J. Plasma Physics (1998), *vol.* 59, *part* 1, *pp.* 125–149. Printed in the United Kingdom

© 1998 Cambridge University Press

Magnetohydrodynamical leaky waves in a layered plasma

ALEJANDRO G. GONZÁLEZ, JULIO GRATTON and SILVIA B. FARINA

INFIP–Instituto de Física del Plasma, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Pabellón I, Ciudad Universitaria, (1428) Buenos Aires, Argentina

(Received 24 July 1996 and in revised form 18 December 1996)

We investigate the existence of MHD leaky waves in compressible and layered plasmas. We consider perturbations of complex frequency in a slab model with three layers. We develop a general method that includes the 'cubic modes' as well as other kinds of leaky waves, and allows us to derive many results in closed form. There are several physical mechanisms that can produce leakage. We give two numerical examples to exhibit the different behaviours that can arise. Finally we comment on the connection of the present results with those obtained by other authors, and correct some errors.

1. Introduction

Leaky waves may appear in structured plasmas where a perturbation is excited in one layer and loses energy transmitting it to another region where it can propagate away. They have been related to the coronal heating of the Sun (Chen and Hasegawa 1974; Ionson 1979). The process is non-dissipative, but analogous to the acoustic damping of a vibrating membrane in air. Similar processes occur in oceanography (Le Blond and Mysak 1978).

Several authors have analysed MHD waves of complex frequency in cylindrical configurations in the thin-tube approximation where the radius of the cylinder is small in comparison with the wavelength of the perturbation; see e.g. Roberts and Webb (1979) and Spruit (1982). Wilson (1981) considered a cylinder of arbitrary radius and allowed for complex frequencies, but his analysis is restricted, since no external magnetic field is included. A more general case was studied by Cally (1986) for an interface between two plasmas of equal β. He found a kind of perturbation called 'cubic modes' that correspond to leaky waves in H_α fibrils in the Sun (Giovanelli 1975). Čadež and Okretič (1989) considered a slab model with three layers with different β, but their analysis of the existence of leaky waves is partial and has important failures that will be discussed below.

In the present paper we consider perturbations of complex frequency in a slab model with three layers similar to that used by Čadež and Okretič. We develop a general method that includes the 'cubic modes' of Cally as well as other kinds of leaky waves, and allows us to derive many results in closed form for arbitrary equilibrium configurations. We discuss the physical mechanisms that lead to the different types of leakage that occur in a three-layer configuration. We

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

apply our method to two numerical cases in order to show the different behaviours that can arise. Finally we comment on the connection of the present results with those of Cally (1986) and Čadež and Okretič (1989).

2. The model

We assume an ideal compressible plasma in equilibrium, stratified along the y axis. We use the MHD equations in a plane slab geometry where

$$\mathbf{B} = B_x(y)\,\mathbf{e}_x + B_z(y)\,\mathbf{e}_z$$

is the magnetic field, $\rho = \rho(y)$ is the density, $p = p(y)$ is the pressure (the unperturbed equilibrium satisfies $d(p+\mathbf{B}\cdot\mathbf{B}/8\pi)/dy = 0$), $C_A = |B|\,(4\pi\rho)^{-1/2}$ and $C_S = (\gamma p/\rho)^{1/2}$. The linear adiabatic perturbations of the basic state satisfy the equation (Gratton *et al.* 1988)

$$\frac{d}{dy}\left[\mathscr{H}\left(\frac{1}{\mathscr{M}}-1\right)\frac{d\zeta}{dy}\right] + k^2\mathscr{H}\zeta = 0, \tag{1}$$

where ζ is the y component of the displacement of a normal mode of the type $\delta\mathscr{Q} = q(y)\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, $\mathbf{k} = (k_x, 0, k_z)$, $\psi(y)$ is the angle between $\mathbf{B}$ and $\mathbf{k}$, and

$$\mathscr{H} = \rho(C_A^2\,k^2\cos^2\psi - \omega^2), \qquad \mathscr{M} = 1 - \frac{k^2}{\omega^2}(C_A^2 + C_S^2) + \frac{C_A^2\,C_S^2\,k^4\cos^2\psi}{\omega^4}. \tag{2}$$

We consider three layers in which ρ, p, T and $\mathbf{B}$ are piecewise-constant, but they may be discontinuous at two interfaces located at $y = 0$ and $y = a$. Note that ρ, p and $\mathbf{B}$ in each layer are arbitrary, subject to the equilibrium conditions $\{(C_S^2/\gamma + \tfrac{1}{2}C_A^2)\,\rho\} = 0$ at each interface ($\{Q\}$ denotes the difference between the values of Q at the two sides of the interface).

The boundary conditions at the interfaces are

$$\left\{\mathscr{H}\left(\frac{1}{\mathscr{M}}-1\right)\frac{d\zeta}{dy}\right\} = 0. \qquad \{\zeta\} = 0. \tag{3}$$

These equations express the continuity of the normal component of the total stress and of the vertical displacement (see e.g. González and Gratton 1991). These conditions can be also obtained integrating (1) across the discontinuity and considering that ζ must be continuous.

In each layer the plasma is uniform and the solution of (1) is

$$\zeta_i = \zeta_{-,i}\exp(-k\Gamma_i y) + \zeta_{+,i}\exp(k\Gamma_i y),$$

where the subscript $i\,(i = 1, 2, 3)$ denotes the region and

$$\Gamma_i = \left[-\frac{(V_i^2 - V_{+,i}^2)\,(V_i^2 - V_{-,i}^2)}{F_i^2(V_i^2 - V_{B,i}^2)}\right]^{1/2}, \tag{4}$$

with $\mathrm{Re}(\Gamma_i) > 0$ or $\mathrm{Im}(\Gamma_i) > 0$ if $\mathrm{Re}(\Gamma_i) = 0$. We have introduced in (4) the following dimensionless quantities:

$$V_i = \frac{\omega}{kC_{S,i}}, \qquad A_i = \frac{C_{A,i}}{C_{S,i}} = \left(\frac{2}{\beta_i\gamma}\right)^{1/2}, \qquad F_i = (1 + A_i^2)^{1/2}, \tag{5}$$

$$V_{A,i} = A_i\cos\psi_i, \qquad V_{B,i} = \frac{A_i\cos\psi_i}{F_i}, \qquad V_{\pm,i}^2 = \tfrac{1}{2}[F_i^2 \pm (F_i^4 - 4V_{A,i}^2)^{1/2}]. \tag{6}$$

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

In these expressions V is the horizontal phase velocity of the perturbation, V_A is the phase velocity of the Alfvén waves and $V_\pm$ are the phase velocities of the magnetosonic waves (fast and slow waves) that propagate parallel to the interface, in units of the sound velocity $C_{S,i}$ in each region. Note that $V_i = V_j/M_{ij}$, where $M_{ij} = C_{S,i}/C_{S,j}$ depends on the ratio of temperatures of the layers i and j. Using these definitions, the continuity of the unperturbed total pressure across an interface can be expressed as $1+\beta_j = f_{ij}(1+\beta_i)$, with $f_{ij} = p_i/p_j$. In the general case the problem involves ten independent dimensionless parameters and variables: two pressure ratios, two temperature ratios, the β of one layer, the three ψ angles, $\alpha \equiv ka$ and V.

The stable modes have real and positive V^2. When $0 < V_i < V_{B,i}$ or $V_{-,i} < V_i < V_{+,i}$, Γ_i is real, and the perturbation grows or decreases exponentially with y. These perturbations are called evanescent or surface waves (denoted by 'E' in the following). If the region is not bounded, the solution may diverge, unless additional constraints are imposed (for example, for region 3 we must take $\zeta_{+,3} = 0$ if we look for bounded solutions). When $V_{B,i} < V_i < V_{-,i}$ or $V_i > V_{+,i}$, Γ_i is imaginary, and the perturbation propagates in the y direction (this is called a 'propagating' wave, and is denoted by 'P' in the following); in this case $\zeta_{+,i}$ and $\zeta_{-,i}$ are the amplitudes of waves whose y components of the phase velocity are along the positive and negative y directions respectively.

3. Dispersion relations

Consider two adjacent uniform media i and j, separated by an interface. Using the boundary conditions (3) and introducing the notation $G_i = (V_{A,i}^2 - V_i^2)/\Gamma_i$, we obtain (González and Gratton 1991)

$$\begin{bmatrix}\zeta_{-,j}\\ \zeta_{+,j}\end{bmatrix} = \frac{1}{2G_j f_{ji}}\begin{bmatrix}\mathscr{T}_{ji} & \mathscr{S}_{ji}\\ \mathscr{S}_{ji} & \mathscr{T}_{ji}\end{bmatrix}\begin{bmatrix}\zeta_{-,i}\\ \zeta_{+,i}\end{bmatrix}, \tag{7}$$

with

$$\mathscr{S}_{ji} = f_{ji}G_j - G_i, \qquad \mathscr{T}_{ji} = f_{ji}G_j + G_i. \tag{8}$$

The matrix defined by (7) and (8) describes the matching of waves at an interface, and allows us to solve problems with an arbitrary number of interfaces. In the present case we use it to match the solutions of regions 1 and 3. The coupling matrix is the product of (i) the matrix that matches the regions 1 and 2, (ii) the displacement matrix from $y = 0$ to $y = a$ and (iii) the matrix that matches regions 2 and 3. The displacement matrix relates the solutions of region 2 in $y = 0$ with those in $y = a$:

$$\begin{bmatrix}\zeta_{-,2}(y=a)\\ \zeta_{+,2}(y=a)\end{bmatrix} = \begin{bmatrix}e^{-\alpha\Gamma_2} & 0\\ 0 & e^{\alpha\Gamma_2}\end{bmatrix}\begin{bmatrix}\zeta_{-,2}(y=0)\\ \zeta_{+,2}(y=0)\end{bmatrix}, \tag{9}$$

with $\alpha \equiv ka$. Then the matching of the solutions in region 1 with those in region 3 is described by

$$\begin{bmatrix}\zeta_{-,3}\\ \zeta_{+,3}\end{bmatrix} = \frac{1}{Z}\begin{bmatrix}\mathscr{V} & \mathscr{U}\\ \mathscr{S} & \mathscr{T}\end{bmatrix}\begin{bmatrix}\zeta_{-,1}\\ \zeta_{+,1}\end{bmatrix}, \tag{10}$$

where

$$Z = 4G_2G_3f_{21}f_{32}\Phi^{1/2}, \qquad \Phi = e^{-2\alpha\Gamma_2}$$

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

Table 1. Dispersion relations for a three-layer plasma. The leaky waves belong to the $\mathscr{S}$, $\mathscr{V}$ and $\mathscr{U}$ branches.

Dispersion relation	Yields	Branches	Behaviour of solution	
			$y\to-\infty$	$y\to+\infty$
$\mathscr{T}\equiv\mathscr{S}_{32}\mathscr{S}_{21}\Phi+\mathscr{T}_{32}\mathscr{T}_{21}=0$	Normal modes	$\mathscr{T}$	Bounded ($\zeta_{-1}=0$)	Bounded ($\zeta_{+3}=0$)
$\mathscr{S}\equiv\mathscr{S}_{32}\mathscr{T}_{21}\Phi+\mathscr{T}_{32}\mathscr{S}_{21}=0$	Pseudomodes	$\mathscr{S}$	Diverges ($\zeta_{+1}=0$)	Bounded ($\zeta_{+3}=0$)
$\mathscr{V}\equiv\mathscr{T}_{32}\mathscr{T}_{21}\Phi+\mathscr{S}_{32}\mathscr{S}_{21}=0$	Pseudomodes	$\mathscr{V}$	Diverges ($\zeta_{+1}=0$)	Diverges ($\zeta_{-3}=0$)
$\mathscr{U}\equiv\mathscr{T}_{32}\mathscr{S}_{21}\Phi+\mathscr{S}_{32}\mathscr{T}_{21}=0$	Pseudomodes	$\mathscr{U}$	Bounded ($\zeta_{-1}=0$)	Diverges ($\zeta_{-3}=0$)

and

$$\mathscr{V}=\mathscr{T}_{32}\mathscr{T}_{21}\Phi+\mathscr{S}_{32}\mathscr{S}_{21},\qquad \mathscr{U}=\mathscr{T}_{32}\mathscr{S}_{21}\Phi+\mathscr{S}_{32}\mathscr{T}_{21},\qquad (11a,b)$$

$$\mathscr{S}=\mathscr{S}_{32}\mathscr{T}_{21}\Phi+\mathscr{T}_{32}\mathscr{S}_{21},\qquad \mathscr{T}=\mathscr{S}_{32}\mathscr{S}_{21}\Phi+\mathscr{T}_{32}\mathscr{T}_{21}.\qquad (11c,d)$$

Equation (10) includes all conceivable matchings between the perturbations in regions 1 and 3. Depending on their behaviour, we shall obtain a particular dispersion relation (DR) that relates the frequency ω with the wavenumber $\mathbf{k}$. Here we are interested in the appearance of leaky waves, so that we look for complex roots of the DR that describe the damping of the perturbation. In this case the Γ_i are complex. Four types of solutions may be distinguished (for $Z\neq 0$ and $\mathrm{Re}(\Gamma_{1,3})\neq 0$):

(i) those that are bounded for $y\to\pm\infty$, i.e. $\zeta_{+,3}=0$ and $\zeta_{-,1}=0$ and as a consequence satisfy $\mathscr{T}=0$ (type $\mathscr{T}$ branches);

(ii) those that are bounded for $y\to+\infty$ but diverge as $y\to-\infty$, i.e. $\zeta_{+,3}=0$ and $\zeta_{+,1}=0$, which implies $\mathscr{S}=0$ (type $\mathscr{S}$ branches);

(iii) those that diverge as $y\to\pm\infty$ (i.e. $\zeta_{-,3}=0$ and $\zeta_{+,1}=0$), so that $\mathscr{V}=0$ (type $\mathscr{V}$ branches);

(iv) those that diverge as $y\to+\infty$ but are bounded for $y\to-\infty$ ($\zeta_{-,3}=0$ and $\zeta_{-,1}=0$), so that $\mathscr{U}=0$ (type $\mathscr{U}$ branches).

Then we have four different DR, and correspondingly four types of branches (Table 1). Note that all the DR can be written in the general form

$$\mathscr{A}_{32}\mathscr{A}_{21}\Phi+\bar{\mathscr{A}}_{32}\bar{\mathscr{A}}_{21}=0,\qquad (12)$$

where

$$\mathscr{A}_{ji}=f_{ji}G_j+\sigma_{ji}G_i,\qquad \bar{\mathscr{A}}_{ji}=f_{ji}G_j-\sigma_{ji}G_i,$$

with

$$\sigma_{32}=-1,\quad \sigma_{21}=1\quad \text{for } \mathscr{S}=0,$$

$$\sigma_{32}=-1,\quad \sigma_{21}=-1\quad \text{for } \mathscr{T}=0,$$

$$\sigma_{32}=1,\quad \sigma_{21}=-1\quad \text{for } \mathscr{U}=0,$$

$$\sigma_{32}=1,\quad \sigma_{21}=1\quad \text{for } \mathscr{V}=0.$$

The formal replacement $G_k\leftrightarrow-G_k$ is equivalent to $\mathscr{T}_{jk}\leftrightarrow\mathscr{S}_{jk}$ and $\mathscr{T}_{ki}\leftrightarrow-\mathscr{S}_{ki}$, so that, starting from any one of the DR, the remaining three can be obtained

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

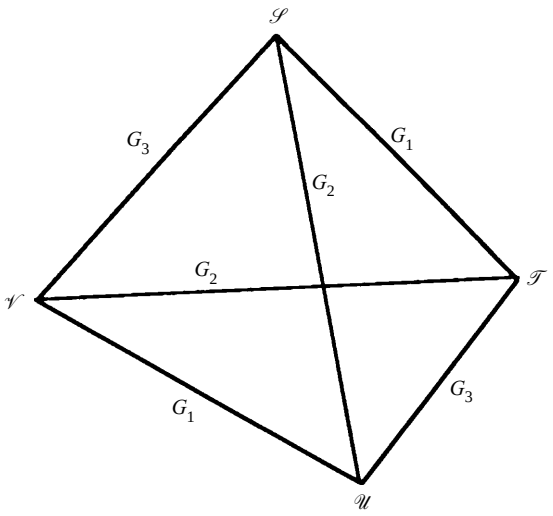


Figure 1. Transformation rules of the dispersion relations.

by changing the signs of G_1, G_2 and G_3. These changes can be visualized with a simple geometrical device (Fig. 1): the four branches are represented by the vertices of a regular tetrahedron, G_1 is represented by the opposite edges that join the pairs $(\mathscr{V},\mathscr{U})$ and $(\mathscr{S},\mathscr{T})$, G_2 is represented by the edges that join $(\mathscr{V},\mathscr{T})$ and $(\mathscr{U},\mathscr{S})$, and G_3 is represented by the edges that join $(\mathscr{V},\mathscr{S})$ and $(\mathscr{U},\mathscr{T})$. Changing the sign of G_i is equivalent to the simultaneous permutation of the vertices of the corresponding edges. Then $G_1 \leftrightarrow -G_1$ yields $(\mathscr{V} \leftrightarrow \mathscr{U})$ and $(\mathscr{S} \leftrightarrow \mathscr{T})$, etc. Taking advantage of these properties, in what follows we shall present formulae only for the $\mathscr{T}$ branch, since those for the other branches can be obtained by the appropriate substitutions $G_k \leftrightarrow -G_k$.

We shall postpone momentarily the discussion about which branches yield the leaky waves. The DR may be analysed in two alternative ways. We can fix $k = (k_x^2 + k_z^2)^{1/2}$, and then solve the equations for ω; the perturbations that carry energy and are damped in time will be found among the complex solutions. The other point of view is to assume a perturbation with a given frequency and then solve the DR for the wavenumber. The leaky perturbations are found among the complex solutions for k. We have adopted the first alternative. The normal modes of our problem belong to type $\mathscr{T}$ branches, and have been studied for the special case of a symmetrical slab configuration by Roberts (1981), Edwin and Roberts (1982) and Rae and Roberts (1983); the related problem in a cylindrical configuration has been considered by Roberts (1991). The solutions of the remaining DR are pseudomodes. The leaky waves in which we are interested are pseudomodes with ω complex, and we shall show that they belong to the $\mathscr{S}$, $\mathscr{V}$ and $\mathscr{U}$ branches. However, for reasons that will be clear below, it is convenient to consider all four DR, including $\mathscr{T} = 0$.

When discussing the spectrum of perturbations for any specific configuration, it is convenient to use the variable $\Omega \equiv V_2^2$, since only even powers of the phase velocity appear in the DR. When one follows the evolution of a branch of solutions of a certain DR as α is varied, it is found that the branch may end abruptly. At these particular values of the parameters a solution of a certain

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

DR becomes a solution of another DR. These transitions between branches of different types occur at predictable values of Ω and α, which will be discussed in Sec. 4. Then, in order to follow the branches, one should solve the four DR so as not to miss parts of the spectrum and to achieve a clear picture of the topology of the ensemble of branches. In addition, real roots may become complex (or vice versa) at certain points (double real roots). Then it is helpful to solve the DR not only for complex frequencies but also for real ones.

The total spectrum is very complicated, with infinite branches that may be real or complex. Moreover, the problem involves a large number of parameters. Nevertheless, we shall develop systematic methods of analysis and highlight the generalities involved, based on general analytical results that allow us to unravel the complex topology of the spectrum.

4. Analytical properties of the spectrum

In this section we shall discuss some important properties of the roots of the four dispersion relations (Table 1). The roots are always finite for any non-vanishing α, but some of them are unbounded for $\alpha \to 0$ (see below).

4.1. Real and complex Ω *roots*

4.1.1. Real roots. When Ω is real, the perturbation may be evanescent (Γ_i and G_i real) or propagating (Γ_i and G_i imaginary) in each region. There are eight possible combinations (EEE, PEE, PEP, EEP, EPE, PPE, EPP and PPP), but, as we shall presently show, the branches with Ω real require that the perturbations have the structure EEE or EPE (we shall write only the formulae for $\mathscr{T} = 0$, since those for the other DR are obtained by the appropriate changes $G_i \leftrightarrow -G_i$).

Consider first the case EEE; we set $G_1 = g_1$, $G_2 = g_2$, $G_3 = g_3$ and $\Gamma_2 = \gamma_2$, with g_1, g_2, g_3 and γ_2 real functions of Ω. Then $\mathscr{T} = 0$ leads to

$$\tanh(\alpha\gamma_2) = -\frac{f_{21}\, g_2 (f_{31}\, g_3 + g_1)}{f_{31}\, g_1\, g_3 + f_{21}^2\, g_2^2}, \tag{13}$$

and there can be branches with real $\Omega(\alpha)$.

To analyse the case EPE (propagating waves trapped in the intermediate region), we set $G_1 = g_1$, $G_2 = ig_2$, $G_3 = g_3$ and $\Gamma_2 = i\gamma_2$; then $\mathscr{T} = 0$ yields

$$\tan(\alpha\gamma_2) = -\frac{f_{21}\, g_2 (f_{31}\, g_3 + g_1)}{f_{31}\, g_1\, g_3 - f_{21}^2\, g_2^2}, \tag{14}$$

so that again real $\Omega(\alpha)$ branches are possible. Note that in this case $\Omega > 0$. Trapped modes of type $\mathscr{T}$ have been considered by Roberts (1981) for some special configurations.

The remaining cases yield only isolated real Ω points (not branches) that will be discussed in detail later on. These roots are not related to the leaky waves, but we shall see that they help in understanding the spectrum. In the cases PPE and EPP we put $G_j = ig_j$, $G_2 = ig_2$, $G_k = g_k$ (with $j = 1, 3$ and $k = 3, 1$ respectively) and $\Phi = \cos\phi + i \sin\phi$; replacing in $\mathscr{T} = 0$ and solving for $\sin\phi$ and $\cos\phi$, we obtain

$$\sin^2\phi + \cos^2\phi = \left(\frac{g_j + f_{2j}\, g_2}{g_j - f_{2j}\, g_2}\right)^2, \tag{15}$$

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

so that perturbations with real Ω are possible only at the isolated points given by $g_j = 0$ or $g_2 = 0$ that correspond to contacts $\mathscr{S} \leftrightarrow \mathscr{T}$ and $\mathscr{U} \leftrightarrow \mathscr{V}$ for PPE and $\mathscr{S} \leftrightarrow \mathscr{V}$ and $\mathscr{T} \leftrightarrow \mathscr{U}$ for EPP. Likewise, isolated real Ω solutions exist for the case PEE at the contact points $\mathscr{S} \leftrightarrow \mathscr{T}$ and $\mathscr{U} \leftrightarrow \mathscr{V}$, and for EEP at the contacts $\mathscr{S} \leftrightarrow \mathscr{V}$ and $\mathscr{T} \leftrightarrow \mathscr{U}$. Finally, for PEP and PPP isolated real Ω solutions exist only at the contact points $\mathscr{T} \leftrightarrow \mathscr{V}$ and $\mathscr{S} \leftrightarrow \mathscr{U}$.

4.1.2. Complex roots. There are no complex roots of $\mathscr{T} = 0$. This is a consequence of a theorem due to Bernstein *et al.* (1957). Essentially, the equation of motion can be written as $\omega_n^2 \rho \xi_n = \mathbf{F}\{\xi_n\}$, where $\xi(\mathbf{r}, t) = \xi_n(\mathbf{r})\, e^{-i\omega_n t}$ is the displacement vector and $\mathbf{F}$ is a vector operator. If $\mathbf{F}$ is self-adjoint, ω_n^2 is real. The necessary and sufficient condition for this is that the solutions remain bounded as $y \to \pm\infty$. Since the solutions of $\mathscr{T} = 0$ have this property, ω_n^2 is always real. On the other hand, the solutions of $\mathscr{V} = 0$, $\mathscr{U} = 0$ and $\mathscr{S} = 0$ diverge as $y \to +\infty$ and/or $y \to -\infty$, and therefore the existence of perturbations with complex Ω is possible. For certain values of α and Ω there are real double roots of these DR; at these points branches of complex-conjugate solutions start. The values of Ω for which these changes occur (double roots) will be denoted in the following by Ω_{d1}, Ω_{d2}, etc., in increasing order of α. Each complex Ω yields two conjugate values of V; leaky waves are found among those with $\mathrm{Im}(V) < 0$.

4.2. Real and complex transitions, crossings and periodicity

Different branches may coincide at certain points (α, Ω), where a real transition, a crossing or a complex transition occurs. All the transitions and crossings occur for real Ω (complex transitions take place at isolated real roots Ω). We call a 'real transition' a point of coincidence where a real branch of solutions of a DR becomes a real branch of another DR (the branches change type), so that there is only *one* branch at each side of the critical point. We call a 'crossing' a point of coincidence where two real branches of different type intersect, and both continue beyond the point of coincidence, maintaining their type. A 'complex transition' occurs when a complex branch and its complex conjugate make jointly a transition to two complex-conjugate branches of a different type.

Let us consider the points of coincidence given by $G_j = 0, \infty$. We can write (12) in the equivalent form

$$f_{j2}\, G_j(\mathscr{A}_{2i}\, \Phi + \overline{\mathscr{A}}_{2i}) + \sigma_{j2}\, G_2(\mathscr{A}_{2i}\, \Phi - \overline{\mathscr{A}}_{2i}) = 0, \tag{16}$$

with $i = 1, 3$ and $j = 3, 1$ respectively. We shall first show that the zeros and poles of Γ_j yield real transitions.

Consider first two branches that coincide at a particular value of (α_0, Ω_0) where Γ_j has a pole, so that $(\mathscr{A}_{2i}\, \Phi - \overline{\mathscr{A}}_{2i})_0 = 0$ (the suffix 0 denotes that the quantities are evaluated at the point of coincidence). Expanding (16) close to the point of coincidence ($G_j = \delta G_j$, $\alpha = \alpha_0 + \delta\alpha$ and $V_j^2 = V_{j0}^2 + \delta V_j^2$) we obtain to the lowest order

$$(-\delta\Omega)^{1/2} = \sigma_{j2}\, \delta\alpha (V_{A,2}^2 - V_{2,0}^2) f_{2j} / C_{j,0}, \tag{17}$$

where $C_{j,0}$ is a real quantity that depends on Ω_0. This means that as $\delta\alpha$ varies and one crosses the point (α_0, Ω_0), the solution of the dispersion relation corresponding to $\sigma_{j2} = 1$ ceases to exist, while that of the DR corresponding to

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

Table 2. Real transitions, periodic crossings and complex transitions and notation for the corresponding Ω.

Conditions	Notation for Ω	Branch coincidences
Real transitions		
$V_1 = V_{+,1}$ $(\Gamma_1 = 0)$	Ω_{+1}	$\mathscr{T} \leftrightarrow \mathscr{S}$, $\mathscr{U} \leftrightarrow \mathscr{V}$
$V_1 = V_{-,1}$ $(\Gamma_1 = 0)$	Ω_{-1}	$\mathscr{T} \leftrightarrow \mathscr{S}$, $\mathscr{U} \leftrightarrow \mathscr{V}$
$V_1 = V_{B,1}$ $(\Gamma_1 = \infty)$	Ω_{b1}	$\mathscr{T} \leftrightarrow \mathscr{S}$, $\mathscr{U} \leftrightarrow \mathscr{V}$
$V_3 = V_{+,3}$ $(\Gamma_3 = 0)$	Ω_{+3}	$\mathscr{U} \leftrightarrow \mathscr{T}$, $\mathscr{S} \leftrightarrow \mathscr{V}$
$V_3 = V_{-,3}$ $(\Gamma_3 = 0)$	Ω_{-3}	$\mathscr{U} \leftrightarrow \mathscr{T}$, $\mathscr{S} \leftrightarrow \mathscr{V}$
$V_3 = V_{B,3}$ $(\Gamma_3 = \infty)$	Ω_{b3}	$\mathscr{U} \leftrightarrow \mathscr{T}$, $\mathscr{S} \leftrightarrow \mathscr{V}$
Crossings		
$V_1 = V_{A,1}$ $(G_1 = 0)$	Ω_{a1}	$\mathscr{T} \leftrightarrow \mathscr{S}$, $\mathscr{U} \leftrightarrow \mathscr{V}$
$V_3 = V_{A,3}$ $(G_3 = 0)$	Ω_{a3}	$\mathscr{U} \leftrightarrow \mathscr{T}$, $\mathscr{S} \leftrightarrow \mathscr{V}$
Crossings or complex transitions		
$\Phi = 1$, $\mathscr{S}_{31} = 0$	Ω_{S+}	$\mathscr{S} \leftrightarrow \mathscr{U}$
$\Phi = -1$, $\mathscr{T}_{21}\mathscr{S}_{32} - \mathscr{T}_{32}\mathscr{S}_{21} = 0$	Ω_{S-}	$\mathscr{S} \leftrightarrow \mathscr{U}$
$\Phi = 1$, $\mathscr{T}_{31} = 0$	Ω_{T+}	$\mathscr{T} \leftrightarrow \mathscr{V}$
$\Phi = -1$, $\mathscr{T}_{21}\mathscr{T}_{32} - \mathscr{S}_{32}\mathscr{S}_{21} = 0$	Ω_{T-}	$\mathscr{T} \leftrightarrow \mathscr{V}$

$\sigma_{j2} = -1$ appears (i.e. we have a real transition). Note that $\delta\Omega \propto \delta\alpha^2$, with $\delta\Omega < 0$ (i.e. $V_j^2 \leqslant V_{B,j}^2$), which corresponds to evanescent perturbations in the external region j. This is the case since real transitions involve always real branches (i.e. case EEE or EPE).

We now consider the points (α_0, Ω_0) where $\Gamma_j = 0$ (i.e. $V_{j,0}^2 = V_{\pm,j}^2$), so that $(\mathscr{A}_{2i}\Phi + \overline{\mathscr{A}_{2i}})_0 = 0$. It can be proved by a similar procedure that these coincidences are real transitions as before, but now $\delta\Omega \propto \mp\delta\alpha^2$, so that we have $V_j^2 \leqslant V_{+,j}^2$ or $V_j^2 \geqslant V_{-,j}^2$, and again the perturbations are evanescent in the external regions.

Summarizing, the real transitions are determined by the zeros and poles of Γ_j $(j = 1, 3)$. For instance, if $V_3 = V_{B,3}$, we have $\mathscr{S}_{32} = -\mathscr{T}_{32}$ and $\mathscr{S}_{31} = -\mathscr{T}_{31}$, so that $\mathscr{S} = -\mathscr{V}$ and $\mathscr{U} = -\mathscr{T}$. Then if $V_3 = V_{B,3}$ is a solution of $\mathscr{S} = 0$ or $\mathscr{U} = 0$, there will be a transition $\mathscr{S} \leftrightarrow \mathscr{V}$ or $\mathscr{U} \leftrightarrow \mathscr{T}$. In a similar way, if $V_1 = V_{\pm,1}$, we have $\mathscr{S}_{31} = -\mathscr{T}_{31}$ and $\mathscr{S}_{21} = -\mathscr{T}_{21}$. In this case $\mathscr{T} \leftrightarrow \mathscr{S}$ or $\mathscr{U} \leftrightarrow \mathscr{V}$ transitions occur. The various possibilities are given in Table 2.

If $V_{j0} = V_{A,j}$ or $\Phi = \pm 1$, one finds crossings and complex transitions. Let us discuss first what happens when $V_{j0} = V_{A,j}$, so that $G_j = 0$ and $(\mathscr{A}_{2i}\Phi - \overline{\mathscr{A}_{2i}})_0 = 0$. Expanding (16) close to the point of coincidence (α_0, Ω_0), we obtain

$$f_{j2}\,\delta G_j + \sigma_{j2}(V_{2,0}^2 - V_{A,2}^2)\,\delta\alpha + C_0\,\delta\Omega = 0, \tag{18}$$

where C_0 is a real quantity that depends on Ω_0 and on the dispersion relation that we are considering. Since $\delta G_j = -\delta\Omega M_{2j}^2/\Gamma_{j,0}$ (note that $\Gamma_{j,0}$ is real), (18) yields $\delta\Omega = D_0\,\delta\alpha$, where D_0 is real, so that two real branches cross each other.

The other crossings and the complex transitions occur if $\Phi = \pm 1$, which requires that the perturbation be propagating in the intermediate region (and Ω is real, as mentioned before). They involve $\mathscr{S}$ and $\mathscr{U}$ branches when Ω is a solution of

$$\mathscr{S}_{31} = 0, \qquad \Phi = 1, \tag{19a}$$

or

$$\mathscr{T}_{21}\mathscr{S}_{32} - \mathscr{T}_{32}\mathscr{S}_{21} = 0, \qquad \Phi = -1. \tag{19b}$$

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

Table 3. Solutions of the dispersion relations for the limiting values of α and notation for the corresponding Ω. Notice that for any ij, the total number of roots Ω_{Tij} and Ω_{Sij} is always 5.

Limit	Condition	Notation for Ω	Coincidence
$\alpha = 0$	$\mathscr{T}_{31} = 0$	Ω_{T31}	$\mathscr{T} \leftrightarrow \mathscr{V}$
	$\mathscr{S}_{31} = 0$	Ω_{S31}	$\mathscr{U} \leftrightarrow \mathscr{S}$
	$V_2 = V_{A,2}$	Ω_{a2}	$\mathscr{T} \leftrightarrow \mathscr{V} \leftrightarrow \mathscr{U} \leftrightarrow \mathscr{S}$
	$V_2 = V_{B,2}$	Ω_{b2}	$\mathscr{T} \leftrightarrow \mathscr{V} \leftrightarrow \mathscr{U} \leftrightarrow \mathscr{S}$
	$\lvert\Psi_{\mathscr{S}}\rvert < 1$	∞	$\mathscr{V} \leftrightarrow \mathscr{S}$
	$\lvert\Psi_{\mathscr{U}}\rvert < 1$	∞	$\mathscr{V} \leftrightarrow \mathscr{U}$
$\alpha \to \infty$	$\mathscr{T}_{21} = 0$	Ω_{T21}	$\mathscr{T} \leftrightarrow \mathscr{U}$
	$\mathscr{S}_{21} = 0$	Ω_{S21}	$\mathscr{S} \leftrightarrow \mathscr{V}$
	$\mathscr{T}_{32} = 0$	Ω_{T32}	$\mathscr{T} \leftrightarrow \mathscr{S}$
	$\mathscr{S}_{32} = 0$	Ω_{S32}	$\mathscr{U} \leftrightarrow \mathscr{V}$
	$V_2 = V_{+,2}$	Ω_{+2}	$\mathscr{T} \leftrightarrow \mathscr{V} \leftrightarrow \mathscr{U} \leftrightarrow \mathscr{S}$
	$V_2 = V_{-,2}$	Ω_{-2}	$\mathscr{T} \leftrightarrow \mathscr{V} \leftrightarrow \mathscr{U} \leftrightarrow \mathscr{S}$

Alternatively, they involve $\mathscr{T}$ and $\mathscr{V}$ branches when Ω is a solution of

$$\mathscr{T}_{31} = 0, \qquad \Phi = 1, \tag{20a}$$

or

$$\mathscr{T}_{21}\mathscr{T}_{32} - \mathscr{S}_{21}\mathscr{S}_{32} = 0, \qquad \Phi = -1. \tag{20b}$$

Equations (19) and (20) can be reduced to two polynomials in Ω, although in a general case of high degree (5 and 10 respectively), by rationalization procedures that introduce additional (spurious) roots. The non-spurious real roots of the resulting polynomials give the points of coincidence for $\Phi = \pm 1$. The notation used for these Ω is given in Table 3. It can be shown that these coincidences will be crossings of real branches when the perturbation there is evanescent in both external regions, or complex transitions if it is propagating everywhere. Since the $\mathscr{T}$ branches are always real, they cannot be involved in complex transitions.

When Ω is real and corresponds to propagating waves in region 2, the solutions are periodic in α with period equal to $\pi/|\Gamma_2|$; in this case we have an infinite set of 'harmonics'. Real transitions will be periodic if they take place in a domain of propagating waves for region 2 (these families of harmonics are related to the presence of trapped waves in the intermediate region). The complex transitions and the crossings for $\Phi = \pm 1$ are always periodic.

4.3. Limiting solutions

4.3.1. Solutions as $\alpha \to \infty$. In the limit as $\alpha \to \infty$ the interfaces (1–2) and (2–3) are isolated. If Γ_2 has a non-vanishing real part, $\Phi \to 0$ and the solutions of $\mathscr{V} = 0$ are roots of $\mathscr{S}_{32} = 0$ or $\mathscr{S}_{21} = 0$, those of $\mathscr{U} = 0$ are roots of $\mathscr{S}_{32} = 0$ or $\mathscr{T}_{21} = 0$, those of $\mathscr{S} = 0$ are roots of $\mathscr{T}_{32} = 0$ or $\mathscr{S}_{21} = 0$, and those of $\mathscr{T} = 0$ are roots of $\mathscr{T}_{32} = 0$ or $\mathscr{T}_{21} = 0$. Then, for α large, some $\mathscr{U}$ and $\mathscr{V}$ branches tend to the roots of $\mathscr{S}_{32} = 0$, some $\mathscr{S}$ and $\mathscr{T}$ branches tend to those of $\mathscr{T}_{32} = 0$, other $\mathscr{S}$ and $\mathscr{V}$ branches tend to those of $\mathscr{S}_{21} = 0$, and finally other $\mathscr{U}$ and $\mathscr{T}$ branches tend to those of $\mathscr{T}_{21} = 0$.

If Γ_2 is imaginary, the exponential factor $e^{-2\alpha\Gamma_2}$ is indeterminate when $\alpha \to \infty$. The only possible case is EPE: setting $G_1 = g_1$, $G_2 = ig_2$, $G_3 = g_3$ and $\Gamma_2 = i\gamma_2$, the DR $\mathscr{T} = 0$ reduces to (14). If $V_2^2 = V_{\pm,2}^2$, γ_2 vanishes but $g_2 \to \infty$, and in order

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

to satisfy (14) $\tan(\alpha\gamma_2)\to 0$. Therefore, as $\alpha\to\infty$, an infinite number of solutions (a family of harmonics) exist for which $V_2^2\to V_{\pm,2}^2$ while $\gamma_2\to 0$ and $\alpha\gamma_2\to n\pi$ (n integer). The analysis of the other DR leads to the same results. Then we conclude that $V_2^2 = V_{\pm,2}^2$ are solutions of all four DR when $\alpha\to\infty$.

4.3.2. Solutions when $\alpha = 0$. This case corresponds to a region 2 of vanishing thickness. The equations $\mathscr{V} = 0$ and $\mathscr{T} = 0$ reduce to

$$G_2\mathscr{T}_{31} = 0, \tag{21}$$

while $\mathscr{U} = 0$ and $\mathscr{S} = 0$ are written as

$$G_2\mathscr{S}_{31} = 0. \tag{22}$$

If we take $Z \neq 0$ then $G_2 \neq 0$, so that when $\alpha = 0$, the solutions of $\mathscr{V} = 0$ and $\mathscr{T} = 0$ are roots of $\mathscr{T}_{31} = 0$, while the solutions of $\mathscr{U} = 0$ and $\mathscr{S} = 0$ are roots of $\mathscr{S}_{31} = 0$.

If $Z = 0$, there are two possibilities: $G_2 = 0$ or $G_3 = 0$. The case in which $G_2 = 0$ occurs when $V_2^2 = V_{A,2}^2$ or $V_2^2 = V_{B,2}^2$. We first study the limit as $V_2^2\to V_{A,2}^2$. Suppose that Γ_2 is real; then the perturbation must also be evanescent in the other regions (case EEE), and (13) holds. As a consequence, $\tanh(\alpha\gamma_2)\to 0$; then, as $\alpha\to 0$, one finds a solution for which $V_2^2\to V_{A,2}^2$. A similar argument can be made if Γ_2 is imaginary (case EPE), leading to an analogous result. Similarly, if $V_2^2 = V_{B,2}^2$ and Γ_2 is real, (13) requires $\tanh(\alpha\gamma_2)\to 0$. Then, as $\alpha\to 0$, there is a solution such that $V_2^2\to V_{B,2}^2$, $\gamma_2\to\infty$ but $\alpha\gamma_2\to 0$. If Γ_2 is imaginary, (14) requires $\tan(\alpha\gamma_2)\to 0$. In consequence, as $\alpha\to 0$, there are infinite solutions for which $V_2^2\to V_{B,2}^2$, and $\gamma_2\to\infty$ but $\alpha\gamma_2\to n\pi$, i.e. a family of harmonics issues from the limiting solution. The other DR lead to the same results. Then we conclude that $V_2^2 = V_{A,2}^2$ and $V_2^2 = V_{B,2}^2$ are solutions of all four DR when $\alpha = 0$. It can be shown that the case $G_3 = 0$, $\alpha = 0$ does not admit a solution.

In addition, it can be shown that as $\alpha\to 0$, there are infinite unbounded complex solutions of $\mathscr{V} = 0$ and of $\mathscr{S} = 0$ or $\mathscr{U} = 0$, of the form

$$\Omega = W(1+iw) \qquad (\alpha\to 0, W, w \text{ real and } W\to\infty, w \text{ finite}). \tag{23}$$

Indeed, substituting the ansatz (23) into (12), we obtain in the limit as $W\to\infty$ the condition

$$\Phi_\infty = \Psi_{\mathscr{X}} \qquad (\mathscr{X} = \mathscr{S}, \mathscr{T}, \mathscr{U}, \mathscr{V}), \tag{24}$$

with

$$\Phi_\infty = \exp\left[-2\alpha\left(\frac{W}{F_2}\right)^{1/2}(b - i\sigma_w a)\right], \tag{25a}$$

$$\Psi_{\mathscr{X}} = -\frac{(f_{32}M_{23}F_3 - \sigma_{32}F_2)(f_{21}F_2 - \sigma_{21}M_{21}F_1)}{(f_{32}M_{23}F_3 + \sigma_{32}F_2)(f_{21}F_2 + \sigma_{21}M_{21}F_1)}, \tag{25b}$$

$$\sigma_w = \mathrm{Sign}(w), \qquad a = (\tfrac{1}{2}[(1+w^2)^{1/2}+1])^{1/2}, \qquad b = (\tfrac{1}{2}[(1+w^2)^{1/2}-1])^{1/2}.$$

Now, since $\Psi_{\mathscr{X}}$ is real, (24) requires that

$$\exp\left(-n\pi\frac{b}{a}\right) = |\Psi_{\mathscr{X}}|, \qquad 2\alpha\left(\frac{W}{F_2}\right)^{1/2} a = n\pi, \tag{26}$$

where $n = 1, 3, 5, \ldots$, if $\Psi_{\mathscr{X}} < 0$ or $n = 2, 4, 6, \ldots$, if $\Psi_{\mathscr{X}} > 0$. The first of the

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

conditions (26) is a transcendental equation that determines w, and obviously can be satisfied for an infinite set of n values, provided that $|\Psi_{\mathscr{X}}| \leqslant 1$. If this is the case then the second condition determines W. It is easy to verify that $|\Psi_{\mathscr{V}}| < 1, |\Psi_{\mathscr{S}}| = 1/|\Psi_{\mathscr{U}}|$ and $|\Psi_{\mathscr{T}}| > 1$. Then there are always infinite $\mathscr{V}$ branches, as well as infinite $\mathscr{S}$ (if $|\Psi_{\mathscr{S}}| < 1$) or $\mathscr{U}$ (if $|\Psi_{\mathscr{U}}| < 1$) branches, of the form (23).

Table 3 gives a list of the solutions of the DR for the limiting values of α, and includes as particular cases those discussed by Edwin and Roberts (1982). To actually compute the solutions of $\mathscr{T}_{ij} = 0$ and $\mathscr{S}_{ij} = 0$, it is convenient to find the roots of the products $\mathscr{T}_{ij}\mathscr{S}_{ij} = 0$ $(i, j = 1, 2, 3,\ i \neq j)$ that lead to polynomial equations of degree 5 in Ω in the general case. Some of the five roots thus obtained will be solutions of $\mathscr{T}_{ij} = 0$ (indicated generically as Ω_{Tij} in Table 3), and the remaining roots will correspond to $\mathscr{S}_{ij} = 0$ (indicated by Ω_{Sij} in Table 3). A simple case that will help in the analysis of leakage mechanisms and in the discussion of the examples occurs if the magnetic field and the pressure are uniform ($\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3 \equiv \mathbf{B}$ and $p_1 = p_2 = p_3 \equiv p$), so that β is uniform. Then $A_1 = A_2 = A_3 \equiv A$, $F_1 = F_2 = F_3 \equiv F$, $V_{\pm,1} = V_{\pm,2} = V_{\pm,3} \equiv V_{\pm}$, etc. In this case $\mathscr{T}_{ij}\mathscr{S}_{ij} = 0$ has a trivial solution $\Omega = 0$.

4.3.3. Complex limiting solutions. In addition to the unbounded complex solutions just discussed, bounded complex solutions may exist in the limits as $\alpha \to 0, \infty$, depending on the parameters of the problem. As will be discussed below, these complex limiting solutions are related to a certain type of leakage, so that some comment is in order. They may appear among the solutions of $\mathscr{T}_{ij} = 0$ and $\mathscr{S}_{ij} = 0$. In the general case, they must be found numerically by solving an associate polynomial equation of degree 5, and no criteria for the existence of complex roots can be given in closed form.

Simple analytical solutions can be found for a constant-β plasma, when $\mathbf{k}$ is parallel to $\mathbf{B}$ ($\psi = 0$), since then $V_A = A, V_- = \mathrm{Min}(1, A)$, $V_+ = \mathrm{Max}(1, A)$, and $V_B = A/F \equiv B$ (no confusion with $|\mathbf{B}|$ should arise). This is a degenerate case, since V_A coincides with V_- or V_+, and $\mathscr{T}_{ij}\mathscr{S}_{ij} = 0$ can be reduced to a polynomial equation of degree 3, which after factoring out the trivial root $\Omega = 0$ yields

$$(V_i^2 - 1)(V_j^2 - 1) + \frac{A^2 - 1}{A^2 + 1} = 0, \tag{27}$$

whose roots are

$$V_i^2 = \tfrac{1}{2}\left(1 + M_{ji}^2 \pm [(1 + M_{ji}^2)^2 - 8B^2 M_{ji}^2]^{1/2}\right). \tag{28}$$

These roots will be complex if $m_- \leqslant M_{ji}^2 \leqslant m_+$, where

$$m_\pm = \frac{3A^2 - 1 \pm [8A^2(A^2 - 1)]^{1/2}}{F^2}. \tag{29}$$

When $A < 1$, there are no complex limiting solutions. On the other hand, if $A > 1$ then $m_\pm$ are real and the limiting solutions are complex if $m_- < M_{ji}^2 < m_+$. Summarizing, for $A > 1$ and appropriate temperature ratios, it is possible to obtain complex roots for $\alpha \to 0, \infty$.

4.3.4. Multiplicity of the limiting solutions. To keep track of all the branches of the spectrum, it is important to know how many branches tend to any given

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

limiting solution. The multiplicity of the bounded limiting solutions depends on the character (E or P) of the perturbation in the intermediate layer as the limit is approached. If it is evanescent then the multiplicity is 2 or 4, according to how many types of branches tend to the limiting solution. If it is propagating, the multiplicity is $2 \times \infty$ or $4 \times \infty$. The unbounded complex solution (23) always has a $2 \times \infty$ multiplicity.

4.4. Topology of the spectrum

The properties we discussed in this section allow us to construct the entire spectrum that comprises all the roots of the four dispersion relations, among which the leaky waves are found. For this purpose, we note that as α varies, if we follow any one of the infinite roots $\Omega(\alpha)$, we shall see that its branch type may change at a real or complex transition point, or that it may change from real to complex (or vice versa) at a double root – but it may never disappear (or appear) suddenly or tend to infinity for finite α. Then, starting from the limiting solutions ($\alpha = 0, \infty$), one can follow numerically any root. In addition, the periodic crossings and transitions we have described allow us to pick up any desired root among the infinite harmonics. In this way, we can be sure that we shall find all the roots, and no significant solutions will be omitted. The branches are analytic functions of the parameters of the problem, except at the transitions and crossings, so that it is not difficult to determine their behaviour as the configuration is changed.

5. Leaky waves

The usual approach to investigating leaky waves is based on Fourier analysis in time. This has the advantage of simplicity, but then the solution of initial value problems is not straightforward. Therefore it is not obvious within this approach why complex solutions of the DR represent leaky waves, and how the leakage occurs. In this section we address this problem heuristically, to avoid lengthy mathematics. Different types of leakage are possible.

5.1. Surface-wave leakage

Consider first a leaky wave that originates from the coupling between a surface wave at the interface (2–3) and a propagating wave in region 1. In the limit as $\alpha \to \infty$ these perturbations are decoupled and their frequency is real. When α is finite, the coupling subtracts energy from the surface wave, which is lost because it is carried to $y = -\infty$ by the propagating wave, so that the frequency becomes complex. The corresponding initial-value problem consists in exciting the surface wave at the interface (2–3) at a given moment and then letting it decay, while the perturbation in region 1 propagates away from the interface (1–2). The perturbed part of region 1 is bounded by a front, whose distance to the interface increases with time. The front originated at the beginning of the process, when the amplitude of the surface wave was larger. In consequence, at a fixed time, the amplitude of the perturbation in region 1 increases with the distance from the interface, and is maximum at the front (see Cally 1986). The solution of this initial-value problem is represented by a packet of Fourier modes whose amplitude increases with y in the region 1 (where the propagating

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

wave for $\alpha \to \infty$ is present). The Fourier components diverge in the limit as $y \to -\infty$ because they exist for any $-\infty < t < \infty$. Then this kind of leakage is related to complex roots of $\mathscr{S} = 0$ that become real as $\alpha \to \infty$. By the same argument, one can show that the complex branches of type $\mathscr{U}$ that become real as $\alpha \to \infty$ are related to leaky waves due to the coupling of a surface wave at the interface (1–2) with a propagating wave in region 3. These types of leakage will be called 'surface-wave leakage'.

However, there are other complex branches that become real for finite values of α, or that remain complex in the limit as $\alpha \to \infty$. They are due to other types of leakage, as we shall presently show; further details will be published elsewhere.

5.2. Single-interface leakage

Let us consider the complex limiting solutions described in Sec. 5.3. The existence of these leaky waves depends on the coupling of perturbations on both sides of a single interface (i–j) (interface (1–3) if $\alpha = 0$, interfaces (1–2) or (2–3) if $\alpha \to \infty$). To understand the physics involved, let us consider the case of uniform pressure and magnetic field introduced in Sec. 4.3.3, with **k** parallel to **B**. Complex roots of the polynomial (27) exist only if $A > 1$; they appear owing to the matching of a slow propagating wave in one region and a fast one in the other. This can easily be verified by considering the case $A = 1+\epsilon$, $M_{ij} = 1+\delta$ and $V_{i,j}^2 = 1+\delta_{i,j}$ ($\epsilon > 0$ and $\epsilon, |\delta|, |\delta_{i,j}| \ll 1$), when the perturbations in both regions propagate almost parallel to the interface and the leakage is small. To the lowest order, we obtain the system $\delta_i \delta_j + \epsilon = 0$, $\delta_i - \delta_j + 2\delta = 0$, which yields the complex roots

$$\delta_i = -\delta \pm i\lambda, \qquad \delta_j = \delta \pm i\lambda, \tag{30}$$

if $\lambda = (\epsilon - \delta^2)^{1/2}$ is real. Very close to the double roots ($\lambda \to 0$), $V_i \approx 1 - \frac{1}{2}\delta$ and $V_j \approx 1 + \frac{1}{2}\delta$. If $\delta > 0$, there is a slow propagating wave in region i and a fast propagating wave in j, with $\Gamma_i \approx \lambda - i\delta$ and $\Gamma_j \approx \lambda + i\delta$. We recall that the y components of the group velocity and of the energy flow vector of a slow mode have a sign opposite to that of $\alpha\,\mathrm{Im}(\Gamma)$, while for a fast propagating wave the signs are equal. Then, if $\alpha > 0$, the energy flows in the positive y direction (if $j > i$, the flow goes from i to j, while if $j < i$, the flow goes from j to i) and the spatial growth of the perturbations has the same direction. On the other hand, if $\alpha < 0$, the energy flow reverses and the perturbations decrease with y. When $\delta < 0$, we have a fast propagating wave in i and a slow one in j, and the directions of the energy flow and of the growth of the perturbations are opposite to those described above.

This type of leakage is related to the 'cubic modes' obtained by Cally, and it does not originate from the existence of a second interface; for this reason, we shall call it 'single-interface leakage'. However, the presence of the other interface (when α is finite and non-vanishing) modifies the pseudomodes (see the numerical examples below).

5.3. Trapped-wave leakage

A third different type of leakage, which we call 'trapped-wave leakage', occurs when a propagating wave trapped in the intermediate region loses energy by coupling with a perturbation in one of the external regions. It can be studied

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

by considering appropriate limiting values of the parameters and performing perturbation expansions similar to that described above. Thus it can be shown that this kind of leakage leads to complex $\mathscr{S}$, $\mathscr{U}$ and $\mathscr{V}$ branches that appear for α finite and non-vanishing (described in the numerical examples). We have verified this analytically in the case of uniform pressure and magnetic field, with **k** parallel to **B**. To find all the varieties of this type of leakage, several limiting cases must be examined. A particular case that is of interest for a numerical example given below corresponds to $A < 1$ and $M_{12} > 1$, when there is a fast propagating wave trapped in the intermediate layer and a perturbation that propagates parallel to the interface in region 1 ($V_1 = 1$), and no perturbation penetrates in region 3 ($V_3 = B$), so that $M_{32} = M_{12}/B$. This requires

$$\alpha = \alpha_0 = \frac{(2n+1)\,\pi}{2}\left[\frac{F^2(M_{12}^2 - B^2)}{(M_{12}^2 - 1)\,(M_{12}^2 - A)}\right]^{1/2} \qquad (n = 0, 1, 2, \ldots),$$

so that the perturbation in the intermediate layer is a standing wave with a node at the interface (1–2) and a antinode at the interface (2–3). If we assume a small departure from this limiting case, so that

$$M_{32} = \frac{M_{12}}{B}(1 - \delta m), \qquad \Omega = M_{12}^2(1 - \delta\Omega), \qquad \alpha = \alpha_0(1 - \delta\alpha) \qquad (\delta m, \delta\Omega, \delta\alpha \ll 1),$$

it can be shown that a damped type $\mathscr{S}$ solution appears, corresponding to leakage of the trapped wave into region 1. The analysis is straightforward but somewhat cumbersome, and will not be given here. Since the trapping of propagating waves in the intermediate region always leads to a family of harmonics, their leakage correspondingly yields a family of complex branches. Then sets of leaky waves connected to periodic real branches originate from 'trapped-wave leakage' in the appropriate limit.

5.4. Lost-insulation leakage

A fourth type of leakage can appear if in one of the external regions there is a wave propagating along the interface, but the perturbation cannot penetrate into the intermediate layer ($\Gamma_2 = \infty$). Then the intermediate layer acts as a perfect insulator, and no exchange of energy between the external regions may take place, even if α is very small (but non-zero). If the parameters of the configuration are slightly changed, the perturbation may penetrate into the intermediate region, and a leakage to the other external region may occur. An example of this type of leakage happens in a symmetrical configuration, in the limiting case as $\alpha \to \epsilon$ ($0 < \epsilon \ll 1$), with $V_2^2 \to V_B^2$ and $\Gamma_1 \to 0$, $\Gamma_3 \to 0$. Precisely in the limit, the perturbations in each of the external regions are decoupled; but close to it there is a leakage from region 1 to region 3 and vice versa, since the perturbation can penetrate into region 2.

5.5. Mixed leakage

The mechanisms just discussed illustrate different ways in which leakage may occur. In the simple limiting cases considered above they appear in pure form, but as soon as the parameters depart significantly from the limiting values, mixtures between them may occur in varying amounts. Then a leaky branch

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

can be driven by entirely different mechanisms at its end points. This is the case of the majority of the leaky waves that appear in the spectrum.

All the complex $\mathscr{S}$, $\mathscr{U}$ and $\mathscr{V}$ roots are related to leaky waves. Since various different types of leakage may occur together in a given configuration, all of them must be considered in applications. Usually it is not possible to classify the leaky waves by a single driving mechanism, except in the neighbourhood of a limiting case. Therefore, when in the examples we ascribe a leakage mechanism to a certain complex branch, it must be understood that we are referring to the neighbourhood of the particular limiting case under consideration. In the present three-layer configuration there are many limiting cases in which leaky waves originate, and consequently the spectrum is very complicated. It is not our aim to give a full discussion here, so that when studying the numerical solutions we shall mention the relevant limiting cases only for a few examples.

However, all the leaky waves of the spectra that we study in the examples can be obtained by analytical continuation, starting from appropriate limiting cases of the types described above, which for this reason, can be considered as the 'basic' leakage mechanisms.

6. Numerical examples

The general problem of MHD leaky waves in a three-layer plasma involves a very large number of parameters, and ultimately the details of the solutions can only be obtained numerically. Therefore it is pointless to deal further with this problem unless one is interested in a specific application. However, to give the reader a feeling of the complexity of the spectra that may exist, and to illustrate the application of our method of analysis, we have worked out in full two cases.

In the examples we assume for simplicity that $(i = 1, 2, 3)$ $\mathbf{B}_i \equiv \mathbf{B}$ and $p_i \equiv p$. Then $A_i \equiv A$, $F_i \equiv F$, β is the same in the three layers and $f_{ij} = 1$. The only quantities that may vary discontinuously at each interface are the density and temperature. If we choose arbitrary values of M_{ij}, the density ratios ρ_i/ρ_j are given by the equation of state. Then a particular configuration is determined by A and two temperature ratios, for instance M_{21} and M_{32}. The third is the product of these two: $M_{31} = M_{32}M_{21}$. Furthermore, we consider only perturbations with $\mathbf{k}$ parallel to $\mathbf{B}$ ($\psi_i \equiv \psi = 0$), so that $V_{A,i} \equiv V_A = A$, $V_{+,i} \equiv V_+ = \mathrm{Max}(1, A)$, $V_{-,i} \equiv V_- = \mathrm{Min}(1, A)$ and $V_{B,i} = A/F \equiv B$ (no confusion with $|\mathbf{B}|$ should arise). Note that when $\psi = 0$, a degeneracy occurs owing to the coincidence of V_A with V_+ (or V_-), as a result of which one of the poles of G_i coalesces with the zero that occurs at $V_i = A$. As a consequence, there are no crossings at $V_{1,3} = A$. Furthermore, we note that for any α there are an $\mathscr{U}$ and an $\mathscr{S}$ branch with $\Omega = 0$, and in addition two $\mathscr{V}$ branches tend to $\Omega = 0$ in the limit as $\alpha \to \infty$.

In the examples we solved the DR for all values of α. The parameters are given in Table 4, and correspond to an equilibrium state. In case I ($A < 1$) all the limiting solutions are real. In case II, $m_- = 0.24$, $m_+ = 4.16$ and we have taken M_{21} and M_{32} such that there are some complex solutions when $\alpha \to \infty$ and real solutions when $\alpha = 0$. The transitions and the crossings are given in Table 5 and the limiting solutions in Table 6.

We first consider real values of Ω. The intervals where real roots may exist can be shown by plotting the type of perturbation in each layer as a function

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

Table 4. The two cases studied numerically.

Case	A	M_{21}	M_{32}
I	$\sqrt{\frac{1}{5}}$	$\sqrt{\frac{3}{5}}$	$\sqrt{5}$
II	2	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$

Table 5. Transitions and crossings for the numerical examples (β uniform and **k** parallel to **B**).

Coincidences	Case I	Case II
Real transitions		
Ω_{+1}	$\frac{5}{3}$	8
Ω_{-1}	$\frac{1}{3}$	2
Ω_{b1}	$\frac{5}{18}$	$\frac{8}{5}$
Ω_{+3}	5	$\frac{4}{3}$
Ω_{-3}	1	$\frac{1}{3}$
Ω_{b3}	$\frac{5}{6}$	$\frac{4}{15}$
Crossings or complex transitions		
Ω_{S+} (period)	6.22 (1.51)	—
Ω_{S-} (period)	5.443 (1.638)	11.906 (2.521)
Ω_{T+} (period)	—	—
Ω_{T-} (period)	1.663 (4.274)	0.875 (3.084)

Table 6. Limiting solutions of the dispersion relations for the numerical examples. We have omitted the $\Omega = 0$ solutions.

Limiting solutions	Case I	Case II
$\alpha = 0$		
Ω_{T31}	0.446	1.709
Ω_{S31}	6.22	0.617
Ω_{a2}	$\frac{1}{5}$	4
Ω_{b2}	$\frac{1}{6}$	$\frac{4}{5}$
$\alpha \to \infty$		
Ω_{T21}	0.227	—
Ω_{S21}	2.438	$1.5 + 0.974i$
Ω_{T32}	0.291	—
Ω_{S32}	5.708	$0.666 + 0.298i$
Ω_{+2}	1	4
Ω_{-2}	$\frac{1}{5}$	1

of Ω; only the cases EEE and EPE are compatible with Ω real. In Figs 2 and 3 the full and dotted lines represent propagating and evanescent waves respectively. The shaded areas mark the intervals where real roots occur (graphs that essentially correspond to ours have been used by Rae and Roberts (1983), for a symmetrical configuration). In case I (Fig. 2) there are three intervals that satisfy the necessary conditions for the existence of real roots. In case II (Fig. 3) we have chosen the parameters so that the combination PEE does not occur, to show that this condition is not necessary for the occurrence of leaky waves, contrarily to the contention of Čadež and Okretič (1989).

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

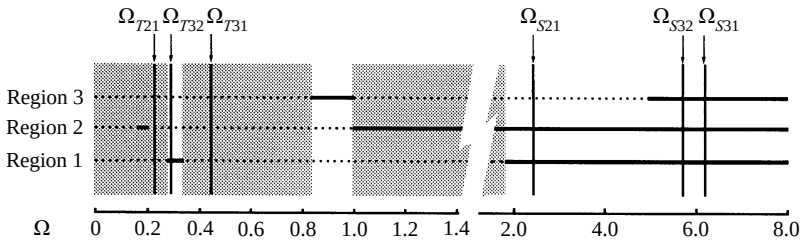


Figure 2. Intervals of existence of real roots as a function of Ω (case I, $A = \sqrt{\frac{1}{5}}$). Full and dotted lines represent propagating and evanescent waves, respectively. The shaded areas mark the intervals where real roots occur.

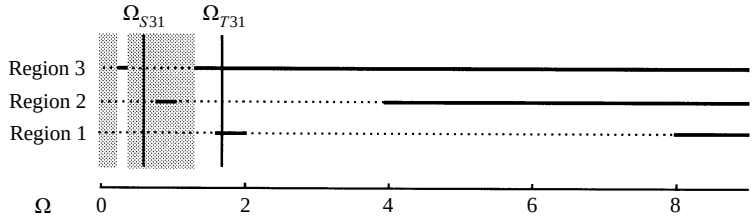


Figure 3. Intervals of existence of real roots as a function of Ω (case II, $A = 2$). Full and dotted lines represent propagating and evanescent waves respectively. The shaded areas mark the intervals where real roots occur.

In Figs 4–8 we plot Re(Ω) (full lines) and Im(Ω) (dotted lines) as functions of α, joining the corresponding lines with arrows. Only Im(Ω) with positive sign has been drawn, but it must be understood that the complex-conjugate solution is also present. Since $\omega \propto \Omega^{1/2}$, the square root must be taken with negative imaginary part to represent damped perturbations. To avoid lengthy circumlocutions when describing the spectra, we introduce a shorthand notation for the label that identifies each branch that shows: (*a*) the type, (*b*) the values of Ω where it starts and ends in increasing order of α, and (*c*) if it is real or complex. Actually, to specify the end points of a branch, we should also give the corresponding α, but for brevity we only use in the notation the value of Ω. The symbol '∞' denotes a complex branch that is unbounded at $\alpha = 0$, and the symbol '0' indicates an horizontal asymptote for $\Omega = 0$. For instance, a real $\mathscr{T}$ branch that starts at Ω_{T31} and ends at Ω_{+1} is labelled $\mathscr{T}$ [T31, +1], while a complex $\mathscr{S}$ branch that has a vertical asymptote in $\alpha = 0$ and ends at Ω_{S-} is labelled $\underline{\mathscr{S}}[\infty, \mathrm{S}-]$ (the underline indicates that a branch is complex). For the branches belonging to a set of harmonics that appear in the bands of periodicity, some further specification may be convenient. It frequently happens that the fundamental branch behaves differently from its harmonics (it may begin and end at different values of Ω). On the other hand, all its harmonics behave in the same way, beginning and ending at the same Ω. To avoid confusion among them, we use, if necessary, an index for the first and upper harmonics; for example in Fig. 4, $\mathscr{U}[\mathrm{b2}, -2]_1$ denotes the first harmonic of a family whose fundamental is the part of $\mathscr{U}[\mathrm{b2}, \mathrm{T21}]$ that lies within the propagation band $\Omega_{b2} < \Omega < \Omega_{-2}$.

We observe that the lines in the graphs are continuous: they begin at $\alpha = 0$ and finite or infinite values of Ω. As α increases, a given branch may change

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

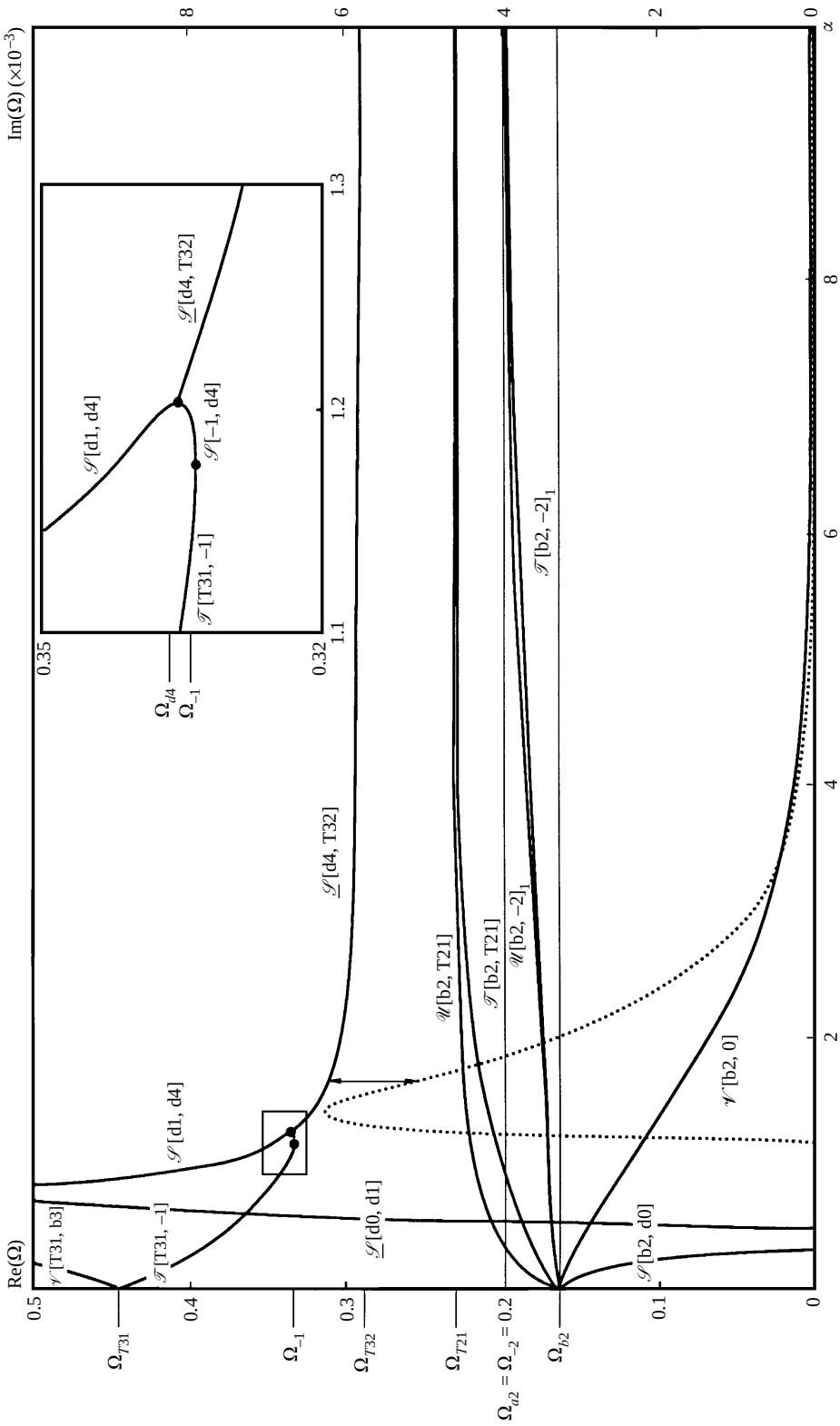


Figure 4. Spectrum of case I ($A = \sqrt{\frac{1}{5}}$): lower part.

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

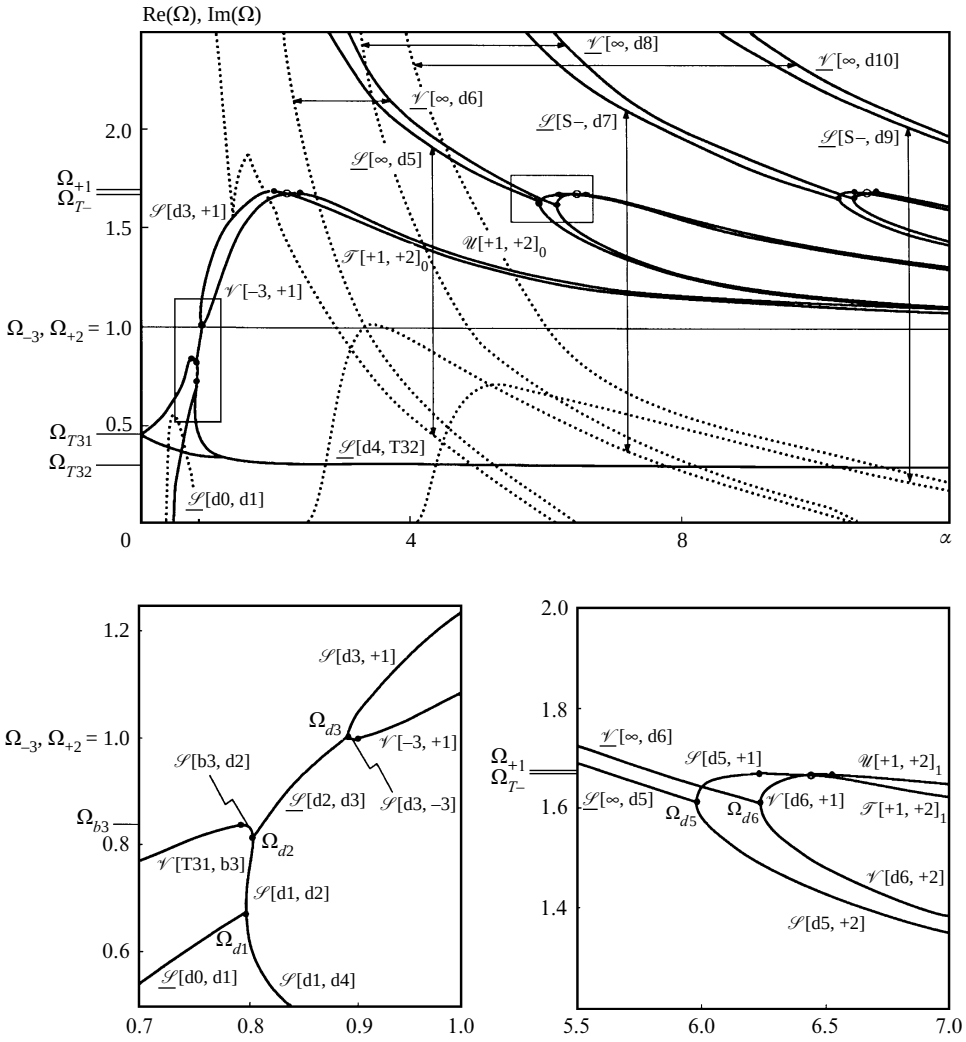


Figure 5. Spectrum of case I ($A = \sqrt{\frac{1}{5}}$): middle part.

from complex to real (and vice versa) or become a branch of another type at a transition point. Finally, it tends to one of the limiting solutions as $\alpha \to \infty$.

6.1. Case I ($A = \sqrt{\frac{1}{5}}$)

The spectrum is shown in Figs 4–6. In Fig. 4 the lower part ($0 \leqslant \mathrm{Re}(\Omega) \leqslant 0.5$) is shown. In Fig. 5 we show the middle part ($0 \leqslant \mathrm{Re}(\Omega) \leqslant 2.5$), omitting several branches already shown in Fig. 4. Finally, Fig. 6 shows the upper part ($2.5 \leqslant \mathrm{Re}(\Omega) \leqslant 7.5$). The horizontal lines mark the Ω boundaries of the propagation bands of region 2. Some branches not related to leaky waves have been omitted. Since $A < 1$, there will be evanescent perturbations in region 2 if $\Omega_{a2} = \Omega_{-2} < \Omega < \Omega_{c2} = \Omega_{+2}$ or $0 < \Omega < \Omega_{b2}$ and propagating waves outside these intervals.

No instances of pure 'single-interface leakage' occur in this case, since $A < 1$, but pure 'surface-wave leakage' occurs for α near infinity, and becomes

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

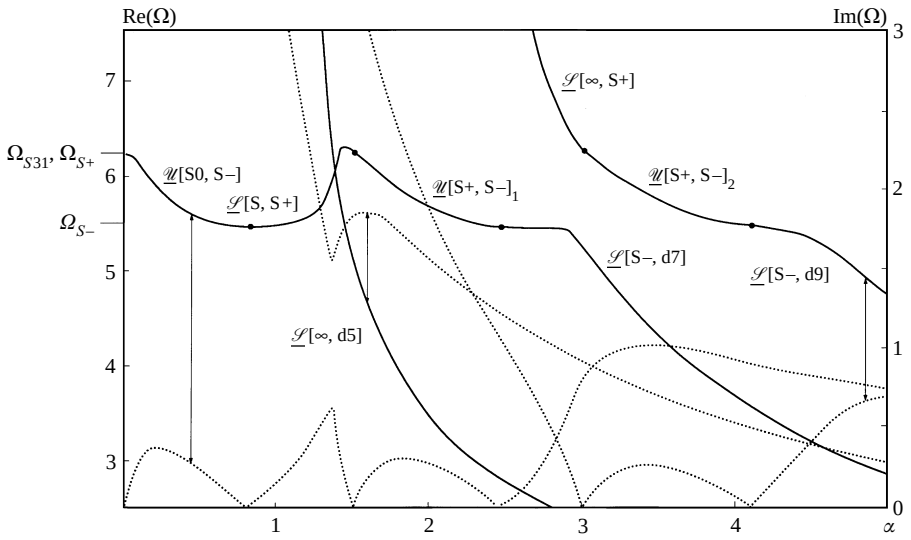


Figure 6. Spectrum of case I ($A = \sqrt{\frac{1}{5}}$): upper part.

mixed as α decreases. This is the case of the complex branch $\underline{\mathscr{L}}$[d4, T32], which tends to Ω_{T32} as $\alpha \to \infty$, and represents leaky waves that originate in the coupling between a surface wave at the interface (2–3) and a propagating wave in region 1 (see Sec. 5). Many other instances of mixed leakage occur, which are related to limiting cases of pure 'trapped-wave leakage'. One of these is the complex branch $\underline{\mathscr{L}}$[d2, d3], which can be obtained by analytical continuation of the limiting case discussed in Sec. 5.3. Note the presence of harmonic branches in the propagation band $\Omega > \Omega_{+2}$ (Fig. 5). The fundamentals $\mathscr{S}$[d3, +1] and $\mathscr{V}$[−3, +1] occur within a finite α interval, and change respectively to $\mathscr{T}$[+1, +2] and $\mathscr{U}$[+1, +2] at Ω_{+1}. The latter two approach Ω_{+2} as $\alpha \to \infty$. All the upper harmonics of the four families extend to $\alpha \to \infty$, and approach Ω_{+2}; they are connected in pairs $\mathscr{S} \leftrightarrow \mathscr{T}$ and $\mathscr{V} \leftrightarrow \mathscr{U}$ at Ω_{+1} (periodic real transition). Associated with each pair, there is a double root of $\mathscr{S}$ or $\mathscr{V}$ where complex branches coalesce. They are related to 'trapped-wave leakage' as $\underline{\mathscr{L}}$[d2, d3]. There is also a periodic crossing Ω_{T-} between the real $\mathscr{T}$ and $\mathscr{V}$ branches. The complex branches of type $\mathscr{S}$ of the upper part of the spectrum have a complicated behaviour (Fig. 6). The first of them is $\underline{\mathscr{L}}$[∞, d5]. The second is $\underline{\mathscr{L}}$[S−, d7], which originates from $\underline{\mathscr{U}}$[S0, S−]. It begins at $\alpha = 0$, and then suffers complex transitions at Ω_{S-} and Ω_{S+}. The third branch, $\underline{\mathscr{L}}$[S−, d9], as well as those related to the higher harmonics of Fig. 5, are connected with complex branches $\underline{\mathscr{L}}[\infty, \mathrm{S}+]_j$ by means of two complex transitions at Ω_{S+} and Ω_{S-}.

6.2. Case II ($A = 2$)

The spectrum is depicted in Fig. 7 ($0 \leqslant \mathrm{Re}(\Omega) \leqslant 2$) and in Fig. 8 ($2 \leqslant \mathrm{Re}(\Omega) \leqslant 20$). The perturbation is evanescent in region 2 for $\Omega_{c2} = \Omega_{-2} < \Omega < \Omega_{+2} = \Omega_{a2}$ or $0 < \Omega < \Omega_{b2}$, and propagating outside these intervals. The branches $\underline{\mathscr{U}}$[d1, S32], $\underline{\mathscr{V}}$[d2, S32] and $\mathscr{V}$[∞, S21], $\underline{\mathscr{L}}$(T31, S21] tend to the complex limiting solutions Ω_{S32} and Ω_{S21}, and in this limit are examples of 'single-interface leakage'. The complex branch $\underline{\mathscr{V}}$[d3, 0] can be obtained by analytical

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

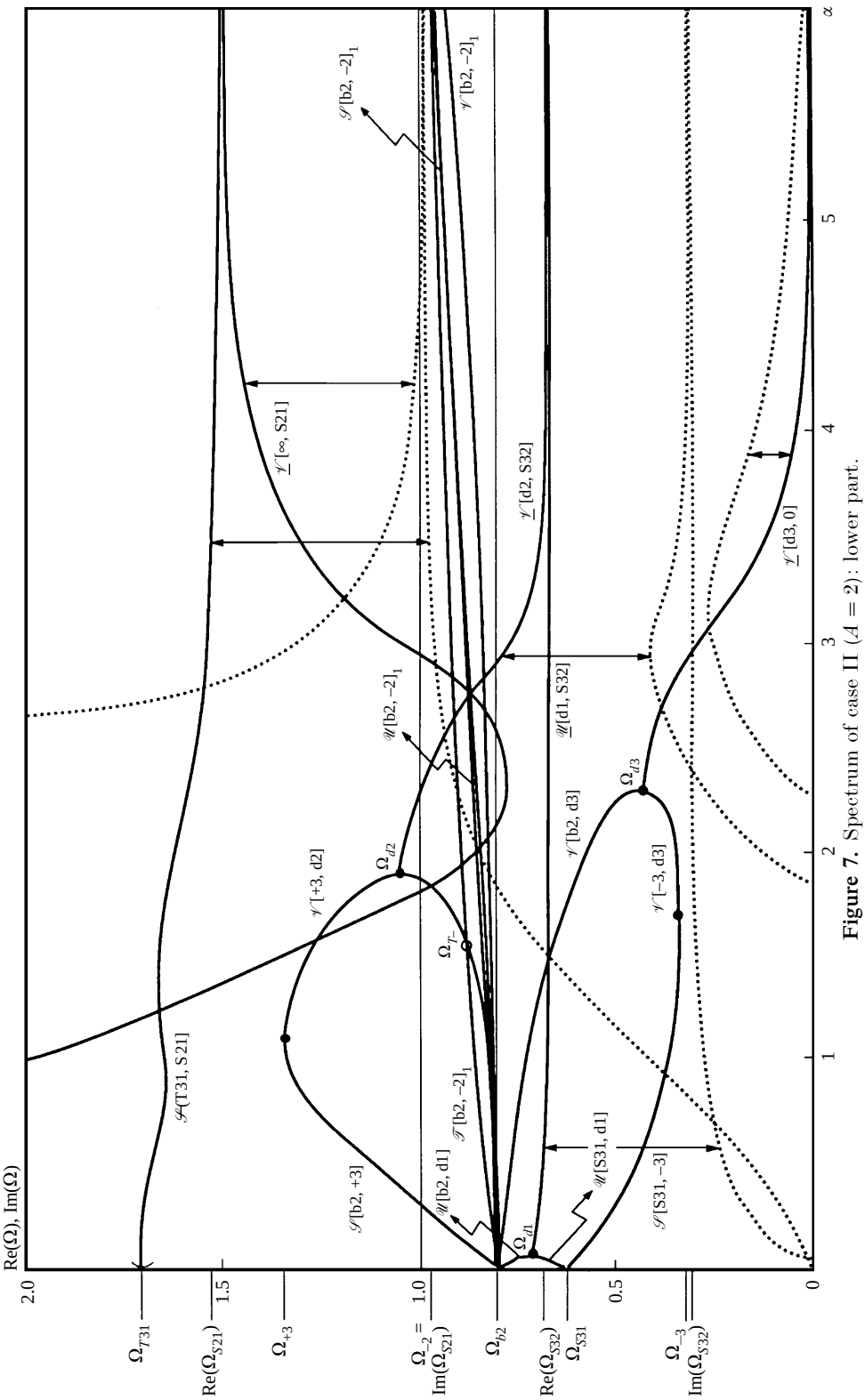


Figure 7. Spectrum of case II ($A = 2$): lower part.

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

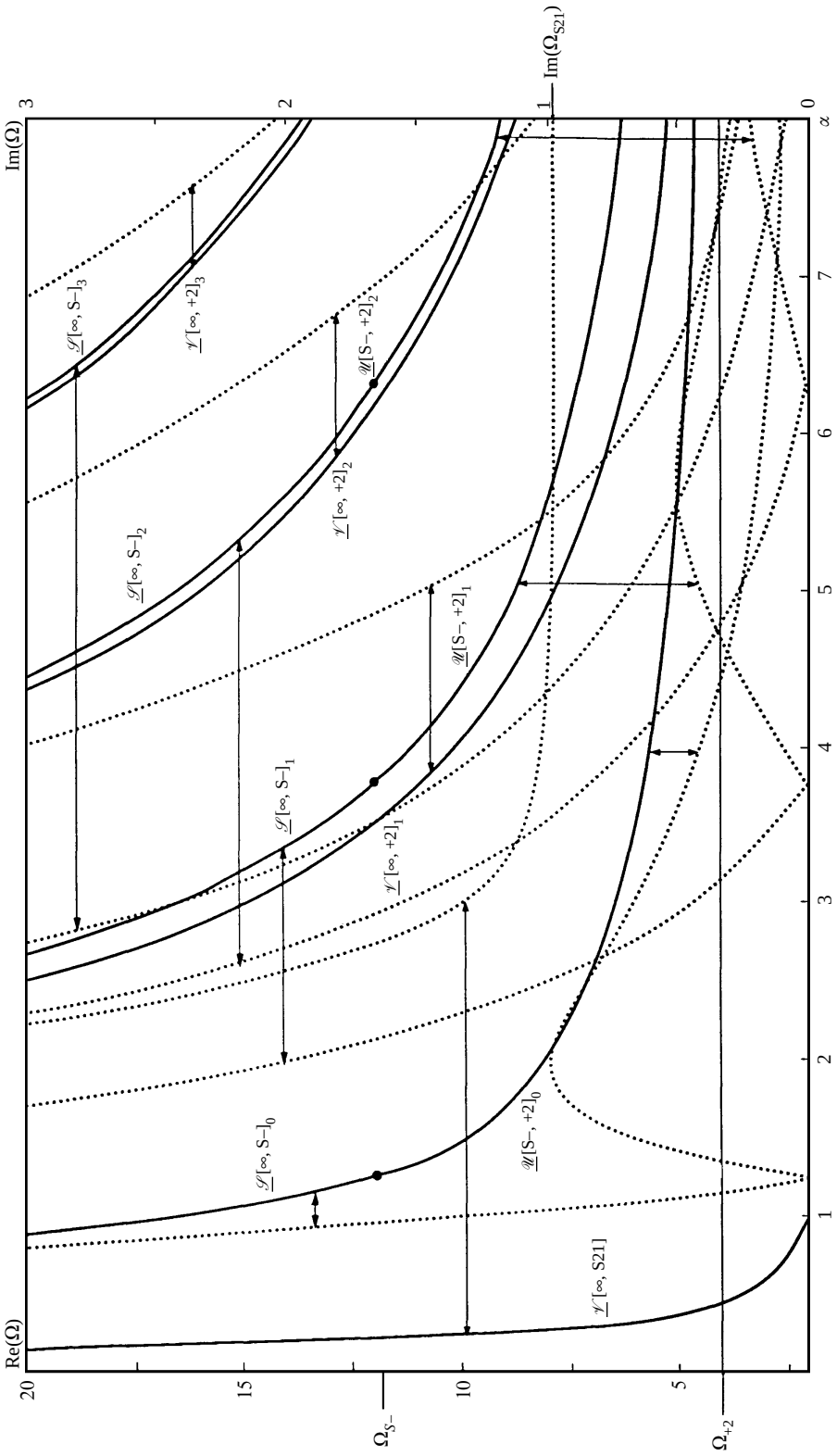


Figure 8. Spectrum of case II ($A = 2$): upper part.

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

continuation, starting from the limiting symmetrical case $\alpha \to \epsilon$, $(0 < \epsilon \ll 1)$, with $\Omega \to B^2$ and $\Gamma_{1,3} \to 0$, in which 'lost insulation leakage' takes place (see Sec. 5.4).

The branch $\underline{\mathscr{L}}(\mathrm{T31}, \mathrm{S21}]$ tends to Ω_{T31} as $\alpha \to 0$, but its behaviour in this limit is singular (this has been denoted by using an opening parenthesis instead of a square bracket). According to Table 3, at $(0, \Omega_{T31})$ the branches $\mathscr{T}$ and $\mathscr{V}$ coincide. But these branches cannot exist in the neighbourhood of $\alpha = 0$. Actually, Ω_{T31} is real, and corresponds to the case PEP (Fig. 2), for which no real branch can exist, except in the special case $e^{-2\alpha\Gamma_2} = 1$, i.e. $\alpha = 0$. For $\alpha \neq 0$ the solution must be complex, but this is not possible for a $\mathscr{T}$ branch. Then the branch $\underline{\mathscr{L}}(\mathrm{T31}, \mathrm{S21}]$ exists for α arbitrarily small, but non-vanishing.

In the upper part of the spectrum one finds the propagation band $\Omega > \Omega_{+2}$, in which there are complex transitions Ω_{S-} between pairs of complex branches $\underline{\mathscr{L}}[\infty, \mathrm{S}-]$ and $\underline{\mathscr{U}}[\mathrm{S}-, +2]$. These $\underline{\mathscr{U}}[\mathrm{S}-, +2]$ branches tend to the real limiting solution Ω_{+2} as $\alpha \to \infty$. In this propagation band there are also branches of the type $\underline{\mathscr{V}}[\infty, +2]$, whose imaginary parts tend to zero as $\alpha \to \infty$, while the real parts tend to Ω_{+2}. We have drawn the first four harmonics of both families. All these leaky waves tend to a particular degenerate case (since $\Omega_{a2} = \Omega_{+2}$) of 'trapped-wave leakage' for very large α.

7. Discussion and conclusions

For a general configuration of three plane plasma slabs in equilibrium, we have investigated the oscillations with complex frequency that describe the temporal decay of a perturbation due to the transfer of energy between different regions, for arbitrary $\mathbf{k}$. We have used a matrix equation that takes into account the matching of the various oscillations at the interfaces. We have then obtained the DR corresponding to the different kinds of perturbations. These relations are transcendental, but we have nevertheless been able to study them analytically. We have obtained relevant information about the different perturbations, including the transitions and crossings between branches and the periodicity bands. We have shown that the branches with real frequency exist only if there are evanescent waves in the external regions.

The branches of complex frequency, which are solutions of the dispersion relations $\mathscr{S} = 0$, $\mathscr{U} = 0$ and $\mathscr{V} = 0$, represent the leaky waves. We have identified four basic leakage mechanisms, which we call 'surface-wave leakage', 'single-interface leakage', 'trapped-wave leakage' and 'lost-insulation leakage'. Several of these may be present in a given configuration. They appear in pure form only in some special limiting cases, and as soon as the parameters depart significantly from the limiting values, mixtures between them may take place in varying amounts, leading to what we call 'mixed leakage'. This is the case for the majority of the leaky waves that appear in the spectrum, as shown by the numerical examples. Therefore it is not possible to classify the leaky waves by a single driving mechanism, except in the neighbourhood of a limiting case. In a three-layer configuration there are many limiting cases in which leaky waves originate, and consequently the spectrum is very complicated.

A plane model of three slabs was studied previously by Čadež and Okretič (1989). Their parameters were chosen to ensure the existence (for some range of Ω) of surface waves in regions 2 and 3 and propagating ones in region 1. Their

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

idea was to excite a surface wave at the interface $y = a$ whose amplitude decays exponentially in time as it transfers its energy to a propagating wave in region 1 (what we call 'surface-wave leakage'). They assumed (incorrectly) that leaky waves could be obtained by solving the dispersion relation $\mathscr{T} = 0$. They began with $\alpha \to \infty$ (i.e. two uncoupled interfaces). For that purpose, they solved the equation $\mathscr{T}_{32} = 0$. They used the resulting root as a starting point to solve $\mathscr{T} = 0$ for finite α, hoping to find a complex frequency whose imaginary part increases (in absolute value) as the distance between the interfaces decreases. They actually found perturbations with complex frequency for α between 3 and 7 approximately, when they should not, since $\mathscr{T} = 0$ does not admit complex roots (see Sec. 4). It is not possible to repeat their calculations, since they do not report the full list of parameters. However, it can be verified that their unperturbed state is not in equilibrium (perhaps this explains why a complex Ω was obtained from $\mathscr{T} = 0$). We conclude that the analysis of Čadež and Okretič is faulty because

(i) the unperturbed state is not in equilibrium;

(ii) for an equilibrium configuration, the dispersion relation $\mathscr{T} = 0$ does not lead to leaky waves;

(iii) the complex branches that are solutions of other dispersion relations were not investigated, although they are precisely those that yield leaky waves;

(iv) in looking only for solutions of the 'surface-wave leakage' type, they implicitly assumed that this is the only driving mechanism, which is false;

(v) based on this implicit assumption, they state an incorrect necessary condition for the existence of leaky waves (namely that the initial configuration must allow evanescent waves in regions 2 and 3 and propagating ones in region 1 for some real Ω).

The 'single-interface leakage' (Sec. 5.2) is related to some results obtained by Cally (1986), who studied the leaky waves in a configuration with cylindrical symmetry and considered two uniform media with a discontinuity at a given radius. When this radius is large compared with the wavelength of the perturbation, he obtains the so-called 'cubic modes'. In this limit the cylindrical interface can be approximated by a plane, and the radial coordinate corresponds to the y coordinate in our problem. In the cylindrical case there is a coupling between the $\mathscr{T}$ and $\mathscr{S}$ branches. For the cubic modes this coupling is weak, and the leaky waves are obtained from the solutions of $\mathscr{S} = 0$. Then our limiting solutions for $\alpha = 0$ are equivalent to the cubic modes of Cally. We checked this by solving the equation $\mathscr{S}_{31} = 0$ for the parameters of Cally ($\psi_{1,3} = 0$, $A_1 = A_3 = 7$ and $M_{31} = 2$), and found perfect agreement ($V_3 = 0.81394 - 0.19365i$).

We have discussed in detail two examples corresponding to different A to show how the methods that we have developed can be used to study specific cases. We have solved the dispersion relations numerically, and found that both cases admit leaky waves for different α but that their decay rates vary considerably. In case I ($A = \sqrt{\frac{1}{5}} < 1$) the solutions for $\alpha = 0$ and $\alpha \to \infty$ are real. In case II ($A = 2$) we chose the configuration so that no evanescent waves exist in regions 2 and 3 and propagating ones in region 1 for any Ω, and we found

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press

leaky waves, in violation of the (incorrect) necessary condition of Čadež and Okretič.

We have indicated appropriate limiting cases of the four basic mechanisms, from which all the leaky waves of the spectra can be obtained by analytical continuation.

The method developed in this paper allows us to investigate the leaky waves in any case of interest involving three layers, since there are no restrictions on the configuration (other than equilibrium) and on **k**. The method can also be extended to more complicated cases (that include, for example, the effect of gravity) that may be of interest in applications.

Acknowledgements

This work was supported by CONICET, and is part of Project EX 246 of the University of Buenos Aires.

References

Bernstein, I. B., Frieman, E. A., Kruskal, M. D. and Kulsrud, R. M. 1957 *Proc. R. Soc. Lond.* **A 244**, 17.

Čadež, V. M. and Okretič, V. K. 1989 *J. Plasma Phys.* **41**, 23.

Cally, P. S. 1986 *Solar Phys.* **103**, 277.

Chen, L. and Hasegawa, A. 1974 *Phys. Fluids* **17**, 1399.

Edwin, P. M. and Roberts, B. 1982 *Solar Phys.* **76**, 239.

Giovanelli, R. G. 1975 *Solar Phys.* **44**, 299.

González, A. G. and Gratton, J. 1991 *Solar Phys.* **134**, 211.

Gratton, J., Gratton, F. and González, A. G. 1988 *Plasma Phys. Contr. Fusion* **30**, 435.

Ionson, J. A. 1978 *Astrophys. J.* **226**, 650.

Le Blond, P. H. and Mysak, L. A. 1978 *Waves in the Ocean*. Elsevier, Amsterdam.

Rae, I. C. and Roberts, B. 1983 *Solar Phys.* **84**, 99.

Roberts, B. 1981 *Solar Phys.* **69**, 39.

Roberts, B. 1991 *Advances in Solar System Magnetohydrodynamics* (ed. E. R. Priest and A. W. Hood), Chap. 6. Cambridge University Press.

Roberts, B. and Webb, A. R. 1979 *Solar Phys.* **65**, 77.

Spruit, H. C. 1982 *Solar Phys.* **75**, 3.

Wilson, P. R. 1981 *Astrophys. J.* **251**, 756.

https://doi.org/10.1017/S0022377897006120 Published online by Cambridge University Press