence $\{\Gamma_n\}_{n\geq 0}$, where Γ_0 is labeled by A.*

Proof. ($\Rightarrow$): Since L is a lamination language, there exists a ribbon graph Γ which codes a lamination $\mathcal{L}$ into L. According to Construction 3.4.1 and Proposition 3.2.5, we know that $\mathrm{Slit}_{\mathcal{L}}^n(\Gamma) = \mathrm{Slit}_L^n(\Gamma)$, for all $n \geq 0$, so that $\{\mathrm{Slit}_{\mathcal{L}}^n(\Gamma)\}_{n\geq 0}$ forms an L-slittable graph sequence.

($\Leftarrow$): By definition, the graphs Γ_n are all inductively embeddable in $\Sigma(\Gamma_0)$. Let us first assume that no isolated cyclic subgraph occurs in $\{\Gamma_n\}_{n\geq 0}$. Let us also assume that Γ_0 is coherent. Thus we can consider regular neighborhoods $N(\Gamma_n)$, here denoted by N_n, and build a sequence $\mathcal{N} = \{N_n\}_{n\geq 0}$ using the following definition: a *rightward* (respectively *leftward*) *inclusion* of N_{n+1} in the interior of N_n is first based on an immersion of the line graph $LG(\Gamma_n)$ (see Construction 3.4.1) in the interior of N_n, such that each vertex v_e of $LG(\Gamma_n)$ corresponding to the edge e of Γ_n is placed in the vertex-trapezium of N_n corresponding to the terminal (respectively initial) vertex of e. Second, as usual, Γ_{n+1} is obtained by pruning off the edges of $LG(\Gamma_n)$, keeping only those corresponding to $\mathrm{Fact}_L(n+1)$. Finally, the corresponding N_{n+1} is built within N_n, so that the N_{n+1}'s vertex-trapezia (respectively edge-rectangles) are included in the interiors of the N_n's vertex-trapezia (respectively in the interior of N_n). Since all the edges of Γ_{n+1} can be assumed to be transverse to N_n's fibers, N_{n+1} can be taken such that its fibers are subarcs of fibers of N_n. Geometrically, a rightward (respectively leftward) inclusion behaves as a neighborhood obtained after a right (respectively left) slitting global step. The sequence $\mathcal{N}$ is defined such that the inclusions of N_{n+1} in N_n are taken alternatively rightward and leftward, for all $n \geq 0$. Here is a rightward inclusion of N_1 in N_0.

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We define a *boundary curve* of a neighborhood N_n as a union of top or bottom sides of quadrilaterals, forming a finite oriented simple curve which starts and ends with edge-rectangle sides. In order to code these boundary curves by Γ_0, we can deform them so that each of their extremities is identified with the vertex of Γ_0 contained in the vertex-trapezia of $N(\Gamma_0)$ the extremity belongs to, accordingly determining a carrying by Γ_0. For instance, in the above figure, ac and bc are codings of boundary curves of N_0 and N_1, where $a, b, c \in A$ are the labels of the involved Γ_0 edges. Now, for each n there is a finite number of boundary curves defined from N_n, so after having modified them as described above, we can make them pairwise disjoint arcs in $N_n \backslash N_{n+1}$ by using homotopies along the transverse fibers. We take the union over n of all these curves, and obtain a set $\mathcal{C}_L$ of finite simple curves, pairwise disjoint, transverse to the fibers of N_0, and unambiguously deformable onto paths of Γ_0. We denote by $L_{\mathcal{C}}$ the set of codings of the curves in $\mathcal{C}_L$ by Γ_0. We claim that $L_{\mathcal{C}} = \mathrm{Fact}_L$.

To prove that $L_{\mathcal{C}} \subseteq \mathrm{Fact}_L$, let e denote any edge of Γ_n, $n \geq 0$. The top and bottom sides of its associated edge-rectangle Q_e in N_n induce curves in $\mathcal{C}_L$ coded by a single edge in Γ_0, thus by a single letter, say $a \in A$. When these sides are prolongated to the right (the reasoning is similar for the left direction), each of them goes along a side of a vertex-trapezium Q_v corresponding to a vertex $v \in \Gamma_n$, and along a side of another edge-rectangle. Consider the prolongation whose sides belong to $Q_{e'}$, coded by $b \in A$, where e' is an edge consecutive to e in Γ_n: (i) if $\partial^+(v) = 1$ (respectively $\partial^-(v) = 1$), then according to the Rauzy graph definition, $ab \in \mathrm{Fact}_L$, since it is the unique way of prolongating e to the right (respectively e' to the left) in Γ_n; (ii) if $\partial^+(v) > 1$ and $\partial^-(v) > 1$, then by construction, e and e' are either the two highest or the two lowest edges incident with v. Consider the case where they are, say, the two highest, and assume $ab \notin \mathrm{Fact}_L(2)$. Since Fact_L is prolongable, there are letters c and d such that ac and db are in $\mathrm{Fact}_L(2)$, where c labels an outgoing edge from v necessarily lower than e', and d labels an incoming edge at v necessarily lower than e. But then, ac and db are labels of paths which cross each other, contradicting the fact that Γ_n is L-slittable.

To prove that $\mathrm{Fact}_L \subseteq L_{\mathcal{C}}$: N_1, being a rightward inclusion in N_0, behaves like a right slitting global step; thus it slits open every edge-rectangle coded by a left special letter in A, producing boundary curves coded by these letters. Next, N_2, being a leftward inclusion in N_1, has the same effect for the right direction. Carrying on this reasoning to all of $\mathcal{N}$, we see that all the prolongations to the left and to the right of these boundary curves produce longer and longer special factors in Fact_L. In fact, all the special factors in Fact_L are produced by this process. Indeed, the prefix of a left special factor being also left special, left special factors are successive prolongations to the right of smaller special factors, and we have an analogous situation for the right special factors. As a result, the only factors in

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Fact_L which fail to be included here are those which are not factors of any special factor of L. Since the alphabet is finite, these factors must be included in periodic words in L, since their prolongations to the left and right must be deterministic. But periodic words do not include special factors of L with arbitrary length. As a consequence, there exists an $n_0 > 0$ such that their factors make up isolated cyclic subgraphs in the Rauzy graphs $\Gamma_{L,n}$, for all $n > n_0$, but for the time being, we have excluded this possibility. Hence, Fact_L is also included in $L_{\mathcal{C}}$.

We are therefore in a position to apply Lemma 2.2.1 to $\mathcal{C}_L$ and Fact_L, and we obtain a lamination $\mathcal{L}$ in the standard surface $\Sigma_0(\Gamma_0)$, coded by Γ_0 into the language L. Now if there exist isolated cyclic subgraphs in $\{\Gamma_n\}_{n\geq 0}$, they correspond to isolated periodic two-way infinite words, which in turn, correspond to pairwise disjoint and non-homotopic compact leaves. These cycles are finite in number since $\Sigma_0(\Gamma_0)$ is of finite type. Therefore we can also apply the preceding procedure to $\{\Gamma_n\}_{n\geq 0}$, except that we exclude the corresponding finite set of closed curves from $\mathcal{C}_L$, as well as their periodic codings from L. We get a lamination $\mathcal{L}'$ from Lemma 2.2.1 as above, to which we add *a posteriori* these closed curves to get $\mathcal{L}$.

If Γ_0 is not coherent, we can apply the same reasoning as above starting from Γ_1, since first, Γ_0 is L-slittable so that $\Gamma_1 = \mathrm{Slit}_L(\Gamma_0)$ is coherent (cf. Proposition 3.4.5); and second, Γ_1 is the first-order Rauzy graph $\Gamma_{L,1}$ (cf. Construction 3.4.1), so that the language L is transformed into L_1 over Γ_1 by a recoding by a length-2 sliding window over $\mathrm{Fact}_L(2)$. As a result, we obtain a lamination $\mathcal{L}$ carried by Γ_1, and coded by L_1. Applying Corollary 3.4.4, $\mathcal{L}$ is also carried and coded by Γ_0, and this coding is the inverse of the recoding of L into L_1, i.e. a projection of each occurrence of $\mathrm{Fact}_L(2)$ to its first letter, that is L itself. □

Note that given a shift L in ${}^{\omega}A^{\omega}$, there exists only a finite number of ribbon graphs which are labeled by A and for which $\mathrm{Fact}_L(2)$ is a set of labels of length-2 admissible paths. Therefore, knowing whether there is no L-slittable graph sequence associated with a given shift L is semi-decidable.

4. *Lamination language combinatorics*

We just have seen that lamination languages are characterized by slittable graph sequences, whose positive terms correspond to their embedded corresponding Rauzy graphs. From this, we can infer results about their combinatorics.

4.1. *Ultimate affine complexity.*

First, let us investigate the complexity functions of lamination languages.

THEOREM B. *Let L be a lamination language. Then L has ultimate affine complexity.*

Proof. Let Γ be a graph which codes a lamination $\mathcal{L}$ into L. Let k be positive, while ensuring that $\mathcal{L}$ is maximal relative to $\Gamma' = \mathrm{Slit}^k_{\mathcal{L}}(\Gamma)$, and Γ' is coherent (see Corollaries 3.3.4 and 3.4.6). By Proposition 3.2.5, the sequence $\mathcal{S} = \{\mathrm{Slit}^n_{\mathcal{L}}(\Gamma)\}_{n\geq k}$ is equal to the sequence of Rauzy graphs $\{\Gamma_{L,n}\}_{n\geq k}$. Now, recall that in a Rauzy graph, each vertex v can be identified to the factor it corresponds to, and that its

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prolongation orders are $m_l(v) = \partial^-(v) - 1$, $m_r(v) = \partial^+(v) - 1$, and $m(v) = |\mathrm{Fact}_L \cap AvA| - (m_l(v) + m_r(v) + 1)$. The graph Γ' being coherent, and $\mathcal{L}$ maximal relative to it, we can apply Proposition 3.3.5 to every v of a graph in $\mathcal{S}$ to get $|\mathrm{Fact}_L \cap AvA| = \partial^-(v) + \partial^+(v) - 1$, i.e. the number of edges of the burst of v, thus $m(v) = 0$. Since Fact_L is prolongable, then $\sum_{v \in \mathrm{Fact}_L(n)} m(v) = p''_L(n)$, for all $n \geq k$ (see §2.3). Hence $p''_L(n) = 0$, i.e. $p_L(n)$ is affine, for all $n \geq k$. □

If a lamination language L is made of a finite number of periodic words (corresponding to a lamination made of a finite number of non-homotopic compact leaves), there exists some $n_0 > 0$, such that L has no length-n special factors for all $n > n_0$. Hence $p'_L(n) = 0$, for all $n > n_0$, and the complexity is ultimately constant. In all the other cases, there are special factors of every length in Fact_L, that is, $p'_L(n) > 0$, for all $n > 0$. More explicit complexity formulae can also be obtained. First, here is a generalization of a result in [**30**].

COROLLARY 4.1.1. *Let L be a lamination language over A, coding a lamination $\mathcal{L}$ by a coherent graph $\Gamma = (V, E)$ relative to which $\mathcal{L}$ is maximal. Then $|E| = |A|$, and:*

$$p_L(n) = (|E| - |V|)n + |V|, \text{ for all } n > 0.$$

Proof. Since Γ is coherent, and $\mathcal{L}$ is maximal relative to Γ, we can apply the case $k = 1$ of the proof of Theorem B, considering the first-order differences, that is, $p'_L(n) = K$, for all $n \geq 1$, $K \geq 0$. But we can also apply Proposition 3.3.5 to Γ to deduce that $|\mathrm{Fact}_L(2)| = \sum_{v \in \Gamma}(\partial^-(v) + \partial^+(v) - 1) = |E_1|$, where $\mathrm{Slit}_{\mathcal{L}}(\Gamma) = \Gamma_{L,1} = (V_1, E_1)$, that is, $2|E| - |V| = |E_1|$. Since on $\Gamma_{L,1}$ we have $p'_L(1) = |E_1| - |V_1| = |E_1| - |E| = K$, we also have that $|E| - |V| = K$. Using that $p_L(1) = |E|$, we obtain $p_L(n) = |E| + (n-1)(|E| - |V|)$, for all $n > 0$. □

A direct consequence of Corollary 4.1.1 is a known property for an iet language L based on an iet T over m intervals and satisfying the idoc [**26**], that is, a coding of a maximal lamination relative to a coherent bouquet of m circles has complexity $p_L(n) = (m-1)n + 1$, for all $n > 0$.

COROLLARY 4.1.2. *Let L be a lamination language over A, coding a lamination $\mathcal{L}$ by a (not necessarily coherent) graph $\Gamma = (V, E)$. Then the complexity p_L is affine if and only if $\mathcal{L}$ is maximal* relative *to* $\mathrm{Slit}_{\mathcal{L}}(\Gamma) = (V_1, E_1)$*, so that $|E| = |A|$ and:*

$$\begin{aligned} p_L(n) &= (|E_1| - |E|)n + (2|E| - |E_1|) \\ &= (|\mathrm{Fact}_L(2)| - |A|)n + (2|A| - |\mathrm{Fact}_L(2)|), \quad \text{for all } n > 0. \end{aligned}$$

Proof. Reusing the arguments from the proof of Corollary 4.1.1, $p'_L(1) = |E_1| - |E|$ and $p_L(1) = |E|$, hence $p_L(n) = |E| + (n-1)(|E_1| - |E|)$, for all $n > 0$. Moreover by definition, E_1 is in bijection with $\mathrm{Fact}_L(2)$, and E with A, hence the second equality. Conversely, if maximality holds only for $\mathrm{Slit}^k_{\mathcal{L}}(\Gamma)$ with $k > 1$, then by Proposition 3.3.5, there is some k_0 with $1 \leq k_0 < k$, and a length-k_0 bispecial v for which $m(v) < 0$, whereas for the others $m(\cdot) \leq 0$. Hence, $p''_L(k_0) < 0$, and the complexity is affine only ultimately. □

Recall that if $\mathcal{L}$ is maximal relative to Γ, it is maximal relative to $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$, and this is how Corollary 4.1.2 is consistent with Corollary 4.1.1. Note that Corollary 4.1.2 is

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useful to obtain the complexities associated with coded laminations maximal relative to non-coherent graphs. This corollary is also generalized as follows.

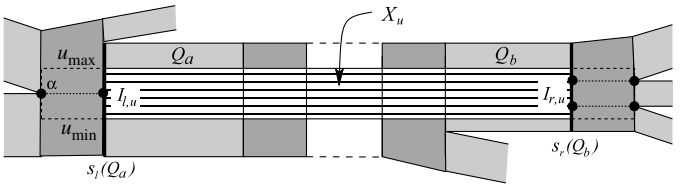
COROLLARY 4.1.3. *Let L be a lamination language over A, coding a lamination $\mathcal{L}$ by a (not necessarily coherent) graph Γ. Let $\mathcal{L}$ be maximal relative to* $\mathrm{Slit}^k_{\mathcal{L}}(\Gamma) = (V_k, E_k)$ *for $k > 0$. Then:*

$$p_L(n) = Dn - (Dk - |V_k|), \text{ for all } n \geq k \text{ where } D = |E_k| - |V_k|.$$

Proof. We have $p'_L(n) = K$, for all $n \geq k$, $K \geq 0$. Since $\mathrm{Slit}^k_{\mathcal{L}}(\Gamma) = \Gamma_{L,k}$, then $p'_L(k) = |E_k| - |V_k| = K$, and $p_L(k) = |V_k|$. Hence, $p_L(n) = |V_k| + (n-k)(|E_k| - |V_k|)$, for all $n \geq k$. □

4.2. *Leaf packets, special factors and infinite special words.* We now analyze the structure of the special factors of lamination languages through their corresponding curve subsets. Let $\mathcal{L}$ be a lamination coded by a coherent carrier graph Γ into a language L over A. Let $N(\Gamma)$ be a fibered neighborhood of Γ, containing $\mathcal{L}$ with all its leaves transverse to the fibers, as usual. In a quadrilateral Q of $N(\Gamma)$, let s be any transverse fiber, and consider two points $x \in (\ell_1 \cap s)$, $y \in (\ell_2 \cap s)$ where ℓ_1, ℓ_2 are leaves in $\mathcal{L}$. Let $I = [x, y] \subset s$ be the interval in s defined by x and y. Let s' be another transverse fiber in Q, and consider $x' \in (\ell_1 \cap s')$ and $y' \in (\ell_2 \cap s')$ such that x, x', and y, y' lie respectively on the same connected component of $\ell_1 \cap Q$ and $\ell_2 \cap Q$. We say then that $I' = [x', y'] \subset s'$ is a *translate* of I along $\mathcal{L}$'s leaves. This definition induces an equivalence relation among these intervals in the fibers of Q, which extends by transitivity to all of $N(\Gamma)$.

Given a factor $u \in \mathrm{Fact}_L$, we define the *right cylinder* $RCyl_u$ as the set of the right one-way infinite half-words of words in L having u as prefix (here, we develop the right direction, the left direction being handled similarly with suffixes and left cylinders). We denote by $s_l(Q)$ and $s_r(Q)$ respectively the left and right sides of the quadrilateral Q of $N(\Gamma)$. Assuming u is a non-empty factor, let $a \in A$ be its first letter and let Q_a be its corresponding edge-rectangle in $N(\Gamma)$. The right cylinder $RCyl_u$ determines a subset $\mathcal{L}_u$ of $\mathcal{L}$ made of *half-leaves*, i.e. subsets of $\mathcal{L}$'s leaves forming one-way infinite curves, each beginning at some point of $s_l(Q_a)$ and having its coding in $RCyl_u$. Let $u_{\min}$ and $u_{\max}$ be respectively the lower and upper bounds of $\mathcal{L}_u$'s origins in $s_l(Q_a)$. The interval $I_{l,u} = [u_{\min}, u_{\max}] \subset s_l(Q_a)$ contains every starting point of the half-leaves in $\mathcal{L}_u$. Indeed, let γ, γ' be two half-leaves in $\mathcal{L}_u$, then all the half-leaves with their origins at $s_l(Q_a)$ and lying between γ and γ' also belong to $\mathcal{L}_u$, since leaves in $\mathcal{L}$ do not intersect. Let $b \in A$ be the last letter of u. Because $u \in \mathrm{Fact}_L$ the interval $I_{l,u}$ can be translated in $N(\Gamma)$ following the quadrilateral sequence corresponding to u until it gets to $s_r(Q_b)$. This translation determines an interval $I_{r,u}$ in $s_r(Q_b)$, together with a set of arcs $X_u \subset \mathcal{L}_u$, called a *leaf packet* of $\mathcal{L}$ [**51**].



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We say that an interval $I_{l,u}$ (respectively $I_{r,u}$) *meets* an intersection point α between edge-rectangles of $N(\Gamma)$, if its left (respectively right) translate in the adjacent vertex-trapezium contains α. For instance, in the above figure, $I_{l,u}$ meets one intersection point, and $I_{r,u}$ meets two. If u is a left special factor in L of order $m_l(u)$, then $I_{l,u}$ meets $m_l(u)$ intersection points in its interior (the same is true for $I_{r,u}$ and the right special factors). These intersection points are the starting points of slitting curves, and any such slitting curve, following the sequence of quadrilaterals associated with u, divides X_u, $I_{l,u}$ and $I_{r,u}$. Let $u' \in \mathrm{Fact}_L$ be a prolongated factor of u to the right, i.e. $u' = uu'' \in \mathrm{Fact}_L$. Then, since $RCyl_{u'} \subseteq RCyl_u$, we have $\mathcal{L}_{u'} \subseteq \mathcal{L}_u$ and $I_{l,u'} \subseteq I_{l,u}$.

The following results about minimal laminations are generalizations of the iet case [**6**].

PROPOSITION 4.2.1. (Special factors with ultimate order one) *Let L be a minimal lamination language over A. Then there exists $n_0 > 0$ such that for every $u \in \mathrm{Fact}_L(n)$, $n \geq n_0$, the prolongation orders of u are at most* 1, *i.e. $m_l(u) \leq 1$ and $m_r(u) \leq 1$.*

Proof. If L is made of a single periodic word, then for every sufficiently long $u \in \mathrm{Fact}_L$, $m_l(u) = m_r(u) = 0$. Thus, let us assume that L is aperiodic. Let Γ be the carrier graph coding a lamination into L, and let us first assume that Γ is coherent. We prove the result for m_l (the proof is similar for m_r). Let $u \in \mathrm{Fact}_L$ be such that $m_l(u) > 1$, and consider $I_{l,u}$ and $I_{r,u}$ in a regular neighborhood $N(\Gamma)$. Accordingly, $I_{l,u}$ meets $m_l(u)$ intersection points. Now, the translates to the right of $I_{r,u}$ eventually meet edge-rectangle intersections because of the aperiodic minimality. Indeed, the complexity of L is ultimately affine but not ultimately constant (see §4.1), that is, there are arbitrarily long right special factors in Fact_L; moreover, by minimality, every factor in Fact_L is necessarily contained in longer right special factors. Let $u' = uu''$ be the shortest one with this property, so that $I_{r,u'}$ meets $m = m_r(u') > 0$ intersection points. Next, we follow m slitting curves from these m points toward the left all along the quadrilateral sequence corresponding to u' until they hit $I_{l,u}$. This hitting defines $m + 1$ subintervals in $I_{l,u}$, each corresponding to a factor $u_i = u'x_i$, where the x_i label the $m + 1$ edge-rectangle prolongations to the right determined by the m intersection points. This also induces partitions of $\mathcal{L}_u$ and $RCyl_u$ into strict subsets. We can inductively apply this dividing process to each of the subintervals I_{u_i}, until each intersection point met by I_u is eventually met by a single subinterval, so as to correspond to a left special factor with $m_l(\cdot) = 1$, whereas the other factors are such that $m_l(\cdot) = 0$.

Now, we apply the above procedure to all the letters $a \in A$ as starting factors, that is, by using a partition $\bigcup_{a \in A} RCyl_a$, until we get a finite set F of factors, all verifying $m_l(\cdot) \leq 1$. Let n_0 be the maximal length of the factors in F. By construction and by minimality, every factor longer than n_0 in Fact_L has a prefix in F, and prolongating such a prefix cannot increase its prolongation order. In case Γ is not coherent, we apply the same arguments to $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$, which is coherent by Proposition 3.4.5. According to Proposition 3.2.5, we can then start the above process with $\mathrm{Fact}_L(2)$ instead of A to reach the same conclusion. □

COROLLARY 4.2.2. *Let L be a minimal lamination language. Then there exists $n_0 > 0$ such that every vertex v in every Rauzy graph $\Gamma_{L,n}$, $n \geq n_0$, has degrees $\partial^-(v) \leq 2$ and $\partial^+(v) \leq 2$.*

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The above results do not hold in the non-minimal case. For instance, consider a loop e in Γ, labeled by a, whose vertex v is such that $\partial^+(v) = 1$ and $\partial^-(v) = m$, with $m > 1$, that is, v has $m - 1$ incoming edges besides e. The leaves of a lamination whose carrying by Γ uses the incoming edges at v, all end up running forever around e. In the corresponding lamination language L, we thus have $a^k \in \mathrm{Fact}_L$, for all $k > 0$, and $m_l(a^k) = m - 1$, for all $k > 0$.

An *infinite left* (respectively *right*) *special word* w with respect to Fact_L, where L is a language of infinite words, is a one-way infinite word such that all its prefixes (respectively suffixes) are left (respectively right) special factors in Fact_L. Its prolongation order $m_l(w)$ (respectively $m_r(w)$) is the smallest of the orders $m_l(\cdot)$ (respectively $m_r(\cdot)$) of all its prefixes (respectively suffixes). If L is a minimal shift, the infinite special words with respect to Fact_L are half-words of words in L.

PROPOSITION 4.2.3. (Finite number of infinite special words) *Let L be a minimal lamination language over A, coding a lamination $\mathcal{L}$ by a graph $\Gamma = (V_0, E_0)$. Let $k \geq 0$ be the smallest such that $\mathcal{L}$ is maximal relative to $\Gamma' = \mathrm{Slit}^k_{\mathcal{L}}(\Gamma) = (V_k, E_k)$ and such that Γ' is coherent. Then there are exactly $|E_k| - |V_k|$ infinite left special words (respectively right special words) with respect to* Fact_L*, all with $m_l(\cdot) = m_r(\cdot) = 1$.*

Proof. If L is made of a single periodic word, there is no infinite special word with respect to Fact_L, and there is $k \geq 0$ such that $\mathrm{Slit}^k_{\mathcal{L}}(\Gamma)$ is a cycle, so that $|E_k| - |V_k| = 0$. Let us then assume L is aperiodic. Let L be transformed into a language L' by a recoding by a length-$(k+1)$ sliding window over $\mathrm{Fact}_L(k+1)$, i.e. L' is the coding of $\mathcal{L}$ by Γ'. We prove the result for the left special words. Consider a neighborhood $N(\Gamma')$, and let α be an edge-rectangle intersection point in $N(\Gamma')$ on the left side of a vertex-trapezium. Let Q_u be the edge-rectangle labeled by $u \in \mathrm{Fact}_L(k+1)$ such that the interval $I_{l,u}$ meets α. A sequence of intervals $\{I_{l,u_i}\}_{i\geq 1}$ in $s_l(Q_u)$, each meeting α, is then built as follows: put $u_1 = u$, and translate I_{r,u_1} to the right until it meets edge-rectangle intersections, which must happen by the same argument as in the proof of Proposition 4.2.1. Following slitting curves starting at these intersection points towards the left until they hit I_{l,u_1} defines subintervals in it. Among these we keep the one meeting α, which exists since by maximality there is no finite slitting curve (see Proposition 3.3.3), and we call it I_{l,u_2}. We inductively obtain the next I_{l,u_i}, such that α meets I_{l,u_i} for every i, and this process goes to infinity because of minimality. The sequence $\{I_{l,u_i}\}_{i\geq 1}$ is decreasing, and its non-empty limit intersection meets at least α. By Proposition 4.2.1 this limit interval meets only α. We obtain a sequence $\{u_i\}_{i\geq 1}$ for which by construction, u_{i+1} is a left special factor of L' with u_i as a strict prefix for every i, that is, u_{i+1} is a prolongation of u_i. Thus there is an infinite left special word w, for which all the u_i are prefixes, and such that $m_l(w) = 1$. Let w' be the inverse of the recoding of w by the sliding window, i.e. the projection of each occurrence of $\mathrm{Fact}_L(k+1)$ to its first letter. Then w' is an infinite left special word with respect to Fact_L, such that $m_l(w') = 1$ too.

Now, we apply the above procedure starting with all the edge-rectangles Q_u of $N(\Gamma')$, where $u \in \mathrm{Fact}_L(k+1)$ is left special, so that the corresponding $I_{l,u}$ meet edge-intersections. We obtain associated sequences $\{u_i^{(u)}\}_{i>0}$. By construction, every left special factor of L' has some $u_i^{(u)}$ as a prefix, while being a prefix of $u_{i+1}^{(u)}$. Thus, the

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sequences $\{u_i^{(u)}\}_{i>0}$ include every sequence of left special factors. The same property holds also on L after the inverse recoding. The number of left edge-rectangle intersections in $N(\Gamma')$ is $|E_k| - |V_k|$, and since the carrying of $\mathcal{L}$ is full and $\mathrm{Fact}_{L'}$ is prolongable, the result follows. □

The above result sheds another light on the affine complexity of a minimal lamination language L: any left special factor in Fact_L of length $>\max(n_0, k)$, where n_0 comes from Proposition 4.2.1 and k from Proposition 4.2.3, belongs to a unique infinite left special word of order one, and all the prefixes of an infinite left special word are left special factors (equivalently for the right special case with suffixes). Therefore, the first difference complexity $p'_L(n)$ must be constant for $n \geq \max(n_0, k)$, hence the ultimate affine complexity of L.

We can also further explain a remark we made about L-slittable graph sequences (see §3.5): the fact that every vertex burst in a Rauzy graph sequence can be independently made into a local planar graph in the embedding surface is not sufficient to yield a lamination language. Examples are given by *Arnoux–Rauzy words* (or *strict episturmian words*) [**4, 20, 44**], which are the minimal words having, for each length, exactly one left and one right special factor with respective prolongation orders $m_r(\cdot) = m_l(\cdot) = |A| - 1$, where A is the alphabet. This constraint gives these words a complexity $p(n) = (|A| - 1)n + 1$, for all $n > 0$, and also that all the bipartite representations of the vertex bursts of their Rauzy graphs have individually biplanar drawings, since each one generates $m_l(\cdot) + m_r(\cdot) + 1$ edges.

COROLLARY 4.2.4. *Arnoux–Rauzy words over A are not lamination words if $|A| > 2$.*

Proof. According to Proposition 4.2.1, minimal lamination words have only special factors which are ultimately of order $m_r(\cdot) \leq 1$, $m_l(\cdot) \leq 1$ as their lengths increase. The case $|A| = 2$ corresponds to *Sturmian words* [**19, 32**], which are known to be symbolic orbits of minimal iets over two intervals, hence are lamination words. □

From the point of view of infinite special words, a difference between Arnoux–Rauzy words and minimal lamination words is the following.

- For every Arnoux–Rauzy word w over A, there is one single infinite left (respectively right) special word w' with respect to Fact_w, such that $m_l(w') = m_r(w') = |A| - 1$.
- For every aperiodic minimal lamination word w over A, there are $h > 0$ infinite left (respectively right) special words with respect to Fact_w, such that $m_l(\cdot) = m_r(\cdot) = 1$ for all of them. If w comes from the coding of a lamination $\mathcal{L}$ maximal relative to $\Gamma = (V, E)$, then $h = |E| - |V| = |A| - |V|$.

4.3. *Another characterization of lamination languages.* We now give another characterization of lamination languages, based on the properties of their bispecial factors. This characterization is an extension of the combinatorial characterization of the iet symbolic orbits as given in [**18**].

Let $\Gamma = (V, E)$ be a directed graph labeled by A. Let $\mathcal{O} = (\leq_{\mathrm{in}}, \leq_{\mathrm{out}})$ denote a pair of partial orders over A, where $\leq_{\mathrm{in}}$ (respectively $\leq_{\mathrm{out}}$) is defined for each vertex $v \in V$ on the subset of A labeling the edges incoming (respectively outgoing) at v. A pair $\mathcal{O}$ induces a

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canonical coherent ribbon graph structure (V, H, h, i, ξ) on Γ as follows: V, H, h, i are inherited from Γ, and the permutation ξ around each vertex $v \in V$ is defined by making a cycle of all the half-edges incident with v, first in the order $\leq_{\text{in}}$ on the incoming half-edges, and then in the opposite order to $\leq_{\text{out}}$ on the outgoing half-edges. Conversely, given a coherent ribbon structure over Γ and an orientation on the ribbon graph surface $\Sigma(\Gamma)$, there is a canonical pair $\mathcal{O}$ defined as follows: at each vertex v, all the incident incoming half-edges are ordered with respect to their sequence in the clockwise direction, and the outgoing ones according to the counterclockwise direction, respectively defining $\leq_{\text{in}}$ and $\leq_{\text{out}}$ around each vertex. Since the edges of Γ are in bijection with A, these partial orders translate to A. For instance, in the first figure of §3.3, we have $a_4 \leq_{\text{in}} a_3 \leq_{\text{in}} a_2 \leq_{\text{in}} a_1$, and $b_3 \leq_{\text{out}} b_2 \leq_{\text{out}} b_1$. Let L be a language over A, and let $\mathcal{O} = (\leq_{\text{in}}, \leq_{\text{out}})$ be also defined over A. Then, a bispecial factor u in Fact_L is said to be *compatible* with $\mathcal{O}$, if for all $a, b, c, d \in A$ with $a \neq b$, $c \neq d$, such that $auc, bud \in \text{Fact}_L$, we have $b \leq_{\text{in}} a \Rightarrow d \leq_{\text{out}} c$.

THEOREM A.2. *Let L be a shift in ${}^{\omega}A^{\omega}$. Then L is a lamination language if and only if there is a ribbon graph Γ labeled by A, and every word in Fact_L is a label of an admissible path of Γ in such a way that:*

- *if Γ is coherent, all the bispecial factors of L are compatible with the canonical pair of partial orders $\mathcal{O} = (\leq_{\text{in}}, \leq_{\text{out}})$ over A induced by Γ;*
- *if Γ is not coherent, then Γ is L-slittable, and all the bispecial factors of the recoding of L on $\text{Slit}_L(\Gamma)$ by a sliding window of length 2 over $\text{Fact}_L(2)$ are compatible with the canonical pair of partial orders $\mathcal{O} = (\leq_{\text{in}}, \leq_{\text{out}})$ over $\text{Fact}_L(2)$ induced by $\text{Slit}_L(\Gamma)$.*

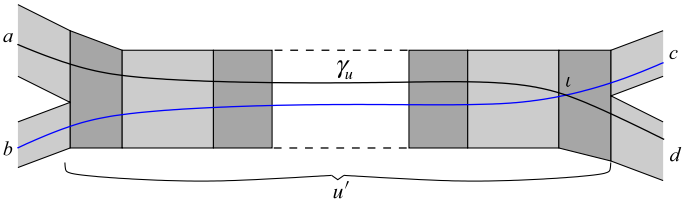
Proof. ($\Rightarrow$): By definition of a lamination language, there exists a ribbon graph Γ coding a lamination $\mathcal{L}$ into L. First, assume that Γ is coherent, and let $\mathcal{O}$ be the canonical pair of orders induced by Γ. Consider a regular neighborhood $N(\Gamma)$ containing $\mathcal{L}$. Let u be any length-n bispecial factor in Fact_L, with $n > 0$. Let auc and bud be any left and right prolongations of u, with $a, b, c, d \in A$, $a \neq b$, $c \neq d$, and with $b \leq_{\text{in}} a$. Using the terminology of §4.2, the pieces of leaves in X_{auc} and X_{bud} in $N(\Gamma)$ do not cross each other, being subsets of $\mathcal{L}$. The same property holds for X_u, and thus also for X_{auc} and X_{bud} within X_u. Hence, since $b \leq_{\text{in}} a$, if a piece of a leaf is in X_{auc}, it lies over all the pieces of leaves in X_{bud} all along its way, i.e. $d \leq_{\text{out}} c$. Hence, u is compatible with $\mathcal{O}$. If $n = 0$, i.e. u is the empty word, then if ac and bd visit the same vertex of Γ, the same reasoning as above applies; otherwise, a, b, c, d are not in relation via $\mathcal{O}$.

If Γ is not coherent, we can apply the same reasoning as above to $\text{Slit}_L(\Gamma)$ since Γ is L-slittable: $\text{Slit}_L(\Gamma)$ is coherent (cf. Proposition 3.4.5), and isomorphic to $\Gamma_{L,1}$ so its edges are labeled over $\text{Fact}_L(2)$ (cf. Construction 3.4.1).

($\Leftarrow$): Assume that Γ is coherent. We build a set of compact curves in $N(\Gamma)$ satisfying the hypothesis of Lemma 2.2.1 as follows: first, we consider an enumeration of the factors in Fact_L. Next, for each factor $u \in \text{Fact}_L$, we consider the path corresponding to u as a curve, denoted by γ_u. Now, let u be the nth factor of Fact_L. We assume that the $n-1$ curves associated with the $n-1$ first factors have been already placed in $N(\Gamma)$ in such a way that they are simple, pairwise disjoint and transverse to the fibers of $N(\Gamma)$, while their extremities have been moved only within the vertex-trapezia they lie in. We

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put γ_u in general position in $N(\Gamma)$, and in minimal intersection position with respect to the other $n-1$ curves and to itself. If there is no intersection, we are done. Otherwise, we push such an intersection ι in the forward direction, so that either we meet an extremity of one of the curves involved, and ι disappears, or there is a vertex-trapezium of $N(\Gamma)$ where the two arcs containing ι cross themselves to enter distinct edge-rectangles, say c and d, with $d \leq_{\text{out}} c$.



We can then attempt to push ι in the other direction, and again we have two cases: either ι disappears, or there is a vertex-trapezium of $N(\Gamma)$ where the two arcs containing ι cross themselves to enter distinct edge-rectangles, say a and b, with $b \leq_{\text{in}} a$. In this case, ι cannot be erased by homotopy. But then, there is a factor u' of u, labeling the path between the two vertex-trapezia where ι has been blocked, and because of the crossing involved, $au'd$ and $bu'c$ are also factors of u, belonging to Fact_L. Thus, u' would be a bispecial factor not compatible with $\mathcal{O}$, which is impossible by hypothesis. Hence, ι can always be erased by homotopy, and γ_u can be made simple and disjoint from the other curves. By inductively applying this process to every factor in Fact_L, we obtain a set of pairwise disjoint curves from Fact_L, carried by Γ when their extremities are taken back to the vertices they come from. Therefore, we can apply Lemma 2.2.1, and obtain a lamination $\mathcal{L}$ coded by Γ into L.

If Γ is not coherent, we can apply the same reasoning as above to $\text{Slit}_L(\Gamma)$, and we conclude by the same arguments as in the last paragraph of the proof of Theorem A.1. □

Like for Theorem A.1, knowing whether there is no ribbon graph with the properties required by Theorem A.2 is semi-decidable. Also, when maximality comes into play, another property based on $\mathcal{O}$ can be proved.

PROPOSITION 4.3.1. (Maximality and special factors) *Let $\mathcal{L}$ be a lamination coded by a coherent ribbon graph Γ into a language L over A. Let $\mathcal{O} = (\leq_{\text{in}}, \leq_{\text{out}})$ be induced by Γ. Then $\mathcal{L}$ is maximal relative to Γ if and only if for every left special factor $u \in \text{Fact}_L$, if $au, bu \in \text{Fact}_L$, where $a, b \in A$ are distinct and consecutive for $\leq_{\text{in}} \in \mathcal{O}$, there exists only one $x \in A$ such that $aux, bux \in \text{Fact}_L$ (a similar result holds for the right special factors and $\leq_{\text{out}}$).*

Proof. ($\Leftarrow$): Let $u \in \text{Fact}_L$ be any left special factor as required. Accordingly, the existence of u implies an intersection α between the two edge-rectangles a and b of $N(\Gamma)$. From α, a slitting curve γ starts, visiting the quadrilaterals of $N(\Gamma)$ following the sequence of labels given by u. According to Proposition 3.3.3, maximality of $\mathcal{L}$ implies that there is no finite slitting curve. Therefore, γ can be prolongated beyond u, so that γ enters an edge-rectangle, say labeled by x, and because the carrying is full, γ has leaves of $\mathcal{L}$ above

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and under it when entering the edge-rectangle x. Thus, the coding of these leaves shares the same continuation letter x, and this letter is the only one because leaves in $\mathcal{L}$ avoid γ.

($\Rightarrow$): The assumptions imply that the burst of each left special factor $u \in \mathrm{Fact}_L$ in its corresponding Rauzy graph is made of $\partial^+(u) + m_l(u)$ edges. Indeed, $m_l(u)$ is the number of slitting curves starting from the left, which is also the number of pairs of consecutive letters $a, b \in A$ such that $au, bu \in \mathrm{Fact}_L$. Because for each such pair there exists $x \in A$ such that $aux, bux \in \mathrm{Fact}_L$, each corresponding slitting must slit in two an edge outgoing from u, thus adding one edge to the burst. Recalling that $m_l(u) = \partial^-(u) - 1$, then according to Proposition 3.3.5, $\mathcal{L}$ must be maximal relative to Γ. □

4.4. *Measure, iet languages and bouquets of circles.* A lamination $\mathcal{L}$ obtained from a shift L using Theorems A.1 or A.2 admits sometimes a transverse measure with full support, in particular in the minimal case. A classic construction exists for geodesic laminations [**9, 10**] that we translate in our context as follows: let Γ be a carrier graph for $\mathcal{L}$, let $N(\Gamma)$ be a fibered neighborhood of Γ, and let $\mathcal{L}$ be considered as a closed subset of $N(\Gamma)$, with all its leaves transverse to the fibers. Let ℓ be a leaf in $\mathcal{L}$ with a base point, and let $\mathcal{Q}$ denote the set of quadrilaterals making $N(\Gamma)$. For each $Q \in \mathcal{Q}$ in intersection with ℓ, and for each transversal s included in Q, we define $\mu_m^Q(s) = (1/2m)|\{\gamma_m \cap s\}|$, where $m \in \mathbb{N}^*$, and where γ_m is the subarc of ℓ going exactly through m quadrilaterals of $N(\Gamma)$ to the left and the right with respect to the base point. Then there exists some sequence $m_i \to \infty$, so that $\{\mu_{m_i}^Q\}$ converges to a transverse measure μ^Q on $\mathcal{L} \cap Q$. Thus, we can successively obtain such transverse measures μ^{Q_j} for all $Q_j \in \mathcal{Q}$, with converging sequences being extracted in such a way that $\mu^{Q_j}(s) = \mu^{Q_k}(s)$ for any transversal s in $Q_j \cap Q_k$ where Q_k in $\mathcal{Q}$, so that the μ^{Q_j} extend to a transverse measure μ on $\mathcal{L}$. If $\mathcal{L}$ is minimal, ℓ is dense in it, and the support of μ is $\mathcal{L}$. Now, here is how Theorems A.1 and A.2 are restated for the iet case.

COROLLARY 4.4.1. *Let L be a minimal shift in ${}^{\omega}A^{\omega}$. Then L is an iet language if and only if there exists an L-slittable graph sequence $\{\Gamma_n\}_{n\geq 0}$, where Γ_0 is a coherent bouquet of circles labeled by A.*

Proof. An iet language L is the coding of a lamination $\mathcal{L}$ by a coherent bouquet of circles Γ_0 (see §2.1), hence Γ_0 is L-slittable, and the graphs $\Gamma_n = \mathrm{Slit}_{\mathcal{L}}^n(\Gamma_0)$ too. Conversely, according to Theorem A.1, from L we obtain a lamination $\mathcal{L}$ carried by Γ_0 in $\Sigma_0(\Gamma_0)$. Since L is assumed minimal, so is $\mathcal{L}$, and then $\mathcal{L}$ has a transverse measure as described above. Let $N(\Gamma_0)$ be a regular neighborhood containing $\mathcal{L}$. Let λ be the vector of the positive measures of a set of transverse fibers, one for each edge-rectangle of $N(\Gamma_0)$, and let π be the permutation defined by the ribbon structure of Γ_0. Then the iet corresponding to L is defined by (π, λ). □

COROLLARY 4.4.2. *Let L be a minimal shift in ${}^{\omega}A^{\omega}$. Then L is an iet language if and only if there is a coherent bouquet of circles Γ labeled by A, and every word in Fact_L is the label of an admissible path in Γ such that all the bispecial factors of L are compatible with the canonical pair of (total) orders $\mathcal{O}$ over A induced by Γ.*

Proof. Similar to Corollary 4.4.1. □

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In Corollary 4.4.2, if we make the additional assumption that the coded iet satisfies the idoc, i.e. the corresponding lamination is maximal, then the property described by Proposition 4.3.1 also holds for the shift L, leading to the same statement as the main result in [**18**].

Other general properties hold about minimal laminations and their relationships with iets. We present them now in context.

PROPOSITION 4.4.3. (Laminations carried by bouquets of circles) *Let $\mathcal{L}$ be a minimal lamination carried by a (not necessarily coherent) graph Γ. Then $\mathcal{L}$ is also carried by a coherent bouquet of circles.*

Proof. Assume first that Γ is coherent. Let v be a vertex of Γ, and let Q_v denote its corresponding vertex-trapezium in a neighborhood $N(\Gamma)$. Consider a set of slitting curves starting from all the edge-rectangle intersections of $N(\Gamma)$: these curves either eventually enter Q_v since $\mathcal{L}$ is minimal, or end up before reaching Q_v, being finite slitting curves. We apply all the possible right (respectively left) slitting basic steps on $N(\Gamma)$ along these slitting curves, until those which reach Q_v hit the left (respectively right) side of Q_v for the first time. We thus obtain a new neighborhood $N(\Gamma')$ with Γ' as a core graph, such that all the edge-rectangle intersections of $N(\Gamma')$ lie in the sides of Q_v. By construction, v is now a vertex of Γ' such that $\partial^-(v) \geq 1$ and $\partial^+(v) \geq 1$, whereas all the other vertices are trivial ones, i.e. vertices with $\partial^-(\cdot) = \partial^+(\cdot) = 1$, which means Γ' is a *bouquet of cycles*. Also, Γ' carries $\mathcal{L}$, and is coherent because Γ is. Erasing the trivial vertices of Γ' yields a bouquet of circles, carrying $\mathcal{L}$ too. In case Γ is not coherent, the same reasoning as above applies to $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ which carries $\mathcal{L}$ too (see Construction 3.4.1), and which is coherent (see Corollary 3.4.6). □

The above result is just a simplified form of the classic result saying that first-return maps of measure-preserving minimal flows on surfaces are essentially iets [**33, 34**]. Here, we can also interpret it into symbolic terms. Let A and B be two finite alphabets. A *substitution* is a map $\theta : A \to B^*$ which is extended to words by sending $w = \ldots w_i w_{i+1} w_{i+2} \ldots$ to $\theta(w) = \ldots \theta(w_i)\theta(w_{i+1})\theta(w_{i+2}) \ldots$. When applied to finite words, a substitution is just a monoid morphism between A^* and B^*. If L is a language of finite words, $\theta(L)$ denotes the language $\{\theta(w) | w \in L\}$. If L is a shift in ${}^{\omega}A^{\omega}$, then $\theta(L)$ denotes the shift generated by $\{\theta(w) | w \in L\}$. A substitution θ is said to be *non-erasing* if the length of $\theta(a)$ is positive, for all $a \in A$.

PROPOSITION 4.4.4. (Minimal lamination languages as iet language recodings) *Let $\mathcal{L}$ be a minimal lamination coded by a (not necessarily coherent) graph Γ into a minimal language L over A. Then there is an iet language L_T over B, and a non-erasing substitution $\theta : B \to A^*$ such that $L = \theta(L_T)$.*

Proof. Assume first that Γ is coherent. Consider the bouquet of cycles Γ' obtained from Γ in the proof of Proposition 4.4.3. Let A' be an alphabet disjoint from A, with as many letters as we want. We define a labeling of Γ' from its construction as follows: first, fixing a letter z in A', we relabel each edge e of Γ with a pair (a, z), where $a \in A$ is the original label of e. Second, following the slitting process of Proposition 4.4.3, each time an edge e

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labeled by $(a, \cdot)$ is slitted by a slitting basic step, we relabel the resulting two edges with (a, z_1) and (a, z_2), with $z_1, z_2 \in A'$, such that all the labels $(a, \cdot)$ in the resulting graph remain distinct. At the end of the slitting process to obtain Γ', each edge of Γ' has a distinct label in $A \times A'$, whose first entry records the label of the edge of Γ it comes from. Thus $\mathcal{L}$ is coded by Γ' into a language L' over $A \times A'$. Let $\theta_0 : A \times A' \to A$ be the substitution defined as $\theta_0((a, \cdot)) = a$, then $L = \theta_0(L')$. Now, each cycle of Γ' is a path with a label in $(A \times A')^*$. Consider the bouquet of circles obtained by erasing all the trivial vertices of the cycles of Γ', and denote it by Γ_T. We bijectively label each edge (circle) of Γ_T by letters in another alphabet B, and we define the substitution $\theta_1 : B \to (A \times A')^*$ by sending each edge letter of Γ_T to the label of its corresponding cycle in Γ'. Let L_T be the coding of $\mathcal{L}$ by Γ_T. Then, $L' = \theta_1(L_T)$, and therefore $L = \theta(L_T)$ where $\theta = \theta_0\theta_1$. Finally, since Γ is coherent, so is Γ_T. Hence there is an iet T corresponding to $\mathcal{L}$, and L_T is an iet language.

If Γ is not coherent, we apply the same reasoning as above to $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ which is coherent (cf. Proposition 3.4.5). The coding of $\mathcal{L}$ by $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ is a recoding by a length-2 sliding window $\kappa : \mathrm{Fact}_L(2) \to A''$, where A'' is another alphabet. Inverting this recoding is also a substitution $\theta_\kappa(a) = fst(\kappa^{-1}(a))$, $a \in A''$, where fst extracts the first letter of the word it applies to. Hence, $\theta = \theta_\kappa\theta_0\theta_1$, and $L = \theta(L_T)$ too. □

Note that the converse of Proposition 4.4.4 is false, that is, a recoding by a substitution of an iet language is not necessarily a lamination language. For instance, one can insert bispecial factors in the images of the letters by a substitution which contradict Theorem A.2. Note also that Arnoux–Rauzy words—i.e. affine complexity words which are generally not lamination words (see Corollary 4.2.4)—are known to be images of substitutions applied to iet symbolic orbits (however by using erasing substitutions) [**4**].

Another remark about the above results is that Proposition 4.4.3 does not ensure that $\mathcal{L}$ is maximal relative to the obtained bouquet of circles. Thus, for instance, the idoc for the corresponding iets does not necessarily hold, and also complexity computations as developed in §4.1 are more involved (one may need to use Corollary 4.1.3). The following result tackles this problem by using a translation in our context of *Rauzy induction* [**43**]. Recall first that a coherent bouquet Γ of m circles induces a permutation π over its circles, labeled $1, 2, \ldots, m$ in the order given by the embedding of its outgoing half-edges. A permutation is said to be *irreducible* if no strict subset of the form $\{1, 2, \ldots, i\}$, $1 \leq i < m$, is left stable by π. If π is reducible, Γ can be decomposed into a disjoint union of bouquets of circles, and if Γ carries a minimal lamination, its permutation π is irreducible.

PROPOSITION 4.4.5. (Maximality, bouquets of circles and Rauzy induction) *Let $\mathcal{L}$ be a lamination carried by a coherent bouquet of circles Γ, such that $\mathcal{L}$ is not maximal relative to Γ, and such that the permutation induced by Γ is irreducible. Then there is another coherent bouquet of circles relative to which $\mathcal{L}$ is maximal.*

Proof. Consider a neighborhood $N(\Gamma)$ with its unique vertex-trapezium Q (in this case necessarily a rectangle). In the left (respectively right) side of Q, let α_l (respectively α_r) denote the highest edge-rectangle intersection point. We say that $\alpha \in \{\alpha_l, \alpha_r\}$ is the highest intersection in Q if a slitting curve starting at α enters the interior of the top edge-rectangle

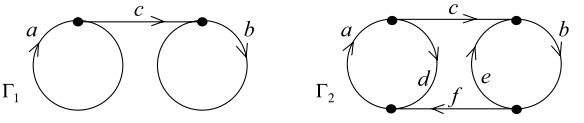
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on the other side of Q. By convention, if a vertex cutting occurs and the slitting curve joins α_l and α_r, we put $\alpha = \alpha_l$. Now, a slitting basic step $\mathrm{slit}_{\alpha,\mathcal{L}}$ yields a neighborhood $\mathrm{slit}_{\alpha,\mathcal{L}}(N(\Gamma))$ whose core graph is a bouquet of cycles. Indeed, let a, b be the two edge-rectangles in intersection at α (with a above b). Then $\mathrm{slit}_{\alpha,\mathcal{L}}$ goes to the other side of Q and slits the top edge-rectangle c into two sub-rectangles c', c'', where c' is above c''. In $\mathrm{slit}_{\alpha,\mathcal{L}}(N(\Gamma))$, c'' is what remains of c after the slitting, thus its core is still a cycle; and c' becomes the consecutive edge-rectangle to a through Q, generating also a cycle in the core graph. In the case of a vertex cutting, i.e. if $\mathrm{slit}_{\alpha,\mathcal{L}}$ stops at the side of Q opposite to the one containing α, the situation is analogous except that we only have the edge-rectangle a prolongated through Q by c, so the number of cycles in the core graph is one less. The slitting basic step $\mathrm{slit}_{\alpha,\mathcal{L}}$ corresponds to a *Rauzy induction step* [**43**]. Now, according to Corollary 3.3.4, we can make $\mathcal{L}$ maximal by performing all the finite slittings on $N(\Gamma)$. But here, to obtain this effect, we first consider a set of slitting curves starting from all the edge-rectangle intersections of $N(\Gamma)$, and we apply slitting basic steps along these curves in a sequence given by Rauzy induction steps as above. All the slitting curves are eventually used in the process since π is irreducible, so that all the finite slittings are eventually performed. Thus, $\mathcal{L}$ becomes maximal relative to a bouquet of cycles. Erasing the trivial vertices off the cycles yields a bouquet of circles. □

5. *Building lamination languages and examples*

5.1. *Building simple lamination languages.* There are cases where carrier graphs are simple enough so that some of their carried laminations can be devised by hand, involving only a few leaves. The simplest cases are given by carrier graphs made of m pairwise disjoint oriented cycles C_i with k_i edges each. These graphs carry laminations made of m isolated compact leaves, whose respective codings are the periodic words ${}^{\omega}(a_{i,1}a_{i,2}\ldots a_{i,k_i})^{\omega}$, where the $a_{i,j}$ are the distinct labels of the consecutive edges of C_i, and where '$(\cdot)$' denotes a factor to be iterated. The complexity of their associated lamination languages L, i.e. the smallest shifts containing these words, is $p_L(n) \equiv \sum_{i=1}^{m} k_i$, for all $n > 0$, as also given by Corollary 4.1.1. Here are less trivial examples of such simple laminations, which all happen to be non-minimal.

Example 5.1.1. Consider the two following carrier graphs.

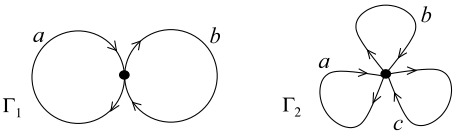


The only carried maximal lamination relative to Γ_1 is made of three leaves with codings ${}^{\omega}a^{\omega}$, ${}^{\omega}b^{\omega}$, ${}^{\omega}acb^{\omega}$. The complexity of its associated lamination language L is $p_L(n) = n + 2$, for all $n > 0$, as given by Corollary 4.1.1. For Γ_2, one of the carried maximal laminations is made of five leaves with codings ${}^{\omega}(ad)^{\omega}$, ${}^{\omega}(eb)^{\omega}$, ${}^{\omega}(acbf)^{\omega}$, ${}^{\omega}(da)c(be)^{\omega}$, ${}^{\omega}(da)(cbfa)^{\omega}$. The complexity of its associated lamination language L is $p_L(n) = 2(n + 2)$, for all $n > 0$.

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Non-coherent bouquets of circles also carry simple finite laminations made essentially of imbricated spiraling leaves.

Example 5.1.2. Consider the two following carrier non-coherent bouquets of circles.



One of the carried maximal laminations relative to Γ_1 is made of five leaves with codings ${}^{\omega}a^{\omega}$, ${}^{\omega}b^{\omega}$, ${}^{\omega}ab^{\omega}$, ${}^{\omega}(ab)^{\omega}$, ${}^{\omega}a(ba)^{\omega}$. The complexity of the associated lamination language L is $p_L(n) = 2n$, for all $n > 0$, as given by Corollary 4.1.2 since $|\mathrm{Fact}_L(2)| = |\{aa, ab, ba, bb\}| = 4$. One of the carried maximal laminations relative to Γ_2 is made of eight leaves with codings ${}^{\omega}a^{\omega}$, ${}^{\omega}b^{\omega}$, ${}^{\omega}c^{\omega}$, ${}^{\omega}ab^{\omega}$, ${}^{\omega}cb^{\omega}$, ${}^{\omega}a(bca)^{\omega}$, ${}^{\omega}c(abc)^{\omega}$, ${}^{\omega}(abc)^{\omega}$. The complexity of the associated lamination language L is $p_L(n) = 4n - 1$, for all $n > 0$, as also given by Corollary 4.1.2, since $|\mathrm{Fact}_L(2)| = |\{aa, bb, cc, ab, cb, bc, ca\}| = 7$.

5.2. *Building lamination languages from measured laminations.* A coherent graph $\Gamma = (V, E)$ is said to be *measured* by a weighting function $\mu : E \to \mathbb{R}^+$ if at every vertex v of Γ, the assigned weights satisfy the *branch equation* $\sum_i \mu(a_i) = \sum_j \mu(b_j)$, where the a_i denote all the incoming edges at v, and the b_j the outgoing ones. Given a measured carrier graph, there is a classic technique to obtain an associated measured lamination [**10, 23, 41**], closely related to the definitions we gave in §3.1.

Construction 5.2.1. (Measured laminations from measured carrier graphs) Let Γ be a coherent ribbon graph measured by μ. Similarly to Construction 3.1.1, to each edge and vertex of Γ we associate a quadrilateral, but here it is always a closed Euclidean *foliated rectangle* given as $I \times I'$, where individual leaves are the sets of the form $I \times \{x\}$, $x \in I'$. The edge-rectangles have heights equal to the edge weights given by μ, so that thanks to the branch equations, the height of each vertex-rectangle is the sum of the heights of its incoming (equivalently, its outgoing) edge-rectangles. Edge-rectangles are matched along the sides of the vertex-rectangles according to the pattern induced by the ribbon structure of Γ, and global leaves are defined by fitting together the individual leaves of the rectangles. We then obtain a Euclidean foliated band complex $N_\mu(\Gamma)$, embeddable into the corresponding ribbon graph surface $\Sigma(\Gamma)$, with singular leaves starting at each edge-rectangle intersection.

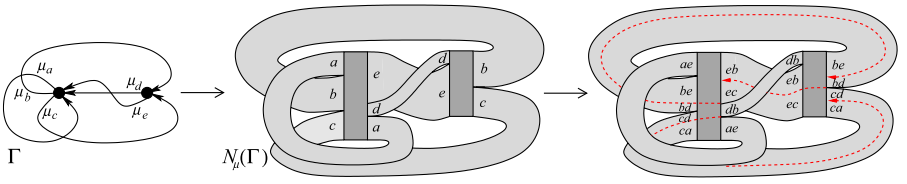
Slitting can then be described like in §3.1, but by using the singular leaves of $N_\mu(\Gamma)$ as slitting curves: slitting basic and global steps, $\mathrm{slit}_{\alpha,\mu}$, $\mathrm{Slit}_{r,\mu}$, $\mathrm{Slit}_{l,\mu}$, are defined similarly to $\mathrm{slit}_{\alpha,\mathcal{L}}$, $\mathrm{Slit}_{r,\mathcal{L}}$, $\mathrm{Slit}_{l,\mathcal{L}}$, using subleaves of singular leaves, horizontally crossing rectangles, and thus without any reference to any existing lamination $\mathcal{L}$. We can then iterate—to infinity if necessary—slitting basic steps $\mathrm{slit}_{\alpha,\mu}$ to exhaust all the singular leaves: like $\mathrm{slit}_{\alpha,\mathcal{L}}$, each $\mathrm{slit}_{\alpha,\mu}$ depends on a separating distance ϵ_i between divided rectangles, and when these distances are globally set to form converging series, a limit set of leaves exists. It is then a classic result that this limit set, under the equivalence relation generated by curve

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isotopy, is a lamination $\mathcal{L}_0$ [**23, 41**]. This lamination is unique up to isotopy and does not depend on the chosen distances ϵ_i. Moreover, $\mathcal{L}_0$ is carried by Γ, and has a transverse measure determined by μ. It is also a fact that $\mathrm{slit}_{\alpha,\mathcal{L}_0}$, $\mathrm{Slit}_{r,\mathcal{L}_0}$, $\mathrm{Slit}_{l,\mathcal{L}_0}$, have respectively the same effect on Γ as $\mathrm{slit}_{\alpha,\mu}$, $\mathrm{Slit}_{r,\mu}$, $\mathrm{Slit}_{l,\mu}$. Note that if Γ is strongly connected, Γ can always be measured by a set of positive weights, so that the carrying of $\mathcal{L}_0$ by Γ is full.

Construction 5.2.2. (Lamination languages from measured carrier graphs) Let Γ be a coherent ribbon graph measured by μ. Using the above Construction 5.2.1, we can obtain step by step the terms of the corresponding Rauzy graph sequence $\{\Gamma_{L,n}\}_{n>0}$, that is, the factors in $\mathrm{Fact}_L(n)$, where L is the coding of the limit lamination $\mathcal{L}_0$. Indeed, according to Proposition 3.2.5, $\Gamma_{L,n}$ corresponds to $\mathrm{Slit}^n_{\mathcal{L}_0}(\Gamma)$, which in turn corresponds to $\mathrm{Slit}^n_{\mu}(\Gamma)$, performed by using the edge-rectangle heights of $N_\mu(\Gamma)$, i.e. the Γ's edge weights.

Example 5.2.3. Let Γ be a measured graph of five edges with weights $\mu_a = 1$, $\mu_b = \sqrt{2}$, $\mu_c = \sqrt[3]{2}$, $\mu_d = \mu_b + \mu_c - \mu_e$, $\mu_e = \sqrt{5}$ (so that $\mathcal{L}_0$ of Construction 5.2.1 is minimal).



The first step of Construction 5.2.2 means to obtain $\mathrm{Fact}_L(2)$, where L codes $\mathcal{L}_0$: to the right of the figure, $\mathrm{Slit}_\mu(N_\mu(\Gamma))$ is applied as a left slitting global step, using the μ_i which satisfy $\mu_a < \mu_e < \mu_a + \mu_b < \mu_e + \mu_d$, and $\mu_d < \mu_b$, on the vertex-rectangle sides. We next use the labeling definition of $\mathrm{Slit}_\mu(N_\mu(\Gamma))$ (see §3.2) and get $\mathrm{Fact}_L(2) = \{ae, be, bd, cd, ca, db, eb, ec\}$ since $\mathrm{Slit}_\mu(\Gamma) = \Gamma_{L,1}$.

It is also possible to exploit some of the results of §4.4.

Construction 5.2.4. (Lamination languages from measured carrier graphs using Rauzy induction) Let Γ be a coherent ribbon graph measured by μ, with μ defined such that the lamination $\mathcal{L}_0$ of Construction 5.2.1 is minimal. Let L be the coding of $\mathcal{L}_0$ by Γ. We can then carry out the following steps.

(1) Let $N_\mu(\Gamma)$ be a foliated neighborhood of Γ. We apply Proposition 4.4.3 to $\mathcal{L}_0$ and Γ, by using slittings associated with the singular leaves of $N_\mu(\Gamma)$. The result is a coherent measured bouquet of circles Γ_T, also carrying $\mathcal{L}_0$. Following Proposition 4.4.4, let L_T be the coding of $\mathcal{L}_0$ by Γ_T over the alphabet B, and let θ be the associated substitution such that $L = \theta(L_T)$.

(2) We apply Construction 5.2.1 to effectively build $\mathcal{L}_0$ from Γ_T to get factors of L_T. But instead of using Construction 5.2.2, we take advantage of the fact that Γ_T is a bouquet of circles, and we use Proposition 4.4.5, that is, we apply Rauzy induction as a slitting process to get longer and longer pieces of leaves of $\mathcal{L}_0$. The relevance of using Rauzy induction is that it has a well-known symbolic translation into a finite set of substitutions. From the proof of Proposition 4.4.5, these substitutions can be computed by looking at the effect produced by each type of slitting basic step on a bouquet of circles starting from its highest intersection. Indeed, each

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such step adds one edge to one of the circles while keeping the other circles fixed, possibly modifying the permutation of the circles. Therefore, such a substitution permutes the letters of B, except one of them whose image is a length-2 word (see the details e.g. in [**29**]). A composition of slitting basic steps as produced by Proposition 4.4.5 translates into a composition of these substitutions. The longer the sequence of slitting basic steps, the longer the substitution composition, hence the longer the generated factors of symbolic orbits of the iet language L_T. If finite slittings occur—i.e. the iet T associated with L_T does not satisfy the idoc—some of the singular leaves are finite, and vertex cuttings eventually occur, each reducing by one the number of subintervals of T, while keeping the resulting graph connected because of minimality. Each time this happens, the associated substitutions must be recomputed on the alphabet minus the label of the erased subinterval.

(3) We finally apply the substitution θ to the factors of L_T generated in the preceding step (2) to obtain factors of the lamination language L.

Summing up the above construction, we see that arbitrarily long factors in Fact_L are obtained, first by composing a finite number of substitutions associated with Rauzy induction, and second, by applying a last substitution representing the first return map (the substitution θ of Proposition 4.4.4). This process is thus related to a classic kind of word generating system in formal language theory, called *HDT0L-systems* [**24, 25, 46**]. Moreover, in the minimal case, a lamination language L can be explicitly described from Fact_L as follows: consider a sequence $\{u_i\}_{i\geq 1}$ of factors in Fact_L, such that u_i is a strict prefix of u_{i+1}, for every i. Then, since u_{i+1} is just a prolongation of u_i, there exists a one-way right infinite word w of L, i.e. a right half lamination word, for which all the u_i are prefixes. Since we are in the minimal case, the corresponding two-way infinite boundary language of w is L itself (see §2.2). Thus, in the case, as here, where Fact_L has been built by composing a finite number of substitutions, this way of generating affine complexity shifts like L is related to *S-adic systems* [**13, 19**].

Now, the simplest instances of the above generating processes are associated with the following technique: let $\theta : A \to A^*$ be a substitution over an alphabet A. A word w is a *fixed point* of θ if $\theta(w) = w$, and is a *periodic point* if there exists $m > 1$ such that $\theta^m(w) = w$. One-way right infinite fixed and periodic points are obtained by iterating θ when $a \in A$ is a strict prefix of $\theta^m(a)$ for some $m \geq 1$, so that $\theta^{mn}(a)$ is a strict prefix of $\theta^{m(n+1)}(a)$, for all $n > 0$; again, this prefix-preserving property produces right one-way infinite words of the form $w = (\theta^m)^\omega(a)$, such that $\theta^m(w) = w$. For instance, the *Fibonacci substitution* [**19**] is defined by $\theta_f(a) = ab$, $\theta_f(b) = a$, and the word $\theta_f^\omega(a) = abaababaabaababaababa \ldots$ is a fixed point of θ_f; incidentally, $\theta_f^\omega(a)$ is known to be a positive symbolic orbit of the iet language corresponding to the minimal iet T_ρ over the two subintervals $[0, \rho)$ and $[\rho, 1)$, where ρ is the inverse of the *golden mean*. Such words obtained by infinitely iterating a substitution are called *purely substitutive* (or *purely morphic*), and they are related to *D0L*-systems [**25, 46**]. Also, words which are images of purely substitutive words by another substitution are called *substitutive* (or *morphic* [**3**]). These words belong to a class strictly larger than the one of purely substitutive words, and they are related to *tag systems* [**3, 15**] and to *HD0L-systems* [**25, 46**]

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(where *D0L*-systems $\subsetneq$ *HD0L-systems* $\subsetneq$ *HDT0L-systems*). In Construction 5.2.4, when Rauzy induction generates periodic compositions of substitutions, we thus get half lamination words of the substitutive kind.

The next section develops a different technique to obtain purely substitutive minimal half lamination words.

5.3. *Building lamination languages from pseudo-Anosov homeomorphisms.* A *Dehn twist* τ is a basic homeomorphism of a surface Σ, associated with a simple closed oriented curve γ, which consists in cutting Σ along γ, and in reglueing it after a full twist turn. Considering a carrier graph Γ embedded in Σ, if τ transforms every edge of Γ into a curve also carried by Γ, then τ induces a *carrier graph map* τ_Γ, sending each edge e of Γ to the admissible path carrying $\tau(e)$ (this is an oriented version of a usual *train track map* [**8**]). From the coding point of view, τ_Γ induces a substitution, sending the label of each edge of Γ to the label of its image by τ_Γ.

Among the homeomorphisms of Σ, the *pseudo-Anosov homeomorphisms* are characterized by the fact that they have one stable and one unstable measured aperiodic minimal lamination. There are cases where a pseudo-Anosov homeomorphism f can be described as a composition of a set of Dehn twists all inducing carrier graph maps on the same graph Γ [**40, 50**]. Composing the corresponding substitutions yields a substitution θ corresponding to f. Then, iterating f on curves carried by Γ produces pieces of curves of its stable lamination, so that iterating θ produces factors of the coding of this lamination [**31**].

We now give a practical description of a technique derived from the above facts to obtain lamination languages. We say that a directed graph $\Gamma = (V, E)$ is *cycle-based* if it is strongly connected and if it can be obtained as the union of $m = k + h$ oriented cycles $\{C_1, \ldots, C_m\}$ as follows:

(i) $\{C_1, \ldots, C_k\}$ is a set of pairwise disjoint cycles with respective non-empty finite sets of vertices V_i, such that $V = \bigcup_{i=1}^{k} V_i$;

(ii) π is a permutation over V such that $v \in V$ is linked to $\pi(v)$ by an edge in E not in $\{C_1, \ldots, C_k\}$, thus determining the other cycles $\{C_{k+1}, \ldots, C_{k+h}\}$.

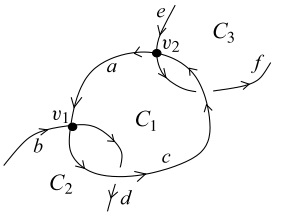
Note that every vertex $v \in V$ has degrees $\partial^-(v) = \partial^+(v) = 2$. Now, to a cycle-based graph Γ on m cycles C_i, we can associate m substitutions. In fact, through this association process, Γ inherits a coherent ribbon structure, inducing embedding surfaces in which these substitutions correspond to graph maps over Γ, which in turn correspond to the Dehn twists associated with each of the C_i. In order to describe these substitutions, here is a method based on a two-dimensional representation.

Construction 5.3.1. (Substitutions from cycle-based graphs) Let $\Gamma = (V, E)$ be a cycle-based graph on $m = k + h$ cycles, and labeled by A. We can then carry out the following steps.

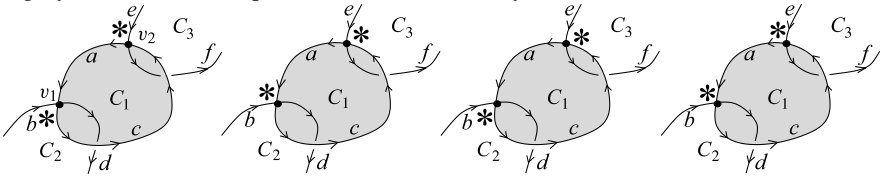
(1) We draw Γ on the plane as follows: we first draw the cycles $\{C_1, \ldots, C_k\}$ in a disjoint way. Next, we draw the cycles $\{C_{k+1}, \ldots, C_{k+h}\}$ in such a way that at each vertex $v \in V$, there is an oriented coherent crossing between exactly two cycles, and such that the crossing orientations are globally consistent over all of Γ (the relative orientations of the edges at any crossing must match when translated along any

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edge path of Γ). These orientations fix a permutation around each vertex, and thus determine a coherent ribbon structure for Γ. A cycle-based graph being generally non-planar, the other crossings which arise in the drawing are just replaced by cuts. For instance, the following figure shows three cycles making two vertex crossings with consistent orientations.



(2) For each vertex crossing, we choose one of the four quadrants it determines, subject to the following constraint: for each cycle C of Γ, the chosen quadrants at the vertices along C must lie in the same complementary region of C in the plane. For instance, here are the four possible choices of quadrants with respect to the above figure for the cycle C_1 and its external complementary region (there are four more patterns for its internal complementary region); the internal complementary region of C_1 is shown in gray, and the chosen quadrants are indicated by stars.



(3) We derive a substitution θ_i from each cycle C_i as follows: let $c_i^{(v)}$ be the finite path label of the cycle C_i starting from the vertex $v \in C_i$. For instance in the above figure, $c_1^{(v_1)} = ca$ and $c_1^{(v_2)} = ac$. At each vertex v, the chosen quadrant is defined by two half-edges. Let $h_i^{(v)}$ denote the half-edge which is not on the cycle C_i, and let $x_i^{(v)}$ be the label of the edge containing $h_i^{(v)}$. For instance, in the four cases in the above figure, $x_1^{(v_1)} = b$ and $x_1^{(v_2)} = e$ (for the four other choices of quadrants, $x_1^{(v_1)} = d$ and $x_1^{(v_2)} = f$). The substitution θ_i is then the identity over all the letters in A, except for the letters $x_i^{(v)}$ where v is a vertex of C_i, for which:

$$\theta_i(x_i^{(v)}) = \begin{cases} x_i^{(v)}\, c_i^{(v)} & \text{if } h_i^{(v)} \text{ is incoming at } v, \\ c_i^{(v)}\, x_i^{(v)} & \text{if } h_i^{(v)} \text{ is outgoing from } v. \end{cases}$$

Intuitively, this scheme represents the dragging of the incident edges with C_i along C_i (topologically this is the carrier graph map corresponding to a Dehn twist application along C_i in the embedding surface, and all the above conditions ensure this fact). For instance, in the four cases in the above figure, θ_1 is the identity except for $x_1^{(v_1)} = b$ and $x_1^{(v_2)} = e$, for which:

$$\theta_1(b) = bca \quad \text{since } c_1^{(v_1)} = ca,$$
$$\theta_1(e) = eac \quad \text{since } c_1^{(v_2)} = ac.$$

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For the four other choices of quadrants, θ_1 is the identity except for $x_1^{(v_1)} = d$ and $x_1^{(v_2)} = f$, for which:

$$\theta_1(d) = cad \quad \text{since } c_1^{(v_1)} = ca,$$
$$\theta_1(f) = acf \quad \text{since } c_1^{(v_2)} = ac.$$

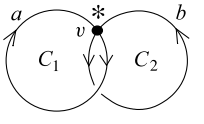
Let $\mathcal{T} = \{\theta_1 \dots \theta_n\}$ be a finite set of substitutions. A composition θ of substitutions in $\mathcal{T}$ is said to be *full* if each $\theta_i \in \mathcal{T}$ appears at least once in θ. Given a cycle-based graph Γ labeled by A, let $\mathcal{T}_\Gamma$ be the set of substitutions induced by all the cycles of Γ as in the above Construction 5.3.1. If the quadrant choices are the same for all the vertices of Γ, any full composition θ of $\mathcal{T}_\Gamma$ is a *primitive substitution* [**42**], i.e. there exists an integer $k \geq 1$ such that for all $a, b \in A$, the word $\theta^k(a)$ contains b. Indeed, a cycle-based graph is strongly connected, and whenever the quadrant choices are uniform, for every letter $a \in A$ there is one of the θ_i in $\mathcal{T}_\Gamma$ which sends a to a word made out of a concatenated with the label of one of the cycles of Γ; thus, iterating θ eventually includes all the cycle labels of Γ. Moreover, a primitive substitution θ is such that there is at least one letter a which is a strict prefix of $\theta^m(a)$ for some $m \geq 1$. In fact, there are always uniform quadrant choices in Construction 5.3.1 so that the full compositions of $\mathcal{T}_\Gamma$ are all such that $m = 1$. Thus, at least one fixed point w of θ can be obtained by iterating θ on some letter a, i.e. $w = \theta^\omega(a)$ (see §5.2), and because of the primitivity of θ, w is necessarily minimal [**42**]. The minimal shift generated by w is then the two-way infinite boundary language of Fact_w (see §2.2). If there are more letters in A with the strict prefix property for θ, they all generate the same minimal shift. Now, the following result proved in [**31**] is an application of the pseudo-Anosov homeomorphism construction from [**16, 40, 50**].

Construction 5.3.2. (Lamination languages from pseudo-Anosov homeomorphisms) Consider a cycle-based graph $\Gamma = (V, E)$ on m cycles, and let $\mathcal{T}_\Gamma = \{\theta_1, \dots, \theta_m\}$ be the set of the associated substitutions obtained from Construction 5.3.1. Then every one-way infinite fixed point w of a full composition substitution of $\mathcal{T}_\Gamma$ is a minimal half lamination word. Moreover, the corresponding two-way infinite boundary L of Fact_w is a minimal lamination language which codes a lamination carried by Γ, which is maximal relative to Γ.

Since the laminations involved in Construction 5.3.2 are maximal relative to their carrier cycle-based graphs, Corollary 4.1.1 applies, and the complexity of their corresponding lamination languages L is $p_L(n) = |V|n + |V|$, for all $n > 0$, over an alphabet of $|E| = 2|V|$ letters. Since these laminations are minimal too, any half lamination word w in L is such that $p_w \equiv p_L$.

Here is the simplest instance of Construction 5.3.2.

Example 5.3.3. (Sturmian case) Let Γ be a cycle-based graph built from one cycle C_1 with one vertex v, and from a trivial π, inducing another cycle C_2, so that Γ is a coherent bouquet of two circles.

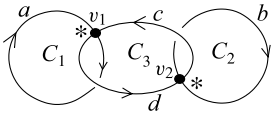


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With the chosen quadrant at v, indicated in the figure by a star, the two corresponding substitutions are: $\theta_1(a) = a$, $\theta_1(b) = ba$, and $\theta_2(a) = ab$, $\theta_2(b) = b$. Any fixed point of any full composition of θ_1, θ_2 is a minimal half lamination word w with complexity $p_w(n) = n + 1$, for all $n > 0$, i.e. a one-way infinite Sturmian word (in fact, θ_1, θ_2, together with the flip substitution $\theta_E(a) = b$, $\theta_E(b) = a$, are generators of the *Sturm monoid* [**32**]).

Here are the two next possible simplest families of examples; both of them are made of cycle-based graphs with two vertices and four edges.

Example 5.3.4. Let Γ be a cycle-based graph built from two cycles C_1, C_2 with one vertex each, respectively v_1, v_2, and from $\pi(v_1) = v_2$, $\pi(v_2) = v_1$, inducing a third cycle C_3.

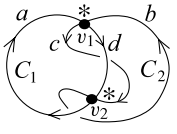


With the quadrants indicated in the figure by stars, the three corresponding substitutions are:

$$\begin{array}{lll} \theta_1(a) = a & \theta_2(a) = a & \theta_3(a) = adc \\ \theta_1(b) = b & \theta_2(b) = b & \theta_3(b) = bcd \\ \theta_1(c) = c & \theta_2(c) = bc & \theta_3(c) = c \\ \theta_1(d) = ad & \theta_2(d) = d & \theta_3(d) = d. \end{array}$$

Any fixed point of any full composition of these substitutions is a minimal half lamination word w over four letters, with complexity $p_w(n) = 2n + 2$, for all $n > 0$. For instance, if $\theta = \theta_1\theta_2\theta_3$, then $\theta(a) = aadbc$, $\theta(b) = bbcad$, $\theta(c) = bc$, $\theta(d) = ad$, and e.g. $w = \theta^\omega(a)$ is such a half lamination word.

Example 5.3.5. Let Γ be a cycle-based graph built from one cycle C_1 with two vertices v_1, v_2, and from $\pi(v_1) = v_2$, $\pi(v_2) = v_1$, inducing a second cycle C_2 (here, the swerving-like aspect of C_2 is necessary to represent the two crossings with consistent local orientations).



With the quadrants indicated in the figure by stars, the two corresponding substitutions are:

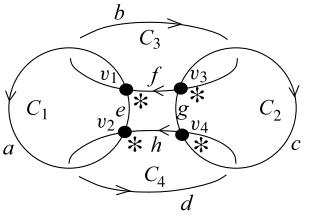
$$\begin{array}{ll} \theta_1(a) = a & \theta_2(a) = acb \\ \theta_1(b) = bda & \theta_2(b) = b \\ \theta_1(c) = cad & \theta_2(c) = c \\ \theta_1(d) = d & \theta_2(d) = dbc. \end{array}$$

Like in Example 5.3.4, any fixed point of any full composition of these substitutions generates minimal half lamination words w over four letters, with complexity $p_w(n) = 2n + 2$, for all $n > 0$.

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Here are two more examples of Construction 5.3.2.

Example 5.3.6. Let Γ be a cycle-based graph built from two cycles C_1, C_2 with two vertices each, respectively v_1, v_2 and v_3, v_4, and from $\pi(v_1) = v_3, \pi(v_3) = v_1, \pi(v_2) = v_4$, $\pi(v_4) = v_2$, inducing two other cycles C_3, C_4.

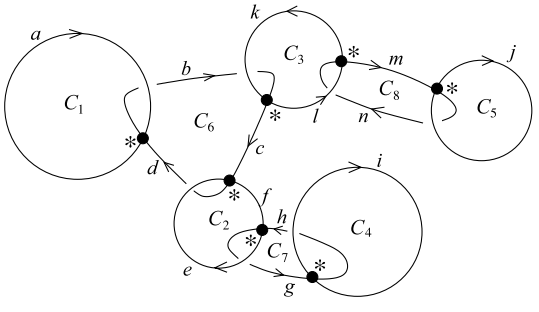


With the quadrants indicated in the figure by stars, the four corresponding substitutions are (we only give the images of the letters which are not the identity):

$$\begin{array}{llll} \theta_1(f) = fae & \theta_2(b) = bcg & \theta_3(g) = gfb & \theta_4(c) = chd \\ \theta_1(h) = hea & \theta_2(d) = dgc & \theta_3(e) = ebf & \theta_4(a) = adh. \end{array}$$

Any fixed point of any full composition of these substitutions is a minimal half lamination word w over eight letters, with complexity $p_w(n) = 4n + 4$, for all $n > 0$.

Example 5.3.7. Let Γ be a cycle-based graph built from five cycles $C_1, \ldots, C_5$ with seven vertices, and from π determining three other cycles C_6, C_7, C_8.



With the quadrants indicated in the figure by stars, the eight corresponding substitutions are (we only give the images of the letters which are not the identity):

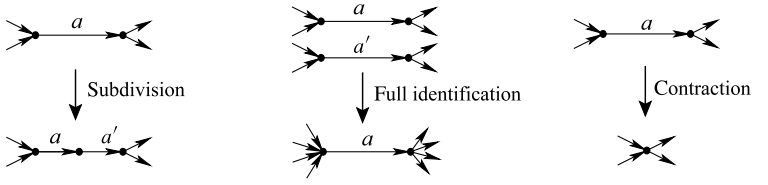
$$\begin{array}{llllll} \theta_1(b) = ab & \theta_3(c) = lkc & \theta_4(h) = ih & \theta_6(a) = bcda & \theta_7(e) = ghe & \theta_8(k) = mnk \\ \theta_2(d) = fed & \theta_3(m) = klm & \theta_5(n) = jn & \theta_6(f) = dbcf & \theta_7(i) = hgi & \theta_8(j) = nmj \\ \theta_2(g) = efg & & & \theta_6(l) = cdbl. & & \end{array}$$

Any fixed point of any full composition of these substitutions is a minimal half lamination word w over fourteen letters, with complexity $p_w(n) = 7n + 7$, for all $n > 0$.

5.4. *Lamination language transformations and carrier graph moves.* Let $\mathcal{L}$ be a lamination carried by a ribbon graph Γ. A *graph move* t of Γ, *compatible* with $\mathcal{L}$, is a transformation of Γ into another graph $t(\Gamma)$, for which there is a surface $\Sigma_t(\Gamma)$ in which

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both Γ and $t(\Gamma)$ are embedded, such that $\mathcal{L}$ is a lamination carried and coded by $t(\Gamma)$ in $\Sigma_t(\Gamma)$. If t is compatible with all the laminations carried by Γ we just call it a *graph move*. For instance, a slitting global step on Γ with respect to a carried lamination $\mathcal{L}$ is a graph move compatible with $\mathcal{L}$ where $\Sigma_t(\Gamma)$ is just the ribbon graph surface $\Sigma(\Gamma)$, and the reverse of such a slitting global step, defined on the same surface, is a graph move (see Corollary 3.4.4). We now define three additional simple graph moves whose main use here is to illustrate how one can transform lamination languages into other lamination languages.



(1) An *edge subdivision* t of an edge a in a graph Γ consists in putting a new vertex v on a, subdividing it into two edges a, a', so that v has degree two. Edge subdivisions take place in $\Sigma_t(\Gamma) = \Sigma(\Gamma)$ and preserve fullness and maximality of the carried laminations, and coherence of Γ.

Let $\mathcal{L}$ be a lamination coded by Γ into a language L over A. Let θ be the substitution over A which is the identity except at a, where $\theta(a) = aa'$, then the language coding $\mathcal{L}$ by $t(\Gamma)$ is $L' = \theta(L)$. If $\mathcal{L}$ is maximal relative to Γ, then since $\mathrm{Fact}_{L'}(2) = \mathrm{Fact}_L(2) + 1$, and the alphabet A' of L' is such that $|A'| = |A| + 1$, by Corollary 4.1.2, we have $p_{L'}(n) = p_L(n) + 1$, for all $n > 0$.

(2) A *full edge identification* t of two edges a and a' in a ribbon graph Γ, is such that a and a' have no vertex in common, and consists first in glueing the upper oriented side of the rectangle corresponding to a in the ribbon graph surface $\Sigma(\Gamma)$ with the lower oriented side of the rectangle corresponding to a', or the converse. The surface obtained after this glueing operation is $\Sigma_t(\Gamma)$, where $t(\Gamma)$ is defined by identifying the edges a and a', using an homotopy within a neighborhood of their glued rectangles containing a and a'. Full edge identifications preserve fullness, but not necessarily coherence of Γ, and never maximality of the carried laminations.

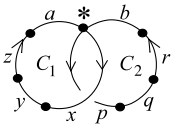
Let $\mathcal{L}$ be a lamination coded by Γ into a language L over A. Let θ be the substitution over A which is the identity except at a', where $\theta(a') = a$, then the language coding $\mathcal{L}$ by $t(\Gamma)$ is $L' = \theta(L)$. If $\mathcal{L}$ is maximal relative to Γ, the identification of a and a' creates one finite slitting curve involving one edge, and by Corollary 4.1.3, we have $p_{L'}(n) = p_L(n)$, for all $n \geq 2$, and $p_{L'}(1) = p_L(1) - 1$, since $\mathrm{Slit}^2_{\mathcal{L}}(t(\Gamma))$ separates the edges a and a' again.

Example 5.4.1. A word w is *quasi-Sturmian* if its complexity function is $p_w(n) = n + h$, for all $n > K$, for some integers $h > 0$ and $K \geq 0$ [**12**]. Let us reconsider Example 5.3.3 and its bouquet of two circles Γ. We subdivide k times each circle of Γ, and we get a graph Γ'. Any leaf coded by Γ is then coded by Γ' into a word w' over an alphabet of $2(k+1)$ letters, and $p_{w'}(n) = n + (1 + 2k)$, for all $n > 0$. Next, considering the edges

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in Γ' with no vertex in common with the ones initially in Γ, we apply $k' < k$ consecutive full edge identifications by pairs between the two cycles, and we get a graph Γ''. A coding w' as above is then transformed into a word w'' over an alphabet of $2(k+1) - k'$ letters, and by Corollary 4.1.3, $p_{w''}(n) = n + (1 + 2k)$, for all $n > k'$, that is, w'' is a quasi-Sturmian word.

For instance, let $k = 3$, so that the two circles a, b of Γ are subdivided three times each. Let x, y, z (in this order) be the new edges given by the subdivision of a, and let p, q, r (in this order) be the new edges given by the subdivision of b, defining a graph Γ' as follows.



Consider any fixed point of any full composition of the substitutions θ_1, θ_2 of Example 5.3.3, and apply to it the substitution $\theta'(a) = xyza$, $\theta'(b) = pqrb$, reflecting the subdivisions of a and b, to get a minimal half lamination word w' over $\{a, b, x, y, z, p, q, r\}$ with complexity $p_{w'}(n) = n + 7$, for all $n > 0$. Next, we apply $k' = 2$ full edge identifications: one identifying q and y, the other one r and z. Let θ'' be the substitution reflecting this graph move, which is the identity except for $\theta''(q) = y$ and $\theta''(r) = z$. Then, $w'' = \theta''(w')$ is a quasi-Sturmian word with $p_{w''}(n) = n + 7$, for all $n > 2$, and $p_{w''}(1) = 6$, $p_{w''}(2) = 8$.

(3) An *edge contraction* t of an edge a linking two distinct vertices v and v' in a ribbon graph Γ, with E_v and $E_{v'}$ as respective sets of incident edges, consists in replacing v and v' with a single vertex, whose set of incident edges is $(E_v \cup E_{v'})\backslash\{a\}$ using the same cyclic order as in Γ, and such that all the edges of E_v (equivalently $E_{v'}$) are consecutive around it. For an edge contraction, $\Sigma_t(\Gamma) = \Sigma(\Gamma)$. Edge contractions preserve fullness, but not necessarily maximality of the carried laminations and coherence of Γ.

Let $\mathcal{L}$ be a lamination coded by Γ into a language L over A. Let θ be the substitution over A which is the identity except that it erases a by sending it to the empty word, then the language coding $\mathcal{L}$ by $t(\Gamma)$ is $L' = \theta(L)$. Let $\Gamma = (V, E)$ and $t(\Gamma) = (V', E')$. If Γ and $t(\Gamma)$ are coherent, and if $\mathcal{L}$ is maximal relative to Γ and $t(\Gamma)$ (the general case is studied in Proposition 5.4.3 below), then since $|V'| = |V| - 1$, $|E'| = |E| - 1$, by Corollary 4.1.1, we have $p_{L'}(n) = p_L(n) - 1$.

Remark 5.4.2. (Contracted carrier graphs and bouquets of circles) Let $\Gamma = (V, E)$ be a connected graph which codes a lamination $\mathcal{L}$ into a language over A. Then there is a bouquet of circles which codes $\mathcal{L}$ into a language over an alphabet of $|A| - |V| + 1$ letters.

Proof. We contract all the edges linking distinct vertices, and since Γ is connected, we get a bouquet of $|E| - |V| + 1$ circles, still carrying $\mathcal{L}$. □

Note that in contrast to the use of slittings in Proposition 4.4.3, the use of edge contractions as in Remark 5.4.2 often leads to non-coherent bouquets of circles (see e.g. Example 5.4.5 below), and laminations carried by them do not correspond to iets.

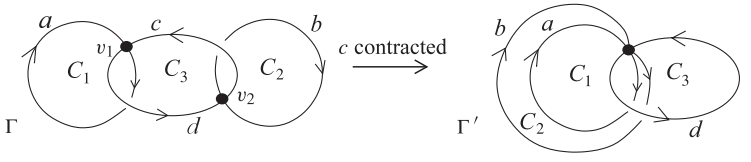
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PROPOSITION 5.4.3. (Maximality and edge contractions) *Let $\mathcal{L}$ be a lamination maximal relative to a graph Γ. Let Γ' be Γ to which a finite number of edge contractions have been applied. Then $\mathcal{L}$ is maximal relative to* $\mathrm{Slit}_{\mathcal{L}}(\Gamma')$.

Proof. We apply an induction on the number of contractions. First, $\mathcal{L}$ is maximal relative to $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ since slitting preserves maximality. Next, we assume that the result is true for the graph resulting from k edge contractions on Γ, denoted by $\Gamma^{(k)}$, and we prove the result for $\Gamma^{(k+1)}$ obtained by contracting an edge e_0 of $\Gamma^{(k)}$. Assume $\mathcal{L}$ is not maximal relative to $\mathrm{Slit}_{\mathcal{L}}(\Gamma^{(k+1)})$. Thus, there is a simple curve γ in the ribbon surface $\Sigma(\Gamma^{(k+1)})$, which is carried by $\mathrm{Slit}_{\mathcal{L}}(\Gamma^{(k+1)})$ but disjoint from $\mathcal{L}$ and not homotopic to any curve of $\mathcal{L}$. By Corollary 3.4.4, γ is also carried by $\Gamma^{(k+1)}$, since any admissible path of $\mathrm{Slit}_{\mathcal{L}}(\Gamma^{(k+1)})$ corresponds to an admissible path of $\Gamma^{(k+1)}$ obtained as a sequence of overlapping length-2 subpaths, each of them used in the carrying of some leaf of $\mathcal{L}$. Let η_γ be the admissible path in $\Gamma^{(k+1)}$ homotopic to γ, and let $\mathcal{P}_\gamma$ be its sequence of overlapping length-2 subpaths. Then, η_γ has a preimage in $\Gamma^{(k)}$ made of the preimages of the subpaths in $\mathcal{P}_\gamma$, which must be of the form either $e'e''$ or $e'e_0e''$. Indeed, the subpaths of the form $e'e_0^m e''$ for $m > 1$ are impossible, since then e_0 would be a loop. Now, a preimage $e'e''$ or $e'e_0e''$ in $\Gamma^{(k)}$ overlaps with adjacent preimage subpaths along e' and e''. Thus, any piece of path $e_1 \dots e_h$, $h > 0$, in η_γ's preimage can be prolongated to the right and to the left, since there always exist subpaths of the above form, overlapping it along e_1 to the left, and along e_h to the right. As a result, η_γ has a preimage on $\Gamma^{(k)}$ which is an admissible path, homotopic to γ. Hence, γ is carried by $\Gamma^{(k)}$, and also by $\mathrm{Slit}_{\mathcal{L}}(\Gamma^{(k)})$, since η_γ's preimage is made of overlapping subpaths all contained in paths used to carry leaves of $\mathcal{L}$. This is a contradiction since $\mathcal{L}$ has been assumed to be maximal relative to $\mathrm{Slit}_{\mathcal{L}}(\Gamma^{(k)})$. □

Thus, according to Corollary 4.1.2 saying that complexity is affine whenever a lamination $\mathcal{L}$ coded by Γ is maximal relative to $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$, edge contractions preserve affine complexity.

Example 5.4.4. Let us reconsider the graph Γ of Example 5.3.4. We apply a contraction of the edge c to obtain a coherent bouquet of three circles Γ'.

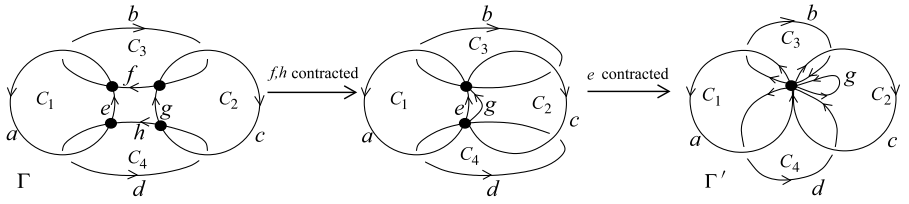


Assume $\mathcal{L}$ is a maximal lamination relative to Γ (like the laminations obtained in Example 5.3.4). Let L be the language coding $\mathcal{L}$ by Γ, and let L' be the language coding $\mathcal{L}$ by Γ'. Let θ be the substitution reflecting the contraction, which is the identity except that it sends the letter c to the empty word, so that $L' = \theta(L)$. We can check that $|\mathrm{Fact}_{L'}(2)| = 5$. Since Γ is a cycle-based graph, its two vertices v_1 and v_2 are such that $\partial^-(\cdot) = \partial^+(\cdot) = 2$. Also by Proposition 3.3.5, since $\mathcal{L}$ is maximal relative to Γ, for each vertex v_i, its burst embedded representation Γ_{v_i} induced by $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ is maximally biplanar, and as such, has three edges (see cases (iii) and (iv) of Example 2.3.4). But then, to contract the edge c in Γ is equivalent to erase the corresponding vertex v_c in $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$,

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which is common to Γ_{v_1} and Γ_{v_2}. This erasing of v_c in $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ corresponds to replacing the edges incident with v_c by the set of edges of all the possible length-2 paths going through v_c while carrying $\mathcal{L}$. In other words, erasing v_c has the same effect as a burst of v_c. By maximality of $\mathcal{L}$ relative to $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$ and by Proposition 3.3.5 again, this burst has an embedded representation which is maximally biplanar, determining the number of involved edges. By inspection of the four cases given by the two possible forms that each Γ_{v_i} can take, we see that the remaining graphs have always five edges. But these graphs correspond to the possible forms of the burst of the vertex of Γ', and therefore $\mathrm{Fact}_{L'}(2) = 5$. Hence, according to Proposition 5.4.3, we can apply Corollary 4.1.2 to deduce that $p_{L'}(n) = 2n + 1$, for all $n > 0$.

Example 5.4.5. Let us reconsider the graph Γ of Example 5.3.6. We apply the contractions of the three edges f, h, e to obtain a non-coherent bouquet of five circles Γ'.



Assume $\mathcal{L}$ is a maximal lamination relative to Γ (like the laminations obtained in Example 5.3.6). Let L and L' be defined as in Example 5.4.4, and let θ be the substitution erasing the letters f, h, e, so that $L' = \theta(L)$. Like in Example 5.4.4, to each vertex of Γ corresponds a burst with three edges in $\mathrm{Slit}_{\mathcal{L}}(\Gamma)$. Also, according to Proposition 5.4.3, at each edge contraction, we can apply the same kind of arguments as in Example 5.4.4 since $\mathcal{L}$ remains maximal relative to the slitted contracted graph $\mathrm{Slit}_{\mathcal{L}}(\cdot)$; in particular, erasing the corresponding vertices in $\mathrm{Slit}_{\mathcal{L}}(\cdot)$ is similar to their bursts where their embedded representations are maximally biplanar. By inspection of all the cases, we see that the remaining graphs have always nine edges, i.e. $|\mathrm{Fact}_{L'}(2)| = 9$. Hence, by Proposition 5.4.3, we can apply Corollary 4.1.2 to deduce that $p_{L'}(n) = 4n + 1$, for all $n > 0$.

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https://doi.org/10.1017/etds.2012.114 Published online by Cambridge University Press