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EARLY STRUCTURAL REASONING. GENTZEN 1932

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Abstract. This paper is a study of the opening section of Gentzen's first publication of 1932, *Über die Existenz unabhängiger Axiomensysteme zu unendlichen Satzsystemen*, a text which shows the relevance of Hertz's work of the 1920's for the young Gentzen. In fact, Gentzen borrowed from Hertz the analysis of the notion of consequence, which was given in terms of the rules of thinning (*Verdünnung*) and cut (*Schnitt*) on sequents (there called "sentences"(*Sätze*)). Moreover, following Hertz again, he also judged it necessary to justify the forms of inference of the system by providing a semantics for them, so that it became possible to make precise the informal notion of consequence, and to show that the inference rules adopted are correct and sufficient.

§1. Introduction. Gentzen's first published paper, *Über die Existenz unabhängiger Axiomensysteme zu unendlichen Satzsystemen*,[1] has recently been stimulating an increasing attention.[2] One of the reasons is that, besides witnessing the relevance of Hertz's work for the young Gentzen, it allows us to know that "structural reasoning" predates the formulation of Natural Deduction and Sequent Calculus. By structural reasoning I mean reasoning based on structural rules which affect the way formulas are arranged in *consequence relations* expressed by sequents, without reference to their internal articulation by means of logical connectives and quantifiers. This means that moving to Sequent Calculus, though strictly linked to difficulties encountered in the attempt to get the *Hauptsatz* within the framework of Natural Deduction, was anyway a step which extended a perspective already present in Gentzen's work.[3]

The paper is organized as follows. Sections 2 and 3 outline the main features of Gentzen's system: the notation he uses and the characterization provided for the notions of inference, proof, and consequence. It is moreover recalled how he defined, with respect to this system, the soundness and completeness notions. In Section 4 we present Gentzen's original proof of the completeness theorem, and some points useful for its theoretical completion.

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[1] See Gentzen (1932).

[2] Many interesting insights into this paper of Gentzen's are for instance provided in Schroeder-Heister (2002), Franks (2010), Franks (2013), and Tennant (2015). The paper Tennant (2015) contains also remarkable generalizations to cases not considered by Gentzen, as to sentences with empty antecedent or succedent, and to consequences of infinite sets of sentences.

[3] In this paper I will deal only with the first of the three sections Gentzen's article consists of. This is a somewhat autonomous section which is devoted to provide a fully general account of the notion of "following by logic"; an account which was meant to be independent of what the elements of a sentence are and of any analysis of the structure of the sentences involved. As a result of the choice to focus on this topic, I do not touch upon the issue that the title of Gentzen's paper alludes to, namely "the existence of independent axiom systems for infinite systems of sentences".

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In Section 5 we present a proof of the completeness theorem which exploits mainly semantic tools, and an alternative proof of Gentzen's normal form theorem that was just mentioned as a possibility in Gentzen' paper. In the last Section 6 we try in conclusion to place Gentzen's completeness theorem within his general proof-theoretic framework.

§2. Sentences. In Gentzen (1932) Gentzen refers to systems of *sentences*. A sentence, a notion which of course predates the notion of *sequent*, is a finite string of *elements* of the form[4]

$$u_1 u_2 \dots u_n \rightarrow v$$

"Elements" can be understood in various ways: as events, as elements in proper set theoretical terms, as properties, and as propositions, in the sense of (the propositional variables of the) propositional calculus. Gentzen stresses that his remarks do not depend on any of these *informal interpretations* (*inhaltliche Deutungen*), but refer just to the formal structure of the sentences. Anyway, here we will refer to the last interpretation, so that the reading of the previous sentence is: "If the propositions $u_1, u_2, \dots, u_n$ (which are the *antecedents*) are true, then the proposition v (which is the *succedent*) is also true". The sets of antecedent(s) and succedent cannot be void, and the succedent must always be a singleton. A sentence with only one antecedent element is called *linear*. A system of finitely many elements is called a *complex* and is denoted by a capital letter: $\mathbf{K}, \mathbf{L}, \mathbf{M}, \dots$ Therefore, a sentence can also be written as $\mathbf{K} \rightarrow v$, and the set-theoretical interpretation entails that writing $\mathbf{KL}$, or $\mathbf{K}u$, or $u\mathbf{K}$ we have to understand their set-theoretical union. If the succedent of a sentence is identical with one of the antecedents, then the sentence is called *trivial*; and a linear trivial sentence is called a *tautologous* sentence.

§3. Inference and proof. Sentences can be arranged in tree-like structures to form proofs by applying one of the following forms of inference, called Thinning, or Weakening (*Verdünnung*), and Cut (*Schnitt*):

$$\frac{\mathbf{L} \rightarrow v}{\mathbf{ML} \rightarrow v}\ \text{Thin} \qquad \frac{\mathbf{L} \rightarrow u \qquad \mathbf{M}u \rightarrow v}{\mathbf{LM} \rightarrow v}\ \text{Cut}$$

Left and right premisses of the Cut are called, respectively, *Lower* and *Upper* sentences.[5] $\mathbf{M}$ may be empty, but u is assumed not to occur in $\mathbf{M}$. Note that two more structural rules are hidden in the set-theoretical understanding of the elements of a sentence: Contraction and Interchange.

The restriction imposed on the Upper premise of the Cut rule could seem puzzling, given the set-theoretical interpretation of the (antecedents of) sentences. It is however dictated by the will to save that precise formal structure of the rule. In fact, if u occurs in $\mathbf{M}$ then the set-theoretical interpretation would produce the following inference schema

$$\frac{\mathbf{L} \rightarrow u \qquad \mathbf{M}^{\{u\}}u \rightarrow v}{\mathbf{LM}^{\{u\}} \rightarrow v}\ \text{Cut}$$

[4] In this paper we adopt Gentzen's notation, even though it diverges from the one used nowadays. Moreover, I chose to quote directly from the German text; the English translation (from Szabo, 1969, sometimes with minor changes) is anyway provided in the footnotes.

[5] We will call them also Lower and Upper premises.

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in which the contexts are not preserved, since $\mathbf{M}$ [$= \mathbf{M}^{\{u\}}u$] occurs in the Upper premise, but not in the conclusion.[6] Moreover, if u occurs in $\mathbf{M}$, then the conclusion of the Cut could be obtained from its Upper premise simply by an application of Thinning

$$\frac{\mathbf{M}^{\{u\}}u \rightarrow v}{\mathbf{ML} \rightarrow v}\ \text{Thin}$$

so that the two inference rules would not be fully independent from each other.

As regards the notion of "proof", it is to be stressed that Gentzen properly does not define this notion but rather the notion

"proof" of a "sentence" from other "sentences"

by stating that

> Unter einem *Beweis* eines Satzes q aus den Sätzen $p_1, \ldots, p_n$ ($n \geq 0$) verstehen wir nunmehr eine geordnete Anzahl von Schlüssen (d.h. Verdünnungen und Schnitten), deren letzer q als Konklusion besitzt, und in der jede Prämisse entweder zu den p gehört oder tautologisch ist, oder mit einer vorangehenden Konklusion übereinstimmt[7] (Gentzen, 1932, p. 331).[8]

Immediately after the definition, Gentzen remarks that he considers tautologous sentences as proved even though this constitutes an exception to the given definition. Thus, *Identity* must be added as a further (maybe improper) inference rule, since tautologous sentence of the form $u \rightarrow u$ are allowed to occur in proofs as top sentences.[9] What is surprising, especially if compared with Gentzen's later work, is that in this paper he rates it necessary to justify the forms of inference of the system by providing a semantics for them, so that it becomes possible to show that, "informally interpreted, they are correct and sufficient". Building on Hertz's approach, Gentzen supplies an exact definition for the vague (*unklar*) notion of *consequence* (*Folgerung*) by stating that if $\mathbf{K}$ is the complex of all (finitely many) elements occurring in the sentences $p_1, ..., p_n, q$, then to assert that q *is a consequence of* $p_1, ..., p_n$ means that for every subset $\mathbf{K_f}$ of $\mathbf{K}$, if $\mathbf{K_f}$ *satisfies* the sentences $p_1, ..., p_n$, then $\mathbf{K_f}$ satifies also q.[10] In turn, to say that a complex *satisfies* a sentence p means that it either does not contain all antecedent elements of p, or alternatively, it contain all of them and also the succedent of p.[11] On this basis, the previously

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[6] Here, by writing $\mathbf{M}^{\{u\}}$ we mean of course that the element u has been cut from the complex $\mathbf{M}$.

[7] [By a *proof* of a sentence q from the sentences $p_1, \ldots, p_n$ ($n \geq 0$) we shall henceforth mean an ordered succession of inferences (i.e., thinnings and cuts) arranged in such a way that the conclusion of the last inference is q, and such that every premiss either belongs to the p's or is a tautology, or coincides with the conclusion of an earlier inference.]

[8] As usual, in this case we will write: $p_1, \ldots, p_n \vdash q$.

[9] Improper since a formal proof must always contain at least one proper rule application, so that a formal proof of $u \rightarrow u$ would be

$$\frac{u \rightarrow u}{u \rightarrow u}\ .$$

[10] As usual, the consequence relation will be denoted by writing: $p_1, \ldots, p_n \models q$. As stressed in Schroeder-Heister (2002), Gentzen makes it explicitly clear that, when considering the question of whether $p_1, ..., p_n \models q$, the underlying domain of elements comprises just the elements occurring in $p_1, ..., p_n, q$. In the terminology of logic programming, this corresponds to considering the (propositional) *Herbrand universe* as a basis.

[11] Thus, a sentence $u_1, \ldots, u_n \rightarrow w$ gets its natural reading as the formula $(u_1 \wedge \ldots \wedge u_n) \supset w$.

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reminded exception to the definition of proof gets immediately its justification: in fact, any complex satisfies a tautologous sentence. Note that only tautologies can occur as top sentences in a proof, and not also trivial sentences even though they share of course the same semantic justification. The reason is that this way Thinning would be a built-in feature of the system, making the presence of the rule superfluous. From the premise $\mathbf{L} \rightarrow \upsilon$ of an application of Thinning, we could in fact obtain its conclusion $\mathbf{ML} \rightarrow \upsilon$ by an application of Cut in which a(n appropriate) trivial sentence occurs as Upper sentence:

$$\frac{\mathbf{L} \rightarrow \upsilon \qquad \mathbf{M}\upsilon \rightarrow \upsilon}{\mathbf{LM} \rightarrow \upsilon}\ \text{Cut}$$

It seems therefore likely that the reason for not allowing the cut element to occur in the context of the Upper premise and for permitting that only tautologous sentences may occur in proofs as top sentences is to provide a thorough analysis of the notion of consequence singling out its most elementary components.

Finally, Gentzen proceeds to state and prove the soundness and completeness of the chosen forms of inference. Before considering the proof of the completeness theorem, it is convenient to retrace Gentzen's steps giving special attention to the terms he employs. Gentzen speaks of "informal interpretation" (*inhaltlich Deutung*) when he refers to sentences, inference schemata and proofs, whereas he denotes as "vague" (*unklar*) the notion of consequence (*Folgerung*). So, in order to be able to show the equivalence between the formally specified notion "being provable from" and the informal notion "being a consequence of", he needs to provide an exact definition of the latter notion. This is what he accomplishes following Hertz's road, and interpreting as propositions (*Aussagen*) the elements a sentence is made of. On this basis, the questions of the informal soundness and completeness can be *formally* expressed as follows: $p_1, \ldots, p_n \vdash q$ iff $p_1, \ldots, p_n \models q$. At the root, however, the two questions continue to be "informal" since depending on a particular interpretation of the elements constituting the involved sentences.

§4. The (semantic) completeness theorem. After having proved, in *Satz* 1, the soundness of his system of rules with respect to the adopted semantics, Gentzen deals with the more challenging task of its completeness. What makes the proof of the completeness theorem remarkable is the fact that it is provided by proving, in *Satz* 2, a stronger result which (together with the soundness theorem) yields a normal form theorem at the same time.

First, a definition:

DEFINITION 4.1. *A proof of a non-trivial sentence q from the sentences* $p_1, \ldots, p_n$, $1 \leq n$, *is a* normal *proof (in this case we will write* $p_1, \ldots, p_n \vdash_n q$*) if it consists of a series of applications of Cut followed by a single, terminal application of Thinning.*

Concretely, a proof in normal form can be written in the following form:

$$\cfrac{\cfrac{l_1 \qquad \cfrac{l_0 \qquad u_0}{u_1}\ \text{Cut}}{\vdots}\ \text{Cut}}{\cfrac{l_{r-1} \qquad u_{r-1}}{\cfrac{u_r}{q}\ \text{Thin}}\ \text{Cut}}$$

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where $0 \leq r$, and if $r = 0$ the entire figure becomes

$$\frac{u_0}{q}\ \text{Thin}$$

In the previous proof figure each step

$$\frac{l_i \qquad u_i}{u_{i+1}}$$

always denotes a Cut with l_i for its Lower premise, u_i for its Upper premise, and u_{i+1} for its conclusion. The Lower premise of a Cut must always be a sentence belonging to $\{p_1, \ldots, p_n\}$. It is immediate to see that in such a normal proof no elements other than those of $p_1, \ldots, p_n$ and q occur. Let us note that in a proof in normal form no application of Thinning occurs before an application of Cut. Thus, in a normal proof no element is introduced (by an application of Thinning) just to be eliminated afterwards (by an application of Cut). In this sense, we can say that a normal proof *does not contain any detour*.

Note that the range of Definition 1 is not at all narrowed by the reference to "non-trivial sentences". In fact, any trivial sentence $\mathbf{K} \rightarrow u, u \in \mathbf{K}$, gets the following obvious one-step *normal* proof

$$\frac{u \rightarrow u}{\mathbf{K} \rightarrow u}\ \text{Thin}$$

The strengthened form of completeness proved by Gentzen is embodied in the following

THEOREM 4.2. *If a non-trivial sentence q is a "consequence" of the sentences $p_1, ..., p_n$, then there exists a "normal proof" for q from $p_1, ..., p_n$. We will write, more concisely, that if $p_1, ..., p_n \models q$ then $p_1, ..., p_n \vdash_n q$.*

Proof. Suppose that the sentence q is of the form

$$\mathbf{L} \rightarrow \upsilon$$

where $\upsilon \notin \mathbf{L}$ (otherwise, it would be a trivial sentence).

Gentzen considers the set of sentences p^* such that:

i) the succedent of p^* is υ

ii) p^* is not trivial, i.e. υ does not occur in the antecedent of p^*

iii) there exists a normal proof of p^* from $p_1, ..., p_n$ which does not exploit the rule of Thinning.

Let's call this set $\mathcal{S}$.[12] Note that q, i.e. the sentence $\mathbf{L} \rightarrow \upsilon$, satisfies both conditions i) and ii) of the definition of $\mathcal{S}$, and what the theorem aims to state is that it satisfies also condition iii), i.e. that $q \in \mathcal{S}$. The proof proceeds by an indirect argument. It is assumed that $q \notin \mathcal{S}$ (which, by previous remarks, amounts to assume that there is no normal proof without Thinning of q from $p_1, ..., p_n$). On this basis, and in contradiction with the main assumption of the theorem, it is then proved that q is not a consequence of $p_1, ..., p_n$.

To this aim, it is necessary to produce a set which is a counter-example to the claim that q is a consequence of $p_1, ..., p_n$; that is, a set which satisfies each $p_i \in \{p_1, ..., p_n\}$, but

[12] About the set $\mathcal{S}$ Gentzen notes only that it is finite, and that any sentence of the set $\{p_1, ..., p_n\}$ which satisfies conditions i) and ii) belongs to $\mathcal{S}$. We will come back to this question later.

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does not satisfy q. The set we are searching for will be defined by induction, through a sequence of steps, such that each of them adjoins an element to the previous set.[13]

1. *Basis step*. Let $\mathbf{N}_0$ be $\mathbf{L}$. Note that clearly $\mathbf{L} \subseteq \mathbf{N}_0$ and $v \notin \mathbf{N}_0$.
2. *Inductive step*. Assume $\mathbf{N}_n$ has been defined. If there exists a $p^* \in \{p_1, ..., p_n\}$ s.t. $\mathbf{N}_n$ does not satisfy p^* and u is the succedent of p^* (meaning that $u \notin \mathbf{N}_n$, whereas the entire antecedent of p^* is contained in $\mathbf{N}_n$), then $\mathbf{N}_{n+1}$ will be equal to $\mathbf{N}_n u$. $\mathbf{N}_{n+1}$, of course, does satisfy p^*.
   Otherwise, $\mathbf{N}_{n+1}$ will be equal to $\mathbf{N}_n$, and the procedure terminates.

Since there are finitely many sentences $p_1, ..., p_n$, the construction terminates after a finite number of steps producing a sequence of sets, each of them properly included in the following one, and eventually reaching a certain set, say $\mathcal{N}_k$, which, it is to be shown, is the set we were searching for. In other words, we have to prove that the set $\mathcal{N}_k$ satisfies any sentence of the set $\{p_1, ..., p_n\}$, but not q. Being q of the form $\mathbf{L} \to v$, the last assertion means that $\mathbf{L} \subseteq \mathcal{N}_k$, whereas $v \notin \mathcal{N}_k$.

Since $\mathbf{L} = \mathbf{N}_0 \subseteq \mathcal{N}_k$, we already know that $\mathbf{L}$ is contained in $\mathcal{N}_k$, and all what we have to show is that $v \notin \mathcal{N}_k$. Once this is done, we will have proved that $\mathcal{N}_k$ does not satisfy q, in contradiction with the assumption that $p_1, ..., p_n \models q$. To this end, the following lemma is proved:

LEMMA 4.3. *For each* $\mathbf{N}_i, i \leq k$, *it holds that* $\mathbf{N}_i$ *satisfies all sentences in* $\mathcal{S}$, *and* $v \notin \mathbf{N}_i$.[14]

Proof. The proof is by induction on the construction of $\mathcal{N}_k$.

- *Base*. Since $\mathbf{N}_0 = \mathbf{L}$, $v \notin \mathbf{N}_0$, otherwise q, i.e. $\mathbf{L} \to v$, would be, against the hypothesis, a trivial sentence. Moreover, $\mathbf{L}$ satisfies any p^* belonging to $\mathcal{S}$. To prove this fact, let us assume that there is a $p^* \in \mathcal{S}$, of the form $\mathbf{Q} \to v$, such that

$$\mathbf{L} \not\models (\mathbf{Q} \to v)$$

  which means that

$$\mathbf{Q} \subseteq \mathbf{L}, \text{ and } v \notin \mathbf{L}.$$

  Since the latter fact is no doubt true, and since p^* is by definition provable from $p_1, ..., p_n$ by a normal proof $\mathcal{D}$ without Thinning, should $\mathbf{Q}$ be a subset of $\mathbf{L}$ we could provide, against the main assumption of the theorem, the following normal proof of q from $p_1, ..., p_n$:

$$\begin{array}{c} \mathcal{D} \\ \vdots \\ \dfrac{\mathbf{Q} \to v}{\mathbf{L} \to v}\ \text{Thin} \end{array}$$

- *Induction step*. Let us assume now as induction hypothesis that $\mathbf{N}_i$ satisfies any $p^* \in \mathcal{S}$, and that $v \notin \mathbf{N}_i$, and let us show that both properties are preserved moving to $\mathbf{N}_{i+1}$. If $\mathbf{N}_i = \mathbf{N}_{i+1}$, we are done. If $\mathbf{N}_i \neq \mathbf{N}_{i+1}$, then, by the construction process, it must be $\mathbf{N}_{i+1} = u\mathbf{N}_i$, for a certain sentence $p^* \in \{p_1, ..., p_n\}$ such that

[13] As stressed in Schroeder-Heister (2002), the construction used by Gentzen is the fixed point construction which is now standard in the theory of logic programming.

[14] Note that, being $\mathbf{L} \subseteq \mathbf{N}_i, i \leq k$, the latter assertion is what is needed to complete the proof. Proving that $\mathbf{N}_i, i \leq k$, satisfies all sentences in $\mathcal{S}$ is subsidiary to the main line of argument.

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$\mathbf{N}_i \not\models p^*$ and u is the succedent of p^*. So, p^* is of the form $(\mathbf{O} \to u)$, for a certain $\mathbf{O}$, and p^* is such that

1. $\mathbf{O} \subseteq \mathbf{N}_i$
2. $u \notin \mathbf{N}_i$
   i.e. $\mathbf{N}_i \not\models (\mathbf{O} \to u)$. Moreover,
3. $v \notin \mathbf{N}_i$
   by induction hypothesis on $\mathbf{N}_i$;
4. $v \notin \mathbf{O}$
   because of 1. and 3. Moreover,
5. $u \neq v$
   otherwise, in fact, $(\mathbf{O} \to u)$ would be a sentence which i) has v as succedent, ii) is not trivial (because of 4.), and iii) would be normally provable from $p_1, ..., p_n$ without Thinning. That is to say, $(\mathbf{O} \to u)$ would be, contrary to the inductive hypothesis on $\mathbf{N}_i$, a sentence of $\mathcal{S}$ which is not satisfied by $\mathbf{N}_i$.
6. $v \notin \mathbf{N}_{i+1}$
   because of 3. and 5., since $\mathbf{N}_{i+1} = u\mathbf{N}_i$.
   Fact 6. is the first half of what we had to show. Now we must prove that

$$\mathbf{N}_{i+1} \text{ satisfies any } p^* \in \mathcal{S}.$$

   Also this step is accomplished through an indirect argument: so let us assume that there exists a $p^* \in \mathcal{S}$ not satisfied by $\mathbf{N}_{i+1}$. Note, however, that by the inductive hypothesis on $\mathbf{N}_i$, such p^* is satisfied by $\mathbf{N}_i$. This sentence p^* will be necessarily of the following form:

$$\mathbf{P}u \to v \text{ (possibly being } \mathbf{P} = \emptyset)$$

   in fact, since $p^* \in \mathcal{S}$, its succedent must be v, and v cannot occur in the antecedent (otherwise it would be trivial). The latter fact is assured by 5. $(u \neq v)$, 6. $(v \notin \mathbf{N}_{i+1})$ and
7. $\mathbf{P}u \subseteq \mathbf{N}_{i+1}$
   Fact 7., in turn, depends on the assumption that p^* is not satisfied by $\mathbf{N}_{i+1}$; which means, by definition, that $\mathbf{N}_{i+1}$ contains the antecedent but not the succedent of p^*. Moreover, $\mathbf{P} \subseteq \mathbf{N}_i$, from which, by 2., it follows:
8. $u \notin \mathbf{P}$.
   Let us now consider the following application of Cut:

$$\frac{\mathbf{O} \to u \qquad \mathbf{P}u \to v}{\mathbf{OP} \to v}\ \text{Cut}$$

   where, we remind, the Lower premise belongs to $\{p_1, ..., p_n\}$, and the Upper premise to $\mathcal{S}$. As regards the conclusion $\mathbf{OP} \to v$ we can note that: i) it has v as succedent; ii) it is not trivial, since $v \notin \mathbf{P}$, by 6. an 7., and $v \notin \mathbf{O}$, by 4; and iii) it has been obtained by a Cut and without Thinning from two sentences belonging, respectively, to $\{p_1, ..., p_n\}$ and to $\mathcal{S}$. Therefore:
9. $(\mathbf{OP} \to v) \in \mathcal{S}$.
   But
10. $(\mathbf{OP} \to v)$ is not satisfied by $\mathbf{N}_i$.
    In fact, $\mathbf{O} \subseteq \mathbf{N}_i$ (by 1.), $\mathbf{P} \subseteq \mathbf{N}_i$ (since $\mathbf{P}u \subseteq \mathbf{N}_{i+1}$, and u is the new element taken into account in the step from $\mathbf{N}_i$ to $\mathbf{N}_{i+1}$), but $v \notin \mathbf{N}_i$ (by 3.).

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Thus, fact 10. is in contradiction with the inductive hypothesis on $\mathbf{N}_i$, i.e. with the assumption that $\mathbf{N}_i$ satisfies any $p^* \in \mathcal{S}$. The contradiction obtained allows us to state that also $\mathbf{N}_{i+1}$ *satisfies any* $p^* \in \mathcal{S}$*, and does not contain* υ. The proof of the Lemma is now complete, and we can state that the same holds also for $\mathcal{N}_k$. □

Therefore, $\mathcal{N}_k$ satisfies any sentence in $\{p_1, ..., p_n\}$ (by construction), but does not satisfy q (since $\mathbf{L} \subseteq \mathcal{N}_k$ and $\upsilon \notin \mathcal{N}_k$). And the proof of the theorem, *Gentzen's original proof*, is now concluded. □

4.1. More details. We recall the definitional clauses of the set $\mathcal{S}$: it is the set of sentences p^* such that:

i) the succedent of p^* is υ;
ii) p^* is not trivial, i.e. υ does not occur in the antecedent of p^*;
iii) there exists a normal proof of p^* from $p_1, ..., p_n$ which does not exploit the rule of Thinning.

Let us first prove that *the set* $\mathcal{S}$ *is finite*:

LEMMA 4.4. $\mathcal{S}$ *is finite.*

Proof. This fact follows immediately from condition iii) in the definition of $\mathcal{S}$: in fact, since any Cut makes an element disappear without giving rise to new elements, by using only Cuts only a finite number of sentences can be obtained from the finite set of sentences $\{p_1, ..., p_n\}$. Note that to this aim the exclusion of Thinning from the considered inferences is essential: Thinning would possibly make $\mathcal{S}$ an infinite set. □

Let us now prove that *the set* $\mathcal{S}$ *is non void*:

LEMMA 4.5. $\mathcal{S} \neq \emptyset$

Proof. First, note that, simply by using tautologous sentences, we can see that any p_i, $i \leq n$, satisfies condition iii). For instance, let p_i be of the form $\mathbf{E} \to t$, then the following is a normal proof of p_i from $p_1, ..., p_n$ which does not exploit the rule of Thinning:

$$\frac{\mathbf{E} \to t \qquad t \to t}{\mathbf{E} \to t}\ \text{Cut}$$

Moreover, some of the $p_1, ..., p_n$ must satisfy condition i). Otherwise, in fact, we would get a contradiction with the assumption that $p_1, ..., p_n \models q$; i.e. that if a finite subset of the complex $\mathbf{K}$ of all (finitely many) elements of $p_1, ..., p_n, q$ satisfies the sentences $p_1, ..., p_n$, then it satisfies also $\mathbf{L} \to \upsilon$. In fact, let $\mathbf{K}^*$ be the subset of $\mathbf{K}$ which contains all the succedents of the premises and the antecedent of q. By the assumptions made, $\upsilon \notin \mathbf{K}^*$, so that $\mathbf{K}^* \not\models q$. On the other hand, for any p_i, $i \leq n$, with antecedent, say, $\mathbf{P}$, it must of course hold that either $\mathbf{K}^*$ is *properly* contained in $\mathbf{P}$ or $\mathbf{P} \subseteq \mathbf{K}^*$. In both cases, $\mathbf{K}^* \models p_i$, contradicting our assumption that $p_1, ..., p_n \models q$.

As regards condition ii), we stress that at least the premises which have υ as succedent cannot be trivial. Otherwise, in fact, also in this case it would be possible to build a counterexample, say $\mathbf{K}^*$, to $p_1, \ldots, p_n \models q$. To give an idea of the procedure that we can adopt, let us assume that $n = 1$, so that just one premise occurs. Let us suppose, besides, that this premise is trivial and has υ as succedent. So, it will be something like $\mathbf{M}\upsilon \to \upsilon$. Then, let the complex $\mathbf{K}^*$ be the set $\{\mathbf{M}, \mathbf{L}\}$. Of course, the premise $\mathbf{M}\upsilon \to \upsilon$

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is satisfied by $\{\mathbf{M}, \mathbf{L}\}$, since $\{\mathbf{M}, \mathbf{L}\}$ does not contain all antecedent elements of $\mathbf{M}v \rightarrow v$. On the other hand, $\{\mathbf{M}, \mathbf{L}\}$ does not satisfy $\mathbf{L} \rightarrow v$ since it contains $\mathbf{L}$ but not v. Thus, we have obtained that, contrary to our assumption, $\mathbf{M}v \rightarrow v \not\models q$. In the general case, we can shape $\mathbf{K}^*$ in an analogous way with respect to other possibly occurring premises which are trivial and have v as succedent, and we adjust in the obvious way the complex $\mathbf{K}^*$ in order to satisfy the remaining premises. So, we would get a counterexample to the assumption that $p_1, \ldots, p_n \models q$.[15] This concludes the proof of the lemma. □

§5. Variations on Gentzen's proof procedures. It could be interesting to note that it is possible to devise a (slightly) different proof procedure, which does not go through the set $\mathcal{S}$. Also in this case we'll argue by contraposition: we assume that there is no normal proof of q from $p_1, ..., p_n$, then concluding that consequently q is not a consequence of $p_1, ..., p_n$. The countermodel to $p_1, ..., p_n \models q$ which we exploit is the same set $\mathcal{N}_k$ used by Gentzen,[16] but we show, in a sense *directly*, that in the construction of $\mathcal{N}_k$ the succedent of q cannot be added to the countermodel without thus making possible the existence of a normal proof of q from $p_1, ..., p_n$.

Firstly, we stress that, by construction, $\mathcal{N}_k$ satisfies each of the sentences $p_1, ..., p_n$. Therefore, what remains to prove is that $\mathcal{N}_k$ does not satisfy q; i. e. that

$$\mathbf{L} \subset \mathcal{N}_k \text{ and } v \notin \mathcal{N}_k.$$

The first half of the claim is guaranteed by the way the set $\mathcal{N}_k$ has been defined. As regards the second half of the claim, we remind that it is obviously satisfied by the *basis* of the construction, since we assumed that $v \notin \mathbf{L}$, which in turn entails that $\mathbf{L} \not\models \mathbf{L} \rightarrow v$. Therefore, in order to prove the theorem we need to show that the second half of the claim (i.e., that $v \notin \mathcal{N}_k$) is preserved along the *steps* of the construction of the set $\mathcal{N}_k$.

Speaking from a general point of view, we observe that

- To drive home the point it is necessary that $\mathbf{L}$ satisfies any sentence p_i, $i \leq n$, *which is non-trivial and has v as succedent*. Otherwise, in fact, the construction of $\mathcal{N}_k$ would reach a step where v is added to the previous set, so that, contrary to what we need, we would get that $v \in \mathcal{N}_k$.
- Then, if $\mathbf{L}$ must satisfy any such p_i, it is necessary that the antecedent of such p_i is not completely contained in $\mathbf{L}$ (since $\mathbf{L}$ does not contain its succedent).
- On the other hand, any trivial p_i having v as succedent is no doubt satisfied by $\mathbf{L}$, since $\mathbf{L}$ does not contain the entire antecedent of p_i.

That granted, let us now proceed to prove that

$$v \notin \mathcal{N}_k$$

Also in this case we will argue by an indirect argument. So, we'll try to obtain a contradiction by the assumption that $v \in \mathcal{N}_k$. In turn, this means of course that in the construction of $\mathcal{N}_k$ there is a certain step $j < k$ where we met a sentence, say $(\mathbf{Q} \rightarrow v)$, which is

[15] Note that this argument could not work with respect to a trivial premise of the form $\mathbf{M}s \rightarrow s$, $s \neq v$, since in this case nothing could rule out the possibility that $v \in \mathbf{M}$ with the consequence that the complex $\{\mathbf{M}, \mathbf{L}\}$ would satisfy also $\mathbf{L} \rightarrow v$.

[16] We preserve throughtout notations used in Section 4.

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non-trivial, has υ as succedent, and it is not satisfied by $\mathbf{N}_j$.[17] Since $\mathbf{N}_j \not\models (\mathbf{Q} \rightarrow \upsilon)$, it holds that $\mathbf{Q} \subset \mathbf{N}_j$ and that $\upsilon \notin \mathbf{N}_j$. By construction, $\mathbf{N}_{j+1}$ is then defined as $\mathbf{N}_j\upsilon$.

Therefore, our next goal is to prove the following lemma:

LEMMA 5.1. *There is no $j \leq k$ such that $\upsilon \in \mathbf{N}_j$.*

Proof. In this case we will argue by induction on the construction of $\mathcal{N}_k$.

Base. The set $\mathbf{N}_j$ we are searching for cannot be $\mathbf{L}$; i.e., j cannot be 0. In fact, this would mean that $\mathbf{L} \not\models (\mathbf{Q} \rightarrow \upsilon)$: in this case, in fact, the construction rule would prescribe to add υ, obtaining $\mathbf{N}_{0+1} = \mathbf{L}\upsilon$. Of course, it must be $\mathbf{L} \subset \mathbf{Q}$ or $\mathbf{Q} \subseteq \mathbf{L}$.

- In the first case, i. e. if $\mathbf{L}$ is properly contained in $\mathbf{Q}$, then, it would hold, contrary to our assumption, that $\mathbf{L} \models (\mathbf{Q} \rightarrow \upsilon)$.
- If $\mathbf{Q} \subseteq \mathbf{L}$ then the following

$$\frac{\mathbf{Q} \rightarrow \upsilon}{\mathbf{L} \rightarrow \upsilon}\ \text{Thin}$$

would be, contrary to the main assumption of the theorem, a normal proof of $\mathbf{L} \rightarrow \upsilon$ from $p_1, ..., p_n$.

Induction step. Let us assume now that there exists an i, $0 < i < k$, s.t. $\mathbf{N}_i \not\models \mathbf{Q} \rightarrow \upsilon$. Again, this means that $\mathbf{Q} \subseteq \mathbf{N}_i$, and $\upsilon \notin \mathbf{N}_i$. Thus, by the production rule we have to expand $\mathbf{N}_i$ obtaining $\mathbf{N}_{i+1} = \mathbf{N}_i\upsilon$, and $\mathbf{N}_{i+1} \models \mathbf{Q} \rightarrow \upsilon$. Of course one of the following alternatives must hold:

1. $\mathbf{Q}$ is properly included in $\mathbf{L}$
2. $\mathbf{Q} = \mathbf{L}$
3. $\mathbf{L}$ is properly included in $\mathbf{Q}$.

In the first two cases it is easily seen that we would get

$$\frac{\mathbf{Q} \rightarrow \upsilon}{\mathbf{L} \rightarrow \upsilon}\ \text{Thin}$$

so that, contrary to the main assumption of the theorem, a normal proof of $\mathbf{L} \rightarrow \upsilon$ from $p_1, ..., p_n$ could be constructed.[18]

Let us consider the last case: $\mathbf{L}$ *is properly included in* $\mathbf{Q}$. Since we tested $\mathbf{Q} \rightarrow \upsilon$ at the step i (> 0), we know that $\mathbf{L} \models \mathbf{Q} \rightarrow \upsilon$, and since, by assumption, $\upsilon \notin \mathbf{L}$, this fact necessarily depends on being $\mathbf{L}$ *properly* contained in $\mathbf{Q}$. On the other hand, since $\mathbf{N}_i$ contains $\mathbf{Q}$, we can assert that the production process has adjoined step by step to $\mathbf{L}$ the elements belonging to $\mathbf{Q}$ and not to $\mathbf{L}$. In other words, in the construction process we must have met finitely many sentences from the set $\{p_1, ..., p_n\}$ which are not satisfied by $\mathbf{L}$. Without any loss of generality, let us assume that these sentences are

$$s, n \rightarrow t$$
$$r \rightarrow m$$
$$l, k \rightarrow o,$$

where, of course, it must hold that $t \neq \upsilon, m \neq \upsilon, o \neq \upsilon$. Spelling out what this means, we can state that

[17] Note that this fact can happen just once in the production process leading to $\mathcal{N}_k$.

[18] In the second case, properly, the rule applied would be rather *Identity*.

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1. $\mathbf{L} \not\models s, n \to t, \mathbf{L} \not\models r \to m, \mathbf{L} \not\models l, k \to o$,
2. $s \in \mathbf{L}, n \in \mathbf{L}, r \in \mathbf{L}, l \in \mathbf{L}, k \in \mathbf{L}$
3. $t \notin \mathbf{L}, m \notin \mathbf{L}, o \notin \mathbf{L}$,
4. $\mathbf{N}_1 = \mathbf{L}t, \mathbf{N}_2 = \mathbf{L}tm, \mathbf{N}_3 = \mathbf{L}tmo = \mathbf{Q}$, and lastly
5. $\mathbf{N}_3 \not\models \mathbf{Q} \to \upsilon$.

It is now possible to construct the following proof

$$\dfrac{l, k \to o \qquad \dfrac{r \to m \qquad \dfrac{s, n \to t \qquad \mathbf{Q} \to \upsilon}{\mathbf{Q}^{\{\mathbf{t}\}} sn \to \upsilon}\,\text{Cut}}{\mathbf{Q}^{\{\mathbf{tm}\}} rsn \to \upsilon}\,\text{Cut}}{\mathbf{Q}^{\{\mathbf{tmo}\}} lkrsn \to \upsilon}\,\text{Cut}$$

where, we remind, by writing $\mathbf{Q}^{\{*\}}$ we mean that the element $*$ has been removed from $\mathbf{Q}$. Now, the previous facts 2. and 3. warrant that writing $\mathbf{Q}^{\{\mathbf{tmo}\}} lkrsn$ is the same as writing $\mathbf{L}lkrsn$, which in turn is the same as writing $\mathbf{L}$. Thus, in conclusion, we have got that, contrary to our main assumption, there is a normal proof for $\mathbf{L} \to \upsilon$ from $p_1, ..., p_n$.

The contradiction we reached allows us to state that there is no $j \leq k$ such that $\upsilon \in \mathbf{N}_j$, and the proof of the lemma is concluded. □

From the lemma follows that

$$\mathbf{L} \subset \mathcal{N}_k \text{ and } \upsilon \notin \mathcal{N}_k$$

so that $\mathcal{N}_k$ meets the required conditions: it satisfies each of the sentences $p_1, ..., p_n$, and does not satisfy q.

The proof of the theorem is now complete: as requested by the indirect argument, in fact, starting from the assumption that $p_1, ..., p_n \not\vdash_n q$, we arrived to prove that $p_1, ..., p_n \not\models q$.

As an easy consequence of the soundness and completeness theorems previously proved, Gentzen obtains the following *normal form* result:

> Satz 3. Ist ein nicht trivialer Satz q aus den Sätzen $p_1, \ldots, p_n$ beweisbar, so gibt es einen Normalbeweis für q aus $p_1, \ldots, p_n$.[19]

Immediately afterwards, Gentzen adds

> Man kann diesen Satz auch ohne den Umweg über den Folgerungsbegriff direkt gewinnen, indem man einen beliebigen Beweis schrittweise in einen Normalbeweis umformt. Wir haben den Umweg gewählt, weil er nicht wesentlich langwieriger ist und wichtige zusätzliche Ergebnisse, nämlich die Richtigkeit und Vollständigkeit unserer Schlußweisen lieferte.[20]

[19] [Theorem III. If a nontrivial sentence q is provable from the sentences $p_1, \ldots, p_n$, then there exists a normal proof for q from $p_1, \ldots, p_n$.]

[20] [This theorem can also be obtained directly without the detour through the notion of consequence by transforming any given proof step by step into a normal proof. We have chosen the detour since it is not essentially more laborious and yet provides us with important additional results, viz. the correctness and completeness of our forms of inference.]

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Two remarkable points are stressed here by Gentzen, and we will deal with them in turn. In the first half of the quotation, when speaking of a reduction process to be performed on a given derivation, in order to transform it in a derivation in normal form, Gentzen of course hints at the method of shifting down Thinning by permuting application of Cut with Thinning. We can make this suggestion explicit by proving the normal form theorem in a purely proof-theoretic way. First, a definition:

DEFINITION 5.2. *Let $\mathcal{D}$ be a derivation and let ρ be an application of Thinning in $\mathcal{D}$: we call* Order *of ρ, say $O(\rho)$, the number of applications of Cut which occur under ρ in $\mathcal{D}$. The order of the derivation, say $O(\mathcal{D})$, is the sum of the orders of all applications of Thinning in $\mathcal{D}$.*

THEOREM 5.3. *If a non trivial sentence q is provable from the sentences $p_1, \ldots, p_n$, then there exists a normal proof of q from $p_1, \ldots, p_n$.*

Proof. The proof is by induction on $O(\mathcal{D})$. We consider two cases.

- $O(\mathcal{D}) = 0$. Then the derivation is already in normal form.
- $O(\mathcal{D}) > 0$. Thus, there is at least an application of Thinning, say ρ, which occurs above an application of Cut. We distinguish two subcases.

  [a] The element introduced by ρ is then eliminated by an application of Cut. The relevant subtree of the derivation $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\vdots}{t \to r} \qquad \dfrac{\dfrac{\vdots}{u \to s}}{r, u \to s}\ \text{Thin}}{t, u \to s}\ \text{Cut}$$

  We get a new derivation $\mathcal{D}^*$ by removing both the application of Thinning and Cut, and by inserting a new application of Thinning, say σ, in order to recover the previous sentence:

$$\dfrac{\dfrac{\vdots}{u \to s}}{t, u \to s}\ \text{Thin}$$

  Of course, $O(\sigma) = O(\rho) - 1$, and the same holds for the order of the two derivations; so, by induction hypothesis, the theorem follows.

  [b] The element introduced by the application ρ of Thinning is not eliminated by a subsequent application of Cut, so that $\mathcal{D}$ looks like:

$$\dfrac{\dfrac{\vdots}{t \to u} \qquad \dfrac{\dfrac{\vdots}{u \to s}}{r, u \to s}\ \text{Thin}}{t, r \to s}\ \text{Cut}$$

  We permute applications of Thinning and Cut by obtaining a new derivation $\mathcal{D}^*$ of the following form:

$$\dfrac{\dfrac{\vdots}{t \to u} \qquad \dfrac{\ldots}{u \to s}}{\dfrac{t \to s}{t, r \to s}\ \text{Thin}}\ \text{Cut}$$

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Calling again σ the new application of Thinning in $\mathcal{D}^*$, we see that $O(\sigma) = O(\rho) - 1$, and the same holds for the order of the two derivations; so, by induction hypothesis, also in this case the theorem follows. □

§6. Some remarks. In the second half of the previous quotation Gentzen explains why he chose the *Umweg über den Folgerungsbegriff*. In order to properly understand this passage we remind that a different route was adopted by Gentzen in his thesis. In Gentzen (1934–1935), in fact, the notion of *completeness* which shapes Gentzen's formulation of the **N**-calculi (and, through the proof of equivalence, also of the **L**-calculi) is deployed in terms of their ability to adequately (*completely*) capture the informally correct inference-patterns usually associated with each individual logical operator. Consequently, the completeness question was not something that could be answered by proving (what we are now accustomed to call) the semantic completeness of a given system of (axioms and) rules. Coping adequately with this question was a task that Gentzen entrusted to the choice of (the axioms and) the rules. And, actually, he was confident to have successfully met this task thanks to the formulation of the **N**-calculi.

As we have seen, the perspective offered in Gentzen (1932) is completely different. In this paper Gentzen presents a "formal" definition of derivability and proves its soundness and completeness with respect to a "formal characterization" of the informal notion of consequence. Such a procedural difference needs of course some explanation. To this end, it is first to be stressed again that the elements which constitute the sentences of Gentzen (1932) bear no logical relation to each other. This bare level of syntactic analysis is able to disclose just the most general features of the relation of consequence; in a way—Gentzen emphasizes—which does not depend on what the elements of a sentence are. So, the aspects by virtue of which sentences follow from one another cannot be the specific features of the elements of a sentence, and the same holds for the ensuing notion of consequence. On the other hand, this is reflected in the semantic analysis which Gentzen borrows from Hertz's work, which does not provide any distinction in kind between the complexes that might satisfy a sentence and the parts of that sentence. In other words, except for identities, no single logical law falls within this kind of analysis, so that no reference to a *realm* of logical truths is assumed in Gentzen (1932). And what Gentzen was confident of, and what he managed successfully to prove, was that an asymmetric version of the rules which were subsequently called the structural rules of the **L**-calculi is able to give a sound and complete description of the most general structural properties of the consequence relation. Because of the normal form theorem something more can be said: what Gentzen thought he had proved is that the Cut rule is the formal inference rule underlying the intuitive consequence relation.[21]

The matter radically changes when expressions from propositional and quantificational logic are taken into account. In this case, in fact, the analysis of their mutual logical relations depends essentially on an accurate inspection of the way their meaning is determined by their *form*; that is, by the way in which their constituent logical operators occur. This way a *realm* of logical truths (that is, of assertions which are true only in virtue of their logical *structure*), and of consequence relations among this kind of expressions is disclosed, and it was impossible to take due account of them by means of the simple semantic analysis provided by Gentzen in Gentzen (1932). As stressed by Curtis Franks in

[21] This feature of Gentzen's work has been duly emphasized by Curtis Franks in Franks (2010) and Franks (2013).

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Franks (2010), the lack of interest for the question of the semantic completeness which Gentzen will exhibit in the thesis stems from the indifference to the kind of semantic analysis which was required in order to provide an appropriate treatment of the (semantic) completeness question. Instead of this analysis, Gentzen, in accordance with the main tendencies of the Hilbert's School, chose an inferential analysis of the meaning of the logical operators. Accordingly, as we said, the question of the semantic completeness will be replaced by the question of warranting that the inferential rules of the system provided are actually able to capture the way mathematicians use assertions built up from those operators.[22]

6.1. Two paradigms at work. Previous remarks are of course pertinent and correct, but maybe something more (interesting) can be said, concerning a possible change of framework occurred in Gentzen's intellectual development. We think that it is worth emphasizing that the question which is at the source of soundness and semantic completeness underwent a significant change in passing from Gentzen (1932) to the thesis. In the former, we stress again, the point of view is given by a structural reasoning which refers to the notion of (logical) consequence, and is couched within the framework of Hertz's calculus for *Satzsysteme* (systems of sentences). The latter was devoted to analyse the independence of axiom systems for systems of sentences, and its interest lays in the interdependencies existing within systems of sentences rather than within sentences themselves. The objects of (structural) reasoning are sentences, intended as specifications of consequence relations, and to reason means to take into account one or several sentences with the aim of getting a new consequence relation, simply thanks to an examination carried out in terms of their structural features. As we learned from Jan von Plato's enquiries,[23] Gentzen tried to accommodate within the structural framework of Gentzen (1932) also the logical meaning. We mean the Logistic Dualistic Classical Calculus **LDK**, where a broadened and symmetric set of structural rules were joined by a set of *ground sequents* intended to cover the usual logical operators.[24] For instance, negation is ruled by the following couple of ground sequents

$$\neg\alpha, \alpha \rightarrow \qquad \text{and} \qquad \rightarrow \neg\alpha, \alpha$$

Together with a single application of Cut they allow us to get the usual rules for introducing negation in the antecedent and in the succedent:

$$\frac{\rightarrow \alpha, \neg\alpha \qquad \alpha, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg\alpha}\ \text{Cut} \qquad \text{and} \qquad \frac{\Gamma \rightarrow \Delta, \alpha \qquad \neg\alpha, \alpha \rightarrow}{\neg\alpha, \Gamma \rightarrow \Delta}\ \text{Cut}$$

The **LDK**-project was characterized by the attempt to recover the logical meaning in the form of basic, or initial, sequents, each of which deploys a relation of a particular complex formula to (some of) its subformula(e). Since these specifications are not derived but declared in the form of specifications, they can only occur as topmost points in a **LDK**-derivation. A characterizing feature of **LDK** is that the Cut rule is obviously not eliminable with respect to the other rules.

However, the project of **LDK** was abandoned by Gentzen. Two reasons, basically, seem to be responsible for this shift. The first one was an inner reason: the ground sequent method encountered significant difficulties in the case of implication (in the intuitionistic

[22] For more details on this subject see also Moriconi (2014).

[23] See von Plato (2009) and von Plato (2012).

[24] Very interesting remarks on this subject can be found in Arndt & Tesconi (2014).

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version) and with the quantifiers. In the latter case, for instance, it is impossible to formulate the *Eigenvariable* condition in the context of a ground sequent, since it is a condition not covered by the specification of the relations of a complex formula to its subformula(e), but concerns the way in which the quantified formula has been derived.

The second reason was an external one: foundational questions coming from the Hilbertian school —consistency and decidability above all— became predominant in Gentzen's investigations. To adequately deal with such topics, Gentzen considered it necessary to take on a radical change of paradigm. The notion of logical consequence of Gentzen (1932) was Hertzian, hence Aristotelian, in nature. The presence of the Cut rule makes that notion *synthetic*: just knowing that a sentence $u_1 u_2 \dots u_n \rightarrow v$ was obtained by an application of Cut does not provide any clue to determine what sentences were used as premises to infer that sentence. On the other hand, given a collection of sentences, it is possible to attempt various pairings of sentences from the given set as premises of a Cut inference in order to obtain new sentences. This is the paradigm of structural reasoning, which was preserved in the intermediate calculus **LDK**, where the Cut rule continues to play a fundamental role. In his thesis Gentzen opted for a different paradigm, which can be defined *analytic*, Leibnizian, so to speak, remembering that when Leibniz started to search for a logic of discovery (*logica inveniendi*) from within his mastery of Aristotelian logic, he maintained that a perfect proof for a given sentence should be built by exploiting only the analysis of the parts occurring in the sentence. Searching for analytic proofs was the new goal, and Gentzen was able to attain it thanks to the *Hauptsatz* proved for that "evolution" of **LDK**-calculi which is constituted by the **LK**-calculi. In the latter calculi, structural reasoning was sharply separated from logical meaning, and the general setting was purely inferential. The contribution of the Cut rule does no more consist in allowing to derive the inference rules for the logical operators, but it now just allows to compose consequence relations without adding further strength to the calculus. This way, the Cut rule becomes of course eliminable, and Gentzen was able to use the *Hauptsatz* to obtain meta-theoretical results which were important for the Hilbertian school, like the decidability of intuitionistic propositional calculus and the consistency of Arithmetic without induction.[25]

6.2. Two paradigms intertwined. Anyway, it is really interesting to note that a sort of trace of the path Gentzen went through survives both in the thesis and in his first published consistency proof. As for Gentzen (1934–1935), the obvious reference is to the paragraph 2.2 of Section III, where he says that if one were to place no importance to the *Hauptsatz*, as he himself did in Gentzen (1932) and in the intermediate **LDK**-calculus, then the **LK**-calculi could be simplified in several respects by replacing some of the logical inference figures by suitable *ground sequents*, which allow us to derive, together with the Cut rule, the suppressed rules.[26]

The same mechanism —ground sequents together with (a generalized form of the) Cut rule— is also working in Gentzen (1936). The framework here is quite different,

[25] Interesting and detailed remarks about the origin of sequent calculus in the work of Hertz and in Gentzen (1932) can be found in the relatively unknown paper Bernays (1965), whose starting point is the examination of the connections between general consequence relations (in the style of Tarski papers of 1930) and conditions on sequents.

[26] Actually, all the rules are replaceable except $\supset$-introduction in the succedent, $\forall$-introduction in the succedent, and $\exists$-introduction in the antecedent; i.e., exactly the rules that disturbed the **LDK**-project.

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since Elementary Arithmetic is formalized using a natural deduction calculus in *sequent formulation*. However, when approaching the proper definition of the concept of a "reduction step on a derivation", in the paragraph 14.1, Gentzen exploits a modified concept of derivation. He introduces ground sequents of both mathematical and logical kind. The latter ones are all sequents of the form $\alpha \to \alpha$ plus any sequent of the form $\alpha_1 \wedge \alpha_2 \to \alpha_i$ $(0 < i \leq 2)$, $\alpha, \beta \to \alpha \wedge \beta$, $\forall x \alpha(x) \to \alpha(t)$, $\neg\neg\alpha \to \alpha$, and $\alpha, \neg\alpha \to 1 = 2$.[27] Besides "Complete Induction", the inference rules still adopted are the rule for introducing the universal quantifier, and a new rule concerning negation:

$$\frac{\Gamma \to \phi(a)}{\Gamma \to \forall x \phi(x)}\,\forall I \qquad \text{and} \qquad \frac{\Gamma, \alpha \to 1 = 2}{\Gamma \to \neg\alpha}\,\neg I$$

A new structural rule is then added, the so-called *Chain rule* (*Kettenschluß*), which can be displayed in the following form:

$$\frac{\begin{array}{c}\Gamma_1 \to \alpha_1 \\ \vdots \\ \Gamma_k \to \alpha_k \\ \vdots \\ \Gamma_n \to \alpha_n\end{array}}{\Gamma_{i_1} \ldots \Gamma_{i_h} \to \alpha_{i_h}}\,K$$

where $1 \leq n$, $i_h \leq n$, and the sequence $\Gamma_{i_1}, \ldots, \Gamma_{i_h}$ contains the antecedent formulas occurring in the sequent of which α_{i_h} is the succedent and in the sequents that preceed that sequent according to the configuration. If the formula α_{i_h} is a false atomic formula, then any other false atomic formula may be taken in its place. An alternative formulation, probably preferred by Gentzen since it displays the inference in a *horizontal* arrangement, is the following:

$$\frac{\Gamma_1 \to \alpha_1 \quad \ldots \quad \Gamma_k \to \alpha_k \quad \ldots \quad \Gamma_n \to \alpha_n}{\Gamma_{i_1} \ldots \Gamma_{i_h} \to \alpha_{i_h}}\,K$$

As we said, the Chain rule is a sort of generalized Cut,[28] and it is a very flexible tool: together with the basic logical sequents, in fact, it allows one to recover the omitted rules of inference. As instances of this procedure we consider $\&I$, $\&E$, and $\forall E$:

$$\frac{\gamma \to \alpha \qquad \delta \to \beta \qquad \alpha, \beta \to \alpha\&\beta}{\gamma, \delta \to \alpha\&\beta}\,K \qquad \frac{\Gamma \to \alpha\&\beta \qquad \alpha\&\beta \to \beta}{\Gamma \to \beta}\,K$$

$$\frac{\Gamma \to \forall x \phi(x) \qquad \forall x \phi(x) \to \phi(t)}{\Gamma \to \phi(t)}\,K$$

It goes without saying that Gentzen's actual train of thought about such issues is only a matter of speculation. Trying to explain why Gentzen gave different answers to a seemingly analogous question, we arrived to single out the occurrence of at least two different paradigms in Gentzen's investigations. Previous remarks, in fact, seem to suggest that, for Gentzen, to get purely analytical proofs, obeying the subformula principle, was an

[27] We remind that symbols $\vee$, $\supset$, and $\exists$ are not primitive in Gentzen (1936), but introduced by definition.

[28] Strongly cognate to the Syllogism rule of P. Hertz. Moreover, it is easy to see that all usual *structural transformations* –Contraction, Cut, and Weakening– are special cases of the chain rule.

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expedient for meta-theoretical investigations, but not something *bonum per se*. He devised also a different paradigm, strictly intertwined with the former, which we called *structural reasoning*, and is characterized by the central role played by the Cut rule in accommodating logical meaning.

What this radical change of paradigm entailed for the question of soundness and semantic completeness which we are thinking about is that in the thesis its target is no more the *structural* reasoning of Gentzen (1932) but the *logical* meaning governed by the inferential rules of **LK**. As we have already suggested, however, Gentzen was not willing to face the amount of questions connected with the project of establishing a semantics for quantificational logic, neither he was interested in them. To say it in a nutshell, he was confident to have solved any adequacy question for such kind of logic by the invention of the calculi of *Natural Deduction*.[29]

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[29] For details on this point see for instance Moriconi (2014), and the references listed there.

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