Ergod. Th. & Dynam. Sys. (2016), **36**, 1067–1105 © *Cambridge University Press, 2015*
doi:10.1017/etds.2014.102

Partially hyperbolic diffeomorphisms with a uniformly compact center foliation: the quotient dynamics

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(*Received* 17 *October* 2012 *and accepted in revised form* 9 *September* 2014)

Abstract. We show that a partially hyperbolic C^1-diffeomorphism $f : M \to M$ with a uniformly compact f-invariant center foliation $\mathcal{F}^c$ is dynamically coherent. Further, the induced homeomorphism $F : M/\mathcal{F}^c \to M/\mathcal{F}^c$ on the quotient space of the center foliation has the shadowing property, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit of center leaves is ϵ-shadowed by an orbit of center leaves. Although the shadowing orbit is not necessarily unique, we prove the density of periodic center leaves inside the chain recurrent set of the quotient dynamics. Other interesting properties of the quotient dynamics are also discussed.

1. *Introduction*

Among chaotic dynamical systems, the dynamics of the hyperbolic diffeomorphisms is now well understood. In particular, they are structurally stable: the nearby diffeomorphisms are conjugate, hence they have the same topological dynamics. This leads to the hope that one might be able to classify their dynamics, up to conjugacy. At the same time the *shadowing lemma* is a very useful tool for understanding the dynamics of hyperbolic systems (for instance, for proving density of periodic orbits in the chain recurrent set) or for proving the structural stability: the orbits of a nearby system are pseudo-orbits of the initial system, hence are shadowed by true orbits.

Here we consider partially hyperbolic systems. A C^1-diffeomorphism $f : M \to M$ on a smooth compact manifold M is called *partially hyperbolic* if its tangent bundle splits into three non-trivial, df-invariant subbundles, called the stable, unstable and center bundle,

$$TM = E^s \oplus E^c \oplus E^u,$$

such that df contracts uniformly vectors in the stable direction, expands uniformly vectors in the unstable direction, and contracts and/or expands in a weaker way vectors in the center direction. Like hyperbolicity, partial hyperbolicity is a C^1-robust property.

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None of these bundles is *a priori* differentiable. However, dynamical arguments ensure that the stable and unstable bundles integrate to unique f-invariant stable and unstable foliations. In contrast, the center bundle might not be integrable, even in a weak sense (see [**Wil98**, **HRHU10**]). If there is a foliation tangent to the center bundle, it is not known if it is unique, and therefore it is not known if there exists an invariant center foliation. Finally, if such an invariant center foliation exists, it may exhibit pathological features: for instance, it may not be absolutely continuous with respect to Lebesgue measure, even for analytic partially hyperbolic diffeomorphisms (see [**SW00**, **Gog12**]).

However, if the center foliation exists and satisfies a condition called *plaque expansivity*, [**HPS70**] proved that it is C^1-*structurally stable*: every diffeomorphism g C^1-close to f admits a center foliation $\mathcal{F}^c_g$ conjugate to $\mathcal{F}^c_f$ by a homeomorphism h so that hgh^{-1} is a C^0-perturbation of f along the center leaves. In other words, the dynamics *transverse to the center bundle* is structurally stable. This leads to the hope of a classification of this transverse dynamics, up to topological conjugacy.

Here we consider a natural setting with an assumption which might seem very restrictive: we restrict ourselves to the class of partially hyperbolic diffeomorphisms where there is an invariant center foliation whose leaves are all compact. This class contains all the partially hyperbolic skew products over an Anosov diffeomorphism. Even assuming that every center leaf is compact, the center foliation may not be a fibration: there are examples of partially hyperbolic diffeomorphisms where the center foliation is a (generalized) Seifert bundle, so there are leaves with non-trivial holonomy (see [**BW05**]). Examples are partially hyperbolic automorphisms of the Heisenberg manifold. It is natural to ask if a center foliation by compact leaves may be even more complicated. This question makes sense as Sullivan constructs an example of a circle foliation on a compact manifold for which the length of the leaves is not bounded ([**Sul76**]; see also [**EV78**]). According to [**Eps76**] this is equivalent to the fact that some leaf has a non-finite holonomy group. Therefore, we call the foliations by compact leaves with finite holonomy *uniformly compact*, and we consider diffeomorphisms for which the center foliation is uniformly compact. It is conjectured that every compact center foliation is uniformly compact, and there are results by Gogolev [**Gog11**] and Carrasco [**Car11**] proving this conjecture under additional assumptions.

In the uniformly compact case [**Eps76**] shows that the quotient space $M/\mathcal{F}^c$ is a compact metric space with respect to the Hausdorff distance between the center leaves. The invariance of the center foliation $\mathcal{F}^c$ means that the diffeomorphism f passes in the quotient to a homeomorphism F. The aim of this paper is the study of this quotient dynamics.

We show that the normal hyperbolicity of the center foliation in M implies some kind of topological hyperbolic behavior for the quotient dynamics. As a first step, we would like to project onto $M/\mathcal{F}^c$ the stable and unstable foliations of f. Such a projection requires some compatibility between the stable, the unstable and the center foliation, called *dynamical coherence*: we need the existence of invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tangent to $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^u \oplus E^c$, respectively, and subfoliated by $\mathcal{F}^c$, and $\mathcal{F}^s$ and $\mathcal{F}^u$, respectively. It should be mentioned that there exist non-dynamically coherent partially hyperbolic systems, even on the 3-torus (see [**HRHU10**]). This is our first result.

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THEOREM 1. (Dynamical coherence) *Let f be a partially hyperbolic diffeomorphism on a compact manifold admitting an invariant uniformly compact center foliation $\mathcal{F}^c$. Then $(f, \mathcal{F}^c)$ is dynamically coherent.*

To prove Theorem 1, one shows that the center leaf $W^c(y)$ through a point $y \in W^s(x)$ is contained in the union of stable leaves through the points $z \in W^c(x)$. An intuitive idea is that, as $y \in W^s(x)$ and as the center foliation is uniformly compact, one easily deduces that the Hausdorff distance $d_H(f^n(W^c(x)),\ f^n(W^c(y)))$ tends to 0 for $n \to \infty$, leading to the intuition that $W^c(y)$ is contained in the stable manifolds through $W^c(x)$. However, this argument is not sufficient: we exhibit in Proposition 8.3 a partially hyperbolic diffeomorphism with a uniformly compact invariant center foliation having two center leaves $W^c(x)$ and $W^c(y)$ so that the Hausdorff distance $d_H(W^c(x), W^c(y))$ tends to 0 under positive iterates, but $W^c(y)$ is disjoint from $\bigcup_{z\in W^c(x)} W^s(z)$. The intuitive argument above is indeed correct if the center foliation is a locally constant fibration (that is, without holonomy); in the general case (with holonomy) we modify the distance by considering lifts of the leaves on holonomy covers. Such holonomy covers only exist locally, leading to many difficulties and a somewhat technical notion of distance, which is one of the key points of this paper.

The dynamical coherence implies that the quotient space is endowed with the quotient of the center stable and center unstable foliations. We would like to use the quotient of these foliations for proving hyperbolic properties of the quotient dynamics. In particular, we aim to recover the *shadowing property* (every pseudo-orbit is shadowed by a true orbit) which is one of the main characteristics of hyperbolicity and an important tool for describing the topological dynamics: for systems satisfying the shadowing property, the non-wandering set coincides with the chain recurrent set; therefore, one may apply Conley theory (dividing the chain recurrent set in chain recurrent classes separated by filtrations) to the non-wandering set. Furthermore, in the case of hyperbolic systems, the system is *expansive*: if the orbits of two points remain close for all time, the two points are equal; thus, the shadowing orbit is unique. This uniqueness leads to an even more precise description of the dynamics, e.g. the density of the periodic orbits in the non-wandering set.

In the hyperbolic case, the proof of the shadowing property makes heavy use of the stable and unstable foliations and the fact that they form a local product structure. However, in our setting the quotient on $M/\mathcal{F}^c$ of the center stable and center unstable foliations may not be foliations, as in [**BW05**, Proposition 4.4], and they may not induce a local product structure on the quotient. In particular, the quotient dynamics may not be expansive, as in the example in [**BW05**]. This induces a difficulty for recovering the shadowing lemma: the shadowing orbit may not be unique, if it exists. Nevertheless, we prove the following theorem†:

THEOREM 2. (Shadowing lemma) *Let $f : M \to M$ be a partially hyperbolic C^1-diffeomorphism with a uniformly compact invariant center foliation $\mathcal{F}^c$. Then the induced*

† Just as we were submitting this paper we learnt that Kryzhevich and Tikhomirov announced in [**KT12**] a shadowing lemma possibly related to ours. Their setting is on the one hand more general as they do not assume a compact foliation, but on the other hand they assume a uniquely integrable center and a stronger version of partial hyperbolicity.

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homeomorphism $F : M/\mathcal{F}^c \to M/\mathcal{F}^c$ has the shadowing property, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit of center leaves is ϵ-shadowed by an orbit of center leaves.

In Theorem 2, infinitely many orbits may shadow the same pseudo-orbit (see the example in Corollary 8.4). The non-uniqueness of the shadowing means that the shadowing property does not imply the existence of periodic orbits (here periodic center leaves). The lack of uniqueness comes from possible choices during the construction caused by the non-trivial holonomy. Theorem 5.10 expresses the fact that we recover the uniqueness of the shadowing orbit if we enrich the pseudo-orbits by these possible choices. Therefore, we get the following result.

THEOREM 3. *Let $f : M \to M$ be a partially hyperbolic C^1-diffeomorphism with a uniformly compact invariant center foliation $\mathcal{F}^c$. Then the periodic center leaves are dense in the chain recurrent set of the induced homeomorphism $F : M/\mathcal{F}^c \to M/\mathcal{F}^c$. In particular, the chain recurrent set coincides with the non-wandering set and with the closure of the set of periodic points.*

Theorem 3 implies that, if the chain recurrent set (or non-wandering set) of the quotient dynamics F is the whole of $M/\mathcal{F}^c$, then F is transitive. This is an important tool in [**Boh13**] for classifying codimension-1 partially hyperbolic diffeomorphisms with a uniformly compact center foliation.

In the case of hyperbolic dynamics, the expansivity is a key ingredient for the structural stability. In the case of partially hyperbolic systems with center foliation, this has been adapted in [**HPS70**] into the notion of *plaque expansivity*: the center foliation is plaque expansive if every pair of pseudo-orbits respecting the center foliation (i.e. with jumps in the local center leaves) remain close for all time, and then they lie in the same center leaf. As already stated, [**HPS70**] proves the structural stability of the center foliation for partially hyperbolic systems with plaque expansive center foliation.

Even, if the quotient dynamics is not expansive, the proof of the shadowing lemma uses an expansiveness property, which in particular implies plaque expansivity.

THEOREM 4. (Plaque expansivity) *Let $f : M \to M$ be a partially hyperbolic C^1-diffeomorphism with a uniformly compact invariant center foliation $\mathcal{F}^c$. Then $(f, \mathcal{F}^c)$ is plaque expansive.*

COROLLARY 1.1. (Structural stability) *Let $f : M \to M$ be a partially hyperbolic C^1-diffeomorphism with a uniformly compact invariant center foliation $\mathcal{F}^c$. Then $(f, \mathcal{F}^c)$ is C^1-structurally stable (in the sense of [**HPS70**]).*

In particular, the regularity of the diffeomorphism f does not interfere with their topological classification.

Remark 1.2. Theorems 1 and 2 have already appeared in [**Boh11**], but were not properly stressed as important results in their own right. Here they are embedded in a diligent and more complete study of the quotient dynamics.

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This work is organized as follows. In §2 we recall the relevant notions and implications from foliation theory with respect to uniformly compact foliations, and in §3 we develop a concept of holonomy cover which suits our purposes. We prove dynamical coherence (Theorem 1) in §4 in several steps, including the proof of the existence of a well-defined notion of unstable projection of a center leaf. All these preparatory steps help to prove the shadowing lemma (Theorem 2) in §5 quite directly, following the classical proof. The subsequent §6 is reserved for the proof of the plaque expansivity (Theorem 4), followed by the very direct implication (Theorem 7.1 in §7) of non-compactness of center stable and center unstable foliations given a uniformly compact center foliation. We conclude with a short discussion in §8 of an example (presented in [**BW05**]) which stresses the important differences of the quotient dynamics compared to the classical hyperbolic behavior.

2. *Preliminaries from foliation theory*

This section recalls the basic concepts of foliation theory used in this paper.

A $C^{1,0+}$*-foliation* is a foliation on a manifold M for which the leaves are C^1-immersed submanifolds and the solution of a distribution defined by a continuous subbundle of TM. Throughout this paper the word 'foliation' refers to $C^{1,0+}$*-foliation*.

2.1. *Holonomy.*

We briefly define the notion of holonomy for a leaf of a foliation. For a more complete definition we refer the reader to the books by Candel and Conlon [**CC00**] and Moerdijk and Mrčun [**MM03**]. See Figure 1.

Consider a foliation $\mathcal{F}$ on a manifold M and a closed path $\gamma : [0, 1] \to L$ with $\gamma(0) = \gamma(1) = x \in L$ which lies entirely inside one leaf $L \in \mathcal{F}$. We define a homeomorphism H_γ on a smooth disk T of dimension $q = \operatorname{codim} \mathcal{F}$ transversely embedded in the foliation $\mathcal{F}$ at x which fixes x and maps intersection points of a nearby leaf L' onto each other following the path γ. We call this the *holonomy homeomorphism along the path* γ. The definition of H_γ (more precisely, of the class of germs of H_γ) only depends on the homotopy class $[\gamma]$ of γ. Hence, we obtain a group homomorphism

$$\pi_1(L, x) \to \operatorname{Homeo}(\mathbb{R}^q, 0),$$

where $\operatorname{Homeo}(\mathbb{R}^q, 0)$ denotes the classes of germs of homeomorphisms $\mathbb{R}^q \to \mathbb{R}^q$ which fix the origin. The image of this group homomorphism is called the *holonomy group* of the leaf L and denoted by $\operatorname{Hol}(L, x)$. By taking the isomorphism class of this group it does not depend on the original embedded disk T in M. It is easily seen that any simply connected leaf has a trivial holonomy group. We say that a leaf has *finite holonomy* if the holonomy group $\operatorname{Hol}(L, x)$ for any $x \in L$ is a finite group, whose order is denoted by $|\operatorname{Hol}(L, x)|$. A foliation is *uniformly compact* if every leaf is compact and has finite holonomy. This is the main object we consider in the following.

A subset of the manifold is called $\mathcal{F}$*-saturated* if it is a union of leaves.

2.2. *Uniformly compact foliations.*

A classical result (proved in [**Hec77**, **EMT77**]) states that leaves with trivial holonomy groups are generic. In the case of a foliation whose leaves are all compact this implies that there exists an open and dense set of leaves with trivial holonomy. The first reason why we assume a uniformly compact foliation are the following equivalences.

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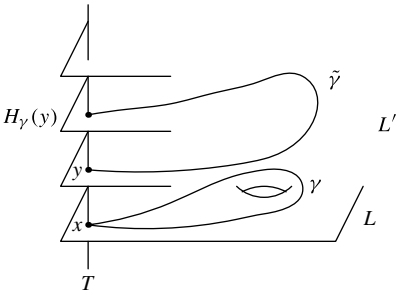


FIGURE 1. Construction of a holonomy homeomorphism $H_\gamma : T \to T$: It fixes $x = \gamma(0) = \gamma(1)$ and maps $y = \tilde{\gamma}(0)$ onto $\tilde{\gamma}(1)$ where $\tilde{\gamma}$ denotes the lift of γ to the nearby leaf L'.

THEOREM 2.1. (Epstein [**Eps76**]) *Let $\mathcal{F}$ be a foliation with all leaves compact of a manifold M. Then the following conditions are equivalent:*

- *The quotient map $\pi : M \to M/\mathcal{F}$ is closed.*
- *Each leaf has arbitrarily small saturated neighborhoods.*
- *The leaf space $M/\mathcal{F}$ is Hausdorff.*
- *If $K \subset M$ is compact then the saturation $\pi^{-1}\pi K$ of K is compact. This means that the set of leaves meeting a compact set is compact.*
- *The holonomy group of every leaf is finite.*

Remark 2.2. Furthermore, the quotient topology is generated by the Hausdorff metric d_H between center leaves in M, and hence $M/\mathcal{F}^c$ is a compact metric space. In the case of a compact foliation with trivial holonomy the resulting leaf space is a topological manifold.

Notation. For every leaf $L \in \mathcal{F}$ and $\delta > 0$, we denote by $B_H(L, \delta)$ the $\mathcal{F}$-saturated open ball of Hausdorff radius δ around L:

$$B_H(L, \delta) := \{L' \in \mathcal{F} \mid d_H(L, L') < \delta\}.$$

According to Theorem 2.1, every leaf admits a basis of saturated neighborhoods. As a consequence of this and of the compactness of the leaf space, one gets the following result.

COROLLARY 2.3. *Let $\mathcal{F}$ be a uniformly compact foliation of a compact manifold M. Then, for given $\alpha > 0$, there exists $\epsilon > 0$ such that, for every $x \in M$, the $\mathcal{F}$-saturate $\mathcal{F}(B(x, \epsilon)) = \bigcup_{y \in B(x,\epsilon)} L(y)$ of an ϵ-ball $B(x, \epsilon)$ is contained in an $\mathcal{F}$-saturated Hausdorff neighborhood $B_H(L(x), \alpha)$ of the leaf $L(x)$, i.e.*

$$\mathcal{F}(B(x, \epsilon)) \subset B_H(L(x), \alpha).$$

2.3. *Neighborhood of a compact leaf.* An important property of uniformly compact foliations is that there exist small saturated neighborhoods $V(L)$ of any leaf L that are foliated bundles $p : V(L) \to L$ and the holonomy group is actually a group. This fact is the content of the Reeb stability theorem which is proved in the present form in [**CC00**, Theorem 2.4.3].

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THEOREM 2.4. (Generalized Reeb stability) *Let L be a compact leaf of a foliation $\mathcal{F}$ of a manifold M. If its holonomy group* $\mathrm{Hol}(L)$ *is finite, then there is a normal neighborhood $p : V \to L$ of L in M such that $(V, \mathcal{F}|_V, p)$ is a fiber bundle with finite structure group* $\mathrm{Hol}(L)$.

Furthermore, each leaf $L'|_V$ is a covering space $p|_{L'} : L' \to L$ with $k \leq |\mathrm{Hol}(L, y)|$ sheets and the leaf L' has a finite holonomy group of order $|\mathrm{Hol}(L, y)|/k$.

2.4. *Transverse disks fields.* Let $\mathcal{F}$ be a foliation of a compact manifold M. We denote by $N_{\mathcal{F}}$ its normal bundle: in an abstract way one can see it as the quotient of the tangent bundle TM by the tangent bundle of $\mathcal{F}$. More concretely, if M is endowed with a Riemannian metric, one could realize the normal bundle as the orthogonal bundle $T\mathcal{F}^{\perp}$ to the foliation. However, we deal here with non-smooth foliations so that the orthogonal bundle is not smooth. For this reason, we denote by $N_{\mathcal{F}}$ a smooth subbundle of TM transverse to the tangent bundle $T\mathcal{F}$ of the foliation and close to the orthogonal bundle.

For any $\rho > 0$ we denote by $N_{\mathcal{F},\rho}$ the ρ-neighborhood of the zero section of $N_{\mathcal{F}}$, that is, the set of vectors in $N_{\mathcal{F}}$ with norm less than or equal to ρ.

For $\rho > 0$ small enough, the exponential θ (associated to the Riemannian metric) defines a submersion from $N_{\mathcal{F},\rho}$ to M and the restriction to each disk

$$N_{\mathcal{F},\rho}(x) = \{u \in N_{\mathcal{F}}(x),\ \|u\| \leq \rho\}$$

is an embedding which is transverse to $\mathcal{F}$. We denote

$$D_{x,\rho} = \theta(N_{\mathcal{F},\rho}(x)).$$

The family $\mathcal{D} = \{D_{x,\rho}\}$ is a family of disks centered at the reference point x, transverse to $\mathcal{F}$ and varying continuously with x in the C^1 topology. Such a family is called a *transverse disks field* . Using the compactness of the manifold, the continuity of the family of disks, and the tranversality, one deduces the following lemma (see, for instance, [**Bon93**, p. 64, §2.2.a] where such transverse disks fields are defined and used in the same way).

LEMMA 2.5. *There is $\rho > 0$ so that:*

- *for every $x \neq y$ in the same leaf L of $\mathcal{F}$ and with $d_{\mathcal{F}}(x, y) < \rho$, we have $D_{x,\rho} \cap D_{y,\rho} = \varnothing$;*
- *for every $x, z \in M$, the intersection of $D_{x,\rho} \cap B_{\mathcal{F}}(z, \rho)$ contains at most one point, where $B_{\mathcal{F}}(z, \rho)$ denotes the ball of radius ρ centered at z inside the leaf of z.*

Definition 2.6. Let L be a leaf of $\mathcal{F}$ and $\gamma : [0, 1] \to L$ a path on the leaf. Denote $x = \gamma(0)$. A path σ is a projection of γ along $\mathcal{D}$ on a leaf of $\mathcal{F}$ if:

- σ is contained in a leaf of $\mathcal{F}$; and
- $\sigma(t) \in D_{\gamma(t),\rho}$ for every t.

LEMMA 2.7. *Given a path $\gamma : [0, 1] \to L$ inside a leaf L of $\mathcal{F}$ and $x = \gamma(0)$, then for every $y \in D_{x,\rho}$ there is at most one projection γ_y of γ on the leaf L_y, starting at y. More precisely, the projection is unique and well defined until the distance $d(\gamma_y(t), \gamma(t))$ becomes larger than ρ.*

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3. *Holonomy covers*

From now on, we suppose that $\mathcal{F}$ is a uniformly compact foliation of a compact manifold M. As most proofs are quite intuitive for the case of trivial holonomy, our main tools are so-called *holonomy covers* which eliminate (locally) the holonomy and whose existence is due to the Reeb stability theorem (Theorem 2.4). In these covers we have to define a suitable well-defined distance between lifted leaves which is equivalent to the canonical Hausdorff distance.

3.1. *Holonomy covers.*

For every leaf L of a foliation $\mathcal{F}$ the holonomy covering map of $\mathcal{F}$ is the covering map associated to the kernel of the holonomy group of L. If $\mathcal{F}$ is uniformly compact, then the holonomy covering is a finite covering. Notice that this covering map extends to any tubular neighborhood of L in M. More precisely, we make the following definition.

Definition 3.1. A closed connected saturated set V *admits a holonomy covering* if there is a finite cover $p : \tilde{V} \to V$ such that the lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ on $\tilde{V}$ has trivial holonomy and if, for every leaf $L \subset V$, it induces the holonomy covering by restriction to every connected component of $p^{-1}(L)$.

Definition 3.2. We say that a finite cover $\mathcal{V} = \{V_i,\ p_i\}$ of M is a *holonomy cover* for the foliation $\mathcal{F}$ if:

- every V_i is a compact saturated set;
- the interiors of the sets V_i cover M, i.e. $M = \bigcup_i \operatorname{int}(V_i)$; and
- each V_i admits a holonomy covering $p_i : \tilde{V}_i \to V_i$.

The existence of a holonomy cover is given by the next lemma, which is a direct consequence of Theorem 2.4.

LEMMA 3.3. *Let $\mathcal{F}$ be a uniformly compact foliation and L a leaf of $\mathcal{F}$. Then there exist a compact saturated neighborhood V of L and a cover $p : \tilde{V} \to V$ such that:*

- *the restriction of p to $p^{-1}(L)$ is the holonomy covering map of L;*
- *for every leaf L' in V and any connected component $\tilde{L}'$ of $p^{-1}(L')$, the restriction of p to $\tilde{L}'$ is conjugate to the holonomy covering of L'.*

Remark 3.4.

- Let $\tilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ on the cover $\tilde{V}$ of V in Lemma 3.3. Then the foliation $\tilde{\mathcal{F}}$ is uniformly compact and with trivial holonomy.
- Any finite cover $q : \hat{V} \to V$, such that the lift of $\mathcal{F}$ on $\hat{V}$ has trivial holonomy, is a finite cover of $\tilde{V}$, i.e. there is a finite cover $r : \hat{V} \to \tilde{V}$ such that $q = p \circ r$.

3.2. *The metric and the holonomy covers.*

We now assume that M is endowed with a Riemannian metric, $\mathcal{F}$ is a uniformly compact foliation on M, and $\mathcal{V} = \{V_i,\ p_i\}$ is a holonomy cover of $\mathcal{F}$. For each V_i, we denote by $p_i : \tilde{V}_i \to V_i$ its holonomy covering. We endow the interior of each $\tilde{V}_i$ with the Riemannian metric obtained by lifting the metric on M by p_i. Therefore the projection p_i is a local isometry if restricted to the interior of each $\tilde{V}_i$.

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LEMMA 3.5. *Let $\tilde{\mathcal{F}}$ denote the lift of the foliation $\mathcal{F}$ on $\tilde{V}_i$. Then the diameter and the volume of the leaves of $\tilde{\mathcal{F}}$ for the lifted Riemannian metric are uniformly bounded independently of the holonomy cover.*

Proof. Let $\tilde{L} \subset \tilde{V}_i$ be a leaf of $\tilde{\mathcal{F}}$. Then $p_i(\tilde{L})$ is a leaf L of $\mathcal{F}$. Furthermore, the projection $p_i : \tilde{L} \to L$ is a finite cover with number of sheets uniformly bounded by the maximal rank of the holonomy groups. As the projection is a local isometry, the diameter of $\tilde{L}$ is bounded by the one of L multiplied by the number of sheets. As $\mathcal{F}$ is uniformly compact, the diameter and volume of its leaves are uniformly bounded, concluding the proof. □

LEMMA 3.6. *Given any holonomy cover $\mathcal{V} = \{V_i, p_i\}$ of a uniformly compact foliation $\mathcal{F}$, there is $\delta = \delta(\mathcal{V}) > 0$ so that for every leaf $L \in \mathcal{F}$ there is i so that:*

- *for every $x \in L$, the ball $B(x, \delta)$ is contained in the interior of V_i and the projection $p_i : \tilde{V}_i \to V_i$ induces an isometry in the restriction to every connected component of $p_i^{-1}(B(x, \delta))$;*
- *in particular, the Hausdorff ball $B_H(L, \delta)$ is contained in the interior of V_i.*

Proof. The interiors of the V_i induce an open cover of the leaf space $M/\mathcal{F}$, endowed with the Hausdorff distance. Therefore, there is a Lebesgue number δ_0 of this cover. In other words, for every leaf $L \in \mathcal{F}$, there is i such that the Hausdorff ball $B_H(L, \delta_0)$ is contained in the interior of V_i. One deduces from Corollary 2.3 the existence of δ_1 for which the union of the balls of radius δ_1 centered at points $x \in L$ is contained in $B_H(L, \delta_0)$. By shrinking δ_1 if necessary, one gets with $\delta := \delta_1$ the isometry condition claimed. □

3.3. *The holonomy covers and the Hausdorff metric.*

Remark 3.7. Let $\mathcal{V} = \{V_i, p_i\}$ be a holonomy cover. We still denote by d_H the Hausdorff distance in each $\tilde{V}_i$.

For any leaves $L_1, L_2 \subset V_i$ and any lifts $\tilde{L}_1, \tilde{L}_2$ in $\tilde{V}_i$ of L_1, L_2,

$$d_H(\tilde{L}_1, \tilde{L}_2) \geq d_H(L_1, L_2).$$

Therefore for every leaf L and i with $B_H(L, \delta) \subset \mathrm{int}(V_i)$, where $\delta = \delta(\mathcal{V})$ is given by Lemma 3.6, and any lift $\tilde{L}$ of L on $\tilde{V}_i$, it follows that $B_H(\tilde{L}, \delta) \subset \mathrm{int}(\tilde{V}_i)$.

LEMMA 3.8. *Let $\mathcal{V} = \{V_i, p_i\}$ be a holonomy cover and $\delta = \delta(\mathcal{V}) > 0$ be given by Lemma 3.6. For every $\delta_0 > 0$ there exists $\delta_1 > 0$ such that, for any leaf L and i with $B_H(L, \delta) \subset V_i$, one has:*

- *for every lift $\tilde{L}$ of L on V_i and every $\tilde{x} \in \tilde{L}$,*

$$\tilde{\mathcal{F}}(B_{\delta_1}(\tilde{x})) \subset B_H(\tilde{L}, \delta_0),$$

  where $\tilde{\mathcal{F}}$ denotes the lift of $\mathcal{F}$ on $\tilde{V}_i$;
- *for every lift $\tilde{L} \subset p_i^{-1}(L)$ and for every leaf $L_2 \subset B_H(L, \delta_1)$ there exists a lift $\tilde{L}_2 \subset p_i^{-1}(L_2)$ such that $\tilde{L}_2 \subset B_H(\tilde{L}, \delta_0)$.*

According to Corollary 2.3 for every uniformly compact foliation $\mathcal{F}$, for every $x \in M$ and every $\alpha > 0$ there exists $\eta > 0$ such that $\mathcal{F}(B_\eta(x)) \subset B_H(L(x), \alpha)$, where $L(x)$ denotes the leaf through x.

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The first item is the same statement, on the $\tilde{V}_i$. The unique difficulty here are *boundary effects*. This is the reason why we require $B_H(L, \delta) \subset V_i$, so that $\tilde{L}$ remains at distance δ from the boundary of $\tilde{V}_i$.

Idea of the proof. We fix a transverse disks field $\mathcal{D} = \{D_{x,\rho}\}$ as in §2.4.

We want to project the leaf $\tilde{L}$ along $\mathcal{D}$ onto the leaf through a point $\tilde{y} \in B_{\delta_1}(\tilde{x})$, and we want to show that, for δ_1 small enough, the distance of the projection remains smaller than δ_0.

We notice that the fact that $\tilde{\mathcal{F}}$ is without holonomy can be expressed as follows: if one projects a path starting at $\tilde{x}$ onto the leaf through $\tilde{y}$, according to Lemma 2.7, then the projection of the end point does not depend on the path (but just on this end point). Therefore, it is enough to consider paths whose length is bounded by the diameter of $\tilde{L}$. According to Lemma 3.5, this diameter is uniformly bounded, so that it is enough to consider paths with *a priori* bounded length.

One covers M by a finite atlas of foliated charts of $\mathcal{F}$ so that each chart is contained in a ball of radius smaller than δ. Therefore each chart meeting L can be lifted isometrically on $\tilde{V}_i$.

As the paths we consider have a uniformly bounded length, they cut a uniformly bounded number of charts. Let K denote this bound.

Now the bound follows from applying K times the uniform continuity of the plaques in each chart: for any $\mu > 0$, there is $\varepsilon > 0$ so that if two plaques in a chart have points at distance less than ε, then the Hausdorff distance between the plaques is less than μ. This proves the first item, choosing δ_1 small enough so that the projection remains at distance less than $\inf(\delta, \delta_0)$.

The second item is a direct consequence: choose a point $x \in L$ and $y \in L_2$ so that $d(x, y) < \delta_1 < \delta$. Let $\tilde{x}$ be a lift of x on $\tilde{L}$. There is a lift $\tilde{y}$ of y for which $d(\tilde{x}, \tilde{y}) < \delta_1$. The leaf $\tilde{L}_2$ is the leaf through $\tilde{y}$. □

The next lemma asserts that the Hausdorff distance of the lifts of leaves does not depend on the open neighborhood V_i such that the leaves remain far from the boundary.

LEMMA 3.9. (Local isometry) *Let $\mathcal{V} = \{V_i, p_i\}$ be a holonomy cover and $\delta = \delta(\mathcal{V}) > 0$ given by Lemma 3.6. Then there exists $\delta_0 > 0$ such that, for every point $x \in M$ and i, j such that $B_H(L(x), \delta) \subset U_i \cap U_j$, for all $x_1 \in p_i^{-1}(x)$, $x_2 \in p_j^{-1}(x)$, there exists an isometry*

$$I : B_H(\tilde{L}(x_1), \delta_0) \to B_H(\tilde{L}(x_2), \delta_0)$$

such that $p_j \circ I = \mathrm{id}_M \circ p_i$. Furthermore, $I(x_1) = x_2$ and I induces a conjugacy between the restrictions of the lifted foliation $\tilde{\mathcal{F}}$ to $B_H(\tilde{L}(x_1), \delta_0)$ and $B_H(\tilde{L}(x_2), \delta_0)$.

Idea of the proof. First notice that the restrictions of p_i and p_j to $\tilde{L}(x_1)$ and $\tilde{L}(x_2)$ are copies of the holonomy cover of $L(x)$. Furthermore, they are local isometries following Lemma 3.6 for $\delta_0 < \delta(\mathcal{V})$. One deduces that there is an isometry $I_L : \tilde{L}(x_1) \to \tilde{L}(x_2)$ with $I_L(x_1) = x_2$.

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We consider the transverse disks field $\mathcal{D}$ of radius ρ given by Lemma 2.5 and smaller than δ. As a consequence, for every $y \in L$ and any lift $\tilde{y}$ of y by p_i or p_j, there is a well-defined lift of $D_{y,\rho}$ centered at $\tilde{y}$, and this lift is an isometry. Let us denote this lift by $D_{\tilde{y},\rho}$.

As $\tilde{\mathcal{F}}$ is without holonomy, one can prove the following claim.

CLAIM 3.10. *For ρ small enough, for every leaf L and i so that $B_H(L, \delta) \subset V_i$, for every leaf $\tilde{L} \subset p_i^{-1}(L)$ and for every $\tilde{x} \neq \tilde{y} \in \tilde{L}$, one has $D_{\tilde{x},\rho} \cap D_{\tilde{y},\rho} = \varnothing$. In other words, $\bigcup_{\tilde{x}\in\tilde{L}} D_{\tilde{x},\rho}$ is a tubular neighborhood of $\tilde{L}$.*

Now, the isometry I is defined as

$$I|_{D_{\tilde{y},\rho}} = (p_j|_{D(I_L(p_i(\tilde{y})),\rho)})^{-1} \circ p_i. \qquad \square$$

The following result is a direct consequence of Lemmas 3.8 and 3.9.

COROLLARY 3.11. *Let $\mathcal{V} = \{V_i\}$ be a holonomy cover. Consider a leaf L and i, j such that $B_H(L, \delta) \subset V_i \cap V_j$. Assume that there are $L_1 \subset V_i$ and lifts $\tilde{L}^i \in p_i^{-1}(L)$ and $\tilde{L}_1^i \in p_i^{-1}(L_1)$ such that*

$$d_H(\tilde{L}^i, \tilde{L}_1^i) < \delta_0.$$

Then, for every lift $\tilde{L}^j \subset p_j^{-1}(L)$, there exists $\tilde{L}_1^j \in p_j^{-1}(L_1)$ such that

$$d_H(\tilde{L}^j, \tilde{L}_1^j) = d_H(\tilde{L}^i, \tilde{L}_1^i) \leq \delta_0.$$

3.4. *A modified Hausdorff distance.* We consider a holonomy cover $\mathcal{V}$ and $\delta = \delta(\mathcal{V})$ given by Lemma 3.6. Let $\delta_0 > 0$ be the constant associated to $\delta/2$ by Lemma 3.9, and $\delta_1 > 0$ associated to $\delta_0/2$ by Lemma 3.8. We can assume that $\delta_1 < \frac{1}{4}\delta$.

To every pair L_1, L_2 of leaves we associate a number $\Delta_H(L_1, L_2)$ as follows:

$$\Delta_H(L_1, L_2) := \begin{cases} \frac{1}{2}\delta_0 & \text{if } d_H(L_1, L_2) \geq \delta_1, \\ \min\{\frac{1}{2}\delta_0, d\} & \text{if } d_H(L_1, L_2) \leq \delta_1, \end{cases}$$

where d is the lower bound of the $d_H(\tilde{L}_1, \tilde{L}_2)$ for all $\tilde{L}_1 \in p_i^{-1}(L_1)$, $\tilde{L}_2 \in p_i^{-1}(L_2)$ such that $B_H(L_1, \delta) \subset V_i$. We denote a ball with respect to Δ_H by

$$B_{\Delta_H}(L, \delta) := \{L_2 \subset M \mid \Delta_H(L, L_2) < \delta\}.$$

As a corollary of Lemma 3.8 we prove the following result.

COROLLARY 3.12. *Δ_H is a distance, and it is topologically equivalent to d_H, i.e. for any leaf L and any $\alpha > 0$ there exist $\beta_1, \beta_2 > 0$ such that*

$$B_H(L, \beta_1) \subset B_{\Delta_H}(L, \alpha) \quad \text{and} \quad B_H(L, \beta_2) \supset B_{\Delta_H}(L, \alpha). \tag{3.1}$$

Proof. The positive definiteness is directly inherited by d_H. The symmetry comes from the fact that the lower bound of $d_H(\tilde{L}_1, \tilde{L}_2)$ for any $\tilde{L}_1 \in p_i^{-1}(L_1)$, $\tilde{L}_2 \in p_i^{-1}(L_2)$ does not depend on the i for which $B_H(L_1, \delta/2) \subset V_i$, according to Lemma 3.9. As we consider i for which $B_H(L_1, \delta) \subset V_i$ and $d_H(L_1, L_2) < \delta_1 < \delta/2$, one also has $B_H(L_2, \delta/2) \subset V_i$, leading to the symmetry.

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For the triangle inequality we consider three leaves L_1, L_2, L_3 and we need to prove

$$\Delta_H(L_1, L_3) < \Delta_H(L_1, L_2) + \Delta_H(L_1, L_3).$$

This is easy if $\Delta_H(L_1, L_2) + \Delta_H(L_1, L_3) \geq \frac{1}{2}\delta_0$. We choose i for which $B_H(L_2, \delta) \subset V_i$ and we consider lifts $\tilde{L}_i$ so that $d_H(\tilde{L}_1, \tilde{L}_2) = \Delta_H(L_1, L_2)$ and $d_H(\tilde{L}_3, \tilde{L}_2) = \Delta_H(L_3, L_2)$. Then $d_H(L_1, L_3) < \Delta_H(L_1, L_2) + \Delta_H(L_1, L_3)$ (triangular inequality for the Hausdorff distance) and $\Delta_H(L_1, L_3) \leq d_H(L_1, L_3)$.

We now show that Δ_H is a topologically equivalent distance to d_H. Consider a leaf L and $\alpha > 0$. Notice that $d_H(\tilde{L}_1, \tilde{L}_2) \geq d_H(L_1, L_2)$ so that $\Delta_H \geq \inf\{\frac{1}{2}\delta_0, d_H\}$. Therefore,

$$B_{\Delta_H}(L, \inf\{\tfrac{1}{4}\delta_0, \alpha\}) \subset B_H(L, \alpha).$$

The converse inclusion follows from Lemma 3.8: there exists $\beta_2 > 0$ such that any $L_2 \subset B_H(L, \beta_2)$ admits a lift so that $d_H(\tilde{L}, \tilde{L}_2) < \alpha$, proving $B_H(L, \beta_2) \subset B_{\Delta_H}(L, \alpha)$. □

4. *Dynamical coherence*

The aim of this section is to prove Theorem 1, i.e. that partially hyperbolic diffeomorphisms with an invariant uniformly compact center foliation are dynamically coherent. In other words, we have to prove firstly that, if x, y are two points in the same stable leaf (that is, $y \in W^s(x)$), then the center leaf $W^c(y)$ through y is contained in the union $\bigcup_{z\in W^c(x)} W^s(z)$ of stable leaves through the center leaf of x. Further, we have to show that for any $x \in M$ the set $\bigcup_{z\in W^c(x)} W^s(z)$ forms a leaf of the center stable foliation.

The idea of the proof is very simple: iterating x and y by f, the distance decreases exponentially. Consequently, the Hausdorff distance $d_H(W^c(f^n(x)), W^c(f^n(y)))$ tends to 0 for $n \to \infty$. On the other hand, if $W^c(y)$ has a point w outside $\bigcup_{z\in W^c(x)} W^s(z)$, its unstable manifold $W^u(w)$ cuts $\bigcup_{z\in W^c(x)} W^s(z)$ at some point w which is, in some sense, the projection of z on $\bigcup_{z\in W^c(x)} W^s(z)$. Keep in mind that this projection might be not unique! If the distance $d_H(W^c(x), W^c(y))$ is very small, the projection distance $d(z, w)$ will be small. At the same time, the distance $d(f^n(z), f^n(w))$ increases exponentially.

If the center foliation $\mathcal{F}^c$ is without holonomy, the projection w of z is uniquely defined, so that we get a contradiction. However, we do not assume $\mathcal{F}^c$ without holonomy so that this projection distance depends on choices. Therefore, to solve this problem,we lift the argument on a holonomy cover.

The argument cannot be so simple, as the example in Proposition 8.3 shows: the fact that $d_H(W^c(f^n(x)), W^c(f^n(y)))$ tends to 0 does not imply that $W^c(y)$ is contained in $\bigcup_{z\in W^c(x)} W^s(z)$. The difficulties come from the fact that each $W^c(f^n(x))$ and $W^c(f^n(y))$ may admit many lifts on a holonomy cover. The key point is to choose, for every lift of $W^c(f^n(x))$, the appropriate lift of $W^c(f^n(y))$.

4.1. *Local product structure.*

We want to prove that the stable and the center foliations are jointly integrable. A first step is to show that the three foliations—stable, center and unstable—form a kind of local product structure: the local unstable manifold of a point intersects at exactly one point the union of the local stable manifold through the local center manifold of a nearby point. That is the aim of the next two lemmas:

- Lemma 4.1 asserts the existence of the intersection point. That is a very general argument, which holds for any triple of transverse foliations.

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- Lemma 4.5 proves the uniqueness of the intersection point. This would be wrong for general $\mathcal{C}^{1,0+}$-foliations: it holds for dynamical reasons using the fact that we deal with the stable, center and unstable foliations of a partially hyperbolic diffeomorphism.

4.1.1. *Triple of pairwise transverse foliations.* Firstly, we establish the existence of an intersection point.

LEMMA 4.1. *Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be three continuous foliations with smooth leaves of a compact smooth manifold M, tangent to continuous distributions E_1, E_2, E_3 such that $TM = E_1 \oplus E_2 \oplus E_3$. Then there exist $\delta_0 > 0$ and $C > 0$ such that, for all $\delta < \delta_0$ and x, y with $d(x, y) < C\delta$, the intersection $F_{1,\delta}(x) \cap \bigcup_{z \in F_{2,\delta}(y)} (F_{3,\delta}(z))$ is non-empty.*

Proof of Lemma 4.1. Let d_1, d_2, d_3 denote the dimensions of the foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ in the lemma. Then $d_1 + d_2 + d_3 = d = \dim M$. Lemma 4.1 follows from Lemmas 4.2 and 4.3 below.

Lemma 4.2 asserts that, in sufficiently small local coordinates, the bundles E_1, E_2 and E_3 can be assumed to be arbitrarily close to the constant bundles $\mathbb{R}^{d_1} \times \{0\}^{d_2+d_3}$, $\{0\}^{d_1} \times \mathbb{R}^{d_2} \times \{0\}^{d_3}$ and $\{0\}^{d_1+d_2} \times \mathbb{R}^{d_3}$.

LEMMA 4.2. *Under the hypothesis of Lemma 4.1, for every $c > 0$ there is a finite family $(\psi_n)_n$ of smooth embeddings $\psi_n : [-4, 4]^d \to M$ such that the $\psi_n(]-1/2, 1/2[^d)$ form a finite open cover of M and, for every n, there are vector fields $e_1, \ldots, e_d$ on $[-4, 4]^d$ such that:*

- *the plane fields $(D\psi_n)^{-1}(E_1)$, $(D\psi_n)^{-1}(E_2)$, and $(D\psi_n)^{-1}(E_3)$ are generated by $(e_1, \ldots, e_{d_1})$, $(e_{d_1+1}, \ldots, e_{d_1+d_2})$ and $(e_{d_1+d_2+1}, \ldots, e_d)$, respectively;*
- *letting $(e_{i,1}(x) \ldots e_{i,d}(x))$ denote the coordinates of $e_i(x)$ in the canonical basis of $\mathbb{R}^d$, then:*
  - *$e_{i,i} = 1$ and $|e_{i,j}| < c$ for $i \neq j$;*
  - *$e_{i,j} = 0$ if $i \neq j$ and $(i, j) \in \{1, \ldots, d_1\}^2$ or $(i, j) \in \{d_1 + 1, \ldots, d_1 + d_2\}^2$ or $(i, j) \in \{d_1 + d_2 + 1, \ldots, d\}^2$.*

The proof of Lemma 4.2 is straightforward and just follows from the compactness of M, the continuity and the transversality of the subbundles E_i.

Lemma 4.3 uses the local coordinates given by Lemma 4.2 for proving the existence of the intersection point mentioned in Lemma 4.1.

LEMMA 4.3. *There is $C_d > 0$ so that, for every c satisfying $0 < c < C_d$ one has the following property. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be three pairwise transverse C^0-foliations of $[-4, 4]^d$ tangent to C^0-bundles E_1, E_2, E_3 of dimensions d_1, d_2, d_3 with $d_1 + d_2 + d_3 = d$ and satisfying the conclusions of Lemma 4.2 for a constant c. That is to say, there are vector fields $e_1, \ldots, e_d$ on $[-4, 4]^d$ such that:*

- *the plane fields $(D\psi_n)^{-1}(E_1)$, $(D\psi_n)^{-1}(E_2)$, and $(D\psi_n)^{-1}(E_3)$ are generated by $(e_1, \ldots, e_{d_1})$, $(e_{d_1+1}, \ldots, e_{d_1+d_2})$ and $(e_{d_1+d_2+1}, \ldots, e_d)$, respectively;*
- *letting $(e_{i,1}(x) \ldots e_{i,d}(x))$ denote the coordinates of $e_i(x)$ in the canonical basis of $\mathbb{R}^d$, then:*

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- $e_{i,i} = 1$ *and* $|e_{i,j}| < c$ *for* $i \neq j$;
- $e_{i,j} = 0$ *if* $i \neq j$ *and* $(i, j) \in \{1, \dots, d_1\}^2$ *or* $(i, j) \in \{d_1 + 1, \dots d_1 + d_2\}^2$ *or* $(i, j) \in \{d_1 + d_2 + 1, \dots, d\}^2$.

Then for any pair of points $x, y \in [-1, 1]^d$, *the leaf of* $\mathcal{F}_1$ *through* x *cuts the union of the leaves of* $\mathcal{F}_3$ *through the points of* $\mathcal{F}_2$ *through* y:

$$\mathcal{F}_1(x) \cap \bigcup_{z \in \mathcal{F}_2(y)} \mathcal{F}_3(z) \neq \varnothing.$$

Proof of Lemma 4.3. One can easily check that, for every c small enough, the leaves $\mathcal{F}_i(x)$ of $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ through points $x \in [-3, 3]^d$ are complete graphs over the cube $[-4, 4]^{d_1} \times \{0\}^{d_2+d_3}$, $\{0\}^{d_1} \times [-4, 4]^{d_2} \times \{0\}^{d_3}$, and $\{0\}^{d_1+d_2} \times [-4, 4]^{d_3}$, respectively. Furthermore, for $x \in [-1, 1]^d$ the intersection of $\mathcal{F}_2(x)$ with $[-3, 3]^d$ is a complete graph over $\{0\}^{d_1} \times [-2, 2]^{d_2} \times \{0\}^{d_3}$.

For every point $y = (y_1, \dots, y_d) \in [-1, 1]^d$ one considers the union $F_{2,3}(y)$ of the $\mathcal{F}_3$-leaves through $\mathcal{F}_2(y) \cap [-3, 3]^d$. These $\mathcal{F}_3$-leaves are pairwise disjoint so that $F_{2,3}$ is a topological embedding of $[-3, 3]^{d_2} \times [-4, 4]^{d_3}$; in particular, it is a $(d_2 + d_3)$-disk. Furthermore, as the constant c tends to 0, $F_{2,3}(y)$ tends uniformly to $\{(y_1, \dots, y_{d_1})\} \times [-3, 3]^{d_2} \times [-4, 4]^{d_3}$ for the C^0-topology.

Then the boundary $\partial F_{2,3}(y)$ is a $(d_2 + d_3 - 1)$-sphere, and tends uniformly to $\{(y_1, \dots, y_{d_1})\} \times \partial([-3, 3]^{d_2} \times [-4, 4]^{d_3})$. In particular, for c small enough, $\partial F_{2,3}(y)$ is disjoint from $[-4, 4]^{d_1} \times [-2, 2]^{d_2+d_3}$ and is homotopic to $\{(y_1, \dots, y_{d_1})\} \times \partial([-3, 3]^{d_2} \times [-4, 4]^{d_3})$ in $[-4, 4]^d \smallsetminus [-4, 4]^{d_1} \times [-2, 2]^{d_2+d_3}$.

In particular, one gets that, for every $x, y \in [-1, 1]^d$, $\partial F_{2,3}(y)$ is homotopic to $\{(y_1, \dots, y_{d_1})\} \times \partial([-3, 3]^{d_2} \times [-4, 4]^{d_3})$ in $[-4, 4]^d \smallsetminus [-4, 4]^{d_1} \times \mathcal{F}_1(x)$.

This implies that $\partial F_{2,3}(y)$ is not homotopic to a point in $[-3, 3]^d \smallsetminus \mathcal{F}_1(x)$. As a consequence, we prove $F_{2,3}(y) \cap F_1(x) \neq \varnothing$, concluding the proof of Lemma 4.3. □

The proof of Lemma 4.3 also completes the proof of Lemma 4.1. □

4.1.2. *Transverse foliations associated to a partially hyperbolic diffeomorphism.* The next lemma ensures that the intersection point is indeed unique. The argument is very similar to that presented in [**Bri03**, Lemma 7] and has been also explained to the second author in a personal communication from Brin.

We now consider a partially hyperbolic diffeomorphism f on a compact manifold M, with f-invariant center foliation $\mathcal{F}^c$. We denote by λ an upper bound for $\|Df\|$. According to [**Gou07**] we can use an equivalent adapted metric such that there exist constants $0 < \alpha < 1 < \beta$ such that the norm $\|Df|_{E^s}\|$ is less than α and $\|Df^{-1}|_{E^u}\|$ is less than β^{-1}.

Remark 4.4.

- As the center foliation $\mathcal{F}^c$ is tangent to a continuous bundle, its leaves are *uniformly* C^1. Therefore, the distance (in the manifold) between points in the same foliated *plaque* is equivalent to the distance in the center leaf. More precisely, we consider a finite foliated atlas $(U_i)_i$ of $\mathcal{F}^c$. The plaques are the leaves of the restrictions of $\mathcal{F}^c$ to the charts U_i. For every $\eta > 0$ there is μ so that, if x, y are points in the same leaf of $\mathcal{F}^c$ such that $d(x, y) < \mu$ and $d^c(x, y) > (1 + \eta)d(x, y)$, then there is i so that

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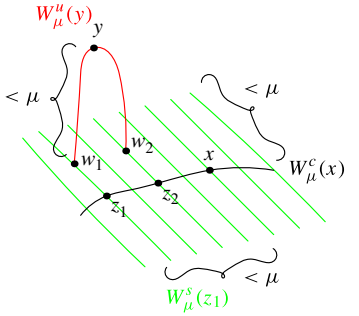


FIGURE 2. Assume, for any $\mu > 0$, that there are points x, y with $d(x, y) < \mu$ such that $W^u_\mu(y)$ cuts $\bigcup_{z \in W^c_\mu(x)} W^s_\mu(z)$ at two distinct points w_1 and w_2.

x and y belong to U_i but are not in the same plaque. As a consequence, there is a constant $\Delta^c > 0$ so that for all pair of points $x, y \in U_i$, but not in the same plaque, we have $d^c(x, y) > \Delta^c$. The same holds for the stable and unstable foliations $\mathcal{F}^s$ and $\mathcal{F}^u$.

- If $\mathcal{F}^c$ is a compact foliation with trivial holonomy, then two points in the same leaf which are nearby in the ambient manifold are in the same plaque. Therefore, the item above can be reformulated as follows: for any $\eta > 0$ there is $\mu > 0$ so that, if x, y are in the same leaf and if $d(x, y) < \mu$, then $d^c(x, y) < (1 + \eta)d(x, y)$.
- The foliations $\mathcal{F}^s, \mathcal{F}^c, \mathcal{F}^u$ are tangent to continuous bundles E^s, E^c, E^u which are in direct sum. As a consequence, for μ small enough, for every $x \in M$ the intersection $W^u_\mu(x) \cap W^c_\mu(x)$ contains at most one point: $W^u_\mu(x) \cap W^c_\mu(x) = \{x\}$.

LEMMA 4.5. *Let $f : M \to M$ be a partially hyperbolic C^1-diffeomorphism with an f-invariant center foliation $\mathcal{F}^c$, and let $C > 0$ be the constant associated to $\mathcal{F}^s, \mathcal{F}^c, \mathcal{F}^u$ by Lemma 4.1. Then:*

- *there exists a constant $\mu_0 > 0$ and, for any $x, y \in M$ with $d(x, y) < \mu_0$, the intersection*
$$W^u_{\mu_0}(y) \cap \left(\bigcup_{z \in W^c_{\mu_0}(x)} W^s_{\mu_0}(z) \right)$$
contains at most one point;
- *furthermore, if $d(x, y) \le \mu_0 / C$ then this intersection is a unique point, which belongs to*
$$W^u_{C\mu}(y) \cap \left(\bigcup_{z \in W^c_{C\mu}(x)} W^s_{C\mu}(z) \right),$$
where $\mu = d(x, y)$.

Proof. One argues by contradiction assuming that, for every $\mu > 0$ there is x, y at a distance $d(x, y) < \mu$ such that $W^u_\mu(y)$ cuts $\bigcup_{z \in W^c_\mu(x)} W^s_\mu(z)$ in two distinct points; see Figure 2. According to Remark 4.4 there is $\mu_* > 0$ so that:

- if two points p, q in the same center leaf are at distance less than μ_* in the manifold, but at distance more than $2d(p, q)$ in the center leaf, then they are at distance larger

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than $8\lambda^2\mu_*$ in the center leaf where $\lambda > 1$ is a upper bound for $\|Df\|$, i.e. for any

$$p, q \in M \text{ with } d(p, q) < \mu_* \text{ and } d^c(p, q) > 2d(p, q) \Rightarrow d^c(p, q) > 8\lambda^2\mu_*; \tag{4.1}$$

- if two points p, q in the same unstable leaf are at distance less than μ_* in the manifold, but at distance more than $2d(p, q)$ in the unstable leaf, then they are at distance larger than $8\lambda^2\mu_*$ in the unstable leaf, i.e. for any

$$p, q \in M \text{ with } d(p, q) < \mu_* \text{ and } d^u(p, q) > 2d(p, q) \Rightarrow d^u(p, q) > 8\lambda^2\mu_*; \tag{4.2}$$

- for every point $x \in M$, one has $W^u_{\mu_*}(x) \cap W^c_{\mu_*}(x) = \{x\}$.

Consider now $\mu < \mu_*/8$. According to our assumption, there are $x = x(\mu)$, $y = y(\mu)$ such that $W^u_\mu(y)$ cuts $\bigcup_{z\in W^c_\mu(x)} W^s_\mu(z)$ at two points w_1, w_2 with $w_i \in W^s_\mu(z_i)$, $z_i \in W^c_\mu(x)$. As $z_1, z_2 \in W^c(x)$ one gets $d^c(z_1, z_2) < 2\mu < \mu_*$.

To get uniqueness, let $0 < \alpha < 1 < \beta$ such that vectors in the stable bundle are contracted by α and vectors in the unstable bundle are expanded by β.

Consider the iterates $f^n(x)$, $f^n(y)$, $f^n(w_1)$, $f^n(w_2)$, $f^n(z_1)$, $f^n(z_2)$ for $n > 0$:

- the stable distance $d^s(z_i, w_i)$ is contracted by at least α at each iteration—that is,

$$d^s(f^n(w_i), f^n(z_i)) \le \alpha^n d^s(w_i, z_i) < \alpha^n\mu < \mu_*/8 \quad \text{for } i = 1, 2;$$

- the unstable distance $d^u(w_1, w_2)$ is expanded by at least β but at most λ at each iteration—that is,

$$\beta^n d^u(w_1, w_2) \le d^u(f^n(w_1), f^n(w_2)) \le \lambda^n d^u(w_1, w_2);$$

- the center distance $d^c(z_1, z_2)$ is at most expanded by λ at each iteration, $d^c(f^n(z_1), f^n(z_2)) \le \lambda^n d^c(z_1, z_2)$.

Therefore, there is $n_\mu > 0$ so that

$$d^c(f^{n_\mu}(z_1), f^{n_\mu}(z_2)) + d^u(f^{n_\mu}(w_1), f^{n_\mu}(w_2)) \in [\lambda^{-1}\mu_*, \mu_*]. \tag{4.3}$$

Let us denote $z_i(\mu) = f^{n_\mu}(z_i)$ and $w_i(\mu) = f^{n_\mu}(w_i)$. Clearly, n_μ tends to ∞ when $\mu \to 0$. One deduces that $d^s(z_i(\mu), w_i(\mu))$ tends to 0 when μ tends to 0.

We consider a sequence μ_k tending to 0 so that the points $z_1(\mu_k)$, $z_2(\mu_k)$, $w_1(\mu_k)$, $w_2(\mu_k)$ converge to points z_1^∞, z_2^∞, w_1^∞, w_2^∞. As $d^s(z_i(\mu_k), w_i(\mu_k))$ tends to 0 one gets $z_i^\infty = w_i^\infty$ for $i = 1, 2$.

As $d^u(w_1(\mu_k), w_2(\mu_k)) < \mu_*$ and $d^c(z_1(\mu_k), z_2(\mu_k)) < \mu_*$, one deduces that $z_1^\infty (= w_1^\infty)$ and $z_2^\infty (= w_2^\infty)$ belong to the same unstable leaf and to the same center leaf and one has

$$d^u(z_1^\infty, z_2^\infty) < \mu_* \quad \text{and} \quad d^c(z_1^\infty, z_2^\infty) < \mu_*.$$

In other words, $z_2^\infty \in W^u_{\mu_*}(z_1^\infty) \cap W^c_{\mu_*}(z_1^\infty)$. The choice of μ_* implies that

$$z_1^\infty = z_2^\infty. \tag{4.4}$$

Therefore we conclude the proof of uniqueness of the intersection point by proving the following claim.

CLAIM 4.6. $z_1^\infty \neq z_2^\infty$. *More precisely,* $d(z_1^\infty, z_2^\infty) \ge \mu_*/4\lambda$.

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Proof. By property (4.3) for $z_i(\mu)$, $w_i(\mu)$, one has

$$d^c(z_1(\mu_n), z_2(\mu_n)) + d^u(w_1(\mu_n), w_2(\mu_n)) \in \left[\frac{\mu_*}{\lambda}, \mu_*\right].$$

By properties (4.1) and (4.2) of μ_*, one has

$$d(z_1(\mu_n), z_2(\mu_n)) \geq \tfrac{1}{2} d^c(z_1(\mu_n), z_2(\mu_n))$$
$$d(w_1(\mu_n), w_2(\mu_n)) \geq \tfrac{1}{2} d^u(w_1(\mu_n), w_2(\mu_n))$$

so that

$$d(z_1(\mu_n), z_2(\mu_n)) + d(w_1(\mu_n), w_2(\mu_n)) \in \left[\frac{\mu_*}{2\lambda}, \mu_*\right].$$

As a consequence one gets the inequality claimed: $2d(z_1^\infty, z_2^\infty) \geq \mu_*/2\lambda$. □

The claim contradicts (4.4) so that our assumption that there exist such points $x(\mu)$, $y(\mu)$ for μ arbitrarily small is wrong. This proves the existence of $\mu_0 > 0$ satisfying the first claim of the lemma.

The second claim of Lemma 4.5 is now a straightforward consequence of the first claim (uniqueness) and Lemma 4.1 (existence). □

Remark 4.7. The same argument given by Brin allows him to get a stronger result: he shows that, for μ small enough, $\bigcup_{z \in W^c_\mu(x)} W^s_\mu(z)$ is a submanifold tangent to $E^c \oplus E^s$. The idea is that every disk tangent to a cone field around E^u with an arbitrarily large angle may cut $\bigcup_{z \in W^c_\mu(x)} W^s_\mu(z)$ at no more than one point; the proof of this fact is precisely our argument. We have not found any published reference for this interesting and useful result.

4.2. *Unstable projection on center stable manifolds.* We define in this section the unstable projection of points onto the center stable manifold. We start with the case of a center foliation with trivial holonomy.

4.2.1. *Center foliation with trivial holonomy.*

LEMMA 4.8. *Let $f : M \to M$ be a partially hyperbolic C^1-diffeomorphism with an f-invariant compact center foliation with trivial holonomy. Let $C > 0$ be given by Lemma 4.5. Then there exists $\mu > 0$ such that, for any $x \in M$ and any leaf $L^c \in \mathcal{F}^c$ with $d(x, L^c) < \mu/C$, the intersection $W^u_\mu(x) \cap (\bigcup_{z \in L^c} W^s_\mu(z))$ consists of a unique point.*

Proof. The proof follows directly from Lemma 4.5. Let us prove the uniqueness. Consider $\mu_0 > 0$ given by Lemma 4.5. Consider $\mu < \mu_0/4C$, and assume that the intersection $W^u_\mu(x) \cap (\bigcup_{z \in L^c} W^s_\mu(z))$ contains two points $w_i \in W^s_\mu(z_i)$, $z_i \in L^c$, $i = 1, 2$. Then the distance $d(z_1, z_2)$ is bounded by 4μ. Assume now that μ is small enough, according to Remark 4.4, so that $d(z_1, z_2) < 4\mu$ implies that $d^c(z_1, z_2) < 5\mu$. Therefore w_1, w_2 belongs to $W^u_\mu(x) \cap (\bigcup_{z \in W^c_{5\mu}} W^s_\mu(z))$. Thus, if $5\mu < \mu_0$, one deduces that $w_1 = w_2$, which concludes the proof. □

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Remark 4.9. One can easily check that:

- the map $(x, L^c) \mapsto w = W^u_\mu(x) \cap \bigcup_{z\in L^c} W^s_\mu(z)$ is continuous, for $d(x, L^c) < \mu/C$ and μ given by Lemma 4.8 above;
- the unstable distance $d^u(x, w)$ is bounded by $Cd(x, L^c)$.

4.2.2. *Unstable projection on the stable manifold of a center leaf in the holonomy cover.* We assume now that f is a partially hyperbolic diffeomorphism with a uniformly compact center foliation $\mathcal{F}^c$. We denote by $\mathcal{V} = \{V_i\}$ a holonomy cover of $\mathcal{F}^c$.

LEMMA 4.10. *Let $f : M \to M$ be a partially hyperbolic C^1-diffeomorphism with an f-invariant uniformly compact center foliation, and let $\mathcal{V} = \{V_i\}$ be a holonomy cover and δ associated to $\mathcal{V}$ by Lemma 3.6. Then there exists $\mu_1 > 0$ such that,*

- *for every leaf L^c of $\mathcal{F}^c$ and all i so that the δ-neighborhood of L^c is contained in V_i,*
- *for every lift $\tilde{L}^c$ of L^c on $\tilde{V}_i$,*
- *for every $\tilde{x} \in \tilde{V}_i$ with $d(\tilde{x}, \tilde{L}^c) < \mu_1/C$ (where C is given by Lemma 4.1),*

the intersection $W^u_{\mu_1}(\tilde{x}) \cap (\bigcup_{z\in\tilde{L}^c} W^s_{\mu_1}(\tilde{z}))$ consists of a unique point.

The proof is identical to that of Lemma 4.8, except for replacing the use of Remark 4.4 by the following remark which is an analogous statement on the holonomy cover.

Remark 4.11. For any $\eta > 0$ there is $\mu > 0$ so that, for any center leaf L and any i so that $B(L, \delta) \subset V_i$, for any lift $\tilde{L}$ in $\tilde{V}_i$, for any $\tilde{x}, \tilde{y} \in \tilde{L}$, one has

$$d(\tilde{x}, \tilde{y}) < \mu \Longrightarrow d^c(\tilde{x}, \tilde{y}) < (1 + \eta)d(\tilde{x}, \tilde{y}).$$

4.2.3. *Projection distance.*

LEMMA 4.12. *There is $\mu_2 > 0$ so that, for any given pair of points $x, y \in M$ with $d(x, y) < \mu_2$, for any given i so that $B_H(W^c(x), \delta) \subset V_i$, for any given lift $\tilde{x} \in \tilde{V}_i$, the following statements hold.*

- *There is a unique lift $\tilde{y} \in \tilde{V}_i$ so that $d(\tilde{x}, \tilde{y}) = d(x, y)$.*
- *For any $\tilde{w} \in W^c(\tilde{y})$, there is a unique point*

$$\Pi_{\tilde{x},\tilde{y}}(w) \in W^u_{\mu_1}(\tilde{w}) \cap W^s_{\mu_1}(W^c(\tilde{x})).$$

- *The map $w \mapsto \Pi_{\tilde{x},\tilde{y}}(w)$ is continuous. One denotes by*

$$\Delta^u_{\text{proj}}(\tilde{x}, \tilde{y}) = \sup_{\tilde{w}\in W^c(\tilde{y})} d^u(\tilde{w}, \Pi_{\tilde{x},\tilde{y}}(w))$$

  the unstable projection distance.
- *The distance $\Delta^u_{\text{proj}}(\tilde{x}, \tilde{y})$ depends neither on the lift $\tilde{x} \in \tilde{V}_i$ nor on i so that $B_H(W^c(x), \delta) \subset V_i$ (hence, it depends only on x, y). We denote $\Delta^u_{\text{proj}}(x, y) = \Delta^u_{\text{proj}}(\tilde{x}, \tilde{y})$.*

Proof. The uniqueness of the lift $\tilde{y}$ near $\tilde{x}$ comes from the fact that the lift is a local isometry on $B(\tilde{x}, \delta)$.

For the existence and uniqueness of the intersection point $W^u_\mu(\tilde{w}) \cap W^s_\mu(W^c(\tilde{x}))$, $\tilde{w} \in W^c(\tilde{y})$, it suffices to choose μ small enough so that $W^c(\tilde{y}) \subset B_H(W^c(\tilde{x}), \mu_1/C)$ where μ_1 and C are given by Lemma 4.10.

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The continuity of $\Delta^u_{\text{proj}}(\tilde{x}, \tilde{y})$ comes from the compactness of $W^u_{\mu_1}(W^c(\tilde{y})) \cap W^s_{\mu_1}(W^c(\tilde{x}))$ together with the existence and uniqueness of the intersection point $W^u_{\mu_1}(\tilde{w}) \cap W^s_{\mu_1}(W^c(\tilde{x}))$, $\tilde{w} \in W^c(\tilde{y})$.

The fact that $\Delta^u_{\text{proj}}(x, y)$ does not depend on the lifts comes from Lemma 3.9: the δ_0-neighborhoods of the lifts of $W^c(x)$ on $\tilde{V}_i$, for $B_H(W^c(c), \delta) \subset V_i$, are all isometric by isometries conjugating the lifted foliations. □

Remark 4.13.

- Lemma 4.12 above assures also the existence of a unique intersection point of the local stable manifold $W^s_{\mu_1}(\tilde{w})$ for any $\tilde{w} \in W^c(\tilde{y})$ with the local center unstable manifold $W^u_{\mu_1}(W^c(\tilde{x}))$ for any two nearby points x, y with $d(x, y) < \mu_2$ if lifted to a holonomy cover. We can therefore analogously define a *stable projection distance* by $\Delta^s_{\text{proj}}(x, y)$.
- For x, y with $d(x, y) < \mu_2$ (for μ_2 given by Lemma 4.12 and μ_1 given by Lemma 4.10) one has

$$\Delta^u_{\text{proj}}(x, y) = 0 \Longleftrightarrow W^c(y) \subset W^s_{\mu_1}(W^c(x)).$$

  As a consequence, the foliations $\mathcal{F}^s$ and $\mathcal{F}^c$ are jointly integrable (that is, there is a center stable foliation subfoliated by $\mathcal{F}^s$ and $\mathcal{F}^c$) if and only if there is $\mu > 0$ so that, for every x, y with $y \in W^s_{\mu}(W^c(x))$, one has $\Delta^u_{\text{proj}}(x, y) = 0$. The analogous statement holds for Δ^s_{proj} and the center unstable manifold.
- By construction it is $\Delta^u_{\text{proj}}(x, y) < \mu_1$ and $\Delta^s_{\text{proj}}(x, y) < \mu_1$.
- The unstable projection distance $\Delta^u_{\text{proj}}(x, y)$ is not a distance: it is not symmetric and may not satisfy the triangle inequality.

4.2.4. *The projection distance and the dynamics.* Recall that $\beta > 1$ denotes the minimum expansion of Df on the unstable bundle E^u.

LEMMA 4.14. *There is $\mu > 0$ so that, for any given pair of points $x, y \in M$ with $d(x, y) < \mu$ and $d(f(x), f(y)) < \mu$, one has $\Delta^u_{\text{proj}}(f(x), f(y)) \geq \beta \Delta^u_{\text{proj}}(x, y)$.*

Proof. Fix $\mu_2 > 0$ of Lemma 4.12 and $C > 0$ of Lemma 4.1. Choose $\mu > 0$ smaller than μ_2 and such that $\mu(1 + 7\lambda C + 3\lambda C^2) < \delta_0$ with $\delta_0 > 0$ from Lemma 3.9 assuring that the holonomy covering maps are local isometries in our setting. Recall that $\lambda > 1$ is the upper bound of $\|Df\|$.

Let $x, y \in M$ with $d(x, y) < \mu$ and $d(f(x), f(y)) < \mu$. Let i, j such that $B_H(W^c(x), \delta_0) \subset V_i$ and $B_H(W^c(f(x)), \delta_0) \subset V_j$. Then, for any lift $\tilde{x} \in p_i^{-1}(x)$, Lemma 4.12 implies the existence of a unique lift $\tilde{y} \in p_i^{-1}(y) \subset \tilde{V}_i$ such that $d(\tilde{x}, \tilde{y}) = d(x, y)$.

Lemma 4.12 asserts that, for any $\tilde{w} \in W^c(\tilde{y})$ there are:

- a unique intersection point $\tilde{z}_{\tilde{w}} = W^u_{\mu_1}(\tilde{w}) \cap W^s_{\mu_1}(W^c(\tilde{x}))$; and
- a unique point $\tilde{u}_{\tilde{w}} \in W^c(\tilde{x})$ so that $\tilde{z}_{\tilde{w}} \in W^s_{\mu_1}(\tilde{u}_{\tilde{w}})$.

Because p_i is a local isometry, the projections $w = p_i(\tilde{w}) \in W^c(y)$, $z_{\tilde{w}} = p_i(\tilde{z}_{\tilde{w}})$, and $u_{\tilde{w}} = p_i(\tilde{u}_{\tilde{w}})$ satisfy

$$d^u(w, z_{\tilde{w}}) = d^u(\tilde{w}, \tilde{z}_{\tilde{w}}) \quad \text{and} \quad d^s(z_{\tilde{w}}, u_{\tilde{w}}) = d^s(\tilde{z}_{\tilde{w}}, \tilde{u}_{\tilde{w}}).$$

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Analogously, for any $\widetilde{f(x)} \in p_j^{-1}(y)$ there exists a unique lift $\widetilde{f(y)} \in p_j^{-1}(f(y)) \subset \tilde{V}_j$, and for any $\tilde{v} \in W^c(\widetilde{f(y)})$ there exist:

- a unique intersection point and $\tilde{z}_{\tilde{v}} = W^u_{\mu_1}(\tilde{v}) \cap W^s_{\mu_1}(W^c(\widetilde{f(x)}))$;
- a unique point $\tilde{u}_{\tilde{v}} \in W^c(\widetilde{f(x)})$ whose local stable manifold contains $\tilde{z}_{\tilde{v}}$.

Once again we denote by $v \in W^c(f(y))$, $z_{\tilde{v}}$ and $u_{\tilde{v}}$ their projection by the local isometry p_j.

Recall that the projection distances are defined by

$$\Delta^u_{\text{proj}}(x, y) = \sup_{\tilde{w} \in W^c(\tilde{y})} \{d^u(\tilde{w}, \tilde{z}_{\tilde{w}})\} \quad \text{and} \quad \Delta^u_{\text{proj}}(f(x), f(y)) = \sup_{\tilde{v} \in W^c(\widetilde{f(y)})} \{d^u(\tilde{v}, \tilde{z}_{\tilde{v}})\}.$$

Notice that, for any $\tilde{w} \in W^c(\tilde{y})$, one has

$$d^u(f(w), f(z_{\tilde{w}})) \geq \beta d^u(w, z_{\tilde{w}}).$$

Using the fact that the projections p_i and p_j are local isometies, we get that $\Delta^u_{\text{proj}}(f(x), f(y)) \geq \beta \Delta^u_{\text{proj}}(x, y)$ (concluding the proof of Lemma 4.14) if we prove the following claim.

CLAIM 4.15. *For any $\tilde{w} \in W^c(\tilde{y})$, there is $\tilde{v} \in W^c(\widetilde{f(y)})$ so that*

$$f(z_{\tilde{w}}) = z_{\tilde{v}}.$$

Proof of Claim 4.15. We start by proving $f(z_{\tilde{y}}) = z_{\widetilde{f(y)}}$. We use the fact that p_j is a local isometry for lifting the point $f(z_{\tilde{y}})$ and show that this point belongs to the local center stable manifold of $\widetilde{f(x)}$ and to the local unstable manifold of $\widetilde{f(y)}$.

One has:

- $d(\tilde{x}, \tilde{y}) = d(x, y) < \mu$;
- $d^s(\tilde{z}_{\tilde{y}}, \tilde{u}_{\tilde{y}}) < 2C\mu$, according to Lemma 4.10;
- $d^u(y, z_{\tilde{y}}) = d^u(\tilde{y}, \tilde{z}_{\tilde{y}}) < C\mu$ and, as a consequence,
- $d(y, u_{\tilde{y}}) = d(\tilde{y}, \tilde{u}_{\tilde{y}}) < 3C\mu$ and, by the triangle inequality,
- $d^c(x, u_{\tilde{y}}) = d^c(\tilde{x}, \tilde{u}_{\tilde{y}}) < C(3C\mu + \mu)$.

Recalling that $\|Df\|$ is bounded by λ, one gets:

- $d(f(x), f(y)) < \mu$ (by hypothesis);
- $d^s(f(z_{\tilde{y}}), f(u_{\tilde{y}})) < 2\lambda C\mu$;
- $d^u(f(y), f(z_{\tilde{y}})) < \lambda C\mu$;
- $d(f(y), f(u_{\tilde{y}})) < 3\lambda C\mu$; and
- $d^c(f(x), f(u_{\tilde{y}})) < \lambda C(3C\mu + \mu)$.

We have chosen μ so that the sum of all these distances is less than the isometry radius δ_0 of the cover. As $d^c(f(x), f(u_{\tilde{y}})) < \delta_0$, there is a unique lift $\widetilde{f(u_{\tilde{y}})}$ so that

$$d^c(\widetilde{f(x)}, \widetilde{f(u_{\tilde{y}})}) = d^c(f(x), f(u_{\tilde{y}})).$$

In the same way there are unique lifts $\widetilde{f(z_{\tilde{y}})}$ and $\widetilde{f(y)}$ so that

$$d^s(\widetilde{f(z_{\tilde{y}})}, \widetilde{f(u_{\tilde{y}})}) = d^s(f(z_{\tilde{y}}), f(u_{\tilde{y}})),$$
$$d^u(\widetilde{f(y)}, \widetilde{f(z_{\tilde{y}})}) = d^u(f(y), f(z_{\tilde{y}})).$$

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By construction, this means that $\widetilde{f(z_{\tilde{y}})}$ is the projection of $\widehat{f(y)}$ on the center stable leaf of $\widetilde{f(x)}$.

One deduces that $\widehat{f(y)}$ is a lift of y with $d(\widetilde{f(x)}, \widehat{f(y)}) < \delta_0$. But $\widetilde{f(y)}$ is the unique lift of y at distance less than δ_0 from $\widetilde{f(x)}$ so that

$$\widehat{f(y)} = \widetilde{f(y)}.$$

As a consequence,

$$\widetilde{f(z_{\tilde{y}})} = \tilde{z}_{\widetilde{f(y)}},$$

and thus we have proved the claim for $\tilde{w} = \tilde{y}$:

$$f(z_{\tilde{y}}) = z_{\widehat{f(y)}}.$$

Now consider any point $\tilde{w} \in W^c(\tilde{y})$. As $W^c(\tilde{y})$ is a connected manifold one can fix a path $\tilde{w}_t \in W^c(\tilde{y})$, $t \in [0, 1]$, with $\tilde{w}_0 = \tilde{y}$ and $\tilde{w}_1 = \tilde{w}$.

The points $u_{\tilde{w}_t} \in W^c(x)$ and $z_{\tilde{w}_t} \in W^u(w_t) \cap W^s(u_{\tilde{w}_t})$ depend continuously on $t \in [0, 1]$ and satisfy:

- $d^s(z_{\tilde{w}_t}, u_{\tilde{w}_t}) = d^s(\tilde{z}_{\tilde{w}_t}, \tilde{u}_{\tilde{w}_t}) \leq 2C\mu$;
- $d^u(w_t, z_{\tilde{w}_y}) = d^u(\tilde{w}_t, \tilde{z}_{\tilde{w}_t}) < C\mu$; and, as a consequence,
- $d(w_t, u_{\tilde{w}_t}) = d(\tilde{w}_t, \tilde{u}_{\tilde{w}_t}) < 3C\mu$.

Applying f, we once more get that the distances between $f(w_t)$, $f(z_{\tilde{w}_t})$ and $f(u_{\tilde{w}_t})$ remain smaller than δ_0 so that the choice of a lift of $f(w_t)$ determines a lift of $f(z_{\tilde{w}_t})$ and $f(u_{\tilde{w}_t})$.

Notice that one can make a continuous choice of lifts $\widetilde{f(w_t)}$ with $\widetilde{f(w_0)} = \widetilde{f(y)}$. This is because f conjugates the holonomy representation of the leaves $W^c(y)$ and $W^c(f(y))$ and because p_i and p_j induces the holonomy cover over $W^c(y)$ and $W^c(f(y))$.

This continuous lift $\widetilde{f(w_t)}$ induces continuous lifts $\widetilde{f(z_{\tilde{w}_t})}$ and $\widetilde{f(u_{\tilde{w}_t})}$ which coincide with $\widetilde{f(z_{\tilde{y}})}$ and $\widetilde{f(u_{\tilde{y}})}$ for $t = 0$. In particular, $\widetilde{f(u_{\tilde{w}_0})}$ belongs to $W^c(\widetilde{f(x)})$ and therefore $\widetilde{f(u_{\tilde{w}_t})}$ belongs to $W^c(\widetilde{f(x)})$. One deduces that, for all $t \in [0, 1]$, has

$$\widetilde{f(z_{\tilde{w}_t})} = \tilde{z}_{\widetilde{f(w_t)}}$$

so that

$$f(z_{\tilde{w}}) = z_{\widetilde{f(w_1)}}, \quad \text{with } \widetilde{f(w_1)} \in W^c(\widetilde{f(y)}).$$

This proves the claim. □

This concludes the proof of Lemma 4.14. □

4.3. *Dynamical coherence: proof of Theorem 1.* We now have the tools to make rigorous the intuitive idea for proving the dynamical coherence: given two center leaves through points w and z in the same stable leaf, the distance between the images by f^n of the leaves tends to 0. One would like to conclude that every point of one leaf belongs to the stable leaf of a point in the other leaf. In our language, this means that the unstable projection distances $\Delta^u_{\text{proj}}(w, z)$ and $\Delta^u_{\text{proj}}(z, w)$ vanish. This is obtained in Corollary 4.17 with the help of Lemma 4.14.

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Indeed, we show† a stronger property than dynamical coherence, called *completeness* in [**Car10**].

PROPOSITION 4.16. *For any $x \in M$,*

$$\bigcup_{w \in W^s(x)} W^c(w) = \bigcup_{z \in W^c(x)} W^s(z).$$

Notice that Proposition 4.16 allows us to define the center stable leaves through a point $x \in M$ as $W^{cs}(x) := \bigcup_{w \in W^s(x)} W^c(w) = \bigcup_{z \in W^c(x)} W^s(z)$. The center stable foliation we obtain is f-invariant as the center foliation $\mathcal{F}^c$ is assumed to be invariant. Applying Proposition 4.16 to f^{-1}, one gets the (invariant) center unstable foliation, and thus the dynamical coherence. In other words, Proposition 4.16 implies Theorem 1. The proof of this proposition is the aim of this section.

We start the proof with the following corollary.

COROLLARY 4.17. *Let μ be the constant chosen in Lemma 4.14. The three following properties are equivalent:*

- *The two center leaves $W^c(x)$ and $W^c(y)$ are such that there exist n_0 and $w \in W^c(x)$ and $z \in W^c(y)$ such that, for any $n \geq n_0$, the distance $d(f^n(w), f^n(z))$ is bounded by μ.*
- *There is $z \in W^c(y)$ so that z belongs to $\bigcup_{w \in W^c(x)} W^s(w)$.*
- *$W^c(y) \subset \bigcup_{w \in W^c(x)} W^s(w)$.*
- *The two center leaves $W^c(x)$ and $W^c(y)$ are such that there exist $w \in W^c(x)$ and $z \in W^c(y)$ such that $\Delta^u_{\text{proj}}(w, z)$ is well-defined and $\Delta^u_{\text{proj}}(w, z) = 0$.*

Proof. The fact that the third item implies the second is trivial. The fact that the second item implies the first is also straightforward.

Let us first show that the first item implies the fourth. According to Lemma 4.12 and Remark 4.13 (because μ is chosen smaller than μ_2) one gets that $\Delta^u_{\text{proj}}(f^n(w), f^n(z))$ is well defined for every $n \geq n_0$ and bounded by μ_1. However, Lemma 4.14 implies that

$$\Delta^u_{\text{proj}}(f^n(w), f^n(z)) \geq \beta^n \Delta^u_{\text{proj}}(w, z),$$

for every $n \geq 0$, where $\beta > 1$ is the expansivity constant of Df on E^u. As a direct consequence it follows that $\Delta^u_{\text{proj}}(w, z) = 0$, the fourth item. This implies the third item:

$$W^c(y) = W^c(z) \subset \bigcup_{u \in W^c(w)} W^s(u) = \bigcup_{w \in W^c(x)} W^s(x).$$

This ends the proof. □

As a direct consequence of Corollary 4.17 one gets the following result.

COROLLARY 4.18. *For any y such that $W^c(y) \cap \bigcup_{z \in W^c(x)} W^s(z) \neq \varnothing$,*

$$\bigcup_{z \in W^c(x)} W^s(z) = \bigcup_{w \in W^c(y)} W^s(w).$$

† In a previous version of this paper, we proved the dynamical coherence without noticing that our proof also gave the completeness. We rewrote the proof (after reading [**Car10**] and another unpublished version of the article [**Car11**]) emphasizing that property.

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Proof. First, Corollary 4.17 implies that $W^c(y) \subset \bigcup_{z\in W^c(x)} W^s(z)$. Then the stable manifold through any point $w \in W^c(y)$ is contained in the stable manifold of a point $z \in W^c(x)$, that is,

$$\bigcup_{w\in W^c(y)} W^s(w) \subset \bigcup_{z\in W^c(x)} W^s(z).$$

To get the reverse inclusion it is now enough to prove

$$W^c(x) \cap \bigcup_{w\in W^c(y)} W^s(w) \neq \varnothing. \tag{4.5}$$

This is just because the hypothesis means there exist $w \in W^c(y)$ and $z \in W^c(x)$ so that $W^s(w) = W^s(z)$. Thus z belongs to $\bigcup_{w\in W^c(y)} W^s(w)$, which is therefore not empty, proving equation (4.5). □

COROLLARY 4.19. *For any $x \in M$,*

$$\bigcup_{w\in W^s(x)} W^c(w) \subset \bigcup_{z\in W^c(x)} W^s(z).$$

Proof. If $y \in W^c(w)$ for some $w \in W^s(x)$, then $d(f^n(w), f^n(x)) \to 0$ as $n \to +\infty$ so that Corollary 4.17 implies that $W^c(y) = W^c(w) \subset \bigcup_{z\in W^c(x)} W^s(z)$. □

We can now finish the proof of the dynamical coherence (and therefore of Theorem 1).

Proof of Proposition 4.16. We just have to prove $\bigcup_{z\in W^c(x)} W^s(z) \subset \bigcup_{w\in W^s(x)} W^c(w)$.

Let $y \in \bigcup_{z\in W^c(x)} W^s(z)$, that is, there is $z \in W^c(x)$ with $y \in W^s(z)$. So for n large $d(f^n(y), f^n(z))$ remains bounded by μ.

According to Corollary 4.17, one deduces $W^c(x) \subset \bigcup_{w\in W^c(y)} W^s(w)$. In particular, there is a $w \in W^c(y)$ so that $x \in W^s(w)$, that is, $w \in W^s(x)$ and $y \in W^c(w)$, which concludes the proof. □

4.4. *Expansivity in the case of trivial center holonomy.* The expansivity of f with respect to orbits of center leaves follows almost immediately from previous arguments for the proof of dynamical coherence if we assume a compact center foliation with *trivial holonomy*.

PROPOSITION 4.20. *Let f be a partially hyperbolic diffeomorphism having a uniformly compact invariant center foliation $\mathcal{F}^c$ without holonomy. Then the homeomorphism $F : M/\mathcal{F}^c \to M/\mathcal{F}^c$ induced by f on the quotient of the center foliation is expansive.*

Remark 4.21.

- The expansivity on the quotient space implies the plaque expansivity of the center foliation $(f, \mathcal{F}^c)$, hence it implies its structural stability (see discussion of these properties in the introduction to this paper).
- Let us recall and emphasize that the conclusion of Proposition 4.20 is wrong without the assumption of trivial holonomy. See an example in [**BW05**], further explained in §8.

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The proof of Proposition 4.20 uses the following lemma.

LEMMA 4.22. *Let f be a partially hyperbolic diffeomorphism having a uniformly compact invariant center foliation $\mathcal{F}^c$ without holonomy. Then there is $\mu > 0$ so that, for every pair of center leaves W_1^c W_2^c whose Hausdorff distance is less than μ, for every pair of pairs (x_1, x_2), $(y_1, y_2) \in W_1^c \times W_2^c$ with $d(x_1, x_2) < \mu$ and $d(y_1, y_2) < \mu$, one has*

$$\Delta^u_{\text{proj}}(x_1, x_2) = \Delta^u_{\text{proj}}(y_1, y_2) \quad \textit{and} \quad \Delta^s_{\text{proj}}(x_1, x_2) = \Delta^s_{\text{proj}}(y_1, y_2).$$

Idea of the proof. Let μ_1, μ_2 be defined by Lemmas 4.10 and 4.12. As we assume trivial holonomy, there is $\mu > 0$ so that, for every leaf W_1^c, W_2^c with Hausdorff distance less than μ, for every $w \in W_2^c$, there is a unique point $\Pi(w) \in W^u_{\mu_1}(w) \cap W^s_{\mu_1}(W_1^c)$; indeed, that is the second claim in Lemma 4.12, noticing that the holonomy covers are trivial.

Therefore for every $(x_1, x_2) \in W_1^c \times W_2^c$ with $d(x_1, x_2) < \mu$,

$$\Delta^u_{\text{proj}}(x_1, x_2) = \sup_{w \in W_2^c} d^u(w, \Pi(w)),$$

which finishes the proof. □

According to Lemma 4.22, there is no ambiguity in denoting

$$\Delta^u_{\text{proj}}(W_1^c, W_2^c) := \Delta^u_{\text{proj}}(x_1, x_2).$$

We are now in a position to prove Proposition 4.20.

Proof of Proposition 4.20. We choose $\mu > 0$ small enough that it satisfies both Lemmas 4.22 and 4.14. We consider two arbitrary center leaves $W^c(x)$ and $W^c(y)$ which remain for all iterates at a distance less than $\mu > 0$, i.e. $d_H(W^c(f^n x), W^c(f^n y)) < \mu$ for all $n \in \mathbb{Z}$.

So, first we have that $d_H(W^c(f^n x), W^c(f^n y)) < \mu$ for all $n \geq 0$. Applying Corollary 4.17, we get $\Delta^u_{\text{proj}}(W^c(x), W^c(y)) = 0$ and $W^c(y) \subset W^s_{\mu_1}(W^c(x))$. This implies that y is the unique intersection point of $W^u_{\mu_1}(y)$ with $W^s_{\mu_1}(W^c(x))$, that is,

$$\{y\} = W^u_{\mu_1}(y) \cap W^s_{\mu_1}(W^c(x)).$$

Now we consider the backward iterates, applying Corollary 4.17 for f^{-1}: we get $\Delta^s_{\text{proj}}(W^c(x), W^c(y)) = 0$ and $W^c(y) \subset W^u_{\mu_1}(W^c(x))$, and thus

$$\{y\} = W^s_{\mu_1}(y) \cap W^u_{\mu_1}(W^c(x)).$$

So there are $w, z \in W^c(x)$ such that $y \in W^s_{\mu_1}(z)$ and $y \in W^u_{\mu_1}(w)$. Thus, according to the notation in the proof of Lemma 4.22, z is the projection $\Pi(w)$ of w on $W^s(W^c(x))$. This implies $z = w$. So $y \in W^s_{\mu_1}(w) \cap W^u_{\mu_1}(w) = \{w\}$.

This implies $y \in W^c(x)$, concluding the proof. □

5. *Shadowing lemma*

In this section we prove the shadowing lemma (Theorem 2) on the leaf space $M/\mathcal{F}^c$ for the homeomorphism F induced by a partially hyperbolic C^1-diffeomorphism $f : M \to M$ with an invariant uniformly compact center foliation $\mathcal{F}^c$.

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5.1. *Some remarks concerning the proof.* First, recall that we defined a modified Hausdorff distance Δ_H on the space of center leaves which is topologically equivalent to the Hausdorff distance (see Corollary 3.12). Therefore, it is enough to prove the shadowing lemma for Δ_H.

Remark 5.1. If a homeomorphism h of a compact metric space K satisfies the shadowing property for finite positive pseudo-orbits, then it satisfies the shadowing property for all bi-infinite pseudo-orbits.

Proof. Consider $\delta > 0$ and $\varepsilon > 0$ so that every finite positive ε-pseudo-orbit is δ-shadowed. Let $(x_i)_{i\in\mathbb{Z}}$ be a bi-infinite ε-pseudo-orbit. For every $n > 0$, let $z^n \in K$ so that its orbit shadows the finite positive ε-pseudo-orbit $(x_{-n}, \ldots, x_n)$; let y^n denote $h^n(z^n)$. By definition, $d(h^i(y^n), x_i) \leq \delta$ for every $i \in \{-n, \ldots, n\}$. By compactness of K one can extract a subsequence y^{n_j} converging to some $y \in K$.

By continuity of h^i for every i one gets $d(h^i(y), x_i) \leq \delta$ for every $i \in \mathbb{Z}$, concluding the proof. □

Note, furthermore, that the shadowing property for a homeomorphism h of a compact metric space is equivalent to the shadowing property for some h^N, $N > 0$.

Thus, up to replacing f by some iterate f^N, we may assume that

$$2C\alpha < 1,$$

where α is an upper bound of $\|Df|_{E^s}\|$ and of $\|Df^{-1}|_{E^u}\|$, and C is the constant given by Lemma 4.1. Recall that $\delta_0 > 0$ is given by Lemma 3.9 and assures that within a Hausdorff ball of radius δ_0 the holonomy covering maps are isometries and that the distances are independent of the choice of the holonomy cover.

The aim of this section is the proof of Proposition 5.2 below which provides the proof for the shadowing lemma:

PROPOSITION 5.2. *With the notation above, for any $\eta > 0$ consider $\varepsilon < \min\{(1-\alpha)\eta/2C, \delta_0\}$. Then for any ε-pseudo-orbit $\{W_i\}_{i\geq 0}$ for Δ_H, where W_i is a center leaf of f, there is a center leaf W such that, for any $i \geq 0$,*

$$\Delta_H(f^i(W), W_i) \leq \eta.$$

5.2. *Adaption of the stable and unstable projection distances.* According to Theorem 1, we know that f is dynamically coherent so that one can use the center stable and center unstable foliations. We will denote by $W^s(W)$ and $W^u(W)$ the center stable and center unstable leaves of a given center leaf W. We have seen (Proposition 4.16) that $W^s(W)$ is the union of the stable leaves through W. We will denote by $W^s_\mu(W)$ the union of the local stable manifolds through the points of W.

Throughout the proof, we use intensively the notion of lifts of center leaves. This refers to lifts on a holonomy cover $\{U_i, p_i\}$ for which a δ_0-neighborhood of the leaves is contained in U_i. The local isometry property (see Lemma 3.9) implies that the quantities we define are independent of the chosen $\{U_i, p_i\}$. However, each leaf may have several lifts to $\tilde{U}_i$ and the estimates for a pair of leaves depend on the lift we choose for each of

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them. For this reason we will often use the expression *given two leaves W_1 and W_2, and given a lift $\widetilde{W}_1$, there is a lift $\widetilde{W}_2$ with the following property*.

In the section above we established the unstable projection distance for any pair of points $x, y \in M$ at a distance bounded by $\mu > 0$ given by Lemma 4.12. Letting (U_i, p_i) be any neighborhood of a holonomy cover such that $B_H(W^c(x), \delta_0) \subset U_i$ and let $\tilde{x} \in p_i^{-1}(x)$, $\tilde{y} \in p_i^{-1}(y)$ be any lifted points with $d(\tilde{x}, \tilde{y}) = d(x, y) < \mu$, we recall the following definition from Lemma 4.12:

$$\Delta^u_{\text{proj}}(x, y) = \sup_{\tilde{w} \in W^c(\tilde{y})} d^u(\tilde{w}, \Pi_{\tilde{x},\tilde{y}}(w))$$

where

$$\Pi_{\tilde{x},\tilde{y}}(w) = W^u_{\mu_1}(\tilde{w}) \cap W^s_{\mu_1}(W^c(\tilde{x})) \quad \text{for all } \tilde{w} \in W^c(\tilde{y}).$$

This definition is independent of the points lifted to a holonomy cover as long as $d_H(\widetilde{W}_1, \widetilde{W}_2) < \mu$ for any choice of two center leaves. So we can denote the unstable projection distance between lifted center leaves by

$$\Delta^u_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2) = \Delta^u_{\text{proj}}(w_1, w_2)$$

where $\widetilde{W}_1$ and $\widetilde{W}_2$ are those lifts of W_1 and W_2 such that $d_H(\widetilde{W}_1, \widetilde{W}_2) < \mu$ and $w_1 \in W_1$, $w_2 \in W_2$ are any points with $d(w_1, w_2) < \mu$. The center leaf $\widetilde{W}_2$ is projected along the unstable foliation onto the center stable leaf $W^s(\widetilde{W}_1)$. In the case of the stable projection distance the center leaf $\widetilde{W}_1$ is projected along the stable foliation onto the center unstable leaf $W^u(\widetilde{W}_2)$. As the projection distance is not symmetric, we write the stable projection distance as

$$\Delta^s_{\text{proj}}(\widetilde{W}_2, \widetilde{W}_1) = \Delta^s_{\text{proj}}(w_2, w_1).$$

We recall (see Remark 4.13) that the projection distance:

- depends on the pair of lifts $\widetilde{W}_1$ and $\widetilde{W}_2$ and has no meaning for the leaves W_1 and W_2;
- even for the lifts, it is not a distance—it is not symmetric (in general, $\Delta^u_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2) \neq \Delta^u_{\text{proj}}(\widetilde{W}_2, \widetilde{W}_1)$) and it does not satisfy the triangle inequality.

Now we can formulate the following lemma to compare the Hausdorff distance with the unstable and stable projection distances on the holonomy cover.

LEMMA 5.3. *Let $\mu > 0$ be given by Lemma 4.12 and $C > 0$ by Lemma 4.1. For all center leaves W_1, W_2 with $\Delta_H(W_1, W_2) < \mu$, there exist lifts $\widetilde{W}_1$, $\widetilde{W}_2$ such that*

$$\frac{1}{C}\Delta^u_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2) \leq d_H(\widetilde{W}_1, \widetilde{W}_2),$$
$$\frac{1}{C}\Delta^s_{\text{proj}}(\widetilde{W}_2, \widetilde{W}_1) \leq d_H(\widetilde{W}_1, \widetilde{W}_2),$$
$$d_H(\widetilde{W}_1, \widetilde{W}_2) \leq \Delta^s_{\text{proj}}(\widetilde{W}_2, \widetilde{W}_1) + \Delta^u_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2).$$

Proof. The first inequality comes from Lemma 4.12: for any pair of points $(\tilde{x}, \tilde{y}) \in \widetilde{W}_1 \times \widetilde{W}_2$ with $d(\tilde{x}, \tilde{y}) < d_H(\widetilde{W}_1, \widetilde{W}_2)$, one has $d^u(\tilde{y}, \Pi_{\tilde{x},\tilde{y}}(\tilde{y})) < Cd(\tilde{x}, \tilde{y})$. The second inequality is proved in a similar way.

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The last inequality is implied by the triangle inequality. Let $\mu_1 > 0$ be given by Lemma 4.10. Given $\tilde{y} \in \widetilde{W}_2$, there is a unique $\tilde{x} \in \widetilde{W}_1$ so that $W^s_{\mu_1}(\tilde{x})$ cuts $W^u_{\mu_1}(\tilde{y})$ at a point $\tilde{w}$. Then

$$d(\tilde{x}, \tilde{y}) < d^s(\tilde{x}, \tilde{w}) + d^u(\tilde{w}, \tilde{y}) \leq \Delta^s_{\text{proj}}(\widetilde{W}_2, \widetilde{W}_1) + \Delta^u_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2). \qquad \square$$

Remark 5.4. If $\widetilde{W}_1 \subset W^u_{\mu_1}(\widetilde{W}_2)$, then $\Delta^s_{\text{proj}}(\widetilde{W}_2, \widetilde{W}_1) = 0$. Thus, by Lemma 5.3,

$$\frac{1}{C}\Delta^u_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2) \leq d_H(\widetilde{W}_1, \widetilde{W}_2) \leq \Delta^u_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2). \tag{5.1}$$

The analogous statement holds for the stable projection distance Δ^s_{proj}.

The following result is a direct consequence of Lemma 4.14.

LEMMA 5.5. *Let $\mu_1 > 0$ be given by Lemma 4.10 and $\mu > 0$ by Lemma 4.12. Let W_1, W_2 be any pair of center leaves and $\widetilde{W}_1$, $\widetilde{W}_2$ be any lifts of these so that:*

- $\widetilde{W}_1 \subset W^s_{\mu_1}(\widetilde{W}_2)$;
- $\Delta^s_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2) < \mu$.

Then. for every $n > 0$. there are lifts $\widetilde{f^n(W_1)}$, $\widetilde{f^n(W_2)}$ such that:

- $\widetilde{f^n(W_1)} \subset W^s_{\mu_1}(\widetilde{f^n(W_2)})$;
- $\Delta^s_{\text{proj}}(\widetilde{f^n(W_1)}, \widetilde{f^n(W_2)}) < \alpha^n \Delta^s_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2)$.

A similar statement can be made for iterates of f^{-1} and Δ^u_{proj}.

Proof. Fix a lift $\widetilde{f^n(W_1)}$. Consider $\tilde{x} \in \widetilde{W}_1$ and let $\tilde{y} \in \widetilde{W}_2$ be the unique point so that $\tilde{y} \in W^s_{\mu_1}(\tilde{x})$. Let x and y be their projections. Then, recalling the definition of the stable projection distance, one gets

$$d^s(f^n(x), f^n(y)) \leq \alpha^n d^s(x, y) \leq \alpha^n \Delta^s_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2).$$

Let $\widetilde{f^n(x)}$ be a lift of $f^n(x)$ in $\widetilde{f^n(W_1)}$. As $d(f^n(x), f^n(y)) \leq \delta_0$, there is a unique lift $\widetilde{f^n(y)}$ so that $d(\widetilde{f^n(x)}, \widetilde{f^n(y)}) < \mu$, and indeed by Lemma 4.12 equal to $d(f^n(x), f^n(y))$. We denote by $\widetilde{f^n(W_2)}$ the lift of $f^n(W_2)$ through $\widetilde{f^n(y)}$.

Let us show that the lift $\widetilde{f^n(W_2)}$ does not depend on the choice of $\tilde{x}$: given any $\tilde{x}_1 \in \widetilde{W}_1$, we fix a path $\tilde{x}_t$ inside $\widetilde{W}_1$ joining $\tilde{x}$ to $\tilde{x}_1$. Then the projections on $\widetilde{W}_2$ along the stable leaves define a path $\tilde{y}_t$ in $\widetilde{W}_2$ with $\tilde{y}_t \in W^s_{\mu_1}(\tilde{x}_t)$.

Denote by x_t, y_t the projections of $\tilde{x}_t$, $\tilde{y}_t$. Then

$$d^s(f^n(x_t), f^n(y_t)) < \alpha^n d^s(\tilde{x}_t, \tilde{y}_t) \leq \alpha^n \Delta^s_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2) \leq \delta_0. \tag{5.2}$$

Consider the continuous lifts $\widetilde{f^n(x_t)} \in \widetilde{f^n(W_1)}$ so that $\widetilde{f^n(x_0)} = \widetilde{f^n(x)}$. Then we have a unique choice of lift $\widetilde{f^n(y_t)}$ at distance less than δ_0 of $\widetilde{f^n(x_t)}$, and these lifts define a path in the lifts of $f^n(W_2)$ starting at $\widetilde{f^n(y_0)}$. Therefore this path is contained in $\widetilde{f^n(W_2)}$. This proves that the lift $\widetilde{f^n(W_2)}$ is independent of the choice of $\tilde{x} \in \widetilde{W}_1$.

By construction we have $\widetilde{f^n(W_1)} \subset W^s_{\mu_1}(\widetilde{f^n(W_2)})$, and inequality (5.2) implies

$$\Delta^s_{\text{proj}}(\widetilde{f^n(W_1)}, \widetilde{f^n(W_2)}) < \alpha^n \Delta^s_{\text{proj}}(\widetilde{W}_1, \widetilde{W}_2). \qquad \square$$

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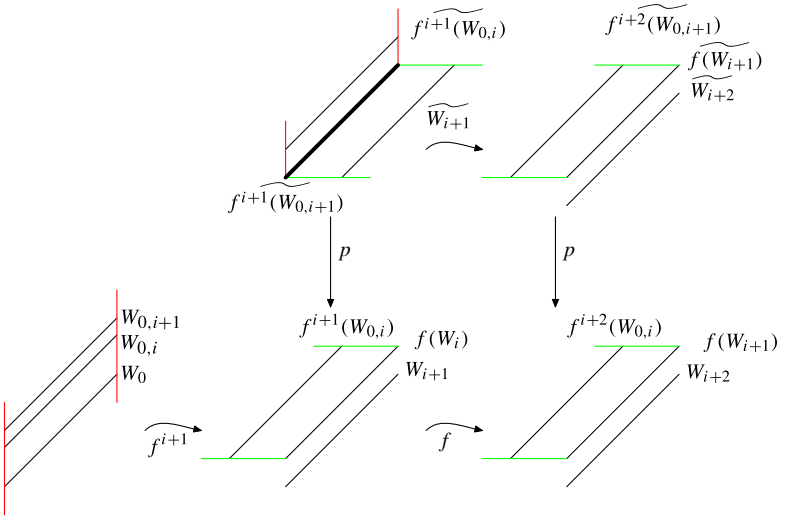


FIGURE 3. Building the sequence $W_{0,i}$ by defining $\widetilde{f^{i+1}(W_{0,i+1})}$ in the holonomy cover.

Remark 5.6. From the considerations above and the proof of Lemma 4.14 we can easily deduce that if $\widetilde{W}_1$ and $\widetilde{W}_2$ are two lifted center leaves with $d_H(\widetilde{W}_1, \widetilde{W}_2) < \mu/\lambda$ ($\mu > 0$ given by Lemma 4.12 and $\lambda > 1$ given as the upper bound of $\|Df\|$) and if $\widetilde{f(W_1)}$ is a lift of $f(W_1)$, then there is a unique lift $\widetilde{f(W_2)}$ with the following property: for every $\tilde{x} \in \widetilde{W}_1$ and $\tilde{y} \in \widetilde{W}_2$ with $d(\tilde{x}, \tilde{y}) < \mu/\lambda$ and if $\widetilde{f(x)}$ is a lift of $f(x)$ on $\widetilde{f(W_1)}$, then there is a lift $\widetilde{f(y)} \in \widetilde{f(W_2)}$ with $d(\widetilde{f(x)}, \widetilde{f(y)}) = d(f(x), f(y))$.

5.3. *Proof of Proposition 5.2 (and of the shadowing lemma).*

Proof. Let $\{W_i\}_{i\geq 0}$ be a sequence of center leaves of f so that $\Delta_H(f(W_i), W_{i+1}) < \varepsilon$, for every i. We construct a converging sequence $\{W_{0,i}\}_{i\geq 0}$ of center leaves in $W^u(W_0)$ so that the limit leaf will shadow the pseudo-orbit $\{W_i\}_{i\geq 0}$. We do so by induction; see Figure 3 for an illustration.

Before we state the exact induction, we construct the first center leaves $W_{0,1}$ and $W_{0,2}$ to motivate the inductive assumption.

By definition of $\Delta_H(f(W_0), W_1)$, we can find lifts $\widetilde{f(W_0)}$ and $\widetilde{W}_1$ such that the minimal Hausdorff distance is attained, that is, $d_H(\widetilde{f(W_0)}, \widetilde{W}_1) = \Delta_H(f(W_0), W_1) < \varepsilon$.

By the choice of ε we then find a unique center leaf $\widetilde{f(W_{0,1})} = W^u_{\mu_1}(\widetilde{f(W_0)}) \cap W^s_{\mu_1}(\widetilde{W}_1)$. Inequality (5.1) implies with $\widetilde{f(W_{0,1})} \subset W^u_{\mu_1}(\widetilde{f(W_0)})$ and $\widetilde{f(W_{0,1})} \subset W^s_{\mu_1}(\widetilde{W}_1)$ the following two estimates:

$$\begin{aligned} \Delta^u_{\text{proj}}(\widetilde{f(W_0)}, \widetilde{f(W_{0,1})}) &\leq C d_H(\widetilde{f(W_0)}, \widetilde{f(W_{0,1})}) < C\varepsilon, \\ \Delta^s_{\text{proj}}(\widetilde{W}_1, \widetilde{f(W_{0,1})}) &\leq C d_H(\widetilde{W}_1, \widetilde{f(W_{0,1})}) < C\varepsilon. \end{aligned} \tag{5.3}$$

Consider points $\widetilde{f(x)} \in \widetilde{f(W_0)}$ and $\widetilde{f(y)} \in \widetilde{f(W_{0,1})}$ so that $\widetilde{f(y)} \in W^u_{\mu_1}(\widetilde{f(x)})$: the

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uniqueness of the projection says that $\widetilde{f(y)}$ is unique in this local unstable manifold and $d^u(\widetilde{f(x)}, \widetilde{f(y)}) < C\varepsilon$.

Let $f(x)$ and $f(y)$ be the projections of these points, and consider $x \in W_0$ and $y \in W_{0,1}$. The distance $d(x, y)$ is bounded by $\alpha d(f(x), f(y)) < C\varepsilon\alpha < \delta_0$. Therefore, any lift of W_0 determines a lift for $W_{0,1}$. According to Lemma 5.5, applied to f^{-1} and the leaves $\widetilde{f(W_0)}$ and $\widetilde{f(W_{0,1})}$, one gets

$$\Delta^u_{\text{proj}}(\widetilde{W_0}, \widetilde{W_{0,1}}) \le \alpha \Delta^u_{\text{proj}}(\widetilde{f(W_0)}, \widetilde{f(W_{0,1})}) < C\varepsilon\alpha. \tag{5.4}$$

Furthermore,

$$\begin{aligned}
&\Delta_H(W_1, f(W_{0,1})) \\
&\quad \le d_H(\widetilde{W_1}, \widetilde{f(W_{0,1})}) \quad \text{(by Corollary 3.12)} \\
&\quad \le \Delta^u_{\text{proj}}(\widetilde{f(W_0)}, \widetilde{f(W_{0,1})}) + \Delta^s_{\text{proj}}(\widetilde{W_1}, \widetilde{f(W_{0,1})}) \quad \text{(by Lemma 5.3)} \\
&\quad = \Delta^s_{\text{proj}}(\widetilde{W_1}, \widetilde{f(W_{0,1})}) \quad \text{(as } \widetilde{f(W_{0,1})} \text{ belongs to the stable manifold of } \widetilde{W_1}) \\
&\quad < C\varepsilon \quad \text{(by inequality (5.3))}.
\end{aligned} \tag{5.5}$$

Now we start the construction of $W_{0,2}$, for which we need to consider the image $f^2(W_{0,1})$.

We fix points $\tilde{w}_1 \in \widetilde{W_1}$ and $\tilde{w}_2 \in \widetilde{f(W_{0,1})}$ such that $d(\tilde{w}_1, \tilde{w}_2) < C\varepsilon$. Then we consider the projections under the covering map. All lengths are preserved as the covering map is a local isometry, so $d(w_1, w_2) < C\varepsilon$.

We consider the images $f^2(W_{0,1})$, $f(W_1)$, $f(w_1)$, $f(w_2)$ under f and the next center leaf W_2 of the pseudo-orbit. We choose lifts $\widetilde{f(W_1)}$ and $\widetilde{W_2}$ to the holonomy cover such that the minimal Hausdorff distance is attained, so that $d_H(\widetilde{f(W_1)}, \widetilde{W_2}) < \varepsilon$.

Further, we choose the lift $\widetilde{f(w_1)}$ which lies in $\widetilde{f(W_1)}$. There is a unique lift $\widetilde{f(w_2)}$ such that

$$d(\widetilde{f(w_1)}, \widetilde{f(w_2)}) = d(f(w_1), f(w_2)).$$

Then we pick the lifted center leaf $\widetilde{f^2(W_{0,1})}$ which contains $\widetilde{f(w_2)}$. According to Lemma 5.5, $\widetilde{f^2(W_{0,1})}$ is contained in the local stable manifold of $\widetilde{f(W_1)}$ and we have, with inequality (5.4),

$$d_H(\widetilde{f(W_1)}, \widetilde{f^2(W_{0,1})}) \le \Delta^s_{\text{proj}}(\widetilde{f(W_1)}, \widetilde{f^2(W_{0,1})}) < C\alpha\varepsilon < \varepsilon \tag{5.6}$$

as α is chosen such that $C\alpha < 1$.

This implies

$$\Delta_H(f^2(W_{0,1}), W_2) \le d_H(\widetilde{f^2(W_{0,1})}, \widetilde{W_2}) < (1 + \alpha C)\varepsilon < 2\varepsilon. \tag{5.7}$$

By the choice of ε there exists a unique center leaf

$$\widetilde{f^2(W_{0,2})} = W^u_{\mu_1}(\widetilde{f^2(W_{0,1})}) \cap W^s_{\mu_1}(\widetilde{W_2}).$$

Thanks to inequalities (5.6) and (5.7), one gets

$$\Delta^u_{\text{proj}}(\widetilde{f^2(W_{0,1})}, \widetilde{f^2(W_{0,2})}) < C\varepsilon(1 + 2C\alpha) < 2C\varepsilon,$$

$$\Delta^s_{\text{proj}}(\widetilde{W_2}, \widetilde{f^2(W_{0,2})}) < C\varepsilon(1 + 2C\alpha) < 2C\varepsilon.$$

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We consider the leaf $W_{0,2}$ whose image by f^2 is the projection of $\widetilde{f^2(W_{0,2})}$. Recall that $f^2(W_{0,2})$ is contained in the local unstable manifold of $f^2(W_{0,1})$. Thus, applying Lemma 5.5, there is a lift $\widetilde{W_{0,2}}$ of $W_{0,2}$ so that

$$\Delta^u_{\text{proj}}(\widetilde{W_{0,2}},\, \widetilde{W_{0,1}}) < \alpha^2 \Delta^u_{\text{proj}}(\widetilde{f^2(W_{0,2})},\, \widetilde{f^2(W_{0,1})}) < 2C\varepsilon\alpha^2. \tag{5.8}$$

With inequalities (5.4) and (5.8), one gets

$$\Delta_H(W_{0,2},\, W_0) \le \Delta^u_{\text{proj}}(\widetilde{W_{0,2}},\, \widetilde{W_0}) < C\varepsilon\alpha + 2C\varepsilon\alpha^2 < 2C\varepsilon \sum_{k=0}^{2} \alpha^k. \tag{5.9}$$

In this way we have constructed the first two elements $W_{0,1}$ and $W_{0,2}$ in the center unstable leaf $W^u(W_0)$. During this construction the following inductive statement became evident, and we have to prove it in order to conclude the construction of the sequence $(W_{0,i})_i$.

LEMMA 5.7. *For any $i \ge 0$ there is $W_{0,i}$ so that:*
(I) $\Delta_H(f^{i+1}(W_{0,i}),\, W_{i+1}) < 2\varepsilon$;
(II) *for all $0 \le j \le i$,*

$$\Delta_H(f^j(W_{0,i}),\, W_j) < 2C\varepsilon \sum_{k=0}^{i-j} \alpha^k < \eta.$$

Proof of Lemma 5.7. For $i = 0$ we set $W_0 = W_{0,0}$. For $i = 1$ we have already proved properties (I) and (II) above. It remains to prove the inductive step: we assume Lemma 5.7 is done for $0, \ldots, i$ and we build $W_{0,i+1}$ satisfying the claimed inequalities.

According to (I), we have

$$\Delta_H(f^{i+1}(W_{0,i}),\, W_{i+1}) < 2\varepsilon.$$

So there are lifts $\widetilde{W_{i+1}}$ and $\widetilde{f^{i+1}(W_{0,i})}$ whose Hausdorff distance realizes the infimum, and is therefore bounded by 2ε.

Consequently, we can construct $W_{0,i+1}$ with the help of the unique center leaf

$$\widetilde{f^{i+1}(W_{0,i+1})} := W^u_{\mu_1}(\widetilde{f^{i+1}(W_{0,i})}) \cap W^s_{\mu_1}(\widetilde{W_{i+1}})$$

which satisfies

$$\Delta^u_{\text{proj}}(\widetilde{f^{i+1}(W_{0,i+1})},\, \widetilde{f^{i+1}(W_{0,i})}) < 2C\varepsilon, \tag{5.10}$$

$$\Delta^s_{\text{proj}}(\widetilde{f^{i+1}(W_{0,i+1})},\, \widetilde{W_{i+1}}) < 2C\varepsilon. \tag{5.11}$$

Then—as a consequence of Claim 4.15—$W_{0,i+1}$ is the center leaf whose image by f^{i+1} is the projection of $\widetilde{f^{i+1}(W_{0,i+1})}$. Let us show that $W_{0,i+1}$ satisfies the inequalities given in (I) and (II). We prove inequality (I) of Lemma 5.7 by means of the following claim.

CLAIM 5.8. $\Delta_H(f^{i+2}(W_{0,i+1}),\, W_{i+2}) < 2\varepsilon$.

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Proof of Claim 5.8. Consider lifts $\widetilde{f(W_{i+1})}$ and $\widetilde{W_{i+2}}$ such that the minimal Hausdorff distance in the holonomy cover is attained, so

$$d_H(\widetilde{f(W_{i+1})}, \widetilde{W_{i+2}}) < \varepsilon.$$

The center leaf $\widetilde{f^{i+1}(W_{0,i+1})}$ lies by construction in the local center stable leaf of $\widetilde{W_{i+1}}$, so we have $\Delta^u_{\mathrm{proj}}(\widetilde{f^{i+1}(W_{0,i+1})}, \widetilde{W_{i+1}}) = 0$. According to Lemma 5.5, there is a lift of $f(f^{i+1}(W_{0,i+1}))$ so that

$$\begin{aligned} d_H(\widetilde{f^{i+2}(W_{0,i+1})}, \widetilde{f(W_{i+1})}) &\le \Delta^s_{\mathrm{proj}}(\widetilde{f^{i+2}(W_{0,i})}, \widetilde{f(W_{i+1})}) \\ &\quad \text{(by Lemma 5.3 and } \Delta^u_{\mathrm{proj}}(\widetilde{f^{i+1}(W_{0,i+1})}, \widetilde{W_{i+1}}) = 0) \\ &\le \alpha \Delta^s_{\mathrm{proj}}(\widetilde{f^{i+1}(W_{0,i+1})}, \widetilde{W_{i+1}}) \text{ (by Lemma 5.5)} \\ &< 2C\varepsilon\alpha \text{ (by inequality (5.10))} \\ &< \varepsilon. \end{aligned}$$

The triangle inequality implies

$$\Delta_H(f^{i+2}(W_{0,i}), W_{i+2}) \le d_H(\widetilde{f^{i+2}(W_{0,i})}, \widetilde{W_{i+2}}) < 2\varepsilon,$$

which proves the claim. □

Let us now prove (II):

CLAIM 5.9. *For all* $0 \le j \le i+1$ *of Lemma 5.7,*

$$\Delta_H(f^j(W_{0,i+1}), W_j) < 2C\varepsilon \sum_{k=0}^{i+1-j} \alpha^k < \eta.$$

Proof of Claim 5.9. For any $0 \le j \le i$, the induction hypothesis implies that there are lifts $\widetilde{f^j(W_{0,i})}$ and $\widetilde{W}_j$ such that

$$d_H(\widetilde{f^j(W_{0,i})}, \widetilde{W}_j) = \Delta_H(f^j(W_{0,i}), W_j) < 2C\varepsilon \sum_{k=0}^{i-j} \alpha^k.$$

On the other hand, one has that $\widetilde{f^{i+1}(W_{0,i+1})}$ is contained in the unstable manifold of $\widetilde{f^{i+1}(W_{0,i})}$ and—due to inequality (5.10)—$\Delta^u_{\mathrm{proj}}(\widetilde{f^{i+1}(W_{0,i+1})}, \widetilde{f^{i+1}(W_{0,i})}) < 2C\varepsilon$.

By Lemma 5.5, there is a lift $\widetilde{f^j(W_{0,i+1})}$ of $f^j(W_{0,i+1})$ so that

$$\begin{aligned} \Delta_H(f^j(W_{0,i+1}), f^j(W_{0,i})) &\le \Delta^u_{\mathrm{proj}}(\widetilde{f^j(W_{0,i+1})}, \widetilde{f^j(W_{0,i})}) \\ &\le \alpha^{i+1-j} \Delta^u_{\mathrm{proj}}(\widetilde{f^{i+1}(W_{0,i+1})}, \widetilde{f^{i+1}(W_{0,i})}) \\ &< 2C\alpha^{i+1-j}\varepsilon. \end{aligned}$$

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As a consequence, the triangle inequality implies

$$\Delta_H(f^j(W_{0,i+1}),\, W_j) \le 2C\varepsilon \sum_{k=0}^{i-j} \alpha^k + 2C\varepsilon\alpha^{i+1-j} = 2C\varepsilon \sum_{k=0}^{i+1-j} \alpha^k.$$

This gives the claimed inequality for $0 \le j \le i$.

For $j = i+1$ the inequality says $\Delta_H(f^{i+1}(W_{0,i+1}),\, W_{i+1}) \le 2C\varepsilon$. This comes from the fact that, according to inequality (5.10),

$$\Delta_H(f^{i+1}(W_{0,i+1}),\, W_{i+1}) \le \Delta^s_{\mathrm{proj}}(\widetilde{f^{i+1}(W_{0,i+1})},\, \widetilde{W_{i+1}}) \le 2C\varepsilon.$$

This finishes the proof of the claim. □

The proof of Claim 5.9 gives us (II) of Lemma 5.7 which completes its proof. □

Let W_∞ be the limit point of the sequence $(W_{0,i})_{i\ge 0}$. Then, for every fixed $j \ge 0$, item (II) of Lemma 5.7 implies

$$\Delta_H(f^j(W_\infty),\, W_j) = \lim_{i\to\infty} \Delta_H(f^j(W_{0,i}),\, W_j) < 2C\varepsilon \frac{1}{1-\alpha} < \eta.$$

Therefore W_∞ is the center leaf whose orbit stays η-close to the positive pseudo-orbit $(W_j)_{j\ge 0}$, i.e. it is its shadowing orbit. This concludes the proof of Proposition 5.2. □

5.4. *Uniqueness of the shadowing and existence of periodic center leaves.* If the center foliation has non-trivial holonomy, there are in general an uncountable set of center leaves δ-shadowing the same ε-pseudo-orbit $\{W_i\}_i$ of center leaves. Therefore the statement of Theorem 2 is not enough to assure the existence of periodic center leaves shadowing the periodic pseudo-orbits.

However, the lack of uniqueness of the shadowing comes from the choice of an appropriate lift which we have to make during the construction of the shadowing orbit. The idea of this section is to enhance the pseudo-orbits by the possible choices, and then to recover the uniqueness in that setting.

THEOREM 5.10. *Let f be a partially hyperbolic diffeomorphism with a uniformly compact invariant center foliation. Let $\{U_j,\, p_j\}_j$ be a holonomy cover and δ_0 be as in Lemma 3.9. Then there is η_0 so that, for any $0 < \eta < \eta_0$, there is $\varepsilon > 0$ such that*

- *for any ε-pseudo-orbit $\{W_i\}_{i\in\mathbb{Z}}$ for the modified Hausdorff distance Δ_H,*
- *for any $i \in \mathbb{Z}$ and any j so that the $B_H(W_i, \delta_0) \subset U_j$,*
- *for any lifts of W_i and $f(W_{i-1})$ with $d_H(\widetilde{W}_i,\, \widetilde{f(W_{i-1})}) < \varepsilon$,*

there is a unique center leaf W admitting lifts $\widetilde{f^i(W)}$ such that:

- *$\Delta_H(\widetilde{f^i(W)},\, W_i) < \eta$;*
- *$d_H(\widetilde{f^i(W)},\, \widetilde{W}_i) < \eta$;*
- *$\widetilde{f^{i+1}(W)}$ is the lift of $f(f^i(W))$ associated to the pair of lifts $(\widetilde{f^i(W)},\, \widetilde{W}_i)$ and the lift $\widetilde{f(W_i)}$ by Remark 5.6.*

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The proof is indeed the proof we wrote for Theorem 2.

We are now ready to prove Theorem 3, which states that the periodic center leaves are dense in the set of center leaves which are chain recurrent for the quotient dynamics.

Proof of Theorem 3. It is enough to prove that any periodic ε-pseudo-orbit of center leaves is η-shadowed by (at least one) periodic center leaf. For that purpose, just consider a periodic choice of lifts of the pseudo-orbit in Theorem 5.10. □

6. *Plaque expansivity*

The aim of this section is to prove the plaque expansivity. Let us recall the precise definition.

Definition 6.1. Let f be a partially hyperbolic diffeomorphism with an f-invariant center foliation $\mathcal{F}^c$. We call $(f, \mathcal{F}^c)$ *plaque expansive* if there is $\eta > 0$ with the following property. Consider η-pseudo-orbits (x_i), (y_i) preserving plaques of the center foliation, i.e. $f(x_i)$ lies in the same plaque as x_{i+1} with $d(f(x_i), x_{i+1}) < \eta$ and $f(y_i)$ in the plaque of y_{i+1}, $d(f(y_i), y_{i+1}) < \eta$. Assume that $d(x_i, y_i) < \eta$ for every $n \in \mathbb{Z}$. Then x_0 lies in the same center plaque as y_0.

Remark 6.2. [**HPS70**] proves that every C^1-center foliation is plaque expansive.

The aim of this section is to prove Theorem 4: any invariant uniformly compact center foliation is plaque expansive. We consider some very small $\eta > 0$ (we will define η later) and η-pseudo-orbits (x_i), (y_i) with central jumps, and remaining at a distance $d(x_i, y_i)$ less than η; see Figure 4 for an illustration and the construction of the pseudo-orbit (z_i). We assume that y_0 does not belong to the center stable manifold of x_0.

We begin with the following lemma.

LEMMA 6.3. *Let $z_i \in W^u_{\mathrm{loc}}(x_i)$ be the projection of x_i on the local center stable manifold of y_i. Then the sequence (z_i) is also a pseudo-orbit with jumps in the center leaves, remaining close to the sequences (x_i) and (y_i).*

Analogously, let $w_i \in W^s_{\mathrm{loc}}(y_i)$ be the projection of y_i on the local center unstable manifold of x_i. Then (w_i) is a pseudo-orbit with jumps in the center leaves, remaining close to (x_i), (y_i), and therefore to (z_i).

Furthermore, for any i, the points z_i and w_i belong to the same center plaque.

Proof. We need to prove that z_{i+1} is on the same center leaf as $f(z_i)$. This is due to the dynamical coherence: the local unstable manifold through the center plaque containing x_{i+1} and $f(x_i)$ intersects the local stable manifold through the center plaque containing y_{i+1} and $f(y_i)$ in a center plaque containing z_{i+1} and at a point which is the intersection of the local unstable manifold of $f(x_i)$ with the local center stable manifold of $f(y_i)$. Notice that these local invariant manifolds are the image of the local invariant manifolds (with size multiplied by a factor bounded by $\|Df\|$). Therefore, these local invariant manifolds intersect on $f(z_i)$. Thus $f(z_i)$ belongs to the same center plaque as z_{i+1}. It works in the same way for (w_i).

Let us now prove the last claim of the lemma. There is u_i in the center plaque of y_i such that $z_i \in W^s_{\mu_1}(u_i)$. Therefore, z_i is the projection of u_i on the center unstable manifold

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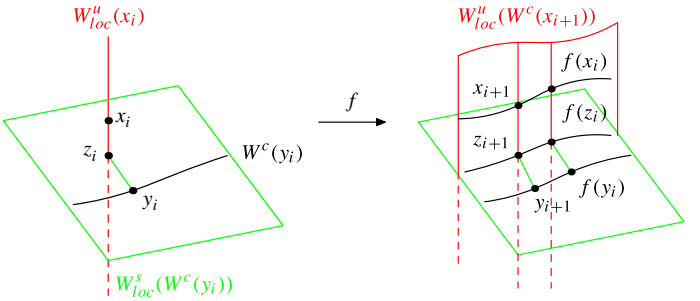


FIGURE 4. Two pseudo-orbits $(x_i)_i$, $(y_i)_i$ and construction of the pseudo-orbit $(z_i)_i$ where $z_i = W^u_{\mathrm{loc}}(x_i) \cap W^s_{\mathrm{loc}}(W^c(y_i)$ is the projection of x_i on the local center stable manifold of y_i.

of x_i. Consider a small path $u_{i,t}$ joining u_i to y_i in the center plaque of y_i. Then the projection of $u_{i,t}$ on the center unstable plaque of x_i defines a small path in the intersection of the local center stable manifold of y_i with the local center unstable manifold of x_i. Therefore this small path is contained in a plaque of a center leaf. Thus w_i and z_i are joined by a path inside a center plaque. □

Now, Theorem 4 is a direct consequence of the following proposition, where $\mu > 0$ is given by Lemma 4.14.

PROPOSITION 6.4.

- *For any pair of μ-pseudo-orbits (x_i), (z_i) preserving the center plaques, and such that $z_i \in W^u_\mu(x_i)$ for any i, one has $x_i = z_i$ for any i.*
- *For any pair of μ-pseudo-orbits (y_i), (w_i) preserving the center plaques, and such that $w_i \in W^s_\mu(y_i)$ for any i, one has $y_i = w_i$ for any i.*

The two items of the proposition are equivalent (up to changing f by f^{-1}), so we prove only the first. Proposition 6.4 is a straightforward consequence of the next lemma.

LEMMA 6.5. *With the notation of Proposition 6.4:*

- *there is $N > 0$ such that, for any $n \geq N$, one has $x_n = z_n$;*
- *for any integer n, if $x_n = z_n$ then $x_{n-1} = z_{n-1}$.*

Proof of Lemma 6.5. By the definition of pseudo-orbits preserving the center plaques, the points $f^{-n}(x_n)$ and $f^{-n}(z_n)$ belong to the center leaf of x_0 and of z_0, respectively. Furthermore, as $z_n \in W^u_\mu(x_n)$, for any $n > 0$ one gets $f^{-n}(z_n) \in W^u_{\alpha^n \mu}(x_n)$ (recall that $\alpha < 1$ is the contraction rate in the stable bundle for an adapted metric). In particular, their distance tends to 0. As $W^c(x_0)$ and $W^c(z_0)$ are compact manifolds which are either disjoint or equal, this implies that $W^c(x_0) = W^c(z_0)$.

As $W^c(x_0)$ is a compact manifold transverse to the unstable foliation (indeed, in general position), this implies that there is η (depending on $W^c(x_0)$) so that every local unstable leaf $W^u_\eta(x)$, $x \in W^c(x_0)$ intersects $W^c(x_0)$ only at the point x. For $\alpha^n \mu < \eta$ this implies $x_n = z_n$, concluding the proof of the first claim of the lemma.

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We now prove the second claim, arguing recursively. Recall that $\mu_2 > 0$ is given by Lemma 4.12 and guarantees that the unstable projection is unique for points at a distance less than μ_2. Assume $x_n = z_n$. Consider x_{n-1} and $z_{n-1} \in W^u_\mu(x_{n-1})$.

By assumption, $f(x_{n-1})$ belongs to $W^c_\mu(x_n)$ and $f(z_{n-1}) \in W^c_\mu(z_n) = W^c_\mu(x_n)$ (as $z_n = x_n$).

Furthermore, $z_{n-1} \in W^u_\mu(x_{n-1})$ so that $f(z_{n-1}) \in W^u_{\lambda\mu}(f(x_{n-1}))$ where λ is a bound of $\|Df\|$. For $\lambda\mu$ smaller than μ_2, the intersection of a local unstable manifold with a local center manifold contains at most one point, so that $f(x_{n-1}) = f(z_{n-1})$ that is $x_{n-1} = z_{n-1}$. □

Remark 6.6. The proof of the first claim shows indeed that $W^c(x_i) = W^c(z_i)$ for any i, that is, x_i and z_i lie in the same center leaf. However, it does not show directly that x_i and z_i are equal or belong to the same center plaque, i.e. the same local center leaf, explaining why we need the second claim.

7. *Non-compactness of center unstable and center stable leaves*

Under the assumption of a uniformly compact invariant center foliation, we proved the dynamical coherence (existence of center stable and center unstable foliations denoted by $\mathcal{F}^{cu}$ and $\mathcal{F}^{cs}$, respectively). The aim of this section is to start the topological study of these foliations.

THEOREM 7.1. *Let $f : M \to M$ be a partially hyperbolic C^1-diffeomorphism with an f-invariant uniformly compact center foliation. Then every leaf of the center unstable foliation $\mathcal{F}^{cu}$ is non-compact. The same holds for every leaf of the center stable foliation $\mathcal{F}^{cs}$.*

This result appeared in [**Car11**]. We include here a detailed proof because the proof is easy and this result is used in [**Boh13**], and [**Car11**] is still unpublished.

LEMMA 7.2. *Let $f : M \to M$ be a partially hyperbolic C^1-diffeomorphism with an f-invariant uniformly compact center foliation. Let n be the maximal (finite) order of the holonomy group of center leaves. Then for every $y, z \in M$,*

$$\sharp\{W^s(y) \cap W^c(z)\} < n.$$

Proof. Notice that any local stable manifold of size μ admits a lift in a holonomy cover $\{U_i, p_i\}$ and the lifted local stable manifold intersects each lifted center leaf at no more than one point. The preimage by p_i of any center manifold consists of at most $\max |\mathrm{Hol}| = \max_{L \in \mathcal{F}^c} |\mathrm{Hol}(L)|$ points: this implies that for any $y, z \in M$,

$$\sharp\{W^s_\mu(y) \cap W^c(z)\} \leq n = \max |\mathrm{Hol}|.$$

Assume now, arguing by contradiction, that there is y, z so that the whole stable manifold $W^s(y)$ cuts $W^c(z)$ in strictly more than n points. Then there is $r > 0$ so that the stable manifold $W^s_r(y)$ of size r cuts $W^c(z)$ at more than n points. Choose $i > 0$ so that $\alpha^i r < \mu$, where α is the contraction rate of the stable bundle.

Then $W^s_{\alpha^i r}(f^i(y)) \subset W^s_\mu(f^i(y))$ cuts $W^c(f^i(z))$ at more than n points, leading to a contradiction. □

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For the proof of Theorem 7.1 we notice the following fact.

LEMMA 7.3. *Let V be a C^1-compact manifold admitting a continuous splitting $TV = F \oplus G$ into transverse subbundles. Assume that there are C^0-foliations $\mathcal{F}$ and $\mathcal{G}$ with C^1-leaves tangent to F and G, respectively. Then for any non-compact leaf L_G of $\mathcal{G}$, there is a leaf L_F of $\mathcal{F}$ such that the intersection $L_F \cap L_G$ is infinite.*

Proof. As L_G is assumed to be non-compact and V is compact, there is a point $x \in V$ which is contained in the limit of an infinite sequence of plaques of L_G. Therefore, the leaf $L_F(x)$ cuts L_G at infinitely many points. □

Now the proof of Theorem 7.1 is straightforward:

Proof of Theorem 7.1. Assume that there exists a compact leaf $W^{cs} \in \mathcal{F}^{cs}$. Therefore, we can apply Lemma 7.3 to $TW^{cs} = E^c|_{W^{cs}} \oplus E^s|_{W^{cs}}$. Every stable leaf $W^s(z)$ for $z \in W^{cs}$ is non-compact. Accordingly, for any leaf $W^s(z)$ there exists a center leaf $W^c \in W^{cs}$ which intersects $W^s(z)$ infinitely many times, contradicting Lemma 7.2. Hence, every center stable leaf is non-compact. A similar argument proves that every center unstable leaf is non-compact, concluding the proof. □

8. *Comments, examples and open questions*

In this section we first present a very simple example from [**BW05**, §4.1.3] which illustrates many of the pathological behaviors one can get for the dynamics on the center leaf space. Then we will ask some questions on the existence of periodic center leaves and on the dynamical characterization of the stable manifolds—in the light provided by this example.

8.1. *Non-expansivity and infinite non-uniqueness of the shadowing.*

Let us return to an example presented in [**BW05**, §4.1.3].

Let $A \in \mathrm{SL}(2, \mathbb{Z})$ be an Anosov matrix, considered as a diffeomorphism of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let M be the compact 3-manifold obtained as the quotient of $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$ by the free involution $\varphi : (r, s, t) \mapsto (-r, -s, t + \frac{1}{2})$. The involution φ commutes with the diffeomorphism $\tilde{f} = A \times \mathrm{id}_{\mathbb{S}^1}$ so that $\tilde{f}$ passes to the quotient as a diffeomorphism f of M. The diffeomorphism f is partially hyperbolic, its center foliation is a circle foliation which is a Seifert bundle over the sphere S^2, with four exceptional leaves which have a non-trivial holonomy, which is indeed generated by $-\mathrm{id}$.

The space of center leaves is therefore the sphere S^2 naturally endowed with an orbifold structure with four singular points. The stable and unstable foliations of f induce on the quotient transverse singular foliations, with exactly one singularity at each singular point of the orbifold. These singularities are *one-prong singularities of generalized pseudo-Anosov type*.

This implies that, for every $\varepsilon > 0$, for every center leaf W^c close to one of the exceptional fibers, $W^s_\varepsilon(W^c) \cap W^u_\varepsilon(W^c)$ consists of the union of W^c and another center leaf W^c_1; see Figure 5.

As a consequence, for every $n \in \mathbb{Z}$ the distance $d_H(f^n(W^c),\ f^n(W^c_1))$ is bounded by ε.

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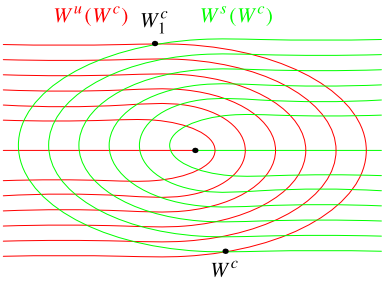


FIGURE 5. Intersection of local stable and unstable leaves in a neighborhood of a one-prong singularity on the quotient space of the center leaves.

Remark 8.1. The argument above shows that the quotient dynamics $A/-\mathrm{id}$ induced by f on the center leaf spaces S^2 is not expansive. This non-expansivity is indeed expected as there are no expansive homeomorphisms on the 2-sphere (see [**Lew89**]).

Let us investigate this lack of expansivity a little further. The Anosov diffeomorphism A admits a dense subset of periodic points. One deduces that there are periodic center leaves W^c which are arbitrarily close to an exceptional leaf. In other words, one may assume that, in the comment above, W^c is a periodic center leaf, and W_1^c is a homoclinic intersection associated to W^c.

One deduces that the maximal invariant set of any small neighborhood of $W^c \cup W_1^c$ contains an invariant compact set, saturated by the center foliation, and which projects onto S^2 as a non-trivial hyperbolic basic set of $A/-\mathrm{id}$. This shows the following result.

LEMMA 8.2. *For any $\varepsilon > 0$, there is a invariant compact set K saturated for the center foliations, homeomorphic to the product of a Cantor set by S^1, and so that, for any leaves W_1, $W_2 \subset K$, one has $d_H(f^n(W_1), f^n(W_2)) < \varepsilon$.*

8.2. *Dynamical classification of invariant manifolds.* For hyperbolic systems, the set of points whose positive orbits remain at a distance less than $\varepsilon > 0$ of the orbit of x is the local stable manifold of x.

This characterization does not hold for the dynamics induced on the center leaf space: in the example above, for any $\varepsilon > 0$, there are center leaves W_1, W_2 whose positive (and indeed negative) orbits remains at a distance less than ε but which are not in the same stable manifold.

PROPOSITION 8.3. *Given an Anosov automorphism A of $\mathbb{T}^2$ the quotient $\varphi = A/-\mathrm{id}$ admits points x, y so that the distance $d(\varphi^n(x), \varphi^n(y))$ tends to 0 when $n \to \pm\infty$, but the lifts of x and y on $\mathbb{T}^2$ do neither lie in the same stable nor unstable manifold.*

As a direct corollary one gets the following result.

COROLLARY 8.4. *Let f be the partially hyperbolic diffeomorphisms associated to an Anosov automorphism A of $\mathbb{T}^2$ in §8.1. Then there exist leaves W_1, W_2 such that $d_H(f^n(W_1) f^n(W_2))$ tends to 0 for $n \to \infty$, but $W_2 \not\subseteq W^s(W_1)$.*

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Proof of Proposition 8.3. Fix some $\delta > 0$ and $\varepsilon > 0$ so that every pseudo-orbit of the Anosov map A is uniquely δ-shadowed by a true orbit of A. We will use the following property which comes directly from the proof of the shadowing lemma: any δ-pseudo-orbit $(x_i)_{i\in\mathbb{Z}}$ whose jumps tend to 0 (that is, $d(f(x_i), x_{i+1}) \to 0$ for $|i| \to \infty$) is uniquely δ-shadowed by the orbit of a point x with $d(f^i(x), x_i)$ tending to 0 with $|i| \to \infty$.

Consider a sequence of periodic points p_n^+, $n \in \mathbb{Z}$ tending to 0 as $|n| \to \infty$ and so that $d(p_n^+, 0) < \varepsilon$. Denote $p_n^- = -\mathrm{id}(p_n^+)$: then p_n^+ and p_n^- have the same period π_n, tending to ∞ as $|n| \to \infty$. Let us denote by $\gamma_n^\pm$ the orbit of $p_n^\pm$, with initial point at $p_n^\pm$. Notice that, for every n, m, one has $d(p_n^\pm, p_m^\pm) < \varepsilon$ and $d(p_n^\pm, p_m^\pm) \to 0$ as $|n|, |m| \to \infty$.

For any sequence $\mu = \{\mu_i \in \{+, -\}, i \in \mathbb{Z}\}$ we consider the infinite sequence Γ_μ,

$$\dots p_{-1}^{\mu_{-1}}, \dots, A^{\pi_{-1}}(p_{-1}^{\mu_{-1}}), p_0^{\mu_0}, \dots, A^{\pi_0}(p_0^{\mu_0}), p_1^{\mu_1}, \dots, A^{\pi_1}(p_1^{\mu_1}) \dots,$$

obtained by concatenation of the $\gamma_n^{\mu_n}$. Then each of the $\Gamma_\mu = (\Gamma_{\mu,i})_{i\in\mathbb{Z}}$ is a δ-pseudo-orbit whose jumps tends to 0. Therefore, each ot the Γ_μ is δ-shadowed by the orbit of a point x_μ, and $d(f^i(x_\mu), \Gamma_{\mu,i}) \to 0$ with i.

Now the orbits of p_n^+ and of p_n^- project onto the same orbit of $\varphi = A/-\mathrm{id}$. Therefore all the pseudo-orbits $\{\Gamma_\mu \mid \mu \in \{+, -\}^{\mathbb{Z}}\}$ of A project onto the same pseudo-orbit Γ of φ. Let z_μ be the projection of x_μ. Thus, Γ is δ-shadowed by the orbit of z_μ and $d(\varphi^i(z_\mu), \Gamma_i) \to 0$ as $|i| \to \infty$.

Now consider μ, $\nu \in \{+, -\}^{\mathbb{Z}}$ so that both sets $\{i|\nu_i = \mu_i\}$ and $\{j|\nu_j = -\mu_j\}$ are infinite and neither upper- nor lower-bounded. For every i, denote

$$d_i = \inf\{d(A^i(x_\mu), A^i(x_\nu)), d(A^i(-x_\mu), A^i(x_\nu))\}.$$

Our choice of μ and ν implies that d_i does not tend to 0 as either $i \to +\infty$ or $i \to -\infty$. Therefore, x_ν is neither in the stable nor in the unstable manifold of x_μ or $-x_\mu$.

Thus the orbits of z_μ and z_ν are positively and negatively asymptotic but are not in the same stable nor unstable leaf, concluding the proof. □

Remark 8.5. The proof of Proposition 8.3 gives a little more: given x (corresponding in the proof to the point x_μ), there are indeed uncountably many points $y = x_\nu$ satisfying the conclusion of Proposition 8.3, corresponding to all possible choices of ν.

8.3. *Unique integrability.* We have been unable to prove the uniqueness of the center foliation even with very strong hypotheses.

Question 8.6. Let f be a partially hyperbolic diffeomorphism admitting invariant uniformly compact center foliations $\mathcal{W}^1$ and $\mathcal{W}^2$ (tangent to the same center bundle). Are they equal?

Acknowledgements. We would like to thank the referee for his careful reading and his numerous comments which helped us to improve the presentation of this paper. The paper was partially supported by the Forschungsfond of the Department of Mathematics and the Körperschaftsvermögen of the University of Hamburg. It was further supported by the Université de Bourgogne, in particular the Institut de Mathématiques de Bourgogne, and

https://doi.org/10.1017/etds.2014.102 Published online by Cambridge University Press

the first author would like to thank the IMB for its hospitality during her several visits. The first author benefits from a *bourse postdoc* given by the Région Bourgogne which helped considerably in finishing this article.

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