Some strategies for teaching a course based on *Gödel, Escher, Bach*

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Introduction

In 1979, Douglas Hofstadter published a Pulitzer prizewinning classic, *Gödel, Escher, Bach: an eternal golden braid* [1], henceforward referred to as GEB. I have presented a course at La Salle University based on GEB eleven times during the last twenty-one years. From these experiences, I present below some useful strategies for instructors who are interested in teaching a course where at least half of GEB is covered in some depth.

Structure of GEB

Hofstadter's GEB consists of 20 Dialogue-Chapter pairs, where each Dialogue introduces some of the major themes of the subsequent chapter. These, in turn, are framed by an opening Introduction and a concluding Dialogue, both of which highlight many of the book's major concepts.* Each Dialogue is based (at least nominally) on a musical work by J. S. Bach, and most of them contain several unusual characters. The most prominent (speaking) characters are Achilles and the Tortoise, but we also encounter the Crab, the Sloth, the Anteater, and even the Author himself.†

Chapters I-VI present an introduction to J. S. Bach, M. C. Escher, Kurt Gödel, and the major ideas of the work, including logical paradoxes, Post production systems, consistency and completeness, and recursion.

Chapters VII-IX cover propositional logic and elementary number theory, ending with a first glimpse into the proof of Gödel's Theorem. Chapters X-XII compare the structure and operation of computers to the intricacy of human brains and minds. Chapters XIII-XV present a formal proof of Gödel's Theorem using a computer language (BLooP) of Hofstadter's invention, and some of the theorem's ramifications. Chapter XVI compares the self-reference inherent in Gödel's Theorem to the self-representation in DNA. In the final part of GEB, Chapters XVII-XX introduce the Turing Test for artificial intelligence (AI), advances (at that time) in AI research, and the prospects for attaining AI. These chapters conclude with a discussion of self-consciousness and free will, while posing the question of whether these can ever be attained by an AI machine.

* The Preface added to the 20th Anniversary edition is only of interest to students after they have worked through the rest of GEB.

† There are also many others, including a TV Announcer, a Cop, many Genies, Eta Oin, SHRDLU (a computer program), Dr Tony Earrwig (an anagram for Terry Winograd), Hexachlorophene J. Goodfortune, and the well-known figures Charles Babbage and Alan Turing.

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Overall, GEB is an enormously clever work, containing much humour (especially puns), and many logical puzzles for the reader to work out, a challenge that many students relish.

Prerequisites for students

The GEB-related course at La Salle is intended for students of all major subjects who are in the university's Honors Program, and have an interest in mathematics, logic, and computer science. The nature of the material is quite challenging, so in order to do justice to Hofstadter's book, the students in the course must be strongly motivated to do a lot of reading and reflecting. While the level of the material becomes quite 'deep' in places, the only necessary prerequisites are a knowledge of arithmetic and an ability to think logically. The students must be able to follow the steps of a given mathematical argument — in particular, an argument in symbolic logic (such as those in Chapters VIII-IX) in which strings in a Post production system are manipulated according to certain given rules. A familiarity with computer programming and/or languages is not necessary, but would be an enormous help when debugging computer programs such as those in Chapter XIII.

Incentives for teaching a GEB course

A GEB-based course is truly rewarding since the instructor has an opportunity to introduce students to many intriguing ideas which they might never encounter in any other course. In addition to the life and works of Bach and Escher, GEB contains an unusually large collection of interesting topics ranging from haiku poetry and Zen koans to mazes and labyrinths — including the Drake equation and the search for extra-terrestrial intelligence. And GEB also contains a veritable mathematical feast! A partial list of mathematically-related topics includes:

- Logical paradoxes, such as Zeno's paradox and Russell's paradox
- Whitehead and Russell's attempt in *Principia Mathematica* to construct a mathematical system containing arithmetic that is both consistent and complete, and the 1931 Incompleteness theorem of Kurt Gödel, which demonstrated that such a goal is impossible
- Mathematical conjectures such as the Goldbach conjecture; the '$3N + 1$' conjecture, and Fermat's Last Theorem (still unproved when GEB was published)
- Recursion; self-referential constructions (e.g. 'quining', DNA/RNA) and their relationship to Gödel's theorem
- Fibonacci and Lucas sequences
- Euclid's proof of the infinity of primes
- Cantor's (diagonal) proof that the real numbers are uncountable
- Möbius strips; tilings of the plane

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- Charles Babbage's computers (Difference Engines 1 and 2, and the Analytical Engine)
- Computer languages and algorithms (primitive vs. general vs. partial recursive)
- The Church-Turing thesis (in various formulations)

Challenges in teaching a GEB-based course

Needless to say, a GEB-based course can be quite daunting due to the sheer scope of the work. GEB is vast (742 pages), and quite dense in places. Reading the book is both exhilarating and draining! Hofstadter's style is very entertaining and thought-provoking, but he can easily overwhelm the reader with many tangential issues and distractions. Reading the book for enjoyment is very different than 'fighting through it' to master all of its complexity. The book could easily be two-thirds of its present size with essentially the same overall impact on a novice reader.

In preparing a course based on GEB, I quickly discovered some major issues:

(1) Choices have to be made up front about what to leave out — choices that are not necessarily obvious when teaching from GEB for the first time. Conversely, GEB contains a wealth of interesting related topics, but does not always have the space for an in-depth coverage of each one. The book practically cries out to be supplemented with related articles, artwork, and music.

(2) GEB has no formal exercises for the students as one would find in a 'regular' mathematics textbook. The instructor must prepare problems and essay questions to entice the students to delve thoroughly into the work.

(3) The lives and works of Lewis Carroll,* Alan Turing, and Charles Babbage play important roles in GEB, and I believe that much more needs to be included in the course about their contributions, especially where Carroll is concerned.

Strategies for teaching a GEB-based course

When teaching a GEB course for the first time, I recommend covering only half to two-thirds of the chapters. Portions of the book that could definitely be skipped the first time around are Chapter VI (*The Location of Meaning*) (pages 158-176), and all of the material from the second half of Chapter XIV (the material after the formal proof of Gödel's theorem) to the end of Chapter XVI (*Self-Ref and Self-Rep*) (pages 450-548). In particular, the material in the second half of Chapter XIV as well as all of Chapter XV forms the most difficult and dense portion of the entire work. However, the

* In fact, two of Hofstadter's Dialogues in GEB are actually by Lewis Carroll: *What the Tortoise said to Achilles* (1895), and the classic nonsense poem *Jabberwocky* from *Through the Looking Glass* (along with French and German translations).

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first half of Chapter XVI does illustrate beautiful parallels between the self-reference in Gödel's theorem and the self-replication engendered by DNA, which are particularly interesting to students majoring in biology or chemistry.

To get through most of the book in one semester, and still leave enough quality time for related topics, I assign roughly 20 pages per class to be read before the next class. (However, to get a jump start while the students are still 'fresh', I assign 30 pages to be read for each of the second and third classes.) While Honors students are able to assimilate much of the content on their own, there are certain places in GEB where it is necessary to slow down considerably, and help the students through the text virtually line by line. In particular, an instructor should be prepared to spend a great deal of extra time on:

- Chapter VII *The Propositional Calculus* and Chapter VIII (*Typographical Number Theory*) where there are many derivations of results in propositional logic and arithmetic using Post production rules. These can be mystifying to students who have never seen propositional logic before.
- The Dialogue *A MU Offering*, which is the most dense and complex Dialogue in GEB, where the students need explanations about what is really going on (on several levels) when they are reading this for the first time.
- Pages 260-272 in Chapter IX (*Mumon and Gödel*), in which a first outline of the proof of Gödel's theorem is sketched.
- Chapter XIII (*BLooP and FLooP and GLooP*), which is particularly difficult for students who have never before written and/or traced through the logic of a computer program.
- The first half of Chapter XIV (*On formally undecideable propositions...*), in which Gödel's theorem is proved (in a more complex way than necessary, but deliberately so in order to relate with later material in Chapter XVI).
- Pages 495-536 in Chapter XVI (*Self-Ref and Self-Rep*), in which the complicated molecular biology of DNA/RNA is presented, before establishing a formal isomorphism with the logical components in Gödel's theorem.
- The Dialogue *SHRDLU, Toy of Man's Designing* between Chapters XVII and XVIII needs to be further illustrated by showing the various configurations of the stacked blocks at crucial points in the Dialogue.

It is extremely helpful to create a series of PowerPoint slides highlighting the main themes and particular concepts presented in each chapter of GEB in order to get through material as expeditiously as possible. Naturally, these slides can be posted to a website for the course for the students' reference.

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Connections to Lewis Carroll

After teaching the course for the first time, I realised that not only the two Carroll Dialogues (see footnote above), but all of Carroll's *Alice in Wonderland* and *Through the Looking Glass* mesh incredibly well with GEB. Carroll's frequent allusions to logic, especially logical puzzles and paradoxes, and his humorous puns fit GEB 'like a glove'. Here are just a few examples: Humpty Dumpty's definitions for the unusual words in *Jabberwocky* (presented in GEB) help to make this poem more understandable; the concepts of mirror inversion (in the Looking Glass world) and time inversion (the White Queen lives backwards) match the note inversion and reversals in the canons of Bach's *Musical Offering*; the changes in the size of Carroll's characters match the changes in the size of characters in Hofstadter's *Little Harmonic Labyrinth* Dialogue. In addition, the card motif of *Alice in Wonderland* and the chess motif of *Through the Looking Glass* symbolise the various games discussed by Hofstadter throughout GEB – in fact, Hofstadter treats computer chess- and checker-playing algorithms in Chapter XVIII.

Martin Gardner's *The Annotated Alice* (*the definitive edition*) is a wonderful supplement to GEB. Not only does Gardner's book contain the complete text of both *Alice in Wonderland* and *Through the Looking Glass*, as well as all of the classic Tenniel illustrations, but it also contains copious notes about Carroll's life and customs in Victorian times, which make so many of Carroll's allusions clearer to the students. There is also much discussion of the philosophical arguments raging in Carroll's time and the references to them in the *Alice* books. (*The Annotated Alice* also contains the suppressed episode 'The Wasp in a Wig' from *Through the Looking Glass*, along with an extensive commentary.) Many students who have read the *Alice* books as children discover that there is much more depth to them after Gardner has explained many of the jokes, riddles, and puns, and they are amazed that these works can be read anew with a different type of enjoyment and appreciation now that they are much older.

I normally have the students read *Alice in Wonderland* during the first half of the semester (usually a chapter per class period), and *Through the Looking Glass* in the second half (again, one chapter per class, but a little more slowly, since the second book is more complex). It helps to work out in advance the 'current' chess diagram (that is, the new positions of the characters on the chess board) as you begin each chapter of *Through the Looking Glass*, so the students can more easily follow the thread of the story.

Bach's music

Most students are unfamiliar with J. S. Bach and the music of his era, so it is enormously useful to prepare several handouts about Bach's life and times, including his family life, the various posts that he held (throughout what is now Germany), basic musical notation and forms (e.g. canons,

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preludes and fugues), types of musical composition (e.g. cantatas, concertos, solo keyboard works), musical instruments (e.g. clavichord, harpsichord, piano, organ), and the need for well-tempering of instruments. Reading through at least one classical biography of Bach's life is a great help before teaching a GEB-course (e.g. Geiringer, Spitta, Forkel, Terry). Another particularly helpful reference is Mendel & David's *The Bach Reader*.

The most important musical works by Bach that are referenced in GEB are:

- *Musical Offering*
- *Art of the Fugue*
- *Two- and three-part inventions*
- *Well-Tempered Clavier*
- *Goldberg Variations*
- *Magnificat in D*
- *Air on the G String* (from *Orchestral Suite no. 3*)
- *Jesu, Joy of Man's Desiring* (from *Cantata no. 157*)
- *Little Harmonic Labyrinth*
- *Chromatic Fantasy and Fugue*

It is very helpful to have handouts about the structure of some of these works (particularly the *Musical Offering*, the *Goldberg Variations*, and the *Magnificat in D*). (There are several lines in GEB's *The Magnificrab, Indeed* Dialogue that parallel lines of text in Bach's *Magnificat*.) It is also useful to have recordings of all of these compositions available (reserved, if necessary) in the university library for students, as well as recommendations for good performances of these works. Of course, students can also access performances of all or part of these works from websites on the Internet.

Usually, in the class closest to Bach's birthday (March 21), we have a Bach Party (with a cake and candles indicating Bach's accumulated age) during which we view a concert of his music. This gives the students some insight about how Bach's music is actually performed in practice, rather than simply considering his music as an academic exercise.

Escher's Works

It is useful to acquire a small library of reference books containing Escher's works, including *The World of M. C. Escher* (published by Abrams), Doris Schattschneider's *Visions of Symmetry* (published by Freeman), *M. C. Escher's Legacy: A Centennial Celebration* (published by Springer), and *Escher's Exploring the Infinite* (published by Abrams). Virtually the complete work of Escher is available on a CD-ROM 'Escher Interactive' which is well worth having. The students particularly enjoy looking at Escher puzzle pieces, calendars, and toys.

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It is also helpful to devote at least one class day to a slide show of Escher's life and travels, the processes he used to create his woodcuts and notebooks, and illustrations of many other Escher works not found in GEB. This is an opportunity to point out recurrent themes in Escher's work: symmetry (various types), infinity, tilings/tesselations of the plane and space, geometric transformations (translations, rotations, reflections, and glide reflections), plane/space similarity, and inversion.

Other useful supplements

Aside from the supplements already mentioned, it is useful to acquaint the students with many of Martin Gardner's books and articles (from his 'Mathematical Games' columns for *Scientific American*), as well as Hofstadter's other books and articles (from his 'Metamagical Themas' columns for *Scientific American*). In particular, Gardner's gift for presenting recreational mathematics at a level accessible to students not majoring in mathematics makes many of his articles an ideal introduction to various topics related to GEB (e.g. on the Fibonacci Sequence, on Möbius strips, on the *I Ching* (p. 232 of GEB), Turing machines, etc.).

It is also worthwhile to create handouts listing chronologically highlights in the lives of Bach, Escher, Gödel, and Carroll. Also:

- The *Little Harmonic Labryinth* Dialogue can be supplemented naturally with samples of famous mazes (old and new), and the Trevaux algorithm for maze traversal can be presented at this point.
- For the *Canon on Intervallic Augmentation* Dialogue, it is useful to have a handout listing some of John Cage's other musical works (aside from *4′ 33″*), incorporating some of his own explanations and comments about these pieces.
- If Chapter VI is covered, supplements on the Drake Equation, the messages on the Pioneer 10 & 11 and Voyager 1 & 2 spacecraft, the SETI Project, including the potential use of mathematics to communicate with extra-terrestrials (e.g. Carl Sagan's *Contact*, or *Life Search* in the Time-Life *Voyage through the Universe* series) are all useful.
- Helpful interpretations for most of the Zen koans presented in GEB can be found in *Zen Koans*, by Gyomay Kubose (Regnery, 1973).
- The introduction of Möbius strips (on p. 276, and their bisection, earlier on p. 30) are natural springboards for a discussion of their properties, and also what happens when they are trisected. Martin Gardner's science-fiction short story *No-sided Professor* is an entertaining supplement on this topic.

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- An article on Magritte and the art of Surrealism (e.g. Bennett Schiff's *Smithsonian* article 'The Artist who was Master of the Double take' forms a nice addition to the *Edifying Thoughts of a Tobacco Smoker* Dialogue.
- Supplements on the Human Genome Project, its history and accomplishments, are especially helpful to illustrate the connections between Gödel's Theorem and DNA/RNA in Chapter XVI.
- Although Alan Turing's contributions to deciphering the Enigma code during WWII are not explicitly mentioned in GEB in the discussion of his life on pp. 594-595, more information about this is now available, so it is helpful to have pamphlets on the design of the Enigma machine. These are available from the National Security Agency. The PBS *Nova* video 'The Codebreakers' is also an excellent introduction to this topic.
- Hofstadter's *Scientific American* column 'A Coffeehouse Conversation on the Turing Test' (reprinted in *Metamagical Themas*) is a lengthy, but useful supplement to the discussion of the Turing Test in Chapter XVIII of GEB.
- A handout introducing the basic syntax of the LISP computer program is helpful since Hofstadter discusses LISP at many places in the book, with an actual example of LISP code presented on p. 632. Such a handout is also important to prevent students unfamiliar with computer languages from confusing the structure of LISP with the very different structures of the languages BLooP and FLooP created by Hofstadter in Chapter XIII.
- The 14 Bongard problems (pp. 646-660) presented in GEB can be easily supplemented with others from the full collection of 100 Bongard problems as challenges for homework or exams.
- When Babbage's machines are discussed in GEB's final Dialogue (p. 727 onward), it is helpful to acquaint the students with the February 2003 article from *Scientific American* on the reconstruction of Charles Babbage's Difference Engine no. 2.
- When the topic of artificial intelligence (AI) is covered in some depth in the last few chapters of GEB, the debate on the nature of AI between John Searle and the Churchlands in the January 1990 issue of *Scientific American* is especially relevant. The very different beliefs of Daniel Dennett and Roger Penrose on AI are worth contrasting as well.

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Some Caveats about GEB

- The MU-Puzzle presented in Chapter I is initially left unsolved, but Hofstadter returns to it later in the text (pp. 260-261) to show that a solution is impossible. It is probably best to explain early on why the puzzle is unsolvable, but with 'U' replaced by '3' instead of '0' when initially showing this to students. Also, an important question that Hofstadter does not fully answer is whether all strings whose corresponding M-I-U number is not a multiple of 3 are actually producible in the M-I-U system. In fact, they are, and going through a proof of this result with the students is helpful, particularly before reaching p. 416, where there is a discussion of a computer program to determine whether such a string is producible.
- There is an error in the program 'PRIME?' on p. 413, since the input value $N = 1$ produces the output 'YES' because the central loop will not be executed.
- In the formal proof of Gödel's theorem (pp. 438-448), some of the functions involved need to be 'tests' rather than 'functions' – that is, their titles should end with a question mark in Hofstadter's scheme, because they produce an output of 'True' or 'False' (see bottom of p. 446 and top of p. 447, for example).
- The Crab's musical puzzles in *The Magnificrab, Indeed* Dialogue usually need an explanation, especially the last three (pp. 556-557), which respectively represent: the square root of 2 is irrational, 50 is the smallest positive integer that can be represented as a sum of squares in two different ways, and the Goldbach conjecture (except that the latter unintentionally asserts that each of 0 and 2 can be expressed as the sum of two primes as well — perhaps contributing to the Crab's reluctance about this statement).

Course requirements and assessment

In addition to their daily reading assignment from GEB, the students are given a daily written assignment to be turned in for grading. This assignment varies from class to class: an essay asking for reactions from the readings, or for the solution to a puzzle based on the mathematics and logic presented in the text. These daily assignments count as 20% of the overall grade.

The students also complete a term paper (15 pages in length) on any subject related to the course material, but each student must choose a different topic. The term paper counts as another 20% of the course grade. Over the years, the term paper topics have been incredibly diverse, such as: the history of robotics, uses of Escher's work in elementary and secondary education, particular aspects of Bach's life and music, ambidextrousness in

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nature, unsolved conjectures in mathematics, and aspects of the paranormal in late Victorian literature! I have kept a copy of every student's paper (with my final comments scattered throughout) which helped me enormously when students needed to find sources for similar papers in later years.

The students are given two in-class tests and a final exam, each of which represents 20% of the course grade. The first test covers Chapters I-VIII of GEB along with *Alice in Wonderland*, the second test covers Chapters IX-XIV of GEB and the first half of *Through the Looking Glass*, and the final exam covers Chapters XVI-XX and the second half of *Through the Looking Glass*. (The second half of Chapter XIV and all of Chapter XV are not covered in the course.)

Because the number of topics in this course is quite vast, and because particular students have strengths in certain areas but not others, I have found it useful to give students a choice of questions in these exams. In this way, students can demonstrate that they have acquired depth in some areas without needing to be an expert in every facet of the material that was covered. Each exam consists of 3 parts: (1) Fill-in-the-blanks (where students must select, say, 12 questions to complete out of 16), (2) Short problems (where students must solve, say, 4 problems out of 6), and (3) Essays (where students must complete, say, 3 essays out of 6).

Final thoughts

While this course has presented my students with much new information concerning the nature of thought and AI, I have inevitably learned a great deal from them as well. They continually find new connections and insights that continue to surprise me, even after teaching this course for so many years. It is especially interesting to read their replies when I ask them (in the final exam) to summarise their views on free will, the nature of thought, and whether they believe, after having gone through this course, that an AI machine is possible, likely or inevitable.

Final advice: the most essential preparation for any GEB-based course is that the instructor has plenty of rest beforehand!

Reference

1. D. Hofstadter, *Gödel, Escher, Bach: an Eternal Golden Braid*, (20th Anniversary edn.), Basic Books (1999).

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