Probability in the Engineering and Informational Sciences, **17**, 2003, 459–465. Printed in the U.S.A.

UPPER FIRST-EXIT TIMES OF COMPOUND POISSON PROCESSES REVISITED

W. STADJE
Fachbereich Mathematik/Informatik, University of Osnabrück, 49069 Osnabrück, Germany
E-mail: wolfgang@mathematik.uni-osnabrueck.de

S. ZACKS
Department of Mathematical Sciences, Binghamton University, Binghamton, New York 13902-6000
E-mail: shelly@math.binghamton.edu

For a compound Poisson process (CPP) with only positive jumps, an elegant formula connects the density of the hitting time for a lower straight line with that of the process itself at time t, $h(x;t)$, considered as a function of time and position jointly. We prove an analogous (albeit more complicated) result for the first time the CPP crosses an upper straight line. We also consider the conditional density of the CPP at time t, given that the upper line has not been reached before t. Finally, it is shown how to compute certain moment integrals of h.

1. INTRODUCTION

We consider a compound Poisson process (CPP) $Y = (Y(t))_{t\geq 0}$ given by $Y(t) = \sum_{n=0}^{N(t)} X_n$, where $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) positive random variables with common density f and distribution function (d.f.) F on $[0,\infty)$, $X_0 \equiv 0$, and $(N(t))_{t\geq 0}$ is a simple Poisson counting process, independent of the X_n, with intensity $\lambda > 0$. It is assumed that $\mu = EX_1 < \infty$ and $\rho = \lambda\mu < 1$. The first-exit times

$$T_L(\beta_1) = \inf\{t \geq 0 | Y(t) = t - \beta_1\}, \qquad \beta_1 > 0,$$
$$T_U(\beta_2) = \inf\{t \geq 0 | Y(t) \geq t + \beta_2\}, \qquad \beta_2 > 0,$$

© 2003 Cambridge University Press 0269-9648/03 $16.00

https://doi.org/10.1017/S0269964803174025 Published online by Cambridge University Press

are important in queuing theory, insurance mathematics, sequential analysis, and related fields. For example, $T_L(\beta_1)$ can be interpreted as the duration of the busy period of an $M/G/1$ queue with initial workload β_1. In insurance, the "ruin probability" $q(\beta_2) = P(T_U(\beta_2) < \infty)$ and the ruin time $T_U(\beta_2)$ itself are crucial in risk reserve analysis (see, e.g., [1]).

The (defective) density of $Y(t)$ on $(0,\infty)$ is

$$h(x;t) = \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} f^{(n)}(x), \qquad x > 0,$$

where $f^{(n)}$ denotes the n-fold convolution of f. ($Y(t)$ has an atom at zero of size $e^{-\lambda t}$.) It is known that the density $t \mapsto \psi_L(t;\beta_1)$ of $T_L(\beta_1)$ can be given in terms of h as

$$\psi_L(t,\beta_1) = \frac{\beta_1}{t} h(t-\beta_1;t), \qquad t \in (\beta_1,\infty). \tag{1.1}$$

Actually, (1.1) holds for general lower skip-free Lévy processes (see [2, p. 66, Thm. 1] and the recent proof in [3]). The main aim of this article is to derive an analog of the elegant formula (1.1) for the density $\psi_U(t;\beta_2)$ of the upper first-exit time $T_U(\beta_2)$. The connection between ψ_U and h turns out to be more intricate than (1.1); we will show that

$$\begin{aligned}\psi_U(t;\beta_2) = {} & \lambda e^{-\lambda t}\bar{F}(\beta_2+t) + \lambda \int_0^{\beta_2+t} h(x;t)\bar{F}(\beta_2+t-x)\,dx \\ & - \lambda \int_0^t h(x+\beta_2;x)e^{-\lambda(t-x)}\bar{F}(t-x)\,dx \\ & - \lambda \int_0^t \frac{1}{u}\left(\int_0^u (u-x)h(x;u)\bar{F}(u-x)\,dx\right)du,\end{aligned} \tag{1.2}$$

where, as usual, $\bar{F}(t) = 1 - F(t)$. In the course of proving (1.2), we will derive an explicit formula (in terms of h) of the defective density

$$g_{\beta_2}(x;t) = \frac{d}{dx} P(Y(t) \le x,\, T_U(\beta_2) > t).$$

Note that $x \mapsto g_{\beta_2}(x;t)/P(T_U(\beta_2) > t)$ is the conditional density of $Y(t)$, given that Y has not reached the upper straight line before time t.

Our approach was developed in a series of articles (see [5–8]) in which the Laplace–Stieltjes transforms of first-exit distributions for CPPs were derived in terms of certain transforms of the basic function h. However, this article is essentially self-contained. Recently, it was shown that

$$q(\beta_2) = (1-\rho)\int_0^{\infty} h(x+\beta_2;x)\,dx \tag{1.3}$$

https://doi.org/10.1017/S0269964803174025 Published online by Cambridge University Press

and that the two-sided exit probability $P(T_L(\beta_1) < T_U(\beta_2))$ can be reduced to $q(\cdot)$ as follows:

$$P(T_L(\beta_1) < T_U(\beta_2)) = e^{-\lambda\beta_1}\frac{q(\beta_2) - q(\beta_1+\beta_2)}{1-q(\beta_1+\beta_2)} + \frac{1-q(\beta_2)}{1-q(\beta_1+\beta_2)} \tag{1.4}$$

(see [7]; (1.3) was also proved in [4] by another method).

The derivation of our main result (1.2) and related identities are given in Section 2. In Section 3, we develop explicit formulas for the integrals

$$\int_0^\infty t^m h(t;t+\beta)\,dt, \qquad m \in \mathbb{Z}_+, \beta \geq 0.$$

It is surprising that these integrals depend on the jump size d.f. G in a simple way and only through the corresponding moments.

2. DENSITIES ψ_U AND g_β

In the following two lemmas, we express g_β in terms of h, first for $\beta = 0$ and then for $\beta > 0$.

LEMMA 1: *The density $g_0(x;t)$ is given by*

$$g_0(x;t) = \frac{t-x}{t}h(x;t), \qquad t > x > 0. \tag{2.1}$$

PROOF: Considering, for arbitrary $\beta > 0$, how the CPP Y can get to $t-\beta$ at time t, it is not difficult to see that h and ψ_L satisfy the renewal-type equation

$$h(t-\beta;t) = \psi_L(t;\beta) + e^{-\lambda\beta}h(t-\beta;t-\beta) + \int_\beta^t \psi_L(u;\beta)h(t-u;t-u)\,du. \tag{2.2}$$

Using (1.1), this yields

$$\beta\int_\beta^t \frac{1}{u}h(u-\beta;u)h(t-u;t-u)\,du = \frac{t-\beta}{t}h(t-\beta;t) - e^{-\lambda\beta}h(t-\beta;t-\beta). \tag{2.3}$$

By a similar argument, it follows that the function $x \mapsto g_0(x;x+y)$, $y > 0$, satisfies

$$g_0(x;x+y) = h(x;x+y) - h(x;x)e^{-\lambda y} - \int_0^x h(u;u)g_0(x-u;x-u+y)\,du. \tag{2.4}$$

https://doi.org/10.1017/S0269964803174025 Published online by Cambridge University Press

Setting $\tilde{g}_0(x) = yh(x; x+y)/(x+y)$ (with $y > 0$ fixed), (2.3) implies that $\tilde{g}_0(x)$ solves the renewal equation (2.4), whose unique solution is $x \mapsto g_0(x, x+y)$. Thus, $\tilde{g}_0(x) = g_0(x, x+y)$, yielding (2.1). ■

LEMMA 2: *For every $\beta > 0$, we have $g_\beta(x;t) = h(x;t)$, if $x \in [0,\beta]$, and*

$$g_\beta(x;t) = h(x;t) - h(x;x-\beta)e^{-\lambda(t-x+\beta)} - (t-x+\beta)\int_{t-x+\beta}^{t} \frac{1}{u} h(u-t+x-\beta;u)h(t-u+\beta;t-u)\,du, \quad x \in (\beta, t+\beta). \quad \textbf{(2.5)}$$

PROOF: The renewal equation (2.4) can clearly be extended to the case $\beta > 0$:

$$h(x;t) = g_\beta(x;t) + h(x;x-\beta)e^{-\lambda(t-x+\beta)} + \int_\beta^x h(y;y-\beta)g_0(x-y;t-y+\beta)\,dy, \qquad x \in (\beta, t+\beta) \quad \textbf{(2.6)}$$

(see also [8]). By (2.1), $g_0(x-y;t-y+\beta) = ((t-x+\beta)/(t-y+\beta))h(x-y; t-y+\beta)$, and a short calculation yields (2.5). ■

Remarks:

(a) By Lemma 2, the distribution function of $T_U(\beta)$ can be expressed as follows:

$$\begin{aligned} P(T_U(\beta) > t) &= e^{-\lambda t} + \int_0^{\beta+t} g_\beta(x;t)\,dx \\ &= H(t+\beta;t) - \int_0^t h(x+\beta;x)e^{-\lambda(t-x)}\,dx \\ &\quad - \int_0^t \frac{1}{u} h(t-u+\beta;t-u)\left(\int_0^u yh(u-y;u)\,dy\right)du. \end{aligned} \quad \textbf{(2.7)}$$

Note that

$$\begin{aligned} \frac{1}{u}\int_0^t uh(u-y;u)\,du &= \frac{1}{u}\int_0^t (u-y)h(y;u)\,du \\ &= E\left(\left(1 - \frac{Y(u)}{u}\right)1_{\{Y(u)\le t\}}\right) \end{aligned} \quad \textbf{(2.8)}$$

and

$$h(t-u+\beta, t-u) = \frac{\partial}{\partial x} H(x;t-u)|_{x=t-u+\beta}. \quad \textbf{(2.9)}$$

https://doi.org/10.1017/S0269964803174025 Published online by Cambridge University Press

Inserting (2.8) and (2.9) in (2.7), we obtain

$$P(T_U(\beta) > t) = H(t+\beta;t) - \int_0^t h(x+\beta;x)e^{-\lambda(t-x)}\,dx - \frac{\partial}{\partial x}\int_0^t H(x;t-u)E\left(\left(1-\frac{Y(u)}{u}\right)1_{\{Y(u)\le t\}}\right)du. \quad \textbf{(2.10)}$$

This result is similar to that in [2, p. 74].

(b) The conditional density of $Y(t)$ given that $T_U(\beta) > t$ is equal to

$$g_\beta(x;t)/P(T_U(\beta) > t),$$

and we have now expressed its numerator and its denominator in terms of h.

Now, we are in a position to derive a formula for $\psi_U(t;\beta_2)$.

THEOREM 1: *The defective density $\psi_U(t;\beta_2)$ of $T_U(\beta_2)$ is given by*

$$\psi_U(t;\beta_2) = \lambda e^{-\lambda t}\bar{F}(\beta_2+t) + \lambda\int_0^{\beta_2+t} h(x;t)\bar{F}(\beta_2+t-x)\,dx - \lambda\int_0^t h(x+\beta_2;x)e^{-\lambda(t-x)}\bar{F}(t-x)\,dx - \lambda\int_0^t \frac{1}{u}\left(\int_0^u (u-x)h(x;u)\bar{F}(u-x)\,dx\right)du. \quad \textbf{(2.11)}$$

PROOF: Conditioning on the time and the size of the last jump of Y before time t, we obtain

$$\psi_U(t;\beta_2) = \lambda e^{-\lambda t}\bar{F}(t+\beta_2) + \lambda\int_0^{\beta_2+t} g_{\beta_2}(x;t)\bar{F}(t+\beta_2-x)\,dx. \quad \textbf{(2.12)}$$

By (2.5), the integral on the right-hand side of (2.12) is equal to

$$\lambda\left[\int_0^{\beta_2+t} h(x;t)\bar{F}(\beta_2+t-x)\,dx - \int_0^t h(x+\beta_2;x)e^{-\lambda(t-x)}\bar{F}(t-x)\,dx - \int_0^t (t-x)\bar{F}(t-x)\left(\int_0^t \frac{h(x-y;t-y)}{t-y}\,dy\right)dx\right],$$

and (2.11) follows easily. ■

https://doi.org/10.1017/S0269964803174025 Published online by Cambridge University Press

3. MOMENT IDENTITIES

Clearly, the moments of $T_L(\beta)$ can be computed as

$$E(T_L(\beta)^k) = e^{-\lambda\beta}\beta^k + \int_\beta^\infty t^k \psi_L(t;\beta)\,dt$$

$$= e^{-\lambda\beta}\beta^k + \beta\int_\beta^\infty t^{k-1}h(t-\beta;t)\,dt, \qquad k = 0,1,2,\ldots, \tag{3.1}$$

where the second equation follows from (1.1). For $k = 0$, (3.1) yields the interesting identity

$$\int_0^\infty \frac{h(t;t+\beta)}{t+\beta}\,dt = \beta^{-1}(1-e^{-\lambda\beta}), \qquad \beta > 0. \tag{3.2}$$

Now, let us consider (3.1) for $k \in \mathbb{N}$. Let $\varphi(\theta) = \int_0^\infty e^{-\theta x} f(x)\,dx$, $\theta \geq 0$, be the Laplace transform of f. The moments on the left-hand side of (3.1) can be derived by applying the well-known martingale equation

$$E(e^{-\alpha T_L(\beta)}) = e^{-\beta\theta(\alpha)}, \qquad \alpha \geq 0, \tag{3.3}$$

where $\theta = \theta(\alpha) \geq 0$ is defined by the relation

$$\alpha = \theta - \lambda(1-\varphi(\theta)),\ \alpha \geq 0. \tag{3.4}$$

Hence,

$$\begin{aligned} E(T_L(\beta)) &= -\frac{d}{d\alpha}E(e^{\alpha T_L(\beta)})|_{\alpha=0} \\ &= \beta\theta'(0)e^{-\beta\theta(0)} \\ &= \beta(1+\lambda\varphi'(0))^{-1} \\ &= \beta(1-\rho)^{-1}. \end{aligned} \tag{3.5}$$

From (3.5) and (3.1), for $k = 1$ we can conclude that

$$\int_0^\infty h(t;t+\beta)\,dt = (1-\rho)^{-1} - e^{-\lambda\beta}. \tag{3.6}$$

In particular,

$$\int_0^\infty h(t;t)\,dt = \frac{\rho}{1-\rho}. \tag{3.7}$$

Equations (3.6) and (3.7) were established in [5] by a different method.

https://doi.org/10.1017/S0269964803174025 Published online by Cambridge University Press

Larger values of m can be dealt with in a similar fashion. For example, it follows from (3.3) and (3.4) that

$$E(T_L(\beta)^2) = \frac{\beta^2}{(1-\rho)^2} + \frac{\beta\lambda\mu_2}{(1-\rho)^3},$$

where $\mu_2 = E(X_1^2)$. Thus, by (3.1),

$$\int_0^\infty th(t; t+\beta)\, dt = \frac{\beta}{(1-\rho)^2} + \frac{\lambda\mu_2}{(1-\rho)^3} - \beta e^{-\lambda\beta}.$$

References

1. Asmussen, S. (2000). *Ruin probabilities*. Advanced Series on Statistical Science and Applied Probability Vol. 2. Singapore: World Scientific.
2. Borovkov, A.A. (1976). *Stochastic processes in queueing theory*. New York: Springer-Verlag.
3. Borovkov, K. & Burq, Z. (2001). Kendall's identity for the first crossing time revisited. *Communications on Probability* 6: 91–94.
4. Gerber, H. (1988). Mathematical fun with ruin theory. *Insurance: Mathematics and Economics* 7: 15–23.
5. Perry, D., Stadje, W., & Zacks, S. (1999). Contributions to the theory of first-exit times for some compound processes in queueing theory. *Queueing Systems* 33: 369–379.
6. Perry, D., Stadje, W., & Zacks, S. (2000). Busy period analysis of some queues with restricted accessibility. *Operations Research Letters* 27: 163–174.
7. Perry, D., Stadje, W., & Zacks, S. (2002). Hitting and ruin probabilities for compound processes and the cycle maximum of $M/G/1$. *Stochastic Models* 18: 553–564.
8. Zacks, S., Perry, D., Bshouty, D., & Bar-Lev, S. (1999). Distributions of stopping times for compound Poisson processes with positive jumps and linear boundaries. *Stochastic Models* 15: 89–101.

https://doi.org/10.1017/S0269964803174025 Published online by Cambridge University Press