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A Second-Order Finite Difference Method for Two-Dimensional Fractional Percolation Equations

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Abstract. A finite difference method which is second-order accurate in time and in space is proposed for two-dimensional fractional percolation equations. Using the Fourier transform, a general approximation for the mixed fractional derivatives is analyzed. An approach based on the classical Crank-Nicolson scheme combined with the Richardson extrapolation is used to obtain temporally and spatially second-order accurate numerical estimates. Consistency, stability and convergence of the method are established. Numerical experiments illustrating the effectiveness of the theoretical analysis are provided.

AMS subject classifications: 65M06, 26A33, 35R11

Key words: Fractional percolation equation, second-order finite difference method, Crank-Nicolson scheme, Richardson extrapolation, stability analysis.

1 Introduction

In this paper, we are concerned with the development of finite difference methods for the two-dimensional fractional percolation problem which seeks unknown pressure function $p(x,y,t)$ satisfying

$$\frac{\partial p}{\partial t}=\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\left(k_x(x,y)\frac{\partial^{\alpha_1}p}{\partial x^{\alpha_1}}\right)+\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}\left(k_y(x,y)\frac{\partial^{\alpha_2}p}{\partial y^{\alpha_2}}\right)+f(x,y,t),\quad (x,y)\in\Omega,\quad 0<t\leq T, \tag{1.1}$$

$$p(x,y,0)=p_0(x,y),\qquad (x,y)\in\Omega, \tag{1.2}$$

$$p(x_L,y,t)=p(x,y_L,t)=0, \tag{1.3}$$

$$p(x_R,y,t)=v_x(y,t),\quad p(x,y_R,t)=v_y(x,t),\qquad (x,y)\in\Omega,\quad 0\leq t\leq T,$$

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where $\Omega=\{(x,y)|x_L\le x\le x_R,y_L\le y\le y_R\}$,$0<\alpha_1,\alpha_2<1$,$0<\beta_1,\beta_2\le 1$, $f(x,y,t)$ is the source term, positive $k_x(x,y)$ and $k_y(x,y)$ are percolation coefficients along the x and y direction, respectively.

The fractional partial derivatives in (1.1) are defined in the Riemann-Liouville form. Generally, for any $\gamma>0$, the Riemann-Liouville fractional partial derivatives $\frac{\partial^\gamma w(x,y)}{\partial x^\gamma}$ and $\frac{\partial^\gamma w(x,y)}{\partial y^\gamma}$ of order γ are defined by [28,32,35]

$$\frac{\partial^\gamma w(x,y)}{\partial x^\gamma}=\frac{1}{\Gamma(n-\gamma)}\frac{\partial^n}{\partial x^n}\int_{x_L}^{x}\frac{w(\xi,y)}{(x-\xi)^{\gamma+1-n}}d\xi \tag{1.4}$$

and

$$\frac{\partial^\gamma w(x,y)}{\partial y^\gamma}=\frac{1}{\Gamma(n-\gamma)}\frac{\partial^n}{\partial y^n}\int_{y_L}^{y}\frac{w(x,\eta)}{(y-\eta)^{\gamma+1-n}}d\eta, \tag{1.5}$$

where n is an integer such that $n-1<\gamma\le n$. If γ is an integer, then the above definitions give the standard integer partial derivatives.

The percolation equations have been applied successfully in groundwater hydraulics, groundwater dynamics and fluid dynamics in porous media [7, 31, 40]. Under the assumptions of seepage flow continuity and the traditional Darcy's law, the traditional percolation equation [24, 34] for two-dimensional seepage flow in porous media is just the special case of equation (1.1) that $\alpha_i=\beta_i=1$ for $i=1,2$. In view of the limitations of these two assumptions above, He [12] proposed the modified Darcy's law with Riemann-Liouville fractional derivatives

$$q_x=k_x(x,y)\frac{\partial^{\alpha_1}p}{\partial x^{\alpha_1}},\qquad q_y=k_y(x,y)\frac{\partial^{\alpha_2}p}{\partial y^{\alpha_2}},\qquad 0<\alpha_1,\alpha_2<1, \tag{1.6}$$

as a generalization of Darcy's law for realistically describing the movement of solute in a non-homogeneous porous medium. Furthermore, considering the fact that the seepage flow is neither continued nor rigid body motion, He employed the fractional differential operators $\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}$ and $\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}$ where $0<\beta_1,\beta_2\le 1$ in the percolation equation, and then the fractional percolation model (1.1) was obtained, which is the focus of this paper.

As analytic solutions of most fractional differential equations cannot be obtained explicitly, numerical methods become major ways and a wide variety of techniques have been developed, including finite difference methods [25–27, 29, 37–39, 49–51], finite element methods [2,8–10,16], finite volume methods [42,44,45,48], spectral methods [15,17, 47], and mesh-free methods [11,20]. Recently, Liu et al. [21] proposed a first-order alternating direction implicit scheme for the three-dimensional non-continued seepage flow in uniform media and a second-order method which combined modified Douglas scheme with Richardson extrapolation for the three-dimensional continued seepage flow in non-uniform media. Chen et al. [5] developed an implicit finite difference method for the initial-boundary value problem of one-dimensional fractional percolation equation with left-sided mixed Riemann-Liouville fractional derivative, and they [6] considered an alternating direction implicit difference method for the two-dimensional case by a similar

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technique. The numerical simulation of variable-order fractional percolation equation in non-homogeneous porous media is discussed by Chen et al. in [4]. The numerical methods in [4–6] were proved to be first-order accurate in time and in space.

For solving high dimensional fractional differential equations, some numerical methods have been considered by many authors. Liu et al. [22] developed an implicit finite difference scheme for the two-dimensional fractional FitzHugh-Nagumo monodomain model, and they proved that the numerical method is unconditionally stable and converges linearly. A spatially second-order accurate semi-alternating direction method for the two-dimensional fractional FitzHugh-Nagumo monodomain model on an approximate irregular domain is proposed in [23]. Yu et al. [46] presented an alternating direction method for the space and time fractional Bloch-Torrey equation in three-dimensions, and a spatially second-order accurate implicit finite difference method with fractional centered derivative approximating the Riesz fractional derivative for the same problem is derived by Song et al. in [36].

Moreover, for fractional diffusion equations, a second-order finite difference method based on the classical Crank-Nicolson scheme and the Richardson extrapolation is developed by Tadjeran et al. in [39]. A second-order method which combines the alternating direction implicit approach with the Crank-Nicolson discretization and the Richardson extrapolation for the two-dimensional fractional diffusion equations is studied in [38]. By analyzing the Toeplitz-like structure of the full coefficient matrix, Basu and Wang [1] proposed a fast second-order finite difference method for fractional diffusion equations, which reduces the computational work from $\mathcal{O}(N^3)$ to $\mathcal{O}(N\log^2 N)$ per time step and reduces the memory requirement from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$. Lin et al. [18] developed two preconditioned iterative methods to solve the relevant linear systems of a second-order finite difference scheme for fractional diffusion equations, which require $\mathcal{O}(N\log N)$ operations per iteration and $\mathcal{O}(N)$ storage. Yang et al. [45] established a finite volume scheme with preconditioned Lanczos method as an attractive and high-efficiency approach for the two-dimensional space fractional reaction-diffusion equations. Tian et al. [41] developed a class of second-order weighted and shifted Grünwald difference operators for Riemann-Liouville fractional derivatives and then proposed the corresponding unconditionally stable difference schemes for the fractional diffusion equations with constant coefficients in one and two dimensions. A quasi-compact difference scheme which is third-order accurate in space is established for the fractional diffusion equations by Zhou et al. [50]. Liu et al. [19] developed an alternating direction implicit method for the two-dimensional Riesz space fractional diffusion equations with a nonlinear reaction term. Zeng et al. [47] proposed a Crank-Nicolson alternating direction implicit Galerkin-Legendre spectral method for the two-dimensional Riesz space fractional nonlinear reaction-diffusion equations. A second-order finite difference-spectral method for the fractional diffusion equations is considered by Huang et al. [13]. However, second-order accurate numerical methods for two-dimensional fractional percolation equations are still limited. This motivates us to examine a numerical approach which combines the classical Crank-Nicolson scheme with the Richardson extrapolation for the initial-

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boundary value problem (1.1) (1.2) (1.3).

The rest of the paper is organized as follows. In Section 2 we establish a general approximation for the mixed fractional derivatives in (1.1). In Section 3 we present a Crank-Nicolson scheme and consider its consistency. In Section 4 we prove stability and convergence of the scheme. In Section 5 we improve the order of convergence of the Crank-Nicolson scheme by the Richardson extrapolation. In Section 6 we carry out numerical experiments to evaluate the performance of the method. Finally, we draw our conclusions in Section 7.

2 Approximation for the mixed fractional derivatives

We begin with some definitions and properties.

Definition 2.1 ([28, 32, 35])**.** For any $\gamma>0$, the Liouville fractional integral $I^{n-\gamma}u(x)$ of order $n-\gamma$ is defined by

$$I^{n-\gamma}u(x)=\frac{1}{\Gamma(n-\gamma)}\int_{-\infty}^{x}\frac{u(\xi)}{(x-\xi)^{\gamma+1-n}}d\xi \tag{2.1}$$

and the Liouville fractional derivative $D^{\gamma}u(x)$ of order γ is defined by

$$D^{\gamma}u(x)=\frac{d^n}{dx^n}I^{n-\gamma}u(x), \tag{2.2}$$

where n is an integer such that $n-1<\gamma\leq n$.

For any positive integer r, let $W^{r,1}(\mathbb{R})$ denote the collection of functions in $C^r(\mathbb{R})$ whose derivatives up to order r belong to $L^1(\mathbb{R})$. For any $u(x)\in L^1(\mathbb{R})$, we use the definition of the Fourier transform as

$$\mathcal{F}[u(x)](\omega)=\hat{u}(\omega):=\int_{\mathbb{R}}e^{\mathbf{i}\omega x}u(x)dx, \tag{2.3}$$

where **i** is the imaginary unit. Three useful properties of the Fourier transform are listed in the following lemma.

Lemma 2.1 ([35])**.** *(i) If $u(x)\in L^1(\mathbb{R})$ and $h\in\mathbb{R}$, then*

$$\mathcal{F}[u(x-h)](\omega)=e^{\mathbf{i}h\omega}\hat{u}(\omega). \tag{2.4}$$

(ii) If $u(x),v(x),u(x)v(x)\in L^1(\mathbb{R})$, then

$$\mathcal{F}[u(x)v(x)](\omega)=\frac{1}{2\pi}\hat{u}(\omega)*\hat{v}(\omega)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{u}(\omega-\tau)\hat{v}(\tau)d\tau. \tag{2.5}$$

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(iii) If $\gamma>0$, n is an integer such that $n-1<\gamma\leq n$, $u(x)\in W^{n,1}(\mathbb{R})$, then

$$\mathcal{F}[D^{\gamma}u(x)](\omega)=(-\mathbf{i}\omega)^{\gamma}\hat{u}(\omega). \tag{2.6}$$

Under the assumption that $u(x_L)=0$, the Riemann-Liouville mixed fractional derivative $\frac{\partial^{\beta}}{\partial x^{\beta}}\left(k(x)\frac{\partial^{\alpha}}{\partial x^{\alpha}}\right)$ and $D^{\beta}(k(x)D^{\alpha}u(x))$ in the Liouville form are equivalent in view of the zero-extension of $u(x)$ for $x<x_L$. Notice the Dirichlet boundary condition (1.3) of the fractional percolation problem, it is sufficient for us to study a general approximation for the mixed fractional derivative $D^{\beta}(k(x)D^{\alpha}u(x))$.

For functions $k(x)$ and $u(x)$, we define the operator $\Delta^{\alpha,\beta}_{h,q,\tilde{q}}$ as follow:

$$\Delta^{\alpha,\beta}_{h,q,\tilde{q}}(k(x),u(x))=\frac{1}{h^{\alpha+\beta}}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}g_l^{(\beta)}g_j^{(\alpha)}k(x-(l-\tilde{q})h)u(x-(j+l-q)h), \tag{2.7}$$

where $0<\alpha<1$, $0<\beta\leq 1$, h is the grid size, q and $\tilde{q}$ are any real numbers, $g_j^{(\alpha)}$ and $g_l^{(\beta)}$ are the alternating fractional binomial coefficients related to α and β, respectively, e.g.,

$$\begin{cases} g_0^{(\alpha)}=1, \\ g_j^{(\alpha)}=\frac{(-1)^j}{j!}\alpha(\alpha-1)\cdots(\alpha-j+1), \qquad j=1,2,3,\cdots. \end{cases} \tag{2.8}$$

It has been proved (see Theorem 1 in [5]) that $\Delta^{\alpha,\beta}_{h,q,0}(k(x),u(x))$ is a first-order accurate approximation for $D^{\beta}(k(x)D^{\alpha}u(x))$, i.e.,

$$\Delta^{\alpha,\beta}_{h,q,0}(k(x),u(x))=D^{\beta}(k(x)D^{\alpha}u(x))+\mathcal{O}(h) \tag{2.9}$$

uniformly in $x\in\mathbb{R}$ as $h\to 0$. With the help of the Fourier transform, we have a more general expansion of $\Delta^{\alpha,\beta}_{h,q,\tilde{q}}(k(x),u(x))$ for any real numbers q and $\tilde{q}$.

Theorem 2.1. *Assume that $0<\alpha<1$, $0<\beta\leq 1$, n is a positive integer, $k(x)\in W^{n+3,1}(\mathbb{R})$, $u(x)\in W^{2n+3,1}(\mathbb{R})$ and $I^{1-\alpha}u(x)\in W^{2n,1}(\mathbb{R})$, then for any q, $\tilde{q}\in\mathbb{R}$, there exist some constants $a_l^{(\alpha)}$ and $a_l^{(\beta)}$ independent of h,$k(x)$ and $u(x)$, such that*

$$\begin{aligned} &\Delta^{\alpha,\beta}_{h,q,\tilde{q}}(k(x),u(x)) \\ &=D^{\beta}(k(x)D^{\alpha}u(x))+\sum_{j=1}^{n-1}\sum_{l=0}^{j}a_l^{(\alpha)}a_{j-l}^{(\beta)}D^{\beta+j-l}\Big(k(x)D^{\alpha+l}u(x)\Big)h^j+\mathcal{O}(h^n) \end{aligned} \tag{2.10}$$

uniformly in $x\in\mathbb{R}$ as $h\to 0$.

To prove Theorem 2.1, we need the following lemmas.

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Lemma 2.2. *Under the assumption of Theorem* 2.1*, for positive integers m_k and m_u which satisfy $m_k<n+2$ and $m_k+m_u<2n+2$, there exists a constant $C(m_k,m_u)$ such that*

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\hat{k}(\omega-\tau)\hat{u}(\tau)(-\mathbf{i}\tau)^{m_u}(-\mathbf{i}\omega)^{m_k}\right|d\tau d\omega\leq C(m_k,m_u), \tag{2.11}$$

where $\hat{k}(\omega)$ and $\hat{u}(\omega)$ are the Fourier transform of $k(x)$ and $u(x)$, respectively.

Proof. Since $k(x)\in W^{n+3,1}(\mathbb{R})$ and $u(x)\in W^{2n+3,1}(\mathbb{R})$, according to the Riemann-Lebesgue lemma, we have

$$|\hat{k}(\omega)|\leq C_k(1+|\omega|)^{-(n+3)} \tag{2.12}$$

and

$$|\hat{u}(\omega)|\leq C_u(1+|\omega|)^{-(2n+3)} \tag{2.13}$$

for all $\omega\in\mathbb{R}$, where C_k and C_u are constants. Hence $w^{m_k}\hat{k}(\omega)$, $w^{m_u}\hat{u}(\omega)$ and $w^{m_k+m_u}\hat{u}(\omega)$ belong to $L^1(\mathbb{R})$. Let

$$C(m_k,m_u)=2^{m_k-1}\left(\|\hat{k}(\omega)\|_{L^1}\|\omega^{m_k+m_u}\hat{u}(\omega)\|_{L^1}+\|\omega^{m_k}\hat{k}(\omega)\|_{L^1}\|\omega^{m_u}\hat{u}(\omega)\|_{L^1}\right), \tag{2.14}$$

then we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\hat{k}(\omega-\tau)\hat{u}(\tau)(-\mathbf{i}\tau)^{m_u}(-\mathbf{i}\omega)^{m_k}\right|d\tau d\omega\\
\leq&\int_{\mathbb{R}}|\hat{u}(\tau)||\tau|^{m_u}d\tau\int_{\mathbb{R}}|\hat{k}(\omega-\tau)||\omega|^{m_k}d\omega\\
\leq&2^{m_k-1}\int_{\mathbb{R}}|\hat{u}(\tau)||\tau|^{m_u}d\tau\int_{\mathbb{R}}|\hat{k}(\omega-\tau)|(|\tau|^{m_k}+|\omega-\tau|^{m_k})d\omega\\
=&C(m_k,m_u).
\end{aligned} \tag{2.15}$$

The proof of Lemma 2.2 is completed. □

Lemma 2.3. *Under the assumption of Theorem* 2.1*, for $0\leq j,l\leq n-1$, we have*

$$\mathcal{F}^{-1}\left(\frac{1}{2\pi}\int_{\mathbb{R}}\hat{k}(\omega-\tau)\hat{u}(\tau)(-\mathbf{i}\tau)^{\alpha+l}(-\mathbf{i}\omega)^{\beta+j}d\tau\right)=D^{\beta+j}\left(k(x)D^{\alpha+l}u(x)\right). \tag{2.16}$$

Proof. Since $D^{\alpha+l}u(x)=\frac{d^{l+1}}{dx^{l+1}}I^{1-\alpha}u(x)\in W^{2n-l-1,1}(\mathbb{R})$ and $k(x)\in W^{n+3,1}(\mathbb{R})$, it is easy to verify that

$$\begin{aligned}
&\mathcal{F}\left[D^{\beta+j}\left(k(x)D^{\alpha+l}u(x)\right)\right](\omega)\\
=&(-\mathbf{i}\omega)^{\beta+j}\mathcal{F}\left[k(x)D^{\alpha+l}u(x)\right](\omega)\\
=&(-\mathbf{i}\omega)^{\beta+j}\frac{1}{2\pi}\left(\hat{k}(\omega)*\mathcal{F}\left[D^{\alpha+l}u(x)\right](\omega)\right)\\
=&(-\mathbf{i}\omega)^{\beta+j}\frac{1}{2\pi}\left(\hat{k}(\omega)*(-\mathbf{i}\omega)^{\alpha+l}\hat{u}(\omega)\right)\\
=&\frac{1}{2\pi}\int_{\mathbb{R}}\hat{k}(\omega-\tau)\hat{u}(\tau)(-\mathbf{i}\tau)^{\alpha+l}(-\mathbf{i}\omega)^{\beta+j}d\tau.
\end{aligned} \tag{2.17}$$

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By Lemma 2.2, we have $\int_{\mathbb{R}}\hat{k}(\omega-\tau)\hat{u}(\tau)(-\mathbf{i}\tau)^{\alpha+l}(-\mathbf{i}\omega)^{\beta+j}d\tau\in L^1(\mathbb{R})$, then (2.16) follows. □

We are now turning to the proof of Theorem 2.1.

Proof of Theorem 2.1. First, we consider a decomposition of the Fourier transform of $\Delta^{\alpha,\beta}_{h,q,\tilde{q}}(k(x),u(x))$. Note that

$$(1-z)^{\gamma}=\sum_{j=0}^{\infty}g_j^{(\gamma)}z^{\gamma} \tag{2.18}$$

converges absolutely for $|z|\leq 1$, where $\gamma\in\{\alpha,\beta\}$. Hence $\Delta^{\alpha,\beta}_{h,q,\tilde{q}}(k(x),u(x))\in L^1(\mathbb{R})$ and therefore

$$\begin{aligned}
&\mathcal{F}\left[\Delta^{\alpha,\beta}_{h,q,\tilde{q}}(k(x),u(x))\right](\omega)\\
&=\frac{1}{h^{\alpha+\beta}}\sum_{l=0}^{\infty}g_l^{(\beta)}\mathcal{F}\left[\sum_{j=0}^{\infty}g_j^{(\alpha)}k(x-(l-\tilde{q})h)u(x-(j+l-q)h)\right](\omega)\\
&=\frac{1}{h^{\alpha+\beta}}\sum_{l=0}^{\infty}g_l^{(\beta)}\frac{1}{2\pi}\left\{\mathcal{F}[k(x-(l-\tilde{q})h)](\omega)*\sum_{j=0}^{\infty}g_j^{(\alpha)}\mathcal{F}[u(x-(j+l-q)h)](\omega)\right\}\\
&=\frac{1}{h^{\alpha+\beta}}\sum_{l=0}^{\infty}g_l^{(\beta)}\frac{1}{2\pi}\left\{\left[e^{\mathbf{i}\omega(l-\tilde{q})h}\hat{k}(\omega)\right]*\sum_{j=0}^{\infty}g_j^{(\alpha)}e^{\mathbf{i}\omega(j+l-q)h}\hat{u}(\omega)\right\}\\
&=\frac{1}{h^{\alpha+\beta}}\sum_{l=0}^{\infty}g_l^{(\beta)}\frac{1}{2\pi}\int_{\mathbb{R}}e^{\mathbf{i}(\omega-\tau)(l-\tilde{q})h}\hat{k}(\omega-\tau)\sum_{j=0}^{\infty}g_j^{(\alpha)}e^{\mathbf{i}\tau(j+l-q)h}\hat{u}(\tau)d\tau\\
&=\frac{1}{h^{\beta}}\sum_{l=0}^{\infty}g_l^{(\beta)}e^{\mathbf{i}\omega(l-\tilde{q})h}\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{h^{\alpha}}\sum_{j=0}^{\infty}g_j^{(\alpha)}e^{\mathbf{i}\tau(j-q+\tilde{q})h}\hat{k}(\omega-\tau)\hat{u}(\tau)d\tau\\
&=\frac{1}{h^{\beta}}(1-e^{\mathbf{i}\omega h})^{\beta}e^{-\mathbf{i}\omega\tilde{q}h}\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{h^{\alpha}}(1-e^{\mathbf{i}\tau h})^{\alpha}e^{-\mathbf{i}\tau(q-\tilde{q})h}\hat{k}(\omega-\tau)\hat{u}(\tau)d\tau\\
&=(-\mathbf{i}\omega)^{\beta}\left(\frac{1-e^{\mathbf{i}\omega h}}{-\mathbf{i}\omega h}\right)^{\beta}e^{-i\omega\tilde{q}h}\frac{1}{2\pi}\int_{\mathbb{R}}(-\mathbf{i}\tau)^{\alpha}\left(\frac{1-e^{\mathbf{i}\tau h}}{-\mathbf{i}\tau h}\right)^{\alpha}e^{-\mathbf{i}\tau(q-\tilde{q})h}\hat{k}(\omega-\tau)\hat{u}(\tau)d\tau.
\end{aligned}\tag{2.19}$$

Define

$$W_{\gamma,\varsigma}(z)=\left(\frac{1-e^{-z}}{z}\right)^{\gamma}e^{\varsigma z}, \tag{2.20}$$

for $\varsigma\in\mathbb{R}$ and $\gamma\in\{\alpha,\beta\}$. Then the previous expression (2.19) may be rewritten as

$$\begin{aligned}
&\mathcal{F}\left[\Delta^{\alpha,\beta}_{h,q,\tilde{q}}(k(x),u(x))\right](\omega)\\
=&\frac{1}{2\pi}\int_{\mathbb{R}}\hat{k}(\omega-\tau)\hat{u}(\tau)(-\mathbf{i}\tau)^{\alpha}(-\mathbf{i}\omega)^{\beta}W_{\alpha,q-\tilde{q}}(-\mathbf{i}\tau h)W_{\beta,\tilde{q}}(-\mathbf{i}\omega h)d\tau.
\end{aligned}\tag{2.21}$$

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Since $W_{\alpha,q-\tilde{q}}(z)$ and $W_{\beta,\tilde{q}}(z)$ are analytic in some neighborhood of the origin, we have the power series expansions

$$W_{\alpha,q-\tilde{q}}(z)=\sum_{l=0}^{\infty}a_l^{(\alpha)}z^l \tag{2.22}$$

and

$$W_{\beta,\tilde{q}}(z)=\sum_{l=0}^{\infty}a_l^{(\beta)}z^l \tag{2.23}$$

which converge absolutely for all $|z|\leq R$ with some $R>0$. Note that $a_0^{(\alpha)}=a_0^{(\beta)}=1$, and that $a_1^{(\alpha)}=q-\tilde{q}-\frac{\alpha}{2}$, $a_1^{(\beta)}=\tilde{q}-\frac{\beta}{2}$. For $x\in\mathbb{R}$, let

$$R_n^{(\alpha)}(x)=W_{\alpha,q-\tilde{q}}(-\mathbf{i}x)-\sum_{l=0}^{n-1}a_l^{(\alpha)}(-\mathbf{i}x)^l, \tag{2.24}$$

$$R_n^{(\beta)}(x)=W_{\beta,\tilde{q}}(-\mathbf{i}x)-\sum_{l=0}^{n-1}a_l^{(\beta)}(-\mathbf{i}x)^l. \tag{2.25}$$

According to Theorem 3.1 in [39], there exist positive constants C_1 and C_2 such that

$$|R_n^{(\alpha)}(x)|\leq C_1|x|^n \tag{2.26}$$

and

$$|R_n^{(\beta)}(x)|\leq C_2|x|^n \tag{2.27}$$

uniformly in $x\in\mathbb{R}$. Thus,

$$\begin{aligned}
&W_{\alpha,q-\tilde{q}}(-\mathbf{i}\tau h)W_{\beta,\tilde{q}}(-\mathbf{i}\omega h)\\
&=\sum_{j=0}^{n-1}a_j^{(\alpha)}(-\mathbf{i}\tau h)^j\sum_{l=0}^{n-1}a_l^{(\beta)}(-\mathbf{i}\omega h)^l+R_n^{(\beta)}(\omega h)\sum_{j=0}^{n-1}a_j^{(\alpha)}(-\mathbf{i}\tau h)^j\\
&\quad+R_n^{(\alpha)}(\tau h)\sum_{l=0}^{n-1}a_l^{(\beta)}(-\mathbf{i}\omega h)^l+R_n^{(\alpha)}(\tau h)R_n^{(\beta)}(\omega h)\\
&=\sum_{j=0}^{n-1}\sum_{l=0}^{j}a_l^{(\alpha)}a_{j-l}^{(\beta)}(-\mathbf{i}\tau h)^l(-\mathbf{i}\omega h)^{j-l}+\sum_{j=1}^{n-1}\sum_{l=n-j}^{n-1}a_j^{(\alpha)}a_l^{(\beta)}(-\mathbf{i}\tau h)^j(-\mathbf{i}\omega h)^l\\
&\quad+R_n^{(\beta)}(\omega h)\sum_{j=0}^{n-1}a_j^{(\alpha)}(-\mathbf{i}\tau h)^j+R_n^{(\alpha)}(\tau h)\sum_{l=0}^{n-1}a_l^{(\beta)}(-\mathbf{i}\omega h)^l+R_n^{(\alpha)}(\tau h)R_n^{(\beta)}(\omega h)\\
&=:\phi_1(\tau,\omega,h)+\phi_2(\tau,\omega,h)+\phi_3(\tau,\omega,h)+\phi_4(\tau,\omega,h)+\phi_5(\tau,\omega,h).
\end{aligned} \tag{2.28}$$

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Substitute (2.28) into (2.21), we obtain

$$\begin{aligned}
&\mathcal{F}\left[\Delta^{\alpha,\beta}_{h,q,\tilde{q}}(k(x),u(x))\right](\omega)\\
=&\sum_{s=1}^{5}\frac{1}{2\pi}\int_{\mathbb{R}}\hat{k}(\omega-\tau)\hat{u}(\tau)(-\mathbf{i}\tau)^{\alpha}(-\mathbf{i}\omega)^{\beta}\phi_s(\tau,\omega,h)d\tau\\
=:&\sum_{s=1}^{5}\hat{\psi}_s(\omega,h).
\end{aligned}\tag{2.29}$$

Next, we analyze the Fourier inversion of $\hat{\psi}_s(\omega,h)$ on the right hand side of (2.29). By Lemma 2.2, the Fourier inversion of $\hat{\psi}_1(\omega,h)$ may be estimated as

$$\begin{aligned}
&\left|\mathcal{F}^{-1}\left[\hat{\psi}_1(\omega,h)\right](x)\right|\\
=&\left|\frac{1}{2\pi}\int_{\mathbb{R}}e^{-\mathbf{i}\omega x}\hat{\psi}_1(\omega,h)d\omega\right|\\
=&\left|\frac{1}{4\pi^2}\int_{\mathbb{R}}e^{-\mathbf{i}\omega x}\sum_{j=0}^{n-1}\sum_{l=0}^{j}a_l^{(\alpha)}a_{j-l}^{(\beta)}h^j\int_{\mathbb{R}}\hat{k}(\omega-\tau)\hat{u}(\tau)(-\mathbf{i}\tau)^{\alpha+l}(-\mathbf{i}\omega)^{\beta+j-l}d\tau d\omega\right|\\
\le&\frac{1}{4\pi^2}\sum_{j=0}^{n-1}\sum_{l=0}^{j}\left|a_l^{(\alpha)}\right|\left|a_{j-l}^{(\beta)}\right|h^j\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\hat{k}(\omega-\tau)\hat{u}(\tau)(-\mathbf{i}\tau)^{\alpha+l}(-\mathbf{i}\omega)^{\beta+j-l}\right|d\tau d\omega\\
\le&\frac{1}{4\pi^2}\sum_{j=0}^{n-1}\sum_{l=0}^{j}\left|a_l^{(\alpha)}\right|\left|a_{j-l}^{(\beta)}\right|C(\alpha+l,\beta+j-l)h^j.
\end{aligned}\tag{2.30}$$

Similarly, due to (2.26) and (2.27), we have that

$$\left|\mathcal{F}^{-1}\left[\hat{\psi}_2(\omega,h)\right](x)\right|\le\frac{1}{4\pi^2}\sum_{j=1}^{n-1}\sum_{l=n-j}^{n-1}\left|a_j^{(\alpha)}\right|\left|a_l^{(\beta)}\right|C(\alpha+j,\beta+l)h^{j+l},\tag{2.31}$$

$$\left|\mathcal{F}^{-1}\left[\hat{\psi}_3(h,\omega)\right](x)\right|\le\frac{C_2}{4\pi^2}\sum_{j=0}^{n-1}\left|a_j^{(\alpha)}\right|C(\alpha+j,\beta+n)h^{j+n},\tag{2.32}$$

$$\left|\mathcal{F}^{-1}\left[\hat{\psi}_4(h,\omega)\right](x)\right|\le\frac{C_1}{4\pi^2}\sum_{l=0}^{n-1}\left|a_l^{(\beta)}\right|C(\alpha+n,\beta+l)h^{n+l},\tag{2.33}$$

and

$$\left|\mathcal{F}^{-1}\left[\hat{\psi}_5(h,\omega)\right](x)\right|\le\frac{C_1C_2}{4\pi^2}C(\alpha+n,\beta+n)h^{2n}\tag{2.34}$$

uniformly in $x\in\mathbb{R}$, respectively.

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Finally, we compute the Fourier inversion of $\mathcal{F}\left[\Delta^{\alpha,\beta}_{h,q,\tilde{q}}(k(x),u(x))\right](\omega)$ to obtain the expansion (2.10). Using Lemma 2.3, we have

$$\begin{aligned}
&\mathcal{F}^{-1}\left[\hat{\psi}_1(\omega,h)\right](x)\\
=&\sum_{j=0}^{n-1}\sum_{l=0}^{j}a_l^{(\alpha)}a_{j-l}^{(\beta)}\mathcal{F}^{-1}\left[\frac{1}{2\pi}\int_{\mathbb{R}}\hat{k}(\omega-\tau)\hat{u}(\tau)(-\mathbf{i}\tau)^{\alpha+l}(-\mathbf{i}\omega)^{\beta+j-l}d\tau\right]h^j\\
=&\sum_{j=0}^{n-1}\sum_{l=0}^{j}a_l^{(\alpha)}a_{j-l}^{(\beta)}D^{\beta+j-l}\Big(k(x)D^{\alpha+l}u(x)\Big)h^j. \qquad (2.35)
\end{aligned}$$

Thus we conclude that

$$\begin{aligned}
&\Delta^{\alpha,\beta}_{h,q,\tilde{q}}(k(x),u(x))\\
=&\sum_{j=0}^{n-1}\sum_{l=0}^{j}a_l^{(\alpha)}a_{j-l}^{(\beta)}D^{\beta+j-l}\Big(k(x)D^{\alpha+l}u(x)\Big)h^j+\sum_{s=2}^{5}\mathcal{F}^{-1}\left[\hat{\psi}_s(\omega,h)\right](x)\\
=&D^{\beta}(k(x)D^{\alpha}u(x))+\sum_{j=1}^{n-1}\sum_{l=0}^{j}a_l^{(\alpha)}a_{j-l}^{(\beta)}D^{\beta+j-l}\Big(k(x)D^{\alpha+l}u(x)\Big)h^j+\mathcal{O}(h^n) \qquad (2.36)
\end{aligned}$$

uniformly in $x\in\mathbb{R}$ as $h\to 0$. This completes the proof. □

3 A Crank-Nicolson scheme and its consistency

For the numerical approximation scheme, define $t_m=m\Delta t$ to be the integration time $0\le t_m\le T$, h_x is the grid size in x-direction, $h_x=(x_R-x_L)/(N_x+1)$, with $x_i=x_L+ih_x$ for $i=0,\cdots,N_x+1$; h_y is the grid size in y-direction, $h_y=(y_R-y_L)/(N_y+1)$, with $y_j=y_L+jh_y$ for $j=0,\cdots,N_y+1$. Denote by $P^m_{i,j}$ the exact solution p at the mesh point (x_i,y_j,t_m). Let $p^m_{i,j}$ to be the numerical approximation to $P^m_{i,j}$ and define $k^x_{i,j}=k_x(x_i,y_j)$, $k^y_{i,j}=k_y(x_i,y_j)$, $f^m_{i,j}=f(x_i,y_j,t_m)$.

According to Theorem 2.1 and the Dirichlet boundary condition $p(x_L,y,t)=0$ for $y_L\le y\le y_R$, $0<t\le T$, the mixed fractional derivative with respect to x in (1.1) can be discretized with $q=\tilde{q}=1$ and $n=2$ as:

$$\begin{aligned}
&\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\left(k_x(x,y)\frac{\partial^{\alpha_1}p}{\partial x^{\alpha_1}}\right)\Big|_{(x_i,y_j,t_m)}\\
=&\frac{1}{h_x^{\alpha_1+\beta_1}}\sum_{l=0}^{i}g_l^{(\beta_1)}k^x_{i-l+1,j}\sum_{s=0}^{i-l}g_s^{(\alpha_1)}P^m_{i+1-s-l,j}+\zeta_x(x_i,y_j,t_m)h_x+\mathcal{O}(h_x^2), \qquad (3.1)
\end{aligned}$$

for $i=1,\cdots,N_x$, $j=1,\cdots,N_y$, $m>0$, where

$$\zeta_x(x,y,t)=\left(\frac{\beta_1}{2}-1\right)\frac{\partial^{\beta_1+1}}{\partial x^{\beta_1+1}}\left(k_x(x,y)\frac{\partial^{\alpha_1}p}{\partial x^{\alpha_1}}\right)+\frac{\alpha_1}{2}\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\left(k_x(x,y)\frac{\partial^{\alpha_1+1}p}{\partial x^{\alpha_1+1}}\right). \qquad (3.2)$$

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Similarly, the mixed fractional derivative with respect to y in (1.1) can be discretized as

$$\begin{aligned}&\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}\left(k_y(x,y)\frac{\partial^{\alpha_2}p}{\partial y^{\alpha_2}}\right)\Big|_{(x_i,y_j,t_m)}\\&=\frac{1}{h_y^{\alpha_2+\beta_2}}\sum_{l=0}^{j}g_l^{(\beta_2)}k^y_{i,j-l+1}\sum_{s=0}^{j-l}g_s^{(\alpha_2)}P^m_{i,j+1-s-l}+\zeta_y(x_i,y_j,t_m)h_y+\mathcal{O}(h_y^2),\end{aligned}\tag{3.3}$$

for $i=1,\cdots,N_x$, $j=1,\cdots,N_y$, $m>0$, where

$$\zeta_y(x,y,t)=\left(\frac{\beta_2}{2}-1\right)\frac{\partial^{\beta_2+1}}{\partial y^{\beta_2+1}}\left(k_y(x,y)\frac{\partial^{\alpha_2}p}{\partial y^{\alpha_2}}\right)+\frac{\alpha_2}{2}\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}\left(k_y(x,y)\frac{\partial^{\alpha_2+1}p}{\partial y^{\alpha_2+1}}\right).\tag{3.4}$$

Substitute (3.1) and (3.3) into the fractional percolation equation (1.1) to get the Crank-Nicolson type numerical scheme, then the resulting two-dimensional implicit finite difference equations may be expressed as follows:

$$\begin{aligned}&\frac{p^m_{i,j}-p^{m-1}_{i,j}}{\Delta t}=\frac{1}{2}\left(\delta_x p^m_{i,j}+\delta_x p^{m-1}_{i,j}+\delta_y p^m_{i,j}+\delta_y p^{m-1}_{i,j}\right)+\frac{1}{2}\left(f^m_{i,j}+f^{m-1}_{i,j}\right),\\&\quad i=1,\cdots,N_x,\quad j=1,\cdots,N_y,\quad m>0,\end{aligned}\tag{3.5}$$

$$\begin{aligned}&p^m_{0,j}=0,\quad p^m_{N_x+1,j}=v_x(y_j,t_m),\quad j=0,\cdots,N_y+1,\quad m>0,\\&p^m_{i,0}=0,\quad p^m_{i,N_y+1}=v_y(x_i,t_m),\quad i=0,\cdots,N_x+1,\quad m>0,\end{aligned}\tag{3.6}$$

$$p^0_{i,j}=p_0(x_i,y_j),\quad i=1,\cdots,N_x,\quad j=1,\cdots,N_y,\tag{3.7}$$

where the operators δ_x and δ_y are defined as

$$\delta_x p^m_{i,j}=\frac{1}{h_x^{\alpha_1+\beta_1}}\sum_{l=0}^{i}g_l^{(\beta_1)}k^x_{i-l+1,j}\sum_{s=0}^{i-l}g_s^{(\alpha_1)}p^m_{i+1-s-l,j},\tag{3.8}$$

$$\delta_y p^m_{i,j}=\frac{1}{h_y^{\alpha_2+\beta_2}}\sum_{l=0}^{j}g_l^{(\beta_2)}k^y_{i,j-l+1}\sum_{s=0}^{j-l}g_s^{(\alpha_2)}p^m_{i,j+1-s-l}.\tag{3.9}$$

By exchanging the order of summations above, (3.8) and (3.9) can be rewritten as

$$\delta_x p^m_{i,j}=\frac{1}{h_x^{\alpha_1+\beta_1}}\sum_{s=1}^{i+1}\left[\sum_{l=0}^{i+1-s}g_l^{(\beta_1)}g^{(\alpha_1)}_{i+1-s-l}k^x_{i-l+1,j}\right]p^m_{s,j},\tag{3.10}$$

$$\delta_y p^m_{i,j}=\frac{1}{h_y^{\alpha_2+\beta_2}}\sum_{s=1}^{j+1}\left[\sum_{l=0}^{j+1-s}g_l^{(\beta_2)}g^{(\alpha_2)}_{j+1-s-l}k^y_{i,j-l+1}\right]p^m_{i,s}.\tag{3.11}$$

Denote the local truncation error by $R^m_{i,j}$. Notice the fact that the center divided difference in t provides an $\mathcal{O}((\Delta t)^2)$ temporal error component. It follows from (3.1) and (3.3)

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that

$$\begin{aligned}R_{i,j}^m&=\frac{P_{i,j}^m-P_{i,j}^{m-1}}{\Delta t}-\frac{1}{2}\left(\delta_x P_{i,j}^m+\delta_x P_{i,j}^{m-1}+\delta_y P_{i,j}^m+\delta_y P_{i,j}^{m-1}\right)-\frac{1}{2}\left(f_{i,j}^m+f_{i,j}^{m-1}\right),\\&=-\frac{(\Delta t)^2}{12}\frac{\partial^3 P}{\partial t^3}\bigg|_{(x_i,y_j,t_{m-\frac{1}{2}})}+\mathcal{O}((\Delta t)^3)+\frac{h_x}{2}(\zeta_x(x_i,y_j,t_m)+\zeta_x(x_i,y_j,t_{m-1}))\\&\quad+\frac{h_y}{2}(\zeta_y(x_i,y_j,t_m)+\zeta_y(x_i,y_j,t_{m-1}))+\mathcal{O}(h_x^2)+\mathcal{O}(h_y^2)\\&=\mathcal{O}((\Delta t)^2)+\mathcal{O}(h_x)+\mathcal{O}(h_y),\quad i=1,\cdots,N_x,\quad j=1,\cdots,N_y,\quad m=1,\cdots,M.\end{aligned}\tag{3.12}$$

This implies that the Crank-Nicolson scheme defined by (3.5) is consistent.

4 Stability and convergence

Let $r_1=\frac{\Delta t}{h_x^{\alpha_1+\beta_1}}, r_2=\frac{\Delta t}{h_y^{\alpha_2+\beta_2}}$, $\mathbf{p}^m$ and $\mathbf{f}^m$ be the N-dimensional vectors defined by

$$\mathbf{p}^m:=\left[p_{1,1}^m,\cdots,p_{N_x,1}^m,p_{1,2}^m,\cdots,p_{N_x,2}^m,\cdots,p_{1,N_y}^m,\cdots,p_{N_x,N_y}^m\right]^T,\tag{4.1}$$

$$\begin{aligned}\mathbf{f}^m:=&\left[f_{1,1}^m,\cdots,f_{N_x-1,1}^m,\tilde{f}_{N_x,1}^m,f_{1,2}^m,\cdots,f_{N_x-1,2}^m,\tilde{f}_{N_x,2}^m,\cdots,\right.\\&\left.f_{1,N_y-1}^m,\cdots,f_{N_x-1,N_y-1}^m,\tilde{f}_{N_x,N_y-1}^m,\tilde{f}_{1,N_y}^m,\cdots,\tilde{f}_{N_x-1,N_y-1}^m,\tilde{f}_{N_x,N_y}^m\right]^T,\end{aligned}\tag{4.2}$$

where

$$\begin{aligned}\tilde{f}_{N_x,j}^m&=f_{N_x,j}^m+\frac{1}{h_x^{\alpha_1+\beta_1}}k_{N_x+1,j}^x p_{N_x+1,j}^m,\quad 1\le j\le N_y-1,\\\tilde{f}_{i,N_y}^m&=f_{i,N_y}^m+\frac{1}{h_y^{\alpha_2+\beta_2}}k_{i,N_y+1}^y p_{i,N_y+1}^m,\quad 1\le i\le N_x-1,\\\tilde{f}_{N_x,N_y}^m&=f_{N_x,N_y}^m+\frac{1}{h_x^{\alpha_1+\beta_1}}k_{N_x+1,N_y}^x p_{N_x+1,N_y}^m+\frac{1}{h_y^{\alpha_2+\beta_2}}k_{N_x,N_y+1}^y p_{N_x,N_y+1}^m.\end{aligned}\tag{4.3}$$

Let $\mathbf{I}$ and $\mathbf{A}$ be the N-by-N identity and stiffness matrices, respectively. Then the finite difference equation is expressed in the matrix form

$$\left(\mathbf{I}-\frac{1}{2}\mathbf{A}\right)\mathbf{p}^m=\left(\mathbf{I}+\frac{1}{2}\mathbf{A}\right)\mathbf{p}^{m-1}+\frac{\Delta t}{2}(\mathbf{f}^m+\mathbf{f}^{m-1}).\tag{4.4}$$

The N-by-N stiffness matrix $\mathbf{A}$ in (4.4) can be expressed as an N_y-by-N_y block matrix with each N_x-by-N_x block $A_{j,\tilde{s}}$ defined by

$$(A_{j,j})_{i,s}:=r_1\sum_{l=0}^{i+1-s}g_l^{(\beta_1)}g_{i+1-s-l}^{(\alpha_1)}k_{i-l+1,j}^x,\quad s\le i-1,\tag{4.5a}$$

$$(A_{j,j})_{i,i}:=r_1\sum_{l=0}^{1}g_l^{(\beta_1)}g_{1-l}^{(\alpha_1)}k_{i-l+1,j}^x+r_2\sum_{l=0}^{1}g_l^{(\beta_2)}g_{1-l}^{(\alpha_2)}k_{i,j-l+1}^y,\tag{4.5b}$$

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$$(A_{j,j})_{i,i+1} := r_1 k^x_{i+1,j}, \tag{4.5c}$$

$$(A_{j,j})_{i,s} := 0, \quad s > i+1, \tag{4.5d}$$

$$(A_{j,\tilde{s}})_{i,s} := \delta_{i,s} r_2 \sum_{l=0}^{j+1-\tilde{s}} g_l^{(\beta_2)} g_{j+1-\tilde{s}-l}^{(\alpha_2)} k^y_{i,j-l+1}, \quad \tilde{s} \le j-1, \tag{4.5e}$$

$$(A_{j,j+1})_{i,s} := \delta_{i,s} r_2 k^y_{i,j+1}, \tag{4.5f}$$

$$(A_{j,\tilde{s}})_{i,s} := 0, \quad \tilde{s} > j+1, \tag{4.5g}$$

where $\delta_{i,s} = 1$ if $i = s$ or 0 otherwise.

To discuss the convergence of the numerical method, we define

$$e^m_{i,j} = P^m_{i,j} - p^m_{i,j}, \quad 0 \le i \le N_x+1, \quad 0 \le j \le N_y+1, \quad 0 \le m \le M \tag{4.6}$$

with corresponding vector

$$\mathbf{e}^m = \left[e^m_{1,1}, \cdots, e^m_{N_x,1}, e^m_{1,2}, \cdots, e^m_{N_x,2}, \cdots, e^m_{1,N_y}, \cdots, e^m_{N_x,N_y}\right]^T, \tag{4.7}$$

and denote the local truncation error vector by

$$\mathbf{R}^m = \left[R^m_{1,1}, \cdots, R^m_{N_x,1}, R^m_{1,2}, \cdots, R^m_{N_x,2}, \cdots, R^m_{1,N_y}, \cdots, R^m_{N_x,N_y}\right]^T. \tag{4.8}$$

From (3.5) and (3.12), we have the error equation

$$\left(\mathbf{I} - \frac{1}{2}\mathbf{A}\right)\mathbf{e}^m = \left(\mathbf{I} + \frac{1}{2}\mathbf{A}\right)\mathbf{e}^{m-1} + \Delta t \mathbf{R}^m, \quad 1 \le m \le M, \tag{4.9}$$

and $\mathbf{e}^0 = \mathbf{0}$.

For any N-dimensional vector

$$\mathbf{u} = \left[u_{1,1}, \cdots, u_{N_x,1}, u_{1,2}, \cdots, u_{N_x,2}, \cdots, u_{1,N_y}, \cdots, u_{N_x,N_y}\right]^T, \tag{4.10}$$

we let the discrete L^2 norm of $\mathbf{u}$ in this paper be

$$\|\mathbf{u}\|_2 = \left(h_x h_y \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} |u_{i,j}|^2\right)^{\frac{1}{2}}. \tag{4.11}$$

We summarize some useful properties of the alternating fractional binomial coefficient $g_j^{(\alpha)}$ in the following lemma [3, 5, 25, 26, 43], and establish the stability of the Crank-Nicolson scheme under the assumption of continuity of seepage flow ($\beta_1 = \beta_2 = 1$).

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Lemma 4.1. *Let $g_j^{(\alpha)}$ be defined in (2.8). We have*

$$\begin{cases} \sum_{j=0}^{\infty} g_j^{(\alpha)} = 0, \quad g_0^{(\alpha)} = 1, \quad g_1^{(\alpha)} = -\alpha, & \text{for } \alpha > 0; \\ g_1^{(\alpha)} < g_2^{(\alpha)} < \cdots < 0, & \text{for } 0 < \alpha < 1; \\ g_2^{(\alpha)} > g_3^{(\alpha)} > \cdots > 0, & \text{for } 1 < \alpha < 2; \\ \sum_{j=0}^{n} g_j^{(\beta)} g_{n-j}^{(\alpha)} = g_n^{(\alpha+\beta)}, \quad n = 0,1,\cdots, & \text{for } \alpha,\beta > 0. \end{cases} \tag{4.12}$$

Theorem 4.1. *If $\beta_1 = \beta_2 = 1$, $k_x(x,y)$ decreases monotonically with respect to x and $k_y(x,y)$ decreases monotonically with respect to y in Ω, then the Crank-Nicolson scheme defined by* (4.4) *is unconditionally stable.*

Proof. Our first goal is to show that the eigenvalues of the matrix **A** have negative real parts. For $\beta = 1$, we have $g_0^{(\beta)} = 1$, $g_1^{(\beta)} = -1$ and $g_j^{(\beta)} = 0 (j \geq 2)$. Then the entries of matrix **A** may be rewritten as

$$(A_{j,j})_{i,s} := r_1 \left(g_{i+1-s}^{(\alpha_1)} k_{i+1,j}^x - g_{i-s}^{(\alpha_1)} k_{i,j}^x \right), \quad s \leq i-1, \tag{4.13a}$$

$$(A_{j,j})_{i,i} := r_1 \left(g_1^{(\alpha_1)} k_{i+1,j}^x - g_0^{(\alpha_1)} k_{i,j}^x \right) + r_2 \left(g_1^{(\alpha_2)} k_{i,j+1}^y - g_0^{(\alpha_2)} k_{i,j}^y \right), \tag{4.13b}$$

$$(A_{j,j})_{i,i+1} := r_1 k_{i+1,j}^x, \tag{4.13c}$$

$$(A_{j,j})_{i,s} := 0, \quad s > i+1, \tag{4.13d}$$

$$(A_{j,\tilde{s}})_{i,s} := \delta_{i,s} r_2 \left(g_{j+1-\tilde{s}}^{(\alpha_2)} k_{i,j+1}^y - g_{j-\tilde{s}}^{(\alpha_2)} k_{i,j}^y \right), \quad \tilde{s} \leq j-1, \tag{4.13e}$$

$$(A_{j,j+1})_{i,s} := \delta_{i,s} r_2 k_{i,j+1}^y, \tag{4.13f}$$

$$(A_{j,\tilde{s}})_{i,s} := 0, \quad \tilde{s} > j+1. \tag{4.13g}$$

Due to the monotonicity of $k_x(x,y)$ and $k_y(x,y)$, we have $0 < k_{i+1,j}^x < k_{i,j}^x$ and $0 < k_{i,j+1}^y < k_{i,j}^y$ for $i = 1,\cdots,N_x$, $j = 1,\cdots,N_y$. From Lemma 4.1, we have

$$\begin{cases} g_{i+1-s}^{(\alpha_1)} k_{i+1,j}^x - g_{i-s}^{(\alpha_1)} k_{i,j}^x > 0, \quad s \leq i-1, \\ g_{j+1-\tilde{s}}^{(\alpha_2)} k_{i,j+1}^y - g_{j-\tilde{s}}^{(\alpha_2)} k_{i,j}^y > 0, \quad \tilde{s} \leq j-1. \end{cases} \tag{4.14}$$

Therefore $(A_{j,j})_{i,s}$, $(A_{j,\tilde{s}})_{i,i} \geq 0$, for $i \neq s$, $j \neq \tilde{s}$, and

$$(A_{j,j})_{i,i} = -r_1(\alpha_1 k_{i+1,j}^x + k_{i,j}^x) - r_2(\alpha_2 k_{i,j+1}^y + k_{i,j}^y) < 0. \tag{4.15}$$

According to the Gerschgorin theorem (see [14]), the eigenvalues of the matrix **A** lie in the union of disks centered at $(A_{j,j})_{i,i}$ with radius

$$\sum_{s=1,s\neq i}^{N_x} |((A_{j,j})_{i,s})| + \sum_{\tilde{s}=1,\tilde{s}\neq j}^{N_y} \sum_{s=1}^{N_x} |(A_{j,\tilde{s}})_{i,s}|.$$

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We estimate

$$\begin{aligned}
&\sum_{s=1,s\neq i}^{N_x}|((A_{j,j})_{i,s})|+\sum_{\tilde{s}=1,\tilde{s}\neq j}^{N_y}\sum_{s=1}^{N_x}|(A_{j,\tilde{s}})_{i,s}|\\
=&\sum_{s=1,s\neq i}^{N_x}|((A_{j,j})_{i,s})|+\sum_{\tilde{s}=1,\tilde{s}\neq j}^{N_y}|(A_{j,\tilde{s}})_{i,i}|\\
=&r_1\left(\sum_{s=1}^{i-1}\left(g_{i+1-s}^{(\alpha_1)}k_{i+1,j}^x-g_{i-s}^{(\alpha_1)}k_{i,j}^x\right)+k_{i+1,j}^x\right)+r_2\left(\sum_{\tilde{s}=1}^{j-1}\left(g_{j+1-\tilde{s}}^{(\alpha_2)}k_{i,j+1}^y-g_{j-\tilde{s}}^{(\alpha_2)}k_{i,j}^y\right)+k_{i,j+1}^y\right)\\
=&r_1\left(\sum_{s=0,s\neq 1}^{i}g_s^{(\alpha_1)}k_{i+1,j}^x-\sum_{s=1}^{i-1}g_s^{(\alpha_1)}k_{i,j}^x\right)+r_2\left(\sum_{\tilde{s}=0,\tilde{s}\neq 1}^{j}g_{\tilde{s}}^{(\alpha_2)}k_{i,j+1}^y-\sum_{\tilde{s}=1}^{j-1}g_{\tilde{s}}^{(\alpha_2)}k_{i,j}^y\right)\\
=&r_1\left(-\left(\sum_{s=i+1}^{\infty}g_s^{(\alpha_1)}+g_1^{(\alpha_1)}\right)k_{i+1,j}^x-\sum_{s=1}^{i-1}g_s^{(\alpha_1)}k_{i,j}^x\right)\\
&+r_2\left(-\left(\sum_{\tilde{s}=j+1}^{\infty}g_{\tilde{s}}^{(\alpha_2)}+g_1^{(\alpha_2)}\right)k_{i,j+1}^y-\sum_{\tilde{s}=1}^{j-1}g_{\tilde{s}}^{(\alpha_2)}k_{i,j}^y\right)\\
<&r_1\left(-g_1^{(\alpha_1)}k_{i+1,j}^x-\sum_{s=1,s\neq i}^{\infty}g_s^{(\alpha_1)}k_{i,j}^x\right)+r_2\left(-g_1^{(\alpha_2)}k_{i,j+1}^y-\sum_{\tilde{s}=1,\tilde{s}\neq j}^{\infty}g_{\tilde{s}}^{(\alpha_2)}k_{i,j}^y\right)\\
<&r_1\left(-g_1^{(\alpha_1)}k_{i+1,j}^x+g_0^{(\alpha_1)}k_{i,j}^x\right)+r_2\left(-g_1^{(\alpha_2)}k_{i,j+1}^y+g_0^{(\alpha_2)}k_{i,j}^y\right)\\
=&-(A_{j,j})_{i,i}.
\end{aligned}$$

This implies that all of the Greschgorin disks of matrix $\mathbf{A}$ are within the left half of the complex plane. Thus, the eigenvalues of the matrix $\mathbf{A}$ have negative real-parts.

Next, λ is an eigenvalue of matrix $\mathbf{A}$ if and only if $1-\lambda/2$ is an eigenvalue of the matrix $(\mathbf{I}-\frac{1}{2}\mathbf{A})$, if and only if $(1+\lambda/2)/(1-\lambda/2)$ is an eigenvalue of the matrix $(\mathbf{I}-\frac{1}{2}\mathbf{A})^{-1}(\mathbf{I}+\frac{1}{2}\mathbf{A})$. We observe that the first part of this statement implies that all the eigenvalues of the matrix $(\mathbf{I}-\frac{1}{2}\mathbf{A})$ have a magnitude larger than 1, and thus this matrix is invertible. Furthermore, since the real part of λ is negative, it is easy to see that

$$\left|\frac{1+\frac{\lambda}{2}}{1-\frac{\lambda}{2}}\right|<1.$$

Therefore, the spectral radius of the system matrix $(\mathbf{I}-\frac{1}{2}\mathbf{A})^{-1}(\mathbf{I}+\frac{1}{2}\mathbf{A})$ is less than one. Thus, the Crank-Nicolson scheme (4.4) is unconditionally stable. □

The stability of the Crank-Nicolson scheme is discussed in Theorem 4.2 under the assumption that $k_x(x,y)$ and $k_y(x,y)$ are constant functions in Ω.

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Theorem 4.2. *If $1<\alpha_l+\beta_l<2$ for $l=1,2$, $k_x(x,y)=K_x$ and $k_y(x,y)=K_y$ for $(x,y)\in\Omega$, where K_x and K_y are positive constants, then the Crank-Nicolson scheme defined by* (4.4) *is unconditionally stable, and there exists a constant C independent of Δt, h_x and h_y such that*

$$\|\mathbf{e}^m\|_2 \le C((\Delta t)^2+h_x+h_y), \tag{4.16}$$

for $1\le m\le M$.

Proof. From Lemma 4.1, the entries of matrix **A** may be rewritten as

$$(A_{j,j})_{i,s} := r_1 g_{i+1-s}^{(\alpha_1+\beta_1)} K_x, \quad s\le i-1, \tag{4.17a}$$
$$(A_{j,j})_{i,i} := r_1 g_1^{(\alpha_1+\beta_1)} K_x + r_2 g_1^{(\alpha_2+\beta_2)} K_y, \tag{4.17b}$$
$$(A_{j,j})_{i,i+1} := r_1 K_x, \tag{4.17c}$$
$$(A_{j,j})_{i,s} := 0, \quad s>i+1, \tag{4.17d}$$
$$(A_{j,\tilde{s}})_{i,s} := \delta_{i,s} r_2 g_{j+1-\tilde{s}}^{(\alpha_2+\beta_2)} K_y, \quad \tilde{s}\le j-1, \tag{4.17e}$$
$$(A_{j,j+1})_{i,s} := \delta_{i,s} r_2 K_y, \tag{4.17f}$$
$$(A_{j,\tilde{s}})_{i,s} := 0, \quad \tilde{s}>j+1, \tag{4.17g}$$

For $1<\alpha_l+\beta_l<2$ $(l=1,2)$, we have

$$g_1^{(\alpha_l+\beta_l)} = -(\alpha_l+\beta_l), \quad g_s^{(\alpha_l+\beta_l)} > 0 \ \ (s\neq 1), \quad \sum_{s=0}^{n} g_s^{(\alpha_l+\beta_l)} < 0,$$

where n is a positive integer. By the Gerschgorin theorem, the eigenvalues of the matrix **A** lie in the union of disks centered at $(A_{j,j})_{i,i}$ with radius

$$\begin{aligned}
&\sum_{s=1,s\neq i}^{N_x} |((A_{j,j})_{i,s})| + \sum_{\tilde{s}=1,\tilde{s}\neq j}^{N_y} \sum_{s=1}^{N_x} |(A_{j,\tilde{s}})_{i,s}| \\
&= r_1 \sum_{s=0,s\neq 1}^{i} g_s^{(\alpha_1+\beta_1)} K_x + r_2 \sum_{\tilde{s}=0,\tilde{s}\neq 1}^{j} g_{\tilde{s}}^{(\alpha_2+\beta_2)} K_y \\
&< -\left(r_1 g_1^{(\alpha_1+\beta_1)} K_x + r_2 g_1^{(\alpha_2+\beta_2)} K_y \right) \\
&= -(A_{j,j})_{i,i}.
\end{aligned} \tag{4.18}$$

Hence every eigenvalue of the matrix **A** has negative real-part. Similar to the proof of Theorem 4.1, we can conclude the results of Theorem 4.2.

We now consider the convergence of the Crank-Nicolson scheme (4.4). From (4.17),

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we have an inequality similar to (4.18) as follow

$$\begin{aligned}
&\sum_{s=1,s\neq i}^{N_x}|(A_{j,j})_{s,i}|+\sum_{\tilde{s}=1,\tilde{s}\neq j}^{N_y}\sum_{s=1}^{N_x}|(A_{\tilde{s},j})_{s,i}|\\
=&r_1\sum_{s=0,s\neq 1}^{i}g_s^{(\alpha_1+\beta_1)}K_x+r_2\sum_{\tilde{s}=0,\tilde{s}\neq 1}^{j}g_{\tilde{s}}^{(\alpha_2+\beta_2)}K_y\\
<&-\left(r_1g_1^{(\alpha_1+\beta_1)}K_x+r_2g_1^{(\alpha_2+\beta_2)}K_y\right)\\
=&-(A_{j,j})_{i,i}.
\end{aligned}\tag{4.19}$$

Combining (4.18) and (4.19) and using the Gerschgorin theorem, we have that the eigenvalues of the matrix $\mathbf{A}+\mathbf{A}^T$ are negative. Since $\mathbf{A}+\mathbf{A}^T$ is symmetric, it is negative definite. Thus we know

$$\begin{aligned}
\left\|\left(\mathbf{I}-\frac{1}{2}\mathbf{A}\right)\mathbf{e}^m\right\|_2^2&=\|\mathbf{e}^m\|_2^2-\frac{h_xh_y}{2}(\mathbf{e}^m)^T\left(\mathbf{A}+\mathbf{A}^T\right)\mathbf{e}^m+\frac{1}{4}\|\mathbf{A}\mathbf{e}^m\|_2^2\\
&\geq\|\mathbf{e}^m\|_2^2+\frac{1}{4}\|\mathbf{A}\mathbf{e}^m\|_2^2,
\end{aligned}\tag{4.20}$$

$$\begin{aligned}
\left\|\left(\mathbf{I}+\frac{1}{2}\mathbf{A}\right)\mathbf{e}^{m-1}\right\|_2^2&=\|\mathbf{e}^{m-1}\|_2^2+\frac{h_xh_y}{2}\left(\mathbf{e}^{m-1}\right)^T\left(\mathbf{A}+\mathbf{A}^T\right)\mathbf{e}^{m-1}+\frac{1}{4}\left\|\mathbf{A}\mathbf{e}^{m-1}\right\|_2^2\\
&\leq\|\mathbf{e}^{m-1}\|_2^2+\frac{1}{4}\left\|\mathbf{A}\mathbf{e}^{m-1}\right\|_2^2
\end{aligned}\tag{4.21}$$

for $1\leq m\leq M$. Taking the discrete L^2 norm on the both sides of (4.9) implies

$$\left\|\left(\mathbf{I}-\frac{1}{2}\mathbf{A}\right)\mathbf{e}^m\right\|_2\leq\left\|\left(\mathbf{I}+\frac{1}{2}\mathbf{A}\right)\mathbf{e}^{m-1}\right\|_2+\Delta t\|\mathbf{R}^m\|_2,\quad 1\leq m\leq M.\tag{4.22}$$

Together with (4.20) and (4.21), we have

$$\sqrt{\|\mathbf{e}^m\|_2^2+\frac{1}{4}\|\mathbf{A}\mathbf{e}^m\|_2^2}\leq\sqrt{\|\mathbf{e}^{m-1}\|_2^2+\frac{1}{4}\left\|\mathbf{A}\mathbf{e}^{m-1}\right\|_2^2}+\Delta t\|\mathbf{R}^m\|_2.\tag{4.23}$$

Summing up (4.23) from 1 to m leads to

$$\sqrt{\|\mathbf{e}^m\|_2^2+\frac{1}{4}\|\mathbf{A}\mathbf{e}^m\|^2}\leq\Delta t\sum_{l=1}^{m}\|\mathbf{R}^l\|_2,\quad 1\leq m\leq M,\tag{4.24}$$

where $\mathbf{e}^0=\mathbf{0}$ is considered. Noticing that

$$|R_{i,j}^m|\leq c((\Delta t)^2+h_x+h_y),\tag{4.25}$$

for $1\leq i\leq N_x$, $1\leq j\leq N_y$, $1\leq m\leq M$, we obtain

$$\|\mathbf{e}^m\|_2\leq cm\Delta t((\Delta t)^2+h_x+h_y)\leq C((\Delta t)^2+h_x+h_y).\tag{4.26}$$

This completes the proof. □

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For the case $1<\alpha_l+\beta_l<2$ $(l=1,2)$, where the functions $k_x(x,y)$ and $k_y(x,y)$ are not constants in Ω, some numerical experiments show that the Crank-Nicolson scheme (4.4) is also unconditionally stable and convergent. We provide evidence in Table 2 for Example 6.1 in Section 6.

5 Improving the order of convergence by extrapolation

The stability of the Crank-Nicolson scheme was shown under two assumptions in Section 4, respectively. The method is consistent with a local truncation error which is $\mathcal{O}((\Delta t)^2)+\mathcal{O}(h_x)+\mathcal{O}(h_y)$. By the fact that the errors in the approximations of the mixed fractional derivatives in (1.1) with respect to x and y have the form $\zeta_x(x,y,t)h_x+\mathcal{O}(h_x^2)$ and $\zeta_y(x,y,t)h_y+\mathcal{O}(h_y^2)$, respectively, where $\zeta_x(x,y,t)$ and $\zeta_y(x,y,t)$ do not depend on the grid size h_x or h_y, it follows that the Richardson extrapolation method [14] can be used to obtain a solution with local truncation error $\mathcal{O}((\Delta t)^2)+\mathcal{O}(h_x^2)+\mathcal{O}(h_y^2)$. More specifically, we may compute two Crank-Nicolson solutions $p_{\Delta t,h_x,h_y}(x,y,t_m)$ and $p_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}(x,y,t_m)$ for the grid sizes (h_x,h_y) and $(\frac{h_x}{2},\frac{h_y}{2})$ with the same Δt at each time step respectively, and then compute the extrapolation solution by

$$p^{ex}_{\Delta t,h_x,h_y}(x,y,t_m)=2p_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}(x,y,t_m)-p_{\Delta t,h_x,h_y}(x,y,t_m) \tag{5.1}$$

to obtain the $\mathcal{O}((\Delta t)^2)+\mathcal{O}(h_x^2)+\mathcal{O}(h_y^2)$ accuracy, where (x,y) is a grid point on the coarse spatial grid. In other words, $(x,y)=(x_i,y_j)$ on the coarse grid of size (h_x,h_y), while $(x,y)=(x_{2i},y_{2j})$ on the fine grid of size $(\frac{h_x}{2},\frac{h_y}{2})$.

6 Numerical examples

In this section, we carry out numerical experiments to demonstrate the effectiveness of the second-order accurate finite difference method. For a fixed time step Δt, we use $p_{\Delta t,h_x,h_y}$, $p_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$ and $p^{ex}_{\Delta t,h_x,h_y}$ to denote the Crank-Nicolson solution with grid size (h_x,h_y), the Crank-Nicolson solution with grid size $(\frac{h_x}{2},\frac{h_y}{2})$, and the corresponding extrapolated solution generated by (5.1), respectively. In the numerical experiments, we compute the discrete infinity norms of the error $p-p_{\Delta t,h_x,h_y}$, $p-p_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$ and $p-p^{ex}_{\Delta t,h_x,h_y}$ at time $T=M\Delta t$ as

$$E^{(\infty)}_{\Delta t,h_x,h_y}=\max_{1\le i\le N_x,1\le j\le N_y}|P^M_{i,j}-p_{\Delta t,h_x,h_y}(x_i,y_j,t_M)|, \tag{6.1}$$

$$E^{(\infty)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}=\max_{1\le i\le 2N_x+1,1\le j\le 2N_y+1}|P^M_{i,j}-p_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}(x_i,y_j,t_M)|, \tag{6.2}$$

$$E^{ex,(\infty)}_{\Delta t,h_x,h_y}=\max_{1\le i\le N_x,1\le j\le N_y}|P^M_{i,j}-p^{ex}_{\Delta t,h_x,h_y}(x_i,y_j,t_M)|, \tag{6.3}$$

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and the discrete L^2 norms of the errors as

$$E^{(2)}_{\Delta t,h_x,h_y}=\left(h_xh_y\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}|P^M_{i,j}-p_{\Delta t,h_x,h_y}(x_i,y_j,t_M)|^2\right)^{\frac{1}{2}}, \tag{6.4}$$

$$E^{(2)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}=\left(\frac{h_xh_y}{4}\sum_{i=1}^{2N_x+1}\sum_{j=1}^{2N_y+1}|P^M_{i,j}-p_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}(x_i,y_j,t_M)|^2\right)^{\frac{1}{2}}, \tag{6.5}$$

$$E^{ex,(2)}_{\Delta t,h_x,h_y}=\left(h_xh_y\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}|P^M_{i,j}-p^{ex}_{\Delta t,h_x,h_y}(x_i,y_j,t_M)|^2\right)^{\frac{1}{2}}. \tag{6.6}$$

Let

$$G^{(\kappa)}_{\Delta t,h_x,h_y}=\log_2\frac{E^{(\kappa)}_{2\Delta t,2h_x,2h_y}}{E^{(\kappa)}_{\Delta t,h_x,h_y}},\quad G^{ex,(\kappa)}_{\Delta t,h_x,h_y}=\log_2\frac{E^{ex,(\kappa)}_{2\Delta t,2h_x,2h_y}}{E^{ex,(\kappa)}_{\Delta t,h_x,h_y}},\quad \kappa\in\{2,\infty\}$$

reflect the convergence order of the Crank-Nicolson method and extrapolated Crank-Nicolson method, respectively.

The algorithm was implemented in Fortran 90 on a Thinkpad E420 laptop with configuration: Intel(R) Core(TM) i3-2350M CPU 2.30GHZ and 4GB RAM. All computations were performed in double precision.

Example 6.1. We consider the two-dimensional fractional percolation equation (1.1) with the following data [6]. The spatial domain is $\Omega=\{(x,y)|0\le x\le 1,0\le y\le 1\}$. The percolation coefficients are $k_x(x,y)=2-x^2$ and $k_y(x,y)=2-y^2$, which are decreases monotonically with respect to x and y, respectively. The source term $f(x,y,t)$ is given by

$$\begin{aligned}f(x,y,t)=&-\frac{\Gamma(3)y^2\exp(-t)}{\Gamma(3-\alpha_1)}\left(\frac{2\Gamma(3-\alpha_1)x^{2-\alpha_1-\beta_1}}{\Gamma(3-\alpha_1-\beta_1)}-\frac{\Gamma(5-\alpha_1)x^{4-\alpha_1-\beta_1}}{\Gamma(5-\alpha_1-\beta_1)}\right)\\&-\frac{\Gamma(3)x^2\exp(-t)}{\Gamma(3-\alpha_2)}\left(\frac{2\Gamma(3-\alpha_2)y^{2-\alpha_2-\beta_2}}{\Gamma(3-\alpha_2-\beta_2)}-\frac{\Gamma(5-\alpha_2)y^{4-\alpha_2-\beta_2}}{\Gamma(5-\alpha_2-\beta_2)}\right)-x^2y^2\exp(-t).\end{aligned} \tag{6.7}$$

The exact solution of this problem is

$$p(x,y,t)=\exp(-t)x^2y^2 \tag{6.8}$$

which satisfies the initial condition

$$p_0(x,y)=x^2y^2, \tag{6.9}$$

and the Dirichlet boundary conditions

$$p(0,y,t)=p(x,0,t)=0,\quad p(1,y,t)=\exp(-t)y^2,\quad p(x,1,t)=\exp(-t)x^2,\quad (x,y)\in\Omega. \tag{6.10}$$

This example problem is solved to time $T=1$.

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Table 1: Error behavior for the Crank-Nicolson scheme and the extrapolated Crank-Nicolson scheme for Example 6.1 with $\alpha_1=0.5$, $\beta_1=1$, $\alpha_2=0.5$, $\beta_2=1$, $\Delta t=h_x=h_y$ at time $T=1$.

Δt	$E^{(\infty)}_{\Delta t,h_x,h_y}$	$G^{(\infty)}_{\Delta t,h_x,h_y}$	$E^{(\infty)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$G^{(\infty)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$E^{ex,(\infty)}_{\Delta t,h_x,h_y}$	$G^{ex,(\infty)}_{\Delta t,h_x,h_y}$
2^{-3}	8.813e-3	-	4.620e-3	-	7.981e-4	-
2^{-4}	4.584e-3	0.943	2.338e-3	0.982	1.792e-4	2.154
2^{-5}	2.351e-3	0.963	1.186e-3	0.978	3.399e-5	2.398
2^{-6}	1.190e-3	0.982	5.974e-4	0.990	8.488e-6	2.001
2^{-7}	5.983e-4	0.993	2.997e-4	0.995	2.124e-6	1.998
Δt	$E^{(2)}_{\Delta t,h_x,h_y}$	$G^{(2)}_{\Delta t,h_x,h_y}$	$E^{(2)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$G^{(2)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$E^{ex,(2)}_{\Delta t,h_x,h_y}$	$G^{ex,(2)}_{\Delta t,h_x,h_y}$
2^{-3}	4.124e-3	-	2.132e-3	-	1.941e-4	-
2^{-4}	2.174e-3	0.923	1.099e-3	0.956	4.101e-5	2.242
2^{-5}	1.110e-3	0.969	5.576e-4	0.978	9.969e-6	2.040
2^{-6}	5.604e-4	0.986	2.807e-4	0.990	2.521e-6	1.983
2^{-7}	2.830e-4	0.986	1.414e-4	0.989	6.302e-7	2.000

Table 2: Error behavior for the Crank-Nicolson scheme and the extrapolated Crank-Nicolson scheme for Example 6.1 with $\alpha_1=0.7$, $\beta_1=0.8$, $\alpha_2=0.6$, $\beta_2=0.9$, $\Delta t=h_x=h_y$ at time $T=1$.

Δt	$E^{(\infty)}_{\Delta t,h_x,h_y}$	$G^{(\infty)}_{\Delta t,h_x,h_y}$	$E^{(\infty)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$G^{(\infty)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$E^{ex,(\infty)}_{\Delta t,h_x,h_y}$	$G^{ex,(\infty)}_{\Delta t,h_x,h_y}$
2^{-3}	9.243e-3	-	4.820e-3	-	7.846e-4	-
2^{-4}	4.796e-3	0.946	2.449e-3	0.977	1.770e-4	2.148
2^{-5}	2.465e-3	0.960	1.240e-3	0.982	3.275e-5	2.434
2^{-6}	1.244e-3	0.986	6.245e-4	0.990	8.205e-6	1.997
2^{-7}	6.256e-4	0.993	3.133e-4	0.995	2.051e-6	1.999
Δt	$E^{(2)}_{\Delta t,h_x,h_y}$	$G^{(2)}_{\Delta t,h_x,h_y}$	$E^{(2)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$G^{(2)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$E^{ex,(2)}_{\Delta t,h_x,h_y}$	$G^{ex,(2)}_{\Delta t,h_x,h_y}$
2^{-3}	4.394e-3	-	2.261e-3	-	1.902e-4	-
2^{-4}	2.310e-3	0.927	1.165e-3	0.956	4.021e-5	2.241
2^{-5}	1.178e-3	0.971	5.908e-4	0.979	9.913e-6	2.020
2^{-6}	5.940e-4	0.987	2.974e-4	0.990	2.535e-6	1.967
2^{-7}	2.951e-4	1.009	1.503e-4	0.984	6.327e-7	2.002

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Example 6.2. In this example, we study the case for which the percolation coefficients are constants. Consider the fractional percolation equation (1.1) on the finite domain $\Omega=\{(x,y)|0\leq x\leq 1,0\leq y\leq 1\}$, with the percolation coefficients

$$k_x(x,y)=k_y(x,y)=1, \tag{6.11}$$

the source term

$$\begin{aligned} f(x,y,t)=&-\exp(1-t)\left(x^2(1-x)^2y^2(1-y)^2\right.\\ &+y^2(1-y)^2\left(\frac{\Gamma(3)x^{2-\alpha_1-\beta_1}}{\Gamma(3-\alpha_1-\beta_1)}-\frac{2\Gamma(4)x^{3-\alpha_1-\beta_1}}{\Gamma(4-\alpha_1-\beta_1)}+\frac{\Gamma(5)x^{4-\alpha_1-\beta_1}}{\Gamma(5-\alpha_1-\beta_1)}\right)\\ &\left.+x^2(1-x)^2\left(\frac{\Gamma(3)y^{2-\alpha_2-\beta_2}}{\Gamma(3-\alpha_2-\beta_2)}-\frac{2\Gamma(4)y^{3-\alpha_2-\beta_2}}{\Gamma(4-\alpha_2-\beta_2)}+\frac{\Gamma(5)y^{4-\alpha_2-\beta_2}}{\Gamma(5-\alpha_2-\beta_2)}\right)\right), \end{aligned} \tag{6.12}$$

the initial condition

$$p_0(x,y)=\exp(1)x^2(1-x)^2y^2(1-y)^2 \tag{6.13}$$

for $(x,y)\in\Omega$ and the homogeneous Dirichlet boundary conditions. The exact solution to the corresponding fractional percolation equation is

$$p(x,y,t)=\exp(1-t)x^2(1-x)^2y^2(1-y)^2. \tag{6.14}$$

This example problem is solved to time $T=1$.

We present the discrete infinity norms and discrete L^2 norms of the errors for the numerical solutions of the Crank-Nicolson method and the extrapolated Crank-Nicolson method in Table 1, for Example 6.1 with $\alpha_1=0.5$, $\beta_1=1$, $\alpha_2=0.5$, $\beta_2=1$. Table 2 reports the error behavior for Example 6.1 with $\alpha_1=0.7$, $\beta_1=0.8$, $\alpha_2=0.6$, $\beta_2=0.9$, and Table 3 reports that for Example 6.2 with $\alpha_1=0.85$, $\beta_1=0.85$, $\alpha_2=0.75$, $\beta_2=0.95$.

As can be seen, when the time step Δt is fixed, the extrapolated solution performs well than both the Crank-Nicolson solution on the coarse grid of size (h_x,h_y) and that on the fine grid of size $(\frac{h_x}{2},\frac{h_y}{2})$. From the numerical results under the parameter setting in Table 2 for Example 6.1, we observe that the convergence condition of the Crank-Nicolson scheme can be weaken. In Tables 1-3, the behavior of the maximum error is almost linear when the Crank-Nicolson method is used, this is because the second-order convergence in time t is masked by the linear order of convergence in space. More importantly, the seventh column in each table shows that the convergence of the extrapolated Crank-Nicolson method is of order $\mathcal{O}((\Delta t)^2)+\mathcal{O}(h_x^2)+\mathcal{O}(h_y^2)$, which supports our theoretical analysis.

7 Conclusions

In this paper, we have presented a general approximation for the mixed fractional derivatives and developed a second-order accurate method which combines the Crank-Nicolson

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Table 3: Error behavior for the Crank-Nicolson scheme and the extrapolated Crank-Nicolson scheme for Example 6.2 with $\alpha_1=0.85$, $\beta_1=0.85$, $\alpha_2=0.75$, $\beta_2=0.95$, $\Delta t=h_x=h_y$ at time $T=1$.

Δt	$E^{(\infty)}_{\Delta t,h_x,h_y}$	$G^{(\infty)}_{\Delta t,h_x,h_y}$	$E^{(\infty)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$G^{(\infty)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$E^{ex,(\infty)}_{\Delta t,h_x,h_y}$	$G^{ex,(\infty)}_{\Delta t,h_x,h_y}$
2^{-3}	2.737e-4	-	1.263e-4	-	8.042e-5	-
2^{-4}	1.180e-4	1.213	6.261e-5	1.012	2.080e-5	1.951
2^{-5}	6.198e-5	0.929	3.243e-5	0.948	5.262e-6	1.983
2^{-6}	3.226e-5	0.942	1.651e-5	0.974	1.317e-6	1.998
2^{-7}	1.646e-5	0.970	8.329e-6	0.987	3.384e-7	1.961
Δt	$E^{(2)}_{\Delta t,h_x,h_y}$	$G^{(2)}_{\Delta t,h_x,h_y}$	$E^{(2)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$G^{(2)}_{\Delta t,\frac{h_x}{2},\frac{h_y}{2}}$	$E^{ex,(2)}_{\Delta t,h_x,h_y}$	$G^{ex,(2)}_{\Delta t,h_x,h_y}$
2^{-3}	1.352e-4	-	6.340e-5	-	4.457e-5	-
2^{-4}	6.209e-5	1.122	3.042e-5	1.059	1.029e-5	2.114
2^{-5}	3.036e-5	1.032	1.510e-5	1.010	2.499e-6	2.041
2^{-6}	1.509e-5	1.008	7.540e-6	1.001	6.182e-7	2.015
2^{-7}	7.652e-6	0.979	3.804e-6	0.987	1.624e-7	1.928

scheme with the Richardson extrapolation for the two-dimensional fractional percolation equations. We discussed the consistency and stability of the Crank-Nicolson scheme and proved relevant theorems. Thanks to the expansion of the operator which is used in approximating the mixed fractional spatial derivatives, the convergence order was improved from $\mathcal{O}(\Delta t)+\mathcal{O}(h_x)+\mathcal{O}(h_y)$ to $\mathcal{O}((\Delta t)^2)+\mathcal{O}(h_x^2)+\mathcal{O}(h_y^2)$ by employing the Richardson extrapolation. Numerical experiments verified the accuracy and efficiency of the extrapolated Crank-Nicolson method.

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