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Clifford monoid presentations

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Abstract

Some decidability results for presentations in the variety of Clifford monoids are obtained. These results are applied in the solution of the E-unitary problem for presentations in the variety of inverse monoids.

1. *Preliminaries*

The general terminology and notation are those of Howie [**4**].

Let M be an inverse monoid with semilattice of idempotents $E(M)$. The least group congruence σ on M is the set of all $(a, b) \in M \times M$ such that $ae = be$ for some $e \in E(M)$. We say that M is E-unitary if and only if each element of M satisfying $a\sigma = 1$ lies in $E(M)$. Finally, M is said to be a Clifford monoid if and only if $ae = ea$ for all $a \in M$ and all $e \in E(M)$.

The class of all Clifford monoids forms a variety of inverse monoids (see [**11**], §XII·4). We will denote by **Inv**, **Clf**, **Gp** and **Su**, respectively, the varieties of inverse monoids, Clifford monoids, groups and semilattices with unity. Their free objects on the non-empty set X are denoted respectively by $\mathrm{FIM}(X)$, $\mathrm{FCM}(X)$, $\mathrm{FG}(X)$ and $\mathrm{FSU}(X)$.

Let V be a variety of inverse monoids with free object on X denoted by $\mathrm{FV}(X)$. A presentation in V is an expression $V\langle X; R\rangle$, where X is a non-empty set and R is a relation on $\mathrm{FV}(X)$. The inverse monoid defined by this presentation is the quotient $M = [\mathrm{FV}(X)]/R^{\#}$, where $R^{\#}$ is the congruence on $\mathrm{FV}(X)$ generated by R, that is, the transitive closure of

$$\{(aub, avb) : a, b \in \mathrm{FV}(X) \text{ and } (u, v) \in R \cup R^{-1}\}.$$

If R is finite, we say that the presentation is *finitely related*. If $|R| = 1$, we will use the term one-relator presentation. If both X and R are finite, M is said to be finitely presented and the presentation is said to be finite.

The following lemma establishes strong connections between our varieties.

Lemma 1. (i) *Every finitely presented Clifford monoid has a finite inverse monoid presentation.* (ii) *Every finitely presented group has a finite Clifford monoid presentation.*

Proof. (i) It is well-known that there exists an isomorphism ϕ from $\mathrm{FCM}(X)$ to $[\mathrm{FIM}(X)]/T^{\#}$, where $T = \{(uvv^{-1}, vv^{-1}u) : u, v \in \mathrm{FIM}(X)\}$ (see [**11**], §VIII·2). Let $S = \{(uu^{-1}, u^{-1}u) : u \in \mathrm{FIM}(X)\}$. First we prove that $S^{\#} = T^{\#}$.

Let $u \in \mathrm{FIM}(X)$. Then $(uu^{-1})\,T^{\#} = [(uu^{-1}u)\,u^{-1}]\,T^{\#} = [(u^{-1}u)\,(uu^{-1})]\,T^{\#}$, so that

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$(uu^{-1})T^{\#} \leqslant (u^{-1}u)T^{\#}$. Similarly, $(u^{-1}u)T^{\#} \leqslant (uu^{-1})T^{\#}$, so that $(uu^{-1})T^{\#} = (u^{-1}u)T^{\#}$ and $S \subseteq T^{\#}$.

Now let $u, v \in \mathrm{FIM}(X)$. Then

$$\begin{aligned}(uvv^{-1})S^{\#} &= (uvv^{-1}vv^{-1}u^{-1}u)S^{\#} = [(vv^{-1}u^{-1})(uvv^{-1})u]S^{\#}\\ &= [(vv^{-1}uu^{-1})u]S^{\#} = (vv^{-1}u)S^{\#},\end{aligned}$$

so that $T \subseteq S^{\#}$. Hence $S^{\#} = T^{\#}$.

Identifying X with the corresponding set of generators of $\mathrm{FIM}(X)$, we define

$$S_1 = \{(xx^{-1}, x^{-1}x) : x \in X\}, \quad S_2 = \{(xyy^{-1}x^{-1}, xx^{-1}yy^{-1}) : x, y \in X \cup X^{-1}\}.$$

We want to prove that $S^{\#} = (S_1 \cup S_2)^{\#}$.

The inclusion $S_1 \subseteq S^{\#}$ is trivial. It is also clear that $S_2 \subseteq T^{\#} = S^{\#}$, so that $(S_1 \cup S_2)^{\#} \subseteq S^{\#}$.

Every $u \in \mathrm{FIM}(X)\backslash\{1\}$ can be written in the form $u = x_1 \dots x_n$, with $x_i \in X \cup X^{-1}$ for $i \in \{1, \dots, n\}$. Using induction on n, we will prove that, if $u = x_1 \dots x_n$, then

$$(uu^{-1})(S_1 \cup S_2)^{\#} = (x_1 x_1^{-1} \dots x_n x_n^{-1})(S_1 \cup S_2)^{\#}. \tag{1·1}$$

This is clearly true for $n = 1$. Suppose that it is true for n, and let $u = x_1 \dots x_{n+1}$. Then

$$\begin{aligned}&(uu^{-1})(S_1 \cup S_2)^{\#}\\ &= [(x_1 \dots x_{n-1})(x_n x_{n+1} x_{n+1}^{-1} x_n^{-1})(x_{n-1}^{-1} \dots x_1^{-1})](S_1 \cup S_2)^{\#}\\ &= [x_1 \dots x_{n-1}(x_n x_n^{-1})(x_{n+1} x_{n+1}^{-1}) x_{n-1}^{-1} \dots x_1^{-1}](S_1 \cup S_2)^{\#}\\ &= [x_1 \dots x_{n-1}(x_{n-1}^{-1} \dots x_1^{-1})(x_1 \dots x_{n-1}) x_n x_n^{-1} x_{n+1} x_{n+1}^{-1} x_{n-1}^{-1} \dots x_1^{-1}] \times (S_1 \cup S_2)^{\#}\\ &= [x_1 \dots x_{n-1}(x_n x_n^{-1}) x_{n-1}^{-1} \dots x_1^{-1} x_1 \dots x_{n-1} x_{n+1} x_{n+1}^{-1} x_{n-1}^{-1} \dots x_1^{-1}] \times (S_1 \cup S_2)^{\#}\\ &= (x_1 \dots x_n x_n^{-1} \dots x_1^{-1})(S_1 \cup S_2)^{\#}(x_1 \dots x_{n-1} x_{n+1} x_{n+1}^{-1} x_{n-1}^{-1} \dots x_1^{-1}) \times (S_1 \cup S_2)^{\#}\\ &= (x_1 x_1^{-1} \dots x_n x_n^{-1})(S_1 \cup S_2)^{\#}(x_1 x_1^{-1} \dots x_{n-1} x_{n-1}^{-1} x_{n+1} x_{n+1}^{-1})(S_1 \cup S_2)^{\#}\\ &= (x_1 x_1^{-1} \dots x_{n+1} x_{n+1}^{-1})(S_1 \cup S_2)^{\#}.\end{aligned}$$

Thus (1·1) holds. Therefore

$$\begin{aligned}(u^{-1}u)(S_1 \cup S_2)^{\#} &= (x_n^{-1} x_n \dots x_1^{-1} x_1)(S_1 \cup S_2)^{\#} = (x_1 x_1^{-1} \dots x_n x_n^{-1})(S_1 \cup S_2)^{\#}\\ &= (uu^{-1})(S_1 \cup S_2)^{\#}\end{aligned}$$

and so $S \subseteq (S_1 \cup S_2)^{\#}$. Thus $S^{\#} = (S_1 \cup S_2)^{\#}$.

Now let M denote the Clifford monoid defined by the presentation $\mathrm{Clf}\langle X; R\rangle$. Let $R\phi$ denote the relation on $[\mathrm{FIM}(X)]/T^{\#}$ induced by ϕ. Obviously $M \simeq ([FIM(X)]/T^{\#})/(R\phi)^{\#}$. Since $R\phi$ is finite, there exists a finite relation R' on $\mathrm{FIM}(X)$ such that $(R\phi)^{\#} = (R' \cup T)^{\#}/(T^{\#})$. By the third isomorphism theorem for semigroups ([**4**], §I·5), we have

$$([\mathrm{FIM}(X)]/T^{\#})/(R\phi)^{\#} \simeq [\mathrm{FIM}(X)]/(R' \cup T)^{\#}.$$

Hence $M \simeq [\mathrm{FIM}(X)]/(R' \cup T)^{\#} = [\mathrm{FIM}(X)]/(R' \cup S_1 \cup S_2)^{\#}$ and so M can be defined by the finite inverse monoid presentation $\mathrm{Inv}\langle X; R' \cup S_1 \cup S_2\rangle$.

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(ii) This is completely similar to the previous case, with $\mathrm{FG}(X) \simeq [\mathrm{FCM}(X)]/T^{\#}$, $T = \{(xx^{-1}, 1) : x \in X\}$.

Now we will describe normal forms for $\mathrm{FSU}(X)$ and $\mathrm{FCM}(X)$, which will be used throughout this paper.

We consider $\mathrm{FSU}(X)$ as the set of all finite subsets of X with the operation $(A, B) \mapsto A \cup B$, the empty set being the unity (see [11], §VIII·2). Considering $\mathrm{FG}(X)$ as the set of all reduced words on $X \cup X^{-1}$, we define a map $\xi : \mathrm{FG}(X) \to \mathrm{FSU}(X)$ as follows: for every $g \in \mathrm{FG}(X)$, $\xi(g)$ is the content of g, that is, the set of all $x \in X$ such that x or x^{-1} occurs in g.

By Liber[5] (see also [11], §VIII·2), we can define $\mathrm{FCM}(X)$ to be $\{(A, g) \in \mathrm{FSU}(X) \times \mathrm{FG}(X) : \xi(g) \subseteq A\}$, with the operation given by $(A, g)(B, h) = (A \cup B, gh)$. The unity of $\mathrm{FCM}(X)$ is $(\varnothing, 1)$ and so it is clear that $\mathrm{FCM}(X)$ has trivial group of units. We note that $(A, g)^{-1} = (A, g^{-1})$ for every $(A, g) \in \mathrm{FCM}(X)$. Moreover, $(A, g)^2 = (A, g)$ if and only if $g = 1$. Let $\iota : X \to \mathrm{FCM}(X)$ be the map defined by $x\iota = (\{x\}, x)$. For every Clifford monoid M and every map $\phi : X \to M$, there exists a unique homomorphism $\bar{\phi} : \mathrm{FCM}(X) \to M$ such that $\iota\bar{\phi} = \phi$.

The word problem for a presentation $V\langle X; R\rangle$ can be stated as follows. Is there an algorithm that determines, for all $u, v \in \mathrm{FV}(X)$, whether or not $(u, v) \in R^{\#}$? The idempotent word problem is the restriction of the word problem to the case when $u, v \in E[\mathrm{FV}(X)]$. Finally, the E-unitary problem for a class C of presentations asks if there is an algorithm that determines, for every presentation in C, whether or not the corresponding inverse monoid is E-unitary.

2. *The word problem*

In this section we will show how word problems in **Clf** can be related to word problems in **Su** and **Gp**.

LEMMA 2. *In* **Su**, *every finitely related presentation has decidable word problem.*

Proof. Let R denote a finite relation on $\mathrm{FSU}(X)$. Let Y be the subset of all the elements of X involved in the formulation of the relation R. Since R is finite, Y is also finite. We prove that, for $U, V \in \mathrm{FSU}(X)$, we have

$$UR^{\#} = VR^{\#} \quad \text{if and only if} \quad (U \cap Y)R^{\#} = (V \cap Y)R^{\#} \text{ and } U \backslash Y = V \backslash Y. \tag{2·1}$$

Suppose that $UR^{\#} = VR^{\#}$. Then there exist $W_0, \ldots, W_n \in \mathrm{FSU}(X)$ such that $W_0 = U$, $W_n = V$ and such that for each $i \in \{1, \ldots, n\}$ there exist an $S_i \in \mathrm{FSU}(X)$ and an element (A_i, B_i) of R satisfying $\{W_{i-1}, W_i\} = \{S_i \cup A_i, S_i \cup B_i\}$. It follows that $W_0 \cap Y = U \cap Y$, $W_n \cap Y = V \cap Y$ and, since $A_i \subseteq Y$ and $B_i \subseteq Y$ for every i, we have

$$\{W_{i-1} \cap Y, W_i \cap Y\} = \{(S_i \cap Y) \cup A_i, (S_i \cap Y) \cup B_i\},$$

for each i. Hence $(U \cap Y)R^{\#} = (V \cap Y)R^{\#}$. Similarly we obtain $W_0 \backslash Y = U \backslash Y$, $W_n \backslash Y = V \backslash Y$ and, for $i \in \{1, \ldots, n\}$, $\{W_{i-1} \backslash Y, W_i \backslash Y\} = \{S_i \backslash Y\}$ so that $W_{i-1} \backslash Y = W_i \backslash Y$. Thus we have $U \backslash Y = V \backslash Y$.

Conversely, suppose that $(U \cap Y)R^{\#} = (V \cap Y)R^{\#}$ and $U \backslash Y = V \backslash Y$. Then

$$\begin{aligned} UR^{\#} &= [(U \cap Y) \cup (U \backslash Y)]R^{\#} = (U \cap Y)R^{\#}(U \backslash Y)R^{\#} \\ &= (V \cap Y)R^{\#}(V \backslash Y)R^{\#} = [(V \cap Y) \cup (V \backslash Y)]R^{\#} = VR^{\#} \end{aligned}$$

and so (2·1) holds.

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It follows easily that the word problem for $\mathrm{Su}\langle X;R\rangle$ is equivalent to the word problem for $\mathrm{Su}\langle Y;R\rangle$. But the latter is certainly decidable, since $\mathrm{FSU}(Y)$ is finite, and so the lemma is proved.

Let $\mathrm{Clf}\langle X;R\rangle$ be a Clifford monoid presentation, with $R=\{((A_i,g_i),(B_i,h_i)):i\in I\}$. We define relations R_1 and R_2 on $\mathrm{FSU}(X)$ and $\mathrm{FG}(X)$, respectively, by

$$R_1=\{(A_i,B_i):i\in I\},\quad R_2=\{(g_i,h_i):i\in I\}.$$

For every $U\in\mathrm{FSU}(X)$, we define a relation $R_2(U)$ on $\mathrm{FG}(X)$ by

$$R_2(U)=\{(g_i,h_i)\in R_2:U(R_1)^\#=(U\cup A_i)(R_1)^\#\}.$$

It follows from the definition that if $U\subseteq V$ then $R_2(U)\subseteq R_2(V)$, and if $U(R_1)^\#=V(R_1)^\#$ then $R_2(U)=R_2(V)$.

Finally, we note that, for I finite, there exists $Y\in\mathrm{FSU}(X)$ such that $R_2(Y)=R_2$, namely $Y=\bigcup_{i\in I}A_i$.

LEMMA 3. *Let* $\mathrm{Clf}\langle X;R\rangle$ *be a presentation, with* $R=\{((A_i,g_i),(B_i,h_i)):i\in I\}$. *Let* $(U,u),(V,v)\in\mathrm{FCM}(X)$. *Then* $(U,u)R^\#=(V,v)R^\#$ *if and only if* $U(R_1)^\#=V(R_1)^\#$ *and* $u[R_2(U)]^\#=v[R_2(U)]^\#$.

Proof. Suppose that $(U,u)R^\#=(V,v)R^\#$. Then there exist $(W_0,w_0),\dots,(W_n,w_n)\in\mathrm{FCM}(X)$ such that $(W_0,w_0)=(U,u)$, $(W_n,w_n)=(V,v)$ and such that for each $k\in\{1,\dots,n\}$ there exist $(S_k,s_k),(T_k,t_k)\in\mathrm{FCM}(X)$ and $i_k\in I$ such that

$$\{(W_{k-1},w_{k-1}),(W_k,w_k)\}=\{(S_k,s_k)(A_{i_k},g_{i_k})(T_k,t_k),(S_k,s_k)(B_{i_k},h_{i_k})(T_k,t_k)\}.$$

This clearly yields that $W_0=U$, $W_n=V$, and $\{W_{k-1},W_k\}=\{S_k\cup A_{i_k}\cup T_k,S_k\cup B_{i_k}\cup T_k\}$ for all $k\in\{1,\dots,n\}$. Hence $U(R_1)^\#=V(R_1)^\#$. For every $k\in\{1,\dots,n\}$, we have $U(R_1)^\#=W_k(R_1)^\#$. Moreover, $A_{i_k}\subseteq W_k$ or $B_{i_k}\subseteq W_k$. Suppose that $A_{i_k}\subseteq W_k$. Then

$$U(R_1)^\#=W_k(R_1)^\#=(W_k\cup A_{i_k})(R_1)^\#=(U\cup A_{i_k})(R_1)^\#.$$

Similarly, if $B_{i_k}\subseteq W_k$, then

$$U(R_1)^\#=(U\cup B_{i_k})(R_1)^\#=(U\cup A_{i_k})(R_1)^\#.$$

Hence $(g_{i_k},h_{i_k})\in R_2(U)$. Moreover, $w_0=u,w_n=v$ and $\{w_{k-1},w_k\}=\{s_kg_{i_k}t_k,s_kh_{i_k}t_k\}$ for each $k\in\{1,\dots,n\}$. Therefore $u[R_2(U)]^\#=v[R_2(U)]^\#$.

Conversely, suppose that $U(R_1)^\#=V(R_1)^\#$ and $u[R_2(U)]^\#=v[R_2(U)]^\#$. Since $u[R_2(U)]^\#=v[R_2(U)]^\#$, there exist $z_0,\dots,z_n\in\mathrm{FG}(X)$ such that $z_0=u,z_n=v$ and such that

for each $k\in\{1,\dots,n\}$ there exist $s_k,t_k\in\mathrm{FG}(X)$ and $(g_{i_k},h_{i_k})\in R_2(U)$

$$\text{such that } \{z_{k-1},z_k\}=\{s_kg_{i_k}t_k,s_kh_{i_k}t_k\}. \quad (3{\cdot}1)$$

Let $P=U\cup V\cup A_{i_1}\cup B_{i_1}\cup\ldots\cup A_{i_n}\cup B_{i_n}$. Let $\theta:\mathrm{FG}(X)\to\mathrm{FG}(P)$ be the homomorphism defined by

$$x\theta=\begin{cases}x & \text{if } x\in P\\ 1 & \text{if } x\in X\setminus P.\end{cases}$$

Since $\xi(u),\xi(v)\subseteq P$, we have $u\theta=u$ and $v\theta=v$. Since $\xi(g_{i_k}),\xi(h_{i_k})\subseteq P$, we have $\{z_{k-1}\theta,z_k\theta\}=\{(s_k\theta)g_{i_k}(t_k\theta),(s_k\theta)h_{i_k}(t_k\theta)\}$ for every $k\in\{1,\dots,n\}$. Therefore we can assume that $\xi(s_k),\xi(t_k),\xi(z_k)\subseteq P$ for every k.

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Since $U(R_1)^\# = V(R_1)^\#$, $U(R_1)^\# = (U \cup A_{i_k})(R_1)^\#$ and $A_{i_k}(R_1)^\# = B_{i_k}(R_1)^\#$ for every k, we have $U(R_1)^\# = P(R_1)^\#$. Hence there exist $W_0, \ldots, W_m \in \mathrm{FSU}(X)$ such that $W_0 = U$, $W_m = P$ and such that for all $q \in \{1, \ldots, m\}$ there exist $S_q \in \mathrm{FSU}(X)$ and $j_q \in I$ satisfying $\{W_{q-1}, W_q\} = \{S_q \cup A_{j_q}, S_q \cup B_{j_q}\}$.

This yields immediately that $(W_0, 1) = (U, 1)$, $(W_m, 1) = (P, 1)$ and that

$$\{(W_{q-1}, 1), (W_q, 1)\} = \{(S_q, 1)(A_{j_q}, 1), (S_q, 1)(B_{j_q}, 1)\}$$

for all $q \in \{1, \ldots, m\}$. But for every q we have

$$(A_{j_q}, 1)R^\# = [(A_{j_q}, g_{j_q})(A_{j_q}, g_{j_q})^{-1}]R^\# = [(B_{j_q}, h_{j_q})(B_{j_q}, h_{j_q})^{-1}]R^\# = (B_{j_q}, 1)R^\#$$

and so $(U, 1)R^\# = (P, 1)R^\#$. Similarly, we obtain $(V, 1)R^\# = (P, 1)R^\#$.

However $(P, s_k), (P, t_k), (P, z_k) \in \mathrm{FCM}(X)$ for every k. Moreover $(P, z_0) = (P, u)$, $(P, z_n) = (P, v)$ and, by (3·1),

$$\{(P, z_{k-1}), (P, z_k)\} = \{(P, s_k)(A_{i_k}, g_{i_k})(P, t_k), (P, s_k)(B_{i_k}, h_{i_k})(P, t_k)\}$$

for all $k \in \{1, \ldots, n\}$. Therefore $(P, u)R^\# = (P, v)R^\#$. Thus

$$\begin{aligned}(U, u)R^\# &= (U, 1)R^\#(U, u)R^\# = (P, 1)R^\#(U, u)R^\# = (P, u)R^\# = (P, v)R^\# \\ &= (P, 1)R^\#(V, v)R^\# = (V, 1)R^\#(V, v)R^\# = (V, v)R^\#,\end{aligned}$$

and the lemma is proved.

Now suppose that $R = \{((A_i, g_i), (B_i, h_i)) : i \in \{1, \ldots, n\}\}$ is a finite relation on $\mathrm{FCM}(X)$. We denote $\bigcup_{i=1}^n (A_i \cup B_i)$ by Y and we define $K(R)$ to be $\{R_2(U) : U \in \mathrm{FSU}(Y)\}$.

THEOREM 4. *Let* $\mathrm{Clf}\langle X; R\rangle$ *be a finitely related presentation, with* $R = \{((A_i, g_i), (B_i, h_i)) : i \in \{1, \ldots, n\}\}$. *Then* (i) $K(R)$ *can be effectively determined*; (ii) *the word problem for* $\mathrm{Clf}\langle X; R\rangle$ *is decidable if and only if the word problem for* $\mathrm{Gp}\langle X; K\rangle$ *is decidable for every* $K \in K(R)$.

Proof. (i) Let $U \in \mathrm{FSU}(Y)$. By Lemma 2, we can, for every $i \in \{1, \ldots, n\}$, determine whether or not $U(R_1)^\# = (U \cup A_i)(R_1)^\#$. Therefore we can effectively compute $R_2(U)$. But, since Y is finite, we only have to compute finitely many $R_2(U)$, so $K(R)$ can be effectively determined.

(ii) Suppose that the word problem for $\mathrm{Clf}\langle X; R\rangle$ is decidable. Let $K \in K(R)$. Then $K = R_2(V)$ for some $V \subseteq Y$. Since Y is finite and we can compute $R_2(U)$ for every $U \in \mathrm{FSU}(Y)$, we can assume that V is maximal with respect to inclusion.

Suppose that $(g_i, h_i) \in K$. Then $(V \cup A_i)(R_1)^\# = V(R_1)^\#$ and so $R_2(V) = R_2(V \cup A_i)$. By the maximality of V, we have $A_i \subseteq V$. Similarly, we have $(V \cup B_i)(R_1)^\# = V(R_1)^\#$ and so $R_2(V) = R_2(V \cup B_i)$ and $B_i \subseteq V$. Hence $\xi(g_i), \xi(h_i) \subseteq V$. Thus by [3], §9·3 the word problem for $\mathrm{Gp}\langle X; K\rangle$ is decidable if and only if the word problem for $\mathrm{Gp}\langle V; K\rangle$ is decidable.

Let $v, v' \in \mathrm{FG}(V)$. By Lemma 3, we have $vK^\# = v'K^\#$ if and only if $(V, v)R^\# = (V, v')R^\#$. But, by hypothesis, we can decide whether or not $(V, v)R^\# = (V, v')R^\#$. Hence the word problem for $\mathrm{Gp}\langle V; K\rangle$ is decidable, and so the word problem for $\mathrm{Gp}\langle X; K\rangle$ is decidable.

Conversely, suppose that the word problem for $\mathrm{Gp}\langle X; K\rangle$ is decidable for every $K \in K(R)$. Let $(U, u), (V, v) \in \mathrm{FCM}(X)$. By Lemma 3, $(U, u)R^\# = (V, v)R^\#$ is equivalent to

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$U(R_1)^\# = V(R_1)^\#$ and $u[R_2(U)]^\# = v[R_2(U)]^\#$. By Lemma 2, we can decide whether or not $U(R_1)^\# = V(R_1)^\#$ and so we only need to show that we can decide whether or not $u[R_2(U)]^\# = v[R_2(U)]^\#$.

We prove that $R_2(U) = R_2(U \cap Y)$. Clearly $R_2(U \cap Y) \subseteq R_2(U)$. Now suppose that $i \in \{1, \ldots, n\}$ and $(U \cup A_1)(R_1)^\# = U(R_1)^\#$. Then, by (2·1), we have $[(U \cup A_i) \cap Y](R_1)^\# = (U \cap Y)(R_1)^\#$. But $A_i \subseteq Y$ and so $[(U \cap Y) \cup A_i](R_1)^\# = (U \cap Y)(R_1)^\#$. Thus we have $R_2(U) \subseteq R_2(U \cap Y)$ and so $R_2(U) = R_2(U \cap Y)$.

Hence $R_2(U) \in K(R)$ and, by hypothesis, we can decide whether or not $u[R_2(U)]^\# = v[R_2(U)]^\#$. Thus the word problem for $\mathrm{Clf}\langle X; R\rangle$ is decidable.

Now the case $|R| = 1$ follows easily.

COROLLARY 5. *One-relator Clifford presentations have decidable word problem.*

Proof. This is a consequence of Theorem 4 and the fact that one-relator group presentations have decidable word problem (Magnus [7]). In fact, we have $|R_2| = 1$ and so $|K| \leqslant 1$ for every $K \in K(R)$.

3. *The E-unitary problem*

In this section we will study the E-unitary problem for the class of one-relator Clifford monoid presentations and the class of finite Clifford monoid presentations.

We need preliminary results relating the E-unitary property to the concepts with which we have been working. We keep the notation of Section 2.

LEMMA 6. *Let M denote the Clifford monoid defined by the presentation* $\mathrm{Clf}\langle X; R\rangle$, *with* $R = \{((A_i, g_i), (B_i, h_i)) : i \in I\}$. *Let σ denote the least group congruence on M. Then, for every* $(U, u) \in \mathrm{FCM}(X)$,

$$[(U, u)R^\#]\sigma = 1 \quad \textit{if and only if} \quad u(R_2)^\# = 1(R_2)^\#.$$

Proof. Let $(U, u) \in \mathrm{FCM}(X)$. Suppose that $[(U, u)R^\#]\sigma = 1$. Then there exists $V \in \mathrm{FSU}(X)$ such that $(U, u)R^\#(V, 1)R^\# = (V, 1)R^\#$. Hence $(U \cup V, u)R^\# = (V, 1)R^\#$ and by Lemma 3, we have $u[R_2(U \cup V)]^\# = 1[R_2(U \cup V)]^\#$. Since $R_2(U \cup V) \subseteq R_2$, we have $u(R_2)^\# = 1(R_2)^\#$.

Now suppose that $u(R_2)^\# = 1(R_2)^\#$. Then there exist $w_0, \ldots, w_n \in \mathrm{FG}(X)$ such that $w_0 = u$, $w_n = 1$ and such that for all $k \in \{1, \ldots, n\}$ there exist $r_k, s_k \in \mathrm{FG}(X)$ and $i_k \in I$ with $\{w_{k-1}, w_k\} = \{r_k g_{i_k} s_k, r_k h_{i_k} s_k\}$.

Let $Y = U \cup A_{i_1} \cup \ldots \cup A_{i_n}$. Then $u[R_2(Y)]^\# = 1[R_2(Y)]^\#$, since $(g_{i_k}, h_{i_k}) \in R_2(Y)$ for every $k \in \{1, \ldots, n\}$, and $U \subseteq Y$. By Lemma 3, we have $(Y, u)R^\# = (Y, 1)R^\#$ and so $(U, u)R^\#(Y, 1)R^\# = (Y, 1)R^\#$. Thus $[(U, u)R^\#]\sigma = 1$ and the lemma is proved.

LEMMA 7. *Let M denote the Clifford monoid defined by the presentation* $\mathrm{Clf}\langle X; R\rangle$. *Then M is E-unitary if and only if*

$$\textit{for } u \in \mathrm{FG}(X)\backslash\{1\},\ u(R_2)^\# = 1(R_2)^\# \textit{ implies } u[R_2(\xi(u))]^\# = 1[R_2(\xi(u))]^\#. \qquad (7{\cdot}1)$$

Proof. Suppose that M is E-unitary. Let $u \in \mathrm{FG}(X)\backslash\{1\}$ and suppose that $u(R_2)^\# = 1(R_2)^\#$. By Lemma 6, we have $[(\xi(u), u)R^\#]\sigma = 1$. Since M is E-unitary, we have $(\xi(u), u)R^\# \in E(M)$, and so

$$(\xi(u), u)R^\# = (\xi(u), u)R^\#[(\xi(u), u)^{-1}]R^\# = (\xi(u), 1)R^\#.$$

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Hence, by Lemma 3, we must have $u[R_2(\xi(u))]^\# = 1[R_2(\xi(u))]^\#$.

Now suppose conversely that $u(R_2)^\# = 1(R_2)^\#$ implies $u[R_2(\xi(u))]^\# = 1[R_2(\xi(u))]^\#$ for every $u \in \mathrm{FG}(X)\backslash\{1\}$. Let $(U,u) \in \mathrm{FCM}(X)$ be such that $[(U,u)R^\#]\sigma = 1$. We want to prove that $(U,u)R^\# \in E(M)$. The case $u = 1$ is trivial, so we assume $u \neq 1$. By Lemma 6, we have $u(R_2)^\# = 1(R_2)^\#$ and so, by hypothesis, $u[R_2(\xi(u))]^\# = 1[R_2(\xi(u))]^\#$. Hence, by Lemma 3, $(\xi(u),u)R^\# = (\xi(u),1)R^\#$ and so $(\xi(u),u)R^\# \in E(M)$. But $(U,u)R^\# = (U,1)R^\#(\xi(u),u)R^\#$, so that $(U,u)R^\# \in E(M)$, and the lemma is proved.

We can provide a very simple algorithm which solves completely the one-relator case, but first we need some definitions.

The reduced word $g \in \mathrm{FG}(X)$ is said to be cyclically reduced if g does not have mutually inverse first and last letters.

Let $g \in \mathrm{FG}(X)$. Clearly g can be written in the form $g = aba^{-1}$ for some $a, b \in \mathrm{FG}(X)$ with b cyclically reduced. Hence g has a cyclically reduced conjugate. Suppose that b and b' are two cyclically reduced conjugates of g. We show that $\xi(b) = \xi(b')$. Suppose that $\xi(b) \neq \xi(b')$. Without loss of generality, we can assume that there exists $x \in \xi(b)\backslash\xi(b')$. Since b and b' are conjugates, we have $b = cb'c^{-1}$ for some $c \in \mathrm{FG}(X)$. Obviously $x \in \xi(c)$. Hence we can write $c = c_1 x^\epsilon c_2$, reduced as written, for some $\epsilon \in \{-1, +1\}$ and $c_1, c_2 \in \mathrm{FG}(X)$ with $x \notin \xi(c_2)$. Let $d = c_2 b' c_2^{-1}$. Since $x \notin \xi(d)$, then $b = c_1 x^\epsilon d x^{-\epsilon} c_1^{-1}$ is reduced as written, which contradicts the fact that b is cyclically reduced. Hence $\xi(b) = \xi(b')$ and so any two cyclically reduced conjugates of g have the same content.

We define $\xi_c(g)$ to be the content of a cyclically reduced conjugate of g.

THEOREM 8. *Let M denote the Clifford monoid defined by the one-relator presentation* $\mathrm{Clf}\langle X; R\rangle$, *with* $R = \{((A,g),(B,h))\}$. *Then M is E-unitary if and only if one of the following conditions is satisfied*: (i) $g = h$; (ii) $A \subseteq \xi_c(gh^{-1})$; (iii) $B \subseteq \xi_c(gh^{-1})$.

Proof. Suppose that $g = h$. Then $(R_2)^\#$ is the identity and (7·1) is trivially satisfied. Hence, by Lemma 7, M is E-unitary.

Now assume that $g \neq h$. We prove that

$$M \text{ is } E\text{-unitary if and only if } \xi_c(gh^{-1})(R_1)^\# = [\xi_c(gh^{-1}) \cup A](R_1)^\#. \tag{8·1}$$

Suppose that M is E-unitary. Since $g(R_2)^\# = h(R_2)^\#$, any cyclically reduced conjugate of gh^{-1}, say f, satisfies $f(R_2)^\# = 1(R_2)^\#$. Since $f \neq 1$, we have $f[R_2(\xi(f))]^\# = 1[R_2(\xi(f))]^\#$, by Lemma 7. Hence $R_2(\xi(f)) \neq \varnothing$ and since $|R_2| = 1$ we have $R_2(\xi(f)) = R_2$, that is, $R_2(\xi_c(gh^{-1})) = R_2$. By definition, we obtain that $\xi_c(gh^{-1})(R_1)^\# = [\xi_c(gh^{-1}) \cup A](R_1)^\#$.

Conversely, suppose that $\xi_c(gh^{-1})(R_1)^\# = [\xi_c(gh^{-1}) \cup A](R_1)^\#$. Then $R_2(\xi_c(gh^{-1})) = R_2$. Let $u \in \mathrm{FG}(X)\backslash\{1\}$ be such that $u(R_2)^\# = 1(R_2)^\#$. The Freiheitssatz (Magnus [**6**]) states that $\xi_c(gh^{-1}) \subseteq \xi(u)$. Therefore $R_2(\xi(u)) = R_2$ and so (7·1) holds. By Lemma 7, M is E-unitary. Hence (8·1) holds.

Suppose that (ii) or (iii) holds. Then clearly $\xi_c(gh^{-1})(R_1)^\# = [\xi_c(gh^{-1}) \cup A](R_1)^\#$ and so, by (8·1), M is E-unitary.

Now suppose that neither (ii) nor (iii) holds. Then $[\xi_c(gh^{-1})](R_1)^\# = \{\xi_c(gh^{-1})\}$ and so, by (8·1), M is not E-unitary and the theorem is proved.

The next corollary is immediate.

COROLLARY 9. *Let M denote the Clifford monoid defined by the one-relator presentation* $\mathrm{Clf}\langle X; R\rangle$, *with* $R = \{((A,g),(\varnothing,1))\}$. *Then M is E-unitary.*

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The E-unitary problem for one-relator inverse monoid presentations is still open. Margolis and Meakin [9] formulated a conjecture on the subject. We can successively identify X with the corresponding sets of generators of $\mathrm{FIM}(X)$ and $\mathrm{FG}(X)$. Then we say that $u = x_1 \dots x_n \in \mathrm{FIM}(X)$, $x_i \in X$, is reduced (respectively, cyclically reduced) if $v = x_1 \dots x_n \in \mathrm{FG}(X)$ is reduced (respectively, cyclically reduced). The conjecture states that, for every $u \in \mathrm{FIM}(X)$ with u reduced and $R = \{(u, 1)\}$, $[\mathrm{FIM}(X)]/R^\#$ is E-unitary if and only if u is cyclically reduced. We can provide a counter-example.

Let $X = \{x, y\}$, let $u = xyx^{-2}yxyx^{-1}$ and let $R = \{(u, 1)\}$. We prove that $[\mathrm{FIM}(X)]/R^\#$ is a group. It is clear that, for every $w, z \in \mathrm{FIM}(X)$,

$$(wzw)R^\# = 1R^\# \quad \text{implies} \quad (ww^{-1})R^\# = 1R^\# = (w^{-1}w)R^\#.$$

Let $v = xyx^{-1}$. Since $u = vx^{-1}yv$, we have $(v^{-1}v)R^\# = 1R^\#$. Therefore

$$[x^{-1}(yvxy)x^{-1}]R^\# = (x^{-1}yv^2)R^\# = (v^{-1}vx^{-1}yv^2)R^\# = (v^{-1}uv)R^\# = (v^{-1}v)R^\# = 1R^\#$$

and so $(x^{-1}x)R^\# = 1R^\# = (xx^{-1})R^\#$. Hence

$$[y(x^{-2}yx)y]R^\# = (x^{-1}xyx^{-2}yxyx^{-1}x)R^\# = (x^{-1}ux)R^\# = (x^{-1}x)R^\# = 1R^\#$$

and so $(yy^{-1})R^\# = 1R^\# = (y^{-1}y)R^\#$. Thus $\{(zz^{-1}, 1) : z \in X \cup X^{-1}\} \subseteq R^\#$, and so M is certainly a group, and in particular it is E-unitary.

The question of whether or not $[\mathrm{FIM}(X)]/R^\#$ is E-unitary when $R = \{(u, 1)\}$, for u cyclically reduced, is still open.

Now we consider a more general class of presentations.

THEOREM 10. *The E-unitary problem for finite Clifford monoid presentations is undecidable.*

Proof. It is well-known that there exists a finite group presentation with undecidable word problem (see Boone [2], Novikov [10]). Let $\mathrm{Gp}\langle Z; T\rangle$ be such a presentation, with $T = \{(h_i, 1) : i \in \{1, \dots, n\}\}$, and let a be a new element, distinct from the elements of Z. Let $h_{n+1} = a^2$. Let $X = Z \cup \{a\}$ and $I = \{1, \dots, n+1\}$. Let $g \in \mathrm{FG}(Z)$ and let $w = ag$. Let y denote a new element, distinct from the elements of X.

We define a finite relation R on $\mathrm{FCM}(X \cup \{y\})$ by

$$R = \{((X, h_i), (X, 1)) : i \in I\} \cup \{((X \cup \{y\}, 1), (\{x\}, 1)) : x \in X\} \cup \{((X \cup \{y\}, y), (X \cup \{y\}, w))\}.$$

We have

$$R_1 = \{(X, X)\} \cup \{(X \cup \{y\}, \{x\}) : x \in X\} \cup \{(X \cup \{y\}, X \cup \{y\})\}.$$

There are three $(R_1)^\#$-classes in $\mathrm{FSU}(X \cup \{y\})$: $\{\varnothing\}$, $\{\{y\}\}$ and $\{A \subseteq X \cup \{y\} : X \cap A \neq \varnothing\}$. It is clear from the definitions that $R_2(\varnothing) = R_2(\{y\}) = \varnothing$ and $R_2(A) = R_2$ for every $A \subseteq X \cup \{y\}$ such that $X \cap A \neq \varnothing$. Hence (7·1), with X replaced by $X \cup \{y\}$, is trivially satisfied by every $u \in \mathrm{FG}(X \cup \{y\}) \backslash \{1\}$, except possibly when $\xi(u) = \{y\}$. Since $y^k \neq 1$ for every $k \in \mathbb{Z} \backslash \{0\}$ and $R_2(\{y\}) = \varnothing$, $[\mathrm{FCM}(X)]/R^\#$ is E-unitary if and only if

$$y^k (R_2)^\# \neq 1(R_2)^\# \quad \text{for all } k \in \mathbb{Z} \backslash \{0\}. \tag{10·1}$$

We have $R_2 = \{(h_i, 1) : i \in I\} \cup \{(1, 1)\} \cup \{(y, w)\}$. The presentation $\mathrm{Gp}\langle X \cup \{y\}; R_2\rangle$ is clearly equivalent (by a Tietze transformation) to the presentation $\mathrm{Gp}\langle X; S\rangle$, where $S = \{(h_i, 1) : i \in I\}$. Moreover, (10·1) is equivalent to

$$w^k S^\# \neq 1S^\# \quad \text{for all } k \in \mathbb{Z} \backslash \{0\}. \tag{10·2}$$

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We prove that (10·2) is equivalent to $gT^\# \neq 1T^\#$.

Suppose that $gT^\# = 1T^\#$. Then $gS^\# = 1S^\#$ and so $wS^\# = aS^\#$. But $aS^\#$ has order 2, so (10·2) does not hold.

Now suppose that $gT^\# \neq 1T^\#$. As we see in [**8**], §4·1, $[\mathrm{FG}(X)]/S^\#$ is the free product (in the variety of groups) of $[\mathrm{FG}(Z)]/T^\#$ and $[\mathrm{FG}(\{a\})]/\{(a^2, 1)\}^\#$. Hence $[\mathrm{FG}(Z)]/T^\#$ embeds canonically in $[\mathrm{FG}(X)]/S^\#$ and so $gS^\# \neq 1S^\#$. Similarly we have $aS^\# \neq 1S^\#$, and so $wS^\#$ is a non-homogeneous element of a free product of two groups, that is, $wS^\#$ is not contained in either of the factor groups. Therefore $wS^\#$ has infinite order and so (10·2) holds.

Thus decidability of the E-unitary problem for all finite Clifford monoid presentations would imply decidability of the word problem for $\mathrm{Gp}\langle Z; T\rangle$. The result follows.

COROLLARY 11. *The E-unitary problem for finite inverse monoid presentations is undecidable.*

Proof. This follows from Theorem 10 and Lemma 1(i).

4. *Other decidability results*

The results of Section 2 provide some general positive answers. By contrast, other problems turn out to be undecidable, as a consequence of analogous results on group presentations.

The following result is a corollary of Lemmas 2 and 3.

THEOREM 12. *The idempotent word problem is decidable for every finitely related Clifford monoid presentation.*

This enables us to decide whether or not a finitely presented Clifford monoid is a group.

COROLLARY 13. *The group problem for finite Clifford monoid presentations is decidable.*

Proof. Let M be the Clifford monoid defined by the presentation $\mathrm{Clf}\langle X; R\rangle$. Then M is a group if and only if M has a single idempotent, that is, if and only if $(A, 1)R^\# = (\varnothing, 1)R^\#$ for every $A \in \mathrm{FSU}(X)$. Since X is finite, this is clearly equivalent to $(X, 1)R^\# = (\varnothing, 1)R^\#$ and so we just apply Theorem 12.

It is known to be undecidable whether or not finitely presented groups are trivial (or finite) (see Adjan [**1**], Rabin [**12**]). This yields some analogous results.

THEOREM 14. *It is undecidable whether or not finitely presented Clifford monoids are* (i) *trivial*, (ii) *finite*, (iii) *semilattices*, (iv) *free*.

Proof. (i) and (ii) follow from Lemma 1(ii). Since a group is a semilattice if and only if it is trivial, the same applies to (iii).

Now we prove (iv). Let G be a group, defined as a Clifford monoid by a finite presentation $\mathrm{Clf}\langle X; R\rangle$. Let Y be a finite non-empty set, disjoint from X. Let S denote the relation R, considered as a relation on $\mathrm{FCM}(X \cup Y)$. We prove that

$$[\mathrm{FCM}(X \cup Y)]/S^\# \text{ is free} \quad \text{if and only if} \quad G \text{ is trivial.} \tag{14·1}$$

Let $\psi : \mathrm{FCM}(X) \to \mathrm{FCM}(X \cup Y)$ denote the inclusion map and let $\bar{\phi}$ be the homomorphism from $\mathrm{FCM}(X \cup Y)$ to $\mathrm{FCM}(X)$ defined by $x\phi = (\{x\}, x)$ for $x \in X$ and

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$y\phi = (\varnothing, 1)$ for $y \in Y$. Obviously $\psi\bar{\phi} = 1$ and so these homomorphisms induce homomorphisms

$$\Psi : [\mathrm{FCM}(X)]/R^\# \to [\mathrm{FCM}(X \cup Y)]/S^\# \quad \text{and} \quad \Phi : [\mathrm{FCM}(X \cup Y)]/S^\# \to [\mathrm{FCM}(X)]/R^\#$$

such that $\Psi\Phi = 1$.

Suppose that $[\mathrm{FCM}(X \cup Y)]/S^\#$ is free. Let $(U, u) \in \mathrm{FCM}(X)$. Since $[(U,u)(U,u)^{-1}]R^\# = (\varnothing, 1)R^\#$, we have

$$[(U,u)(U,u)^{-1}]S^\# = ([(U,u)(U,u)^{-1}]R^\#)\Psi = [(\varnothing,1)R^\#]\Psi = (\varnothing,1)S^\#.$$

Similarly, we obtain $[(U,u)^{-1}(U,u)]S^\# = (\varnothing,1)S^\#$. Hence $(U,u)S^\#$ is a unit of $[\mathrm{FCM}(X \cup Y)]/S^\#$. Since the group of units of a free Clifford monoid is trivial, we have $(U,u)S^\# = (\varnothing,1)S^\#$. Hence

$$(U,u)R^\# = [(U,u)S^\#]\Phi = [(\varnothing,1)S^\#]\Phi = (\varnothing,1)R^\#$$

and so G is trivial.

Conversely, suppose that G is trivial. Let $\bar{\alpha} : \mathrm{FCM}(Y) \to [\mathrm{FCM}(X \cup Y)]/S^\#$ be the homomorphism defined by $y\alpha = (\{y\}, y)S^\#$ for $y \in Y$ and let $\bar{\gamma} : \mathrm{FCM}(X \cup Y) \to \mathrm{FCM}(Y)$ be the homomorphism defined by $x\gamma = (\varnothing, 1)$ for $x \in X$ and $y\gamma = (\{y\}, y)$ for $y \in Y$. Clearly $\bar{\gamma}$ induces a homomorphism $\Gamma : [\mathrm{FCM}(X \cup Y)]/S^\# \to \mathrm{FCM}(Y)$ and $\bar{\alpha}\Gamma = 1$. Since G is trivial, we have

$$(\{x\}, x)S^\# = [(\{x\}, x)R^\#]\Psi = [(\varnothing,1)R^\#]\Psi = (\varnothing,1)S^\# = [(\{x\},x)S^\#]\Gamma\bar{\alpha}$$

for every $x \in X$. Now it follows easily that $\Gamma\bar{\alpha} = 1$ and so $\bar{\alpha}$ is an isomorphism. Thus $[\mathrm{FCM}(X \cup Y)]/S^\#$ is free and (14·1) holds.

Since it is not decidable whether or not G is trivial for every finitely presented group G, it is not decidable whether or not every finitely presented Clifford monoid is free.

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