Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 92 JULY 1982 PART 1

Math. Proc. Camb. Phil. Soc. (1982), **92**, 1
Printed in Great Britain

Supercontinuity

BY J. M. HENLE AND A. R. D. MATHIAS
Smith College, Massachusetts, and Peterhouse, Cambridge

(*Received* 4 *January* 1982)

Abstract. Let $\kappa > \omega$ and $\forall \delta\colon < \kappa (\kappa \to (\kappa)^{\kappa}_{\delta})$. It is known that there then exists a non-trivial κ-additive 2-valued measure, ν_{κ}, on the set $[\kappa]^{\kappa}$ of subsets of κ of cardinality κ.

Definition. $\mathscr{F}\colon [\kappa]^{\kappa} \to [\kappa]^{\kappa}$ is *supercontinuous* on $x \in [\kappa]^{\kappa}$ if $\forall \zeta\colon < \kappa \forall y\colon \in [x]^{\kappa}\ \forall z\colon \in [x]^{\kappa}$ $(y \cap \zeta = z \cap \zeta \to \mathscr{F}(y) \cap \zeta = \mathscr{F}(z) \cap \zeta)$.

THEOREM. *For any function* $\mathscr{F}\colon [\kappa]^{\kappa} \to [\kappa]^{\kappa}$, *the set* $\{x \mid \mathscr{F}$ *is supercontinuous on* $x\}$ *is of* ν_{κ}*-measure one.*

Further structure theorems for functions and applications to the Rudin–Keisler ordering of κ-additive measures are given.

0. *Preliminaries*

The second author proved as theorem 6·1 of *Happy Families* (7) a result about functions from $[\omega]^{\omega}$ to $\mathscr{P}(\omega)$, assuming the axiom of dependent choices, DC, and the partition relation $\omega \to (\omega)^{\omega}_{2}$, which stated roughly that every such function was continuous in a strong sense when restricted to $[x]^{\omega}$ for a suitably chosen $x \in [\omega]^{\omega}$. Here, following the notation of Kleinberg's book (6), $[A]^{\zeta}$, where A is a set of ordinals, is the set of subsets of A of order type ζ, $\mathscr{P}(X)$ is the power set of X, and $\kappa \to (\lambda)^{\xi}_{\eta}$ is the assertion that every function $F\colon [\kappa]^{\xi} \to \eta$ is constant on $[A]^{\xi}$ for some $A \in [\kappa]^{\lambda}$.

In this paper we develop an analogue of that result for those $\kappa > \omega$ for which $\forall \delta\colon < \kappa\ \kappa \to (\kappa)^{\kappa}_{\delta}$. Such, necessarily uncountable, κ are said to *have the strong partition property* or, more succinctly, to be *strong*.

There are many such κ if the axiom of determinacy, AD, holds: for example, $\kappa = \omega_1$. For further results on the strong partition property, the reader should consult Kechris, Kleinberg, Moschovakis and Woodin (2), pp. 75–99, M. Spector (8) and Henle (3), (4).

We shall derive in Section 2 a structure theorem for functions $\mathscr{F}\colon [\kappa]^{\kappa} \to [\kappa]^{\kappa}$, where κ is strong, from a structure theorem for functions $F\colon [\kappa]^{\kappa} \to \kappa$ which we prove in Section 1. In formulating them, which we do as Theorems A and B below, we shall use these definitions:

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Definition 0·0. Let $F:[\kappa]^\kappa \to \kappa$ and $y \in [\kappa]^\kappa$. F is *strongly continuous* on y if

$$\forall z:\in[y]^\kappa\, \forall w:\in[y]^\kappa\, \forall \zeta:<\kappa (\zeta \in z \,\&\, F(z) \leqslant \zeta \,\&\, z \cap (\zeta+1) = w \cap (\zeta+1) \to F(w) = F(z)).$$

Definition 0·1. Let $\mathscr{F}:[\kappa]^\kappa \to [\kappa]^\kappa$ and $y \in [\kappa]^\kappa$. $\mathscr{F}$ is *strongly continuous* on y if

$$\forall z:\in[y]^\kappa\, \forall w:\in[y]^\kappa\, \forall \zeta:<\kappa\, \zeta \in z \,\&\, z \cap (\zeta+1) = w \cap (\zeta+1) \to \mathscr{F}(z) \cap (\zeta+1) = \mathscr{F}(w) \cap (\zeta+1).$$

THEOREM A. *Suppose that κ is strong and $F:[\kappa]^\kappa \to \kappa$. Then $\forall x:\in[\kappa]^\kappa\, \exists y:\in[x]^\kappa$ F is strongly continuous on y.*

THEOREM B. *Suppose that κ is strong and $\mathscr{F}:[\kappa]^\kappa \to [\kappa]^\kappa$. Then $\forall x:\in[\kappa]^\kappa\, \exists y:\in[x]^\kappa$ $\mathscr{F}$ is strongly continuous on y.*

In Section 3 we shall consider these more extreme notions:

Definition 0·2. Let $F:[\kappa]^\kappa \to \kappa$ and $y \in [\kappa]^\kappa$. F is *supercontinuous* on y if

$$\forall z:\in[y]^\kappa\, \forall w:\in[y]^\kappa\, z \cap (F(z)+1) = w \cap (F(z)+1) \to F(w) = F(z).$$

Definition 0·3. Let $\mathscr{F}:[\kappa]^\kappa \to [\kappa]^\kappa$ and $y \in [\kappa]^\kappa$. $\mathscr{F}$ is *supercontinuous* on y if

$$\forall z:\in[y]^\kappa\, \forall w:\in[y]^\kappa\, \forall \zeta:<\kappa\, z \cap \zeta = w \cap \zeta \to \mathscr{F}(z) \cap \zeta = \mathscr{F}(w) \cap \zeta.$$

Definition 0·2 is a special case of 0·3, as was 0·0 of 0·1; and 0·3 results from 0·1 by dropping the restriction to successors of elements of z.

We shall give counterexamples to show that the results of replacing 'strongly continuous' by 'supercontinuous' in Theorem A and in Theorem B are false; but these weaker statements hold:

THEOREM C. *Suppose that κ is strong and $F:[\kappa]^\kappa \to \kappa$. Then $\exists y:\in[\kappa]^\kappa$ F is supercontinuous on y.*

THEOREM D. *Suppose that κ is strong and $\mathscr{F}:[\kappa]^\kappa \to [\kappa]^\kappa$. Then $\exists y:\in[\kappa]^\kappa$ $\mathscr{F}$ is supercontinuous on y.*

Actually Theorems C and D understate the position: we show in Theorem 3·10 that the set of those $y \in [\kappa]^\kappa$ on which F or $\mathscr{F}$ is supercontinuous, while not necessarily dense, is of measure one for a certain κ-additive measure on $[\kappa]^\kappa$.

In Section 4 we give further theorems in analogy to those in (7), and in Section 5 we apply the concept of strong continuity to yield information about the Rudin–Keisler ordering of κ-additive measures.

Our set-theoretical notation is standard. We write $[\lambda]^{<\kappa}$ for $\bigcup\{[\lambda]^\zeta \mid \zeta < \kappa\}$ and $otp(X)$, where X is a well-ordering, for the unique ordinal isomorphic to X. If X is a set of ordinals, the ordering is understood to be the usual ordering of ordinals. If $p \in [\kappa]^\kappa$, we write $\tilde{p}$ for the function from κ to κ that enumerates the members of p in increasing order. Here we differ from Kleinberg (6), who uses p for both p and $\tilde{p}$. All assumptions are given in the statements of Theorems, but standing assumptions may be omitted from the statements of Lemmas and Propositions. The end of a proof is signalled by ▮.

In Section 2 we shall use this result of Henle, proved in (3):

THEOREM 0·4. *If $\kappa \to (\kappa)^\kappa_\delta$ then $\kappa \to (\kappa)^\kappa_{2^{exp\delta}}$.*

It is also shown in (3) that if $\kappa \to (\kappa)^\kappa_2$ but $\exists \delta:<\kappa \not\to (\kappa)^\kappa_\delta$ then $\kappa > \omega$, the least such δ

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carries a δ-additive non-trivial two-valued measure, and, if the axiom of choice for countable families of arbitrary sets be assumed, the least such δ is uncountable and hence measurable. Whether such a δ can exist is unknown.

1. *Proof of Theorem A*

LEMMA 1·0. *If* $\kappa \to (\kappa)^\kappa_2$ *and* $F: [\kappa]^\kappa \to \kappa$ *then*

$$\forall x: \in [\kappa]^\kappa \exists y: \in [x]^\kappa \forall z, w: \in [y]^\kappa (z \subseteq w \to F(w) \leqslant F(z)).$$

Definition 1·1. Such a y is said to *monotonize* F.

Proof of Lemma 1·0. Given $x \in [\kappa]^\kappa$, define $G: [x]^\kappa \to 2$ by

$$G(y) = 0 \quad \text{if} \quad \neg\, \exists w: \in [y]^\kappa F(w) < F(y)$$
$$= 1 \quad \text{otherwise.}$$

Let $y \in [x]^\kappa$ be homogeneous for G, and let $y_1 \in [y]^\kappa$ be such that $\forall z: \in [y]^\kappa F(y_1) \leqslant F(z)$. *A fortiori*, $\forall z: \in [y_1]^\kappa F(y_1) \leqslant F(z)$, so $G(y_1) = 0$, so $G''[y]^\kappa = \{0\}$ by homogeneity, and thus y monotonizes F. ∎

LEMMA 1·2. *Assume that* $\kappa \to (\kappa)^\kappa_2$, $F: [\kappa]^\kappa \to \kappa$ *and that* $x \in [\kappa]^\kappa$ *monotonizes* F. *Then* $\exists y: \in [x]^\kappa \forall z: \in [y]^\kappa$

$$F((x \cap (\bigcap z)) \cup (z \backslash \{\bigcap z\})) < \bigcap z. \tag{1.3}$$

Proof. Define $H: [x]^\kappa \to 2$ by

$$H(z) = 0 \quad \text{if} \quad (1{\cdot}3) \text{ holds;}$$
$$= 1 \quad \text{if not.}$$

Let $y \in [x]^\kappa$ be homogeneous for H, and let $r = \{\tilde{y}(2\nu) \mid \nu < \kappa\}$ be the even-placed members of y: so $r \subseteq y$, and $\bar{\bar{r}} = \bar{\bar{y \backslash r}} = \kappa$. Pick $\zeta \in y \backslash r$ with $\zeta > F(r)$. Set $q = \{\zeta\} \cup (r \backslash \zeta)$. Then $\zeta = \bigcap q$ and $q \backslash \{\bigcap q\} = r \backslash \zeta$.

Then

$$\begin{aligned} F((x \cap (\bigcap q)) \cup (q \backslash \{\bigcap q\})) &= F((x \cap \zeta) \cup (r \backslash \zeta)) \\ &\leqslant F((r \cap \zeta) \cup (r \backslash \zeta)) \quad \text{as } x \text{ monotonizes } F \text{ and } r \cap \zeta \subseteq x \cap \zeta \\ &= F(r) \\ &< \zeta \quad \text{by choice of } \zeta \\ &= \bigcap q. \end{aligned}$$

So $H(q) = 0$, and thus $H''[y]^\kappa = \{0\}$. ∎

LEMMA 1·4. *Assume that* $\kappa \to (\kappa)^\kappa_2$, $F: [\kappa]^\kappa \to \kappa$ *and that* $x \in [\kappa]^\kappa$ *monotonizes* F. *Then* $\exists \bar{x}: \in [x]^\kappa \exists \xi: \in \bar{x}$ *such that* $\bar{x} \cap \xi = x \cap \xi$ *and* $\forall z: \in [\bar{x}]^\kappa (x \cap \xi = z \cap \xi \to F(z) < \xi)$.

Proof. Obtain $y \in [x]^\kappa$ as in Lemma 1·2, and set $\xi = \inf y$, $\bar{x} = (x \cap \xi) \cup (y \backslash \xi)$. Then $\xi \in \bar{x}$ and $\bar{x} \cap \xi = x \cap \xi$. To verify the last property, let $z \subseteq \bar{x}$ with $\bar{x} \cap \xi = z \cap \xi$, and consider two cases:

Case 1. if $\xi \notin z$, set $\bar{z} = \{\xi\} \cup (z \backslash \xi)$. Then $\bigcap \bar{z} = \xi$, so $F(z) = F((x \cap \bigcap \bar{z}) \cup \bar{z} \backslash \{\bigcap \bar{z}\}) < \xi$ by (1·3), as $H(\bar{z}) = 0$.

Case 2. if $\xi \in z$, $F(z) \leqslant F(z \backslash \{\xi\})$ as x monotonizes F; and $F(z \backslash \{\xi\}) < \xi$ by Case 1. ∎

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LEMMA 1·5. *If* $\forall \xi{:}<\kappa\ \kappa\to(\kappa)^\xi_\xi$, $F{:}[\kappa]^\kappa\to\kappa$ *and* $x\in[\kappa]^\kappa$ *monotonizes* F, *then* $\exists y{:}\in[x]^\kappa\exists\xi{:}<\kappa$ *such that*

$$y\cap\xi = x\cap\xi\ \&\ \forall z{:}\in[y]^\kappa\,(y\cap\xi = z\cap\xi\to F(y) = F(z)).$$

Proof. Let $\bar{x},\xi$ be as in Lemma 1·4: then $\bar{x}\cap\xi = x\cap\xi$. Define $G{:}[\bar{x}\backslash\xi]^\kappa\to\xi$ by $G(z) = F((x\cap\xi)\cup z)$, and let $\bar{y}\in[\bar{x}\backslash\xi]^\kappa$ be homogeneous for G. Set $y = (x\cap\xi)\cup\bar{y}$. Then $x\cap\xi = y\cap\xi$, and y is readily seen to have the stated property.

Note that there is no reason for us to have $\xi\in y$. ∎

Definition 1·6. Let $y\in[\kappa]^\kappa$, $F{:}[\kappa]^\kappa\to\kappa$. The *freeze point* of y with respect to F, in symbols $fr_F(y)$, is defined to be the least $\xi\leqslant\kappa$ such that $\forall z{:}\in[y]^\kappa\,(z\cap\xi = y\cap\xi\to F(z) = F(y))$.

Example 1·7. Let $\zeta<\kappa$, and define $F(x) = x(\zeta)$, the ζth element of x. Then for each x, $fr_F(x) = F(x)+1$.

Example 1·8. Let $F(x) =$ the largest limit ordinal $\leqslant\bigcap x$.

For $x\in[\{y\mid 0<\lambda = \bigcup\lambda<\kappa\}]^\kappa$, $F(x) = \bigcap x$ and $fr_F(x) = F(x)+1$.

For $x\in[\{\nu+1\mid\nu<\kappa\}]^\kappa$, $F(x)<\bigcap x$ and $fr_F(x) = (\bigcap x)+1 > F(x)+1$.

For all $x\in[\kappa]^\kappa$, $fr_F(x) = \bigcap(x\backslash F(x))+1$.

The following notation will be increasingly useful:

Definition 1·9. For $y\in[\kappa]^\kappa$, $\xi<\kappa$, set

$$(\xi,y) =_{df}\{w\mid w\cap\xi = y\cap\xi\ \&\ w\in[y]^\kappa\}.$$

For the rest of this section we assume that κ is strong and fix $F{:}[\kappa]^\kappa\to\kappa$.

LEMMA 1·10. *If* $x\in[\kappa]^\kappa$ *monotonizes* F,

$$\exists\bar{x}{:}\in[x]^\kappa\forall y{:}\in[\bar{x}]^\kappa fr_F(y)<\kappa.$$

Proof. Define $G{:}[x]^\kappa\to 2$ by $G(y) = 0$ if $fr_F(y)<\kappa$, $G(y) = 1$ if $fr_F(y) = \kappa$. Let $\bar{x}$ be homogeneous for G. By Lemma 1·5, $\exists y{:}\in[\bar{x}]^\kappa G(y) = 0$, so $G''[\bar{x}]^\kappa = \{0\}$. ∎

That proof may be summarized by saying that Lemma 1·10 follows from Lemma 1·6 *by density*.

LEMMA 1·11. *Suppose that for all* $y\in[x]^\kappa fr_F(y)<\kappa$, *and that* x *monotonizes both* F *and* fr_F. *Then*

$$\forall y{:}\in[x]^\kappa\forall z{:}\in(fr_F(y),x)\,F(z) = F(x).$$

Proof. Given y and z with $z\in(fr_F(y),x)$, put

$$x^* = (y\cap fr_F(y))\cup(x\backslash fr_F(y)).$$

Then $$y\cap fr_F(y) = x^*\cap fr_F(y) = z\cap fr_F(y);$$

but $$fr_F(x^*)\leqslant fr_F(y)\quad\text{as}\quad y\subseteq x^*\quad\text{and } x\text{ monotonizes } fr_F;$$

so $$y\cap fr_F(x^*) = x^*\cap fr_F(x^*) = z\cap fr_F(x^*).$$

As $y\subseteq x^*$ and $z\subseteq x^*$, $F(y) = F(x^*) = F(z)$. ∎

LEMMA 1·12. $$fr_F(x)<\kappa\to\forall y{:}\in(fr_F(x),x)fr_F(y)\leqslant fr_F(x).$$

Proof. Let $\eta = fr_F(x)$. If $z\in(\eta,y)$, then $z\in(\eta,x)$, so

$$F(z) = F(x) = F(y);\quad\text{so}\quad fr_F(y)\leqslant\eta.$$ ∎

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LEMMA 1·13. *If* $fr_F(x) < \kappa$ *and* x *monotonizes* fr_F, *then*

$$\forall y{:}\in (fr_F(x), x) fr_F(y) = fr_F(x).$$

Proof. $fr_F(y) \geqslant fr_F(x)$ by monotonicity; $fr_F(y) \leqslant fr_F(x)$ by Lemma 1·12. ∎

COROLLARY 1·14. *For* x *as in* 1·13, $fr_{fr_F}(x) \leqslant fr_F(x)$.

We now work towards an estimate for fr_F which, Example 1·8 shows, may be optimal.

Definition 1·15. $G(x) =_{df} (\bigcap (x \backslash F(x)) + 1$.

Remark 1·16. $x \cap G(x)$ has a largest element, which is a member of x greater than or equal to $F(x)$.

LEMMA 1·17. *Let* z *monotonize* fr_F, *and let* $\zeta < fr_F(z)$. *Then* $\forall \theta{:} < \kappa \exists w{:}\in (\zeta, z) F(w) > \theta$.

Proof. Suppose false for z, ζ, θ. Then $\forall w{:}\in (\zeta, z) F(w) \leqslant \theta$, so (using $\kappa \to (\kappa)^{\zeta}_{\theta+1}$) $\exists w{:}\in (\zeta, z) \forall v{:}\in (\zeta, w) F(w) = F(v)$. For this w, $fr_F(w) \leqslant \zeta < fr_F(z)$ by choice of ζ, but $fr_F(z) \leqslant fr_F(w)$ by monotonicity! Contradiction. ∎

Definition 1·18. $\mathscr{H}(x) =_{df} (x \cap G(x)) \cup (x \backslash fr_F(x))$.

Remark 1·19. $\mathscr{H}(x) \in [x]^{\kappa}$; indeed $\overline{\overline{x \backslash \mathscr{H}(x)}} < \kappa$.

LEMMA 1·20. *Suppose* y *monotonizes* fr_F, *and that* $G(y) < fr_F(y)$. *Then* $\exists w{:}\in [y]^{\kappa}$ $F(\mathscr{H}(w)) > fr_F(w)$.

Proof. Let $\zeta = G(y)$, $\theta = fr_F(y)$. By Lemma 1·17, $\exists z{:}\in (\zeta, y) F(z) > \theta$. Let $w = (y \cap \theta) \cup (z \backslash \theta)$. Note that $z \subseteq w \subseteq y$. By Lemma 1·13, $fr_F(w) = fr_F(y)$ so $F(w) = F(y)$; and therefore

$$\begin{aligned} G(w) &= (\bigcap (w \backslash F(w)) + 1 \quad \text{by definition,} \\ &= (\bigcap (w \backslash F(y)) + 1 \quad \text{as} \quad F(w) = F(y) \\ &= \bigcap ((w \cap \theta) \backslash F(y)) + 1 \quad \text{by Remark 1·16 and the fact that } \theta > G(y) \\ &= \bigcap ((y \cap \theta) \backslash F(y)) + 1 \quad \text{as} \quad w \cap \theta = y \cap \theta \\ &= G(y) = \zeta. \end{aligned}$$

so

$$\begin{aligned} \mathscr{H}(w) &= (w \cap G(w)) \cup (w \backslash fr_F(w)) \\ &= (w \cap \zeta) \cup (w \backslash \theta) \\ &= (z \cap \zeta) \cup (z \backslash \theta) \\ &\subseteq z. \end{aligned}$$

Hence

$$\begin{aligned} F(\mathscr{H}(w)) &\geqslant F(z) \quad \text{by monotonicity} \\ &> \theta \quad \text{by choice of } z \\ &= fr_F(y) = fr_F(w). \end{aligned}$$ ∎

The next lemma establishes the promised estimate for fr_F and completes the proof of Theorem A.

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LEMMA 1·21. $\forall x{:}\in[\kappa]^\kappa \exists y{:}\in[x]^\kappa (\forall z{:}\in[y]^\kappa fr_F(z) \leqslant G(z)$: *i.e. F is strongly continuous on y).*

Proof. By density it suffices to show that there is no $y\in[\kappa]^\kappa$ such that $\forall z{:}\in[y]^\kappa$ $fr_F(z) > G(z)$. Let y be a counter-example: we may assume by further density arguments and Lemmas 1·0, 1·20 that y monotonizes F and fr_F and that $\forall z{:}\in[y]^\kappa$ $G(z) < fr_F(z) < F(\mathscr{H}(z))$.

We define a sequence $p_\nu(\nu < \kappa)$ thus:

$$p_0 = y$$
$$p_{\nu+1} = \mathscr{H}(p_\nu)$$
$$p_\lambda = \bigcap_{\nu<\lambda} p_\nu \quad \text{for} \quad 0 < \lambda = \cup\lambda < \kappa.$$

Repeated application of Remark 1·19 shows that $\forall \nu < \kappa\, \overline{\overline{y\backslash p_\nu}} < \kappa$, and so each $p_\nu\in[y]^\kappa$ and the above definition is sound for all $\nu < \kappa$.

For each $\nu < \kappa$,

$$F(p_\nu) < G(p_\nu) < fr_F(p_\nu) < F(p_{\nu+1}) < G(p_{\nu+1}) < fr_F(p_{\nu+1}),$$

so $p_\nu \cap G(p_\nu) \subseteq p_{\nu+1} \cap G(p_{\nu+1})$. Further, for $\nu < \lambda = \bigcup\lambda < \kappa$,

$$G(p_\lambda) = (\bigcap (p_\lambda\backslash F(p_\lambda)) + 1 \geqslant \bigcap (p_\nu\backslash F(p_\lambda)) \geqslant \bigcap (p_\nu\backslash F(p_\nu))$$

(by monotonicity of F) $= G(p_\nu)$.

So $\nu < \eta \to p_\nu \cap G(p_\nu) \subseteq p_\lambda \cap G(p_\lambda)$. As $\delta < \nu \to p_\delta \supseteq p_\nu$, $p_\nu \cap G(p_\nu) \subseteq p_\delta$; so, setting $p_\kappa = \bigcap_{\nu<\kappa} p_\nu$,

$$p_\kappa \supseteq \bigcap_{\nu<\kappa} (p_\nu \cap G(p_\nu)).$$

By Remark 1·16, $\bigcup p_\kappa \geqslant \bigcup_{\nu<\kappa} F(p_\nu) = \kappa$.

Thus $\bar{\bar{p}}_\kappa = \kappa$. Further $\nu < \eta \to F(p_\nu) < F(p_\eta)$, by construction at successors, and by monotonicity at limits, so $\forall \nu{:} < \kappa F(p_\nu) \geqslant \nu$. By monotonicity $F(p_\kappa) \geqslant \kappa$, which is absurd. ∎

If $F(x)\in x, G(x) = F(x)+1$: so Lemma 1·21 for F such that $\forall x F(x)\in x$ yields a y such that $\forall z{:}\in[y]^\kappa fr_F(z) \leqslant F(z)+1$; that is, on which F is supercontinuous. Hence

COROLLARY 1·22. $\forall x{:}\in[\kappa]^\kappa ((\forall x'{:}\in[x]^\kappa F(x')\in x') \to \exists y{:}\in[x]^\kappa$ (F *is supercontinuous on* y)).

2. *Proof of Theorem B*

Assume throughout this section that κ is strong and fix $\mathscr{F}{:}[\kappa]^\kappa\to[\kappa]^\kappa$. For each $\zeta < \kappa$ define $F_\zeta{:}[\kappa]^\kappa\to\kappa$ by

$$F_\zeta(x) = \text{the } \zeta\text{th element of } \mathscr{F}(x). \qquad (2\cdot1)$$

Our strategy for proving Theorem B is to seek an $x\in[\kappa]^\kappa$ on which each F_ζ is strongly supercontinuous. That, taken literally, is probably impossible, but by considering (ζ, y) for various ζ's we can construct a suitable 'diagonal intersection'. We shall need the following relativized versions of definitions and lemmata from Section 1.

Definition 2·2. (ζ, x) *monotonizes* $G{:}[\kappa]^\kappa\to\kappa$ if

$$\forall y{:}\in(\zeta, x)\, \forall z{:}\in(\zeta, y)\, G(z) \geqslant G(y).$$

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LEMMA 2·3. $\forall\zeta\colon <\kappa\,\forall x\colon\in[\kappa]^\kappa\,\exists y\colon\in(\zeta,x)\,(\zeta,y)$ *monotonizes* G.

Definition 2·4. (ζ,x) is *homogeneous* for $\pi\colon[\kappa]^\kappa\to I$ if $\pi''(\zeta,x)$ is a singleton.

LEMMA 2·5. *Given* $H\colon[\kappa]^\kappa\to\kappa$, $x\in[\kappa]^\kappa$ *and* $\gamma\in x$,

$$\exists y\colon\in(\gamma+1,x)\,\forall z\colon\in(\gamma+1,y)\,fr_H(z)\leqslant(\bigcap(z\backslash\max\{\gamma,H(z)\}))+1.$$

Proof. Define $K\colon[x\backslash(\gamma+1)]^\kappa\to\kappa$ by

$$K(w)=H(x\cap(\gamma+1))\cup w)\quad\text{for}\quad w\in[x\backslash(\gamma+1)]^\kappa.$$

$$\exists\overline{w}\colon\in[x\backslash(\gamma+1)]^\kappa\,\forall w\colon\in[\overline{w}]^\kappa\,fr_\kappa(w)\leqslant\bigcap(w\backslash K(w))+1.$$

Set $y=(x\cap(\gamma+1))\cup\overline{w}$, and let $z\in(\gamma+1,y)$. Put

$$w=\overline{w}\cap z\quad\text{and}\quad\theta=(\bigcap(w\backslash K(w)))+1\quad(\theta\geqslant\gamma+1).$$

Suppose $z'\in(\theta,z)$: we shall show that $H(z')=H(z)$. To see that, put $w'=\overline{w}\cap z'$. As $z'\cap\theta=z\cap\theta$, $w'\cap\theta=w\cap\theta$; as $z'\subseteq z$, $w'\subseteq w$; so $K(w')=K(w)$. Hence

$$H(z')=H((x\cap(\gamma+1))\cup w')=K(w')=K(w)=H((x\cap(\gamma+1))\cup w)=H(z).$$

Hence $fr_H(z)\leqslant\theta$. But $\theta=\bigcap(z\backslash H(z))+1$ if $H(z)\geqslant\gamma$, and $\theta=\gamma+1$ if $H(z)<\gamma$. In both cases, $\theta=\bigcap(z\backslash\max\{\gamma,H(z)\})+1$.

LEMMA 2·6. *Let* $f\colon\kappa\to\kappa$ *and suppose that* $(\pi_\gamma\mid\gamma<\kappa)$ *is a sequence of partitions such that* $\pi_\gamma\colon[\kappa]^\kappa\to 2^{f(\gamma)}$. *Then* $\forall x\colon\in[\kappa]^\kappa\,\exists y\colon\in[x]^\kappa\,\forall\gamma\colon\in y(\gamma+1,y)$ *is homogeneous for* π_γ.

Proof. If not, $\exists x\colon\in[\kappa]^\kappa\,\forall y\colon\in[x]^\kappa\,\exists\gamma\colon\in y(\gamma+1,y)$ is not homogeneous for π_γ. Fix such an x, and let $G(y)$ $(y\in[x]^\kappa)$ be the least $\gamma\in y$ with $(\gamma+1,y)$ not homogeneous for π_γ. Pick an $\bar{x}\in[x]^\kappa$ on which G is supercontinuous, and let $G(\bar{x})=\gamma$. By 0·4 there is a $y\in(\gamma+1,\bar{x})$ such that $(\gamma+1,y)$ is homogeneous for π_γ; but then $G(y)\neq\gamma$, contradicting the fact that $fr_G(\bar{x})\leqslant\gamma+1$. ∎

Remark 2·7. A similar argument would strengthen the lemma to read 'homogeneous for all $\pi_\delta(\delta\leqslant\gamma)$' instead of 'homogeneous for π_γ'. Another strengthening is given as Proposition 4·1.

LEMMA 2·8. *Let* $(G_\delta\mid\delta<\kappa)$ *be a sequence of functions with each* $G_\delta\colon[\kappa]^\kappa\to\kappa$. *Then* $\forall x\colon\in[\kappa]^\kappa\,\exists y\colon\in[x]^\kappa\,\forall\zeta\colon\in y\forall\delta\leqslant\zeta\,(\zeta+1,y)$ *monotonizes* G_δ.

Proof. Define $\pi_\gamma\colon[x]^\kappa\to 2^{\gamma+1}$ by

$$\pi_\gamma(y)=\{\delta\leqslant\gamma\mid(\gamma+1,y)\quad\text{does not monotonize }G_\delta\}.$$

By Lemma 2·6, $\exists y\colon\in[x]^\kappa\,\forall\gamma\colon\in y(\gamma+1,y)$ is homogeneous for π_γ. It is therefore enough to show that $\pi_\gamma(y)=\varnothing$ for $\gamma\in y$. If not, let $\delta\in\pi_\gamma(y)$: by Lemma 2·3 $\exists z\colon\in(\gamma+1,y)$ z monotonizes G_δ, so $\delta\notin\pi_\gamma(z)$, contradicting $\pi_\gamma(z)=\pi_\gamma(y)$. ∎

LEMMA 2·9. *For* $\delta<\kappa$ *let* $G_\delta\colon[\kappa]^\kappa\to\kappa$. *Then* $\forall x\colon\in[\kappa]^\kappa$

$$\exists y\colon\in[x]^\kappa\,\forall\gamma\colon\in y\forall\delta\colon\leqslant\gamma\forall z\colon\in(\gamma+1,y)\,fr_{G_\delta}(z)\leqslant(\bigcap(z\backslash\max\{\gamma,G_\delta(z)\}))+1.\quad(2\cdot10)$$

Proof. Fix $x\in[\kappa]^\kappa$. For $y\in[x]^\kappa$ set $\pi_\gamma(y)=\{\delta\leqslant\gamma\mid\exists z\colon\in(\gamma+1,y)$ for which (2·10) fails$\}$.

By Lemma 2·6, $\exists y\colon\in[x]^\kappa\,\forall\gamma\colon\in y(\gamma+1,y)$ is homogeneous for π_γ.

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Suppose for this y $\exists\gamma{:}\in y\exists\delta{:}\leqslant\gamma\,\delta\in\pi_\gamma(y)$. By Lemma 2·5, $\exists y'{:}\in(\gamma+1,\,y)\,\forall z{:}\in(\gamma+1,\,y)$ (2·10) holds. But then $\delta\notin\pi_\gamma(y')$, contradicting the homogeneity of $(\gamma+1, y)$ for π_γ. ∎

By density, Lemma 2·9 is self-improving to:

LEMMA 2·11. *For $\delta<\kappa$ let $G_\delta:[\kappa]^\kappa\to\kappa$. Then*

$$\forall x{:}\in[\kappa]^\kappa\,\exists y{:}\in[x]^\kappa\,\forall z{:}\in[y]^\kappa\,\forall\gamma{:}\in z\forall\delta{:}\leqslant\gamma\, fr_{G_\delta}(z)\leqslant(\bigcap(z\backslash\max\{\gamma, G_\delta(z)\}))+1.$$

We are now in a position to prove Theorem B.

Let $\mathscr{F}:[\kappa]^\kappa\to[\kappa]^\kappa$ and fix $x\in[\kappa]^\kappa$. Let $F_\delta(z)$ be the δth element of $\mathscr{F}(z)$, for $\delta<\kappa$. By Lemma 2·11,

$$\exists y{:}\in[x]^\kappa\,\forall z{:}\in[y]^\kappa\,\forall\gamma{:}\in z\forall\delta{:}\leqslant\gamma fr_{F_\delta}(z)\leqslant(\bigcap(z\backslash\max\{\gamma, F_\delta(z)\}))+1.$$

Suppose that $z\in[y]^\kappa$, $w\in[y]^\kappa$, $\zeta\in z$ and $z\cap(\zeta+1)=w\cap(\zeta+1)$. We shall show that $\mathscr{F}(z)\cap(\zeta+1)=\mathscr{F}(w)\cap(\zeta+1)$.

Let $\xi\in\mathscr{F}(z)\cap(\zeta+1)$. For some $\delta\leqslant\zeta$, $\xi=F_\delta(z)$. By choice of y,

$$\begin{aligned} fr_{F_\delta}(z\cup w) &\leqslant(\bigcap((z\cup w)\backslash\max\{\zeta, F_\delta(z\cup w)\}))+1\\ &=(\bigcap((z\cup w)\backslash\max\{\zeta,\xi\}))+1\\ &=\zeta+1\quad\text{as}\quad\xi\leqslant\zeta\in z\cup w.\end{aligned}$$

As
$$(z\cup w)\cap(\zeta+1)=w\cap(\zeta+1)=z\cap(\zeta+1),$$
$$F_\delta(z\cup w)=F_\delta(w)=F_\delta(z)=\xi,$$

so $\xi\in\mathscr{F}(w)\cap(\zeta+1)$.

By the symmetry of the hypotheses and the arbitrary nature of ξ, $\mathscr{F}(z)\cap(\zeta+1)=\mathscr{F}(w)\cap(\zeta+1)$. ∎

3. *Proofs of Theorems C and D*

Example 1·8 shows that there is an $F:[\kappa]^\kappa\to\kappa$ and an x with F supercontinuous on no $y\in[x]^\kappa$. The following develops the same theme:

THEOREM 3·0. *There is a $\mathscr{G}:[\kappa]^\kappa\to[\kappa]^\kappa$ and an $x\in[\kappa]^\kappa$ such that $\mathscr{G}$ is supercontinuous on no $y\in[x]^\kappa$.*

Proof. Let C be a closed set of limit ordinals $<\kappa$ with $\bigcup C=\kappa$. Define $\mathscr{G}:[\kappa]^\kappa\to[\kappa]^\kappa$ by
$$\mathscr{G}(z)=\{\bigcup(C\cap\zeta)\mid\zeta\in z\backslash\bigcap C\}.$$

Put $x=\kappa\backslash C$. $\overline{\overline{x}}=\kappa$ as $x\supseteq\{\xi+1\mid\xi<\kappa\}$. Note that for $z\in[x]^\kappa$, $\mathscr{G}(z)\cap z=\varnothing$, as $\mathscr{G}(z)\subseteq C$. Suppose $y\in[x]^\kappa$ is such that $\forall\zeta{:}<\kappa\forall w, z{:}\in(\zeta, y)\,\mathscr{G}(w)\cap\zeta=\mathscr{G}(z)\cap\zeta$. Pick ordinals $\gamma_1<\eta\leqslant\xi_1<\gamma_2<\xi_2$ such that

$$\{\gamma_1,\gamma_2\}\subseteq C,\{\xi_1,\xi_2)\subseteq y,\gamma_1=\bigcup(C\cap\xi_1)\text{ and }\gamma_2=\bigcup(C\cap\xi_2).$$

Let
$$z=(y\cap\gamma_1)\cup\{\xi_1\}\cup(y\backslash\xi_2)$$
and
$$w=(y\cap\gamma_1)\cup(y\backslash\xi_2).$$
Then
$$\gamma_1\in\mathscr{G}(x);\gamma_1\notin\mathscr{G}(w);\gamma_1<\eta;\quad\text{but}\quad z\cap\eta=w\cap\eta.$$

In our proofs of Theorems C and D we shall use this concept from Kleinberg (6), p. 39:

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Definition 3·1. For $p \in [\kappa]^\kappa$, let $\tilde{p}: \kappa \to \kappa$ be the enumeration of the elements of p in ascending order, and set

$$ {}_\omega p = \{\bigcup \{\tilde{p}(\omega\zeta + n) \mid n < \omega\} \mid \zeta < \kappa\}, $$

the set of successive ω-sups of elements of p.

The behaviour of the map $p \mapsto {}_\omega p$ is noteworthy:

PROPOSITION 3·2. *Let* $p \in [\kappa]^\kappa$.

(i) $\forall x: \in [{}_\omega p]^\kappa \, \exists q: \in [p]^\kappa \, x = {}_\omega q$.

(ii) $\exists q: \in [p]^\kappa \, {}_\omega q \cap {}_\omega p = \varnothing$.

Proof. (i) The map $g: \kappa \to {}_\omega p$ given by $g(\zeta) = \bigcup \{\tilde{p}(\omega\zeta + n) \mid n < \omega\}$ is 1–1 and onto. If $x \in [{}_\omega p]^\kappa$, $x = {}_\omega q$, where $q = \{\tilde{p}(\omega\zeta + n) \mid n < \omega \,\&\, g(\zeta) \in x\}$.

(ii) If $\xi \in {}_\omega p$,

$$ \exists \zeta: < \kappa \bigcup \{\tilde{p}(\eta) \mid \eta < \omega(\zeta + 1)\} = \xi \leqslant \tilde{p}(\omega(\zeta + 1)). \text{ So let } q = \{\tilde{p}(\omega \,.\, \zeta) \mid \zeta < \kappa\}. $$

$$ \xi \in {}_\omega q \to \exists \zeta: < \kappa \bigcup \{\tilde{p}(\eta) \mid \eta < \omega^2(\zeta + 1)\} = \xi \leqslant \tilde{p}(\omega^2(\zeta + 1)), $$

so ${}_\omega p \cap {}_\omega q = \varnothing$.

Theorem C is a consequence of

THEOREM 3·3. *Assume that* κ *is strong and let* $F: [\kappa]^\kappa \to \kappa$. *Then* $\forall x: \in [\kappa]^\kappa \, \exists y: \in [x]^\kappa \, F$ *is supercontinuous on* ${}_\omega y$.

Proof. Define $G: [\kappa]^\kappa \to \kappa$ by $G(z) = F({}_\omega z)$, and fix $x \in [\kappa]^\kappa$. By successive refinement find a $y \in [x]^\kappa$ such that

(i) G is strongly continuous on y;

(ii) F is strongly continuous on ${}_\omega y$;

and *either* (iii$a$) $\forall q: \in [y]^\kappa \, F({}_\omega q) \in {}_\omega q$

or (iii$b$) $\forall q: \in [y]^\kappa \, F({}_\omega q) \notin {}_\omega q$.

(i) is achieved by using Lemma 1·21, and (iii) by a simple application of $\kappa \to (\kappa)^\kappa_2$. To achieve (ii) note that given $y' \in [x]^\kappa$ there is by Lemma 1·21 a $q \in [{}_\omega y']^\kappa$ on which F is strongly continuous; by Proposition 3·2(i), $q = {}_\omega y$ for some $y \in [y']^\kappa$.

If (iii$a$) holds, then $\forall p: \in [{}_\omega y]^\kappa \, F(p) \in p$ (by Proposition 3·2(i)), so F will be supercontinuous on ${}_\omega y$ by (ii) and Corollary 1·22.

Suppose therefore that (iii$b$) holds, and let $p \in [{}_\omega y]^\kappa$. Let $q \in [y]^\kappa$ be such that ${}_\omega q = p$, and put $\theta = \bigcap (q \backslash G(q)) + 1$. Suppose $s \in [p]^\kappa$ with $s \cap \theta = p \cap \theta$. As $\theta \notin p$, since θ is a successor ordinal, $\exists t: \in [q]^\kappa \, ({}_\omega t = s \,\&\, t \cap \theta = q \cap \theta)$. By (i) and the choice of θ, $G(t) = G(q)$; so $F({}_\omega t) = F({}_\omega q)$; that is, $F(s) = F(p)$.

Thus

$$ \begin{aligned} fr_F(p) &\leqslant \theta \\ &= \bigcap (q \backslash G(q)) + 1 \\ &= \bigcap (q \backslash F({}_\omega q)) + 1 \\ &< \bigcap (p \backslash F(p)), \quad \text{since} \quad F(p) \notin p \quad \text{(by (iii}b\text{))}, \end{aligned} $$

and ω members of q must occur before the next member of $p > F(p)$.

So $fr_F(p) \leqslant \bigcup \{\zeta + 1 \mid \zeta \in p \cap F(p)\} \leqslant F(p)$; p was arbitrary, so F is supercontinuous on ${}_\omega y$, as required. ∎

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Remark 3·4. The above proof brings out the fact that when F is supercontinuous on z and $F(z) \notin z, fr_F(z) \leqslant F(z)$.

In a similar vein, Theorem D is a consequence of

THEOREM 3·5. *Let* $\mathscr{F}: [\kappa]^\kappa \to [\kappa]^\kappa$ *where* κ *is strong. Then* $\forall x{:}\in [\kappa]^\kappa \exists y{:}\in [x]^\kappa \mathscr{F}$ *is supercontinuous on* ${}_\omega y$.

Proof. Define $\mathscr{G}: [\kappa]^\kappa \to [\kappa]^\kappa$ by $\mathscr{G}(p) = \mathscr{F}({}_\omega p)$. Pick $y \in [x]^\kappa$ on which $\mathscr{G}$ is strongly continuous. We assert that $\mathscr{F}$ is supercontinuous on ${}_\omega y$.

For let $\zeta < \kappa$, $z \in [{}_\omega y]^\kappa$, $w \in [{}_\omega y]^\kappa$, with $z \cap \zeta = w \cap \zeta$: we must show that $\mathscr{F}(z) \cap \zeta = \mathscr{F}(w) \cap \zeta$. We distinguish two cases.

First, suppose $\zeta \in {}_\omega y$. Then $cf(\zeta) = \omega$, and $\bigcup (({}_\omega y) \cap \zeta) < \zeta$. ζ is the sup of ordinals ξ such that

$$\bigcup (\zeta \cap {}_\omega y) < \xi < \zeta \quad \text{and} \quad \xi \in y.$$

We shall show that for each such ξ, $\mathscr{F}(z) \cap (\xi + 1) = \mathscr{F}(w) \cap (\xi + 1)$, whence

$$\mathscr{F}(z) \cap \zeta = \mathscr{F}(w) \cap \zeta.$$

For a given ξ as above, there are q, r such that

$$q \in [y]^\kappa, \quad r \in [y]^\kappa, \quad \xi \in q, \quad q \cap (\xi + 1) = r \cap (\xi + 1), \quad {}_\omega q = z \quad \text{and} \quad {}_\omega r = w.$$

By the strong continuity of $\mathscr{G}$ on y, $\mathscr{G}(q) \cap (\xi + 1) = \mathscr{G}(r) \cap (\xi + 1)$; so $\mathscr{F}(z) \cap (\xi + 1) = \mathscr{F}(w) \cap (\xi + 1)$, as required.

Now suppose $\zeta \notin {}_\omega y$. Set $\theta = \bigcap (y \backslash \zeta)$, and notice that there are

$$q, r \in [y]^\kappa \quad \text{with} \quad \theta \in q, q \cap (\theta + 1) = r \cap (\theta + 1), {}_\omega q = z \quad \text{and} \quad {}_\omega r = w.$$

Again by strong continuity, $\mathscr{G}(q) \cap (\theta + 1) = \mathscr{G}(r) \cap (\theta + 1)$, so $\mathscr{F}(z) \cap \zeta = \mathscr{F}(w) \cap \zeta$, by definition of $\mathscr{G}$ and the fact that $\zeta \leqslant \theta$. ∎

Theorems 3·3 and 3·5 admit attractive restatements in terms of a measure defined and studied by the first author in (4). We first make

Definition 3·6. For $p \in [\kappa]^\kappa$, $\langle p \rangle = {}_{df}\{{}_\omega q \mid q \in [p]^\kappa\}$
and list some elementary properties of the function $p \mapsto \langle p \rangle$ in

PROPOSITION 3·7. *For* $p, q \in [\kappa]^\kappa$:

(i) $p \subseteq q \to \langle p \rangle \subseteq \langle q \rangle$
(ii) $\langle {}_\omega p \rangle \subseteq \langle p \rangle$
(iii) $p \in \langle q \rangle \to \langle p \rangle \subseteq \langle q \rangle$
(iv) $p \in \langle q \rangle \to [p]^\kappa \subseteq \langle q \rangle$
(v) $\exists r{:}\in \langle p \rangle \langle r \rangle \subseteq \langle p \rangle \cap \langle q \rangle$

Definition 3·8. For $X \subseteq [\kappa]^\kappa$, set $\nu_\kappa(X) = 1$ if $\langle p \rangle \subseteq X$ for some $p \in [\kappa]^\kappa$, and $= 0$ otherwise.

THEOREM 3·9. (Henle (4), theorem 3·1.) *If* κ *is strong then* ν_κ *is a* κ*-additive two-valued measure on* $[\kappa]^\kappa$ *giving singletons measure* 0 *and* $[\kappa]^\kappa$ *measure* 1.

In terms of ν_κ, Theorem 3·5 takes this form:

THEOREM 3·10. *If* κ *is strong, then for any* $\mathscr{F}: [\kappa]^\kappa \to [\kappa]^\kappa, \nu_\kappa(\{p \in [\kappa]^\kappa \mid \mathscr{F}$ *is supercontinuous on* $p\}) = 1$.

Proof. Set $\pi(z) = 0$ if $\mathscr{F}$ is supercontinuous on ${}_\omega z$, and $= 1$ otherwise; let $x \in [\kappa]^\kappa$ be

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homogeneous for π. By 3·5, $\exists y{:}\in[x]^\kappa$ with $\mathscr{F}$ supercontinuous on ${}_\omega y$, so $\pi(x) = \pi(y) = 0$. Hence $\langle x\rangle \subseteq \{p\in[\kappa]^\kappa \mid \mathscr{F} \text{ is supercontinuous on } p\}$. ∎

4. *Further theorems on functions*

First we present a counterpart to Proposition 6·6 of (7).

THEOREM 4·0. *Suppose that κ is strong and that $\mathscr{F}:[\kappa]^\kappa \to [\kappa]^\kappa$. Then* $\forall x{:}\in[\kappa]^\kappa\, \exists y{:}\in [x]^\kappa\, \forall z{:}\in[y]^\kappa\, (z \neq y \to \mathscr{F}(z) \neq y)$.

To prove that, we establish a variant of Lemma 2·6:

PROPOSITION 4·1. *Let $f:[\kappa]^{<\kappa}\to\kappa$ and for each $s\in[\kappa]^{<\kappa}$, let $F_s:[\kappa]^\kappa \to 2^{f(s)}$. Then* $\forall p{:}\in[\kappa]^\kappa\, \exists q{:}\in[p]^\kappa\, \forall\zeta{:}\in q\forall\delta{:}\leqslant \zeta+1 (q\setminus(\zeta+1)$ *is homogeneous for $F_{q\cap\delta}$).*

Proof. Let the conclusion fail for a certain p. Define $G:[p]^\kappa\to\kappa$ by $G(q) =$ least $\zeta\in q\exists\delta\leqslant\zeta+1 (q\setminus(\zeta+1)$ not homogeneous for $F_{q\cap\delta})$. Let G be supercontinuous on $q\in[p]^\kappa$, and suppose $G(q)=\xi$. Define $H:[q\setminus(\xi+1)]^\kappa\to\xi+1$ by $H(r) =$ least $\delta\leqslant\xi+1$ with r not homogeneous for $F_{q\cap\delta}$, if such δ exists; $=\xi+2$ otherwise. As $\xi+1<\kappa$, $\exists\bar r\in[q\setminus(\xi+1)]^\kappa H$ is constant with value $\bar\delta$ on $[\bar r]^\kappa$. But if $\bar\delta<\xi+2$ some $r\in[\bar r]^\kappa$ is homogeneous for $F_{q\cap\delta}$, so $H(r)\neq\bar\delta$. This shows that $\bar\delta=\xi+2$; in other words, $\forall\delta{:}\leqslant\xi+1\ \bar r$ is homogeneous for $F_{q\cap\delta}$: but then, putting $x=(q\cap(\xi+1))\cup\bar r$, $G(x)\neq\xi$, which contradicts the supercontinuity of G on q, as $x\cap(G(q)+1) = q\cap(G(q)+1)$, $x\subseteq q$, and $G(q)=\xi$. ∎

Remark 4·2. The restriction '$\zeta\in q$' cannot be dropped in the conclusion of 4·1: for define $F_s(x) = \bigcap(x\setminus\bigcup s)$ for $x\in[\kappa]^\kappa$ and $s\in[\kappa]^{<\kappa}$.

By density, Proposition 4·1 is self-strengthening to

PROPOSITION 4·3. *Let $f:[\kappa]^{<\kappa}\to\kappa$ and for each $s\in[\kappa]^{<\kappa}$, let $F_s:[\kappa]^\kappa\to 2^{f(s)}$. Then*

$\forall p{:}\in[\kappa]^\kappa\, \exists\bar q{:}\in[p]^\kappa\, \forall q{:}\in[\bar q]^\kappa\, \forall\zeta{:}\in q\forall\delta{:}\leqslant\zeta+1 (q\setminus(\zeta+1)$ *is homogeneous for $F_{q\cap\delta}$).*

Proof of Theorem 4·0. Given $\mathscr{F}:[\kappa]^\kappa\to[\kappa]^\kappa$ and $x\in[\kappa]^\kappa$, pick $x_0\in[x]^\kappa$ on which $\mathscr{F}$ is strongly continuous. For $t\in[x_0]^{<\kappa}$ with $\bigcup t\,(=\zeta,$ say$)\in t$, set

$$f(t) = \mathscr{F}(t\cup(x_0\setminus(\zeta+1)))\cap(\zeta+1).$$

By definition of strong continuity, we have that for all $z\in[x_0]^\kappa$ with $z\cap(\zeta+1)=t$, $\mathscr{F}(z)\cap(\zeta+1)=f(t)$.

For $s\in[x_0]^{<\kappa}$, $\zeta\in x_0$, $\eta\in x_0$, with $\bigcup\{\xi+1\mid\xi\in s\}\leqslant\zeta<\eta$, set

$$\begin{aligned}\pi_s(\zeta,\eta) &= 0 \quad \text{if} \quad \zeta\in f(s\cup\{\eta\})\\ &= 1 \quad \text{if} \quad \zeta\notin f(s\cup\{\eta\}).\end{aligned}$$

Define $F_s:[\kappa]^\kappa\to 3$ by

$$\begin{aligned}F_s(z) &= 2 \qquad \text{if} \quad z\not\subseteq x_0 \quad \text{or} \quad s\not\subseteq x_0\\ &= \pi_s(\zeta,\eta) \quad \text{if} \quad s\in[x_0]^{<\kappa} \quad \text{and} \quad z\in[x_0]^\kappa,\end{aligned}$$

where ζ and η are the first two elements of z greater than all elements of s.

By Proposition 4·3, there is a $y_0\in[x_0]^\kappa$ such that for all $y_1\in[y_0]^\kappa$ and all $\zeta\in y_1$ and all $\delta\leqslant(\zeta+1)$, $y_1\setminus(\zeta+1)$ is homogeneous for $F_{y_1\cap\delta}$; in particular if $\zeta_0<\zeta_1<\zeta_2<\eta$ and $\{\zeta_0,\zeta_1,\zeta_2,\eta\}\subseteq y_1$, taking $\delta=\zeta=\zeta_0$ and putting $s=y_1\cap\zeta_0$, $z_1=\{\zeta_1,\eta\}\cup(y_1\setminus(\eta+1))$, and $z_2=\{\zeta_2,\eta\}\cup(y_1\setminus(\eta+1))$, we see that

$$F_s(z_1) = F_s(z_2),$$

https://doi.org/10.1017/S0305004100059624 Published online by Cambridge University Press

or, in other words, that

$$\zeta_1 \in f(s \cup \{\eta\}) \leftrightarrow \zeta_2 \in f(s \cup \{\eta\}).$$

So let $\tilde{y}_0 : \kappa \to \kappa$ enumerate the members of y_0 in ascending order, and put $y = \{\tilde{y}_0(3\nu+1) \mid \nu < \kappa\}$. Then $y \in [x]^\kappa$. Now let $z \in [y]^\kappa$ with $z \neq y$: we shall show that $\mathscr{F}(z) \neq y$.

Let $\zeta_1 \in y \backslash z$. $\zeta_1 = \tilde{y}_0(3\nu+1)$ for some $\nu < \kappa$; for that ν, put $\zeta_0 = \tilde{y}_0(3\nu)$, $\zeta_2 = \tilde{y}_0(3\nu+2)$, $\eta = \bigcap (z \backslash \zeta_1)$, and $y_1 = (z \cap \eta_0) \cup (y_0 \backslash \zeta_0)$. Then $\{\zeta_0, \zeta_1, \zeta_2, \eta\} \subseteq y_1$, and $\zeta_0 < \zeta_1 < \zeta_2 < \eta$; so

$$\zeta_1 \in f((y_1 \cap \zeta_0) \cup \{\eta\}) \leftrightarrow \zeta_2 \in f((y_1 \cap \zeta_0) \cup \{\eta\}).$$

But $z \cap (\eta+1) = (y_1 \cap \zeta_0) \cup \{\eta\}$, so

$$\zeta_1 \in \mathscr{F}(z) \leftrightarrow \zeta_2 \in \mathscr{F}(z).$$

As $\zeta_1 \in y$ and $\zeta_2 \notin y$, $\mathscr{F}(z) \neq y$. ∎

The proof of the following, an inexact counterpart to Theorem 7·3 of (7), is left to the reader.

THEOREM 4·4. *If κ is strong and $\mathscr{F} : [\kappa]^\kappa \xrightarrow{\text{onto}} [\kappa]^\kappa$ is such that $\forall x : \in [\kappa]^\kappa\ \mathscr{F}(x) \supseteq x$, then $\forall x : \in [\kappa]^\kappa\ \exists y : \in [x]^\kappa\ \forall z : \in [y]^\kappa\ \mathscr{F}(z) = z$.*

We now consider functions from $[\kappa]^\kappa$ to $[\kappa]^\kappa$.

Definition 4·5. For $\mathscr{F} : [\kappa]^\kappa \to [\kappa]^{<\kappa}$, define $\lambda_{\mathscr{F}} : [\kappa]^\kappa \to \kappa$ and $\mathscr{G}_{\mathscr{F}} : [\kappa]^\kappa \to \kappa$

by $$\lambda_{\mathscr{F}}(z) = \bigcup \{\xi + 1 \mid \xi \in \mathscr{F}(z)\}$$

and $$\mathscr{G}_{\mathscr{F}}(z) = \mathscr{F}(z) \cup \{\zeta \mid \lambda_{\mathscr{F}}(z) < \zeta < \kappa\}.$$

Remark 4·6. $\kappa \backslash \mathscr{G}_{\mathscr{F}}(z)$ has a largest element, namely $\lambda_{\mathscr{F}}(z)$, and $\mathscr{F}(z) = \mathscr{G}_{\mathscr{F}}(z) \cap \lambda_{\mathscr{F}}(z)$.

Given $\mathscr{F} : [\kappa]^\kappa \to [\kappa]^{<\kappa}$ and $x \in [\kappa]^\kappa$, pick a $y \in [x]^\kappa$ on which both $\mathscr{G}_{\mathscr{F}}$ and $\lambda_{\mathscr{F}}$ are strongly continuous. For $z, w \in [y]^\kappa$, put $\xi = \bigcap (z \backslash \lambda_{\mathscr{F}}(z))$; then $\xi \in z$ and $\lambda_{\mathscr{F}}(z) \leqslant \xi$, so if

$$z \cap (\xi+1) = w \cap (\xi+1) \quad \text{then} \quad \lambda_{\mathscr{F}}(z) = \lambda_{\mathscr{F}}(w) \quad \text{and} \quad \mathscr{G}_{\mathscr{F}}(z) \cap (\xi+1) = \mathscr{G}_{\mathscr{F}}(w) \cap (\xi+1),$$

and so $\mathscr{F}(z) = \mathscr{F}(w)$.

Similarly, if we choose $y \in [x]^\kappa$ with $\mathscr{G}_{\mathscr{F}}$ and $\lambda_{\mathscr{F}}$ both supercontinuous on ${}_\omega y$, we shall find that for $z, w \in [{}_\omega y]^\kappa$ with $z \cap (\lambda_{\mathscr{F}}(z)+1) = w \cap (\lambda_{\mathscr{F}}(z)+1)$, $\mathscr{F}(z) = \mathscr{F}(w)$.

We are thus led to the following definitions and theorems:

Definition 4·7. $\mathscr{F} : [\kappa]^\kappa \to [\kappa]^{<\kappa}$ is *strongly continuous* on $y \in [\kappa]^\kappa$ if $z \in [y]^\kappa$, $w \in [y]^\kappa$, $z \cap (\xi+1) = w \cap (\xi+1)$, where $\xi = \bigcap (z \backslash \lambda_{\mathscr{F}}(z))$, always imply $\mathscr{F}(z) = \mathscr{F}(w)$.

Definition 4·8. $\mathscr{F} : [\kappa]^\kappa \to [\kappa]^{<\kappa}$ is *supercontinuous* on $y \in [\kappa]^\kappa$ if $z \in [y]^\kappa$, $w \in [y]^\kappa$, $z \cap (\lambda_{\mathscr{F}}(z)+1) = w \cap (\lambda_{\mathscr{F}}(z)+1)$ always imply $\mathscr{F}(z) = \mathscr{F}(w)$.

THEOREM 4·9. *If κ is strong and $\mathscr{F} : [\kappa]^\kappa \to [\kappa]^{<\kappa}$, $\forall x : \in [\kappa]^\kappa\ \exists y : \in [x]^\kappa\ \mathscr{F}$ is strongly continuous on y.*

THEOREM 4·10. *If κ is strong and $\mathscr{F} : [\kappa]^\kappa \to [\kappa]^{<\kappa}$, $\forall x : \in [\kappa]^\kappa\ \exists y : \in [x]^\kappa\ \mathscr{F}$ is supercontinuous on ${}_\omega y$.*

Discussions of functions from $[\kappa]^\kappa$ to $\mathscr{P}(\kappa)$ reduce by consideration of an appropriate partition to the above two theorems or to the results of Sections 2 and 3.

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5. *Applications to the Rudin–Keisler ordering*

We assume DC throughout this final section. We shall first rehearse the relevant definitions and then demonstrate the lack of fine measures below ν_κ in the Rudin–Keisler ordering. We then apply these results to the case $\kappa = \omega_1$ assuming AD in addition to DC. For further background reading, (1) and (2) are recommended.

PROPOSITION 5·0. *Let κ be an uncountable regular ordinal, let μ_1 be a κ-additive measure on I, and let $f: I_1 \to I_2$. Then $f_*(\mu_1)$, defined by*

$$f_*(\mu_1)(X) = \mu_1(\{a \mid f(a) \in X\})$$

for $X \subseteq I_2$, is a κ-additive measure on I_2.

The Rudin–Keisler (partial pre-) ordering is defined thus:

Definition 5·1. $\mu_2 \leqslant_{RK} \mu_1 \leftrightarrow \exists f\ \mu_2 = f_*(\mu_1)$.

If μ is a κ-additive measure on I, we may for each ordinal ζ form the ultraproduct ζ^I/μ and show, using DC, that it is a well-founded linear ordering and hence isomorphic to an ordinal, namely $otp(\zeta^I/\mu)$. We shall use below the fact that if $\mu_2 = f_*(\mu_1)$ there is, definable from f, an order-preserving embedding of ζ^{I_2}/μ_2 into ζ^{I_1}/μ_1; hence the

PROPOSITION 5·2 (DC). *If $\mu_2 \leqslant_{RK} \mu_1$, and ζ is an ordinal,*

$$otp(\zeta^{I_2}/\mu_2) \leqslant otp(\zeta^{I_1}/\mu_1).$$

One particular measure in which we shall be interested is ν_κ; another, μ_ω, is defined in 5·3 and 5·4:

Definition 5·3. $x \subseteq \kappa$ is *ω-closed* if every ascending ω-sequence of elements of x with $\sup < \kappa$ has its $\sup \in x$.

Definition 5·4. $\mu_\omega(x) = 1$ if x contains an unbounded ω-closed subset of κ; $\mu_\omega(x) = 0$ otherwise.

PROPOSITION 5·5. ((6), theorem 3·2.) *If κ is strong, or more generally, if $\kappa \to (\kappa)^{\omega+\omega}$, μ_ω is a κ-additive measure on κ.*

In the absence of AC, measures on sets other than ordinals become interesting: we shall consider fine measures on $[\kappa_2]^{<\kappa}$ and $[\kappa_2]^\kappa$, where

Definition 5·6. $\kappa_2 =_{df} otp(\kappa^\kappa/\mu_\omega)$.

Definition 5·7. A measure μ on $[\lambda]^{<\kappa}$ (or on $[\lambda]^\kappa$) is *fine* if for all $s \in [\lambda]^{<\kappa}$,

$$\mu(\{t \in [\lambda]^{<\kappa} \mid s \subseteq t\}) = 1$$

(or, respectively, $\mu(\{t \in [\lambda]^\kappa \mid s \subseteq t\}) = 1$).

We are now in a position to state

THEOREM 5·8 (DC). *If κ is strong, no fine measure on $[\kappa_2]^{<\kappa}$ lies below ν_κ in the Rudin–Keisler ordering*

and its:

COROLLARY 5·9 (DC). *If κ is strong, no fine measure on $[\kappa_2]^\kappa$ lies below ν_κ in the Rudin–Keisler ordering.*

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Proof of 5·8. Assume to the contrary that $f_*(\nu_\kappa)$ is fine, where $f:[\kappa]^\kappa \to [\kappa_2]^{<\kappa}$. Then, unravelling definitions 3·8 and 5·7,

$$\forall s{:}\in[\kappa_2]^{<\kappa}\ \exists p{:}\in[\kappa]^\kappa\ \forall q{:}\in\langle p\rangle\ s \subseteq f(q).$$

Define $g:[\kappa]^\kappa \to [\kappa_2]^{<\kappa}$ by

$$g(q) = \bigcap\{f(p) \mid p \in [q]^\kappa\}.$$

Then g has the property that

$$p \in [q]^\kappa \to g(q) \subseteq g(p) \tag{5·10}$$

and $g_*(\nu_\kappa)$ is also a fine measure, for if $\forall p{:}\in\langle q\rangle\ s \subseteq f(p)$, then by Proposition 3·7 (iv), $\forall p{:}\in\langle q\rangle\ s \subseteq \bigcap_{r\in[p]^\kappa} f(r)$.

For $p\in[\kappa]^\kappa$, define $j(p)$ to be the ordinal $< \kappa_2$ corresponding to the position of $\tilde{p}$ in κ^κ/μ_ω.

LEMMA 5·11. $\forall x{:}\in[\kappa]^\kappa\ \exists y{:}\in[x]^\kappa\ (\bigcup g({}_\omega y) \geqslant j(y))$.

Proof. Fix x. As $g_*(\nu_\kappa)$ is fine, there is a $p\in[\kappa]^\kappa$ such that $\forall q{:}\in\langle p\rangle j(x)\in g(q)$. By 3·7 (v) we may assume that $\langle p\rangle \subseteq \langle x\rangle$.

There is a closed unbounded subset q of κ such that for all $\zeta,\eta\in q$, $\zeta<\eta \to p\cap(\tilde{x}(\eta)\backslash\tilde{x}(\zeta)) \neq 0$. Let $y\in[x]^\kappa$ be defined by $\tilde{y}(\nu) = \tilde{x}(\tilde{q}(\nu))$, and let $r\in[p]^\kappa$ be defined by $\tilde{r}(\nu) = \bigcap(p\cap(\tilde{y}(\nu+1)\backslash\tilde{y}(\nu)))$. As $\forall\nu{:}<\kappa\ \tilde{y}(\nu) \leqslant \tilde{r}(\nu) < \tilde{y}(\nu+1)$, ${}_\omega r = {}_\omega y$. Since $\mu_\omega(\{\nu \mid \nu = \tilde{q}(\nu)\}) = 1$, $j(x) = j(y)$. As ${}_\omega r\in\langle p\rangle$, $\bigcup g({}_\omega r) \geqslant j(x)$. Putting these facts together, we have $\bigcup g({}_\omega y) = \bigcup g({}_\omega r) \geqslant j(x) = j(y)$. ∎

Arguments of that kind stem from Kleinberg's shuffle technology, (6) chapter 4.

LEMMA 5·12. $\forall x{:}\in[\kappa]^\kappa\ \exists y{:}\in[x]^\kappa\ (j(y) > \bigcup g({}_\omega x)$ *and* ${}_\omega y \subseteq {}_\omega x)$.

Proof. Given x, pick $z\in[\kappa]^\kappa$ with $j(z) > \bigcup g({}_\omega x)$. Pick $y\in[x]^\kappa$ such that $\forall\nu{:}<\kappa(\tilde{y}(\omega\nu) \geqslant \tilde{z}(\omega\nu)$, and if $\tilde{y}(\omega\nu) = \tilde{x}(\zeta)$, then $\tilde{y}(\omega\nu+k) = \tilde{x}(\zeta+k)$ for $1 \leqslant k \leqslant \omega$). Then $j(y) \geqslant j(z)$ as $\mu_\omega(\{\omega\nu \mid \nu<\kappa\}) = 1$, and ${}_\omega y \subseteq {}_\omega x$. ∎

We shall now complete the proof of 5·8 by reaching a contradiction.

Define $F:[\kappa]^\kappa \to \kappa$ by $F(p) = otp(g({}_\omega p))$. By Lemma 5·11, a density argument and Theorem A, there is an $x\in[\kappa]^\kappa$ on which F is strongly continuous and such that $\forall y{:}\in[x]^\kappa \bigcup g({}_\omega y) \geqslant j(y)$. By Lemma 5·12, there is a $z\in[x]^\kappa$ with ${}_\omega z \subseteq {}_\omega x$ and $j(z) > \bigcup g({}_\omega x)$. Set ζ = the freeze point $fr_F(x)$, and put $v = (x\cap\zeta)\cup z$. Now note these facts:

(*a*) ${}_\omega v \subseteq {}_\omega x$	by choice of z and v
(*b*) $g({}_\omega v) \supseteq g({}_\omega x)$	from (*a*) and (5.10)
(*c*) $\bigcup(g({}_\omega v)) \geqslant j(v)$	by choice of x, as $v\in[x]^\kappa$
(*d*) $j(v) = j(z)$	as $\tilde{v}(\xi) = \tilde{z}(\xi)$ for all $\xi \geqslant \zeta\omega$
(*e*) $j(z) > \bigcup g({}_\omega x)$	by choice of z
(*f*) $\bigcup g({}_\omega v) > \bigcup g({}_\omega x)$	by (*c*), (*d*) and (*e*).

From (*b*), (*f*), $otp(g({}_\omega v)) > otp(g({}_\omega x))$; but $F(v) = F(x)$ by strong continuity of F. ∎

Proof of 5·9. Suppose *per impossibile* that $h_*(\nu_\kappa)$ is a fine measure on $[\kappa_2]^\kappa$ where $h:[\kappa]^\kappa \to [\kappa_2]^\kappa$. Define $f:[\kappa]^\kappa \to \mathscr{P}(\kappa_2)$ by

$$f(p) = \{\zeta < \kappa_2 \mid \forall q{:}\in\langle p\rangle\ \zeta\in h(q)\}.$$

Note that for each p, $\overline{\overline{f(p)}} < \kappa$, for by the fine nature of $h_*(\nu_\kappa)$, $\exists q{:}\in[p]^\kappa \bigcup h({}_\omega q) > \bigcup h({}_\omega p)$, and so $otp(h({}_\omega q)\cap h({}_\omega p)) < \kappa$. So $f:[\kappa]^\kappa \to [\kappa_2]^{<\kappa}$.

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That $f_*(\nu_\kappa)$ is fine, in contradiction to 5·8, is readily checked with the help of Proposition 3·7. |

We now assume AD + DC. Then by a result of Martin (6), theorem 2·7, ω_1 is strong. In this context, for $\kappa = \omega_1, \kappa_2 = \omega_2$ by a result of Solovay.

Denote by μ the fine normal measure defined by Martin on $[\omega_2]^{<\omega_1}$ and discussed in (5), and write ν for ν_{ω_1}. 'Normal' here means that any function f with $f(s) \in s$ on a set of measure 1 is constant on a set of measure 1.

THEOREM 5·13 (AD + DC). *$\mu \not\leqslant_{RK} \nu$ and $\nu \not\leqslant_{RK} \mu$.*

Proof. $\mu \not\leqslant_{RK} \nu$ by 5·8 and the fine nature of μ.

Now μ is the composition of measures on $[\zeta]^{<\omega_1}$ (for $\omega_1 \leqslant \zeta < \omega_\nu$), each equivalent to μ_ω, using one, call it μ^*, of the two normal measures on ω_2. By work of Kunen, $\omega_2^{\omega_2}/\mu^* \cong \omega_3$; so by routine arguments

$$\omega_1^{([\omega_2]^{<\omega_1})}/\mu \cong (\omega_1^{\omega_1}/\mu_\omega)^{\omega_2}/\mu^* \cong \omega_3. \tag{5·14}$$

But by Theorem 3·4 of Henle (4),

$$\omega_1^{([\omega_1]^{\omega_1})/\nu} > \aleph_\omega \tag{5·15}$$

so by Proposition 5·2, $\nu \not\leqslant_{RK} \mu$. |

Under AD, fine measures can be constructed on $[\omega_2]^{\omega_1}$, but whether normal ones exist is unknown. We can, though, say this:

PROPOSITION 5·16 (AD + DC). *If η is a fine normal measure on $I = [\omega_2]^{\omega_1}$, $\omega_1^I/\eta \cong \omega_2$.*

Proof. Let $f: [\omega_2]^{\omega_1} \to \omega_1$. Define $g: [\omega_2]^{\omega_1} \to \omega_2$ by $g(p) = \tilde{p}(f(p)) \in p$. By normality, $\exists$ (unique) $\zeta: < \omega_2$ $\eta(g^{-1''}\{\zeta\}) = 1$. Put $\pi(f) =$ that ζ.

If $\eta(\{s \mid f_1(s) \leqslant f_2(s)\}) = 1$, then $\eta(\{s \mid g_1(s) \leqslant g_2(s)\}) = 1$, so $\pi(f_1) \leqslant \pi(f_2)$, and so π induces an embedding (readily seen to be onto) of ω_1^I/η into ω_2.

COROLLARY 5·17 (AD + DC). *If η is a fine normal measure on $[\omega_2]^{\omega_1}$, then $\nu \not\leqslant_{RK} \eta$, $\eta \not\leqslant_{RK} \nu$ and $\mu \not\leqslant_{RK} \eta$.*

Proof. By 5·2, (5·14), (5·15), 5·16 and 5·9. |

This work was conceived while the first author was enjoying the support of Smith College, during a sabbatical leave in Cambridge.

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