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A WEIGHTED EMPIRICAL INTERPOLATION METHOD: *A PRIORI* CONVERGENCE ANALYSIS AND APPLICATIONS

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Abstract. We extend the classical empirical interpolation method [M. Barrault, Y. Maday, N.C. Nguyen and A.T. Patera, An empirical interpolation method: application to efficient reduced-basis discretization of partial differential equations. *Compt. Rend. Math. Anal. Num.* **339** (2004) 667–672] to a weighted empirical interpolation method in order to approximate nonlinear parametric functions with weighted parameters, *e.g.* random variables obeying various probability distributions. *A priori* convergence analysis is provided for the proposed method and the error bound by Kolmogorov N-width is improved from the recent work [Y. Maday, N.C. Nguyen, A.T. Patera and G.S.H. Pau, A general, multipurpose interpolation procedure: the magic points. *Commun. Pure Appl. Anal.* **8** (2009) 383–404]. We apply our method to geometric Brownian motion, exponential Karhunen–Loève expansion and reduced basis approximation of non-affine stochastic elliptic equations. We demonstrate its improved accuracy and efficiency over the empirical interpolation method, as well as sparse grid stochastic collocation method.

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1. Introduction

The empirical interpolation method [1] was originally developed to approximate the non-affine terms of a partial differential equation in order to effectively decompose the reduced basis method into offline construction and online evaluation procedure. Since its development, many applications and extensions of this method have been considered [4, 10, 12, 13, 18, 20, 22]. In particular, we mention its application and analysis in the context of reduced basis approximation for nonlinear elliptic and parabolic equations [10] and, more recently, its extension to a general, multipurpose interpolation procedure [13], in which *a priori* error estimate compared to Kolmogorov N-width was obtained.

Keywords and phrases. Empirical interpolation method, *a priori* convergence analysis, greedy algorithm, Kolmogorov N-width, geometric Brownian motion, Karhunen–Loève expansion, reduced basis method.

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The basic idea behind empirical interpolation for parametric function $g(x,\mu)$ is to choose the parameter samples $\mu^1,\mu^2,\dots$ and the interpolation nodes $x^1,x^2,\dots$ recursively in a greedy approach according to the criteria that μ^m and x^m selected at each step $m=1,2,\dots$ are the most representative ones in L^∞ norm or the ones where the function is worst approximated by the interpolation formula constructed from the previous steps [1]. This is essentially different from the conventional interpolation construction which requires the interpolation nodes to be chosen *a priori* according to a specific rule, *e.g.* roots of orthogonal polynomials [19]. The so called "magic points" [13] $(\mu^m,x^m), m=1,2,\dots$ obtained by the goal-oriented or function-specified empirical interpolation procedure are supposed to identify an interpolation formula by capturing some specific features (*e.g.* regularity, extreme values) of the given function, thus providing higher interpolation accuracy. Another superiority of the empirical interpolation construction is attributed to the affine expansion of the function given in whatever form, leading to the separation of the variable x and the parameter μ in the following expression [10]

$$g(x,\mu)\approx \mathcal{I}_M[g]=\sum_{j=1}^{M}\Theta_j(\mu)q_j(x), \tag{1.1}$$

which can be efficiently used in conducting mathematical manipulation, *e.g.* numerical integration, reduced basis approximation [10]. By convention, one supposes that the parameter μ, if viewed as a random variable, is uniformly distributed in a bounded space Γ.

However, in many applications, *e.g.* stochastic problems with parametrized random variables obeying normal distribution, the request of the boundedness of the parameter space Γ and that of the uniform distribution of the parameter μ is hard to fulfill. In this situation, the approximation to some quantity of interest (*e.g.* weighted integral or statistics of the function) based on the parameter samples and interpolation nodes selected by the empirical interpolation procedure would not lead to results that are as accurate or efficient as those expected when taking distinct weights of the parameter at different values into account. In this work we propose a weighted empirical interpolation method (wEIM) by considering a weighted optimization problem and analyzing its convergence property by improving the *a priori* error estimate obtained in [13]. To demonstrate numerically its effectiveness and efficiency, we apply the wEIM to approximating nonlinear parametric functions, geometric Brownian motion in one dimension, exponential Karhunen–Loève expansion in multi-dimension as well as reduced basis approximation to non-affine stochastic elliptic problems, and compare it with the conventional empirical interpolation method (EIM) and sparse grid stochastic collocation method. It is worth mentioning that constructing a goal-oriented numerical method is a quite common procedure in adaptive finite element methods [2,9] and has also been applied to construct adaptive reduced basis method [5].

The work is organized as follows: we present the weighted empirical interpolation method in Section 2. *A priori* convergence analysis is carried out in Section 3, followed by Section 4 where different applications of this method are addressed. Some concluding remarks are drawn in Section 5.

2. Weighted empirical interpolation method (wEIM)

For notational convenience, we introduce the spaces $L^\infty(D)$ defined in a bounded physical domain $D\subset\mathbb{R}^d, d\in\mathbb{N}_+$ and $C_w^0(\Gamma)$ defined in a parameter space (not necessarily bounded) $\Gamma\subset\mathbb{R}^K, K\in\mathbb{N}_+$, which are equipped with the following norms: $||g||_{L^\infty(D)}=\operatorname{ess\,sup}_{x\in D}|g(x)|$ and $||g||_{C_w^0(\Gamma)}=\max_{\mu\in\Gamma}w(\mu)|g(\mu)|$ with a positive weight function $w:\Gamma\to\mathbb{R}_+$. We also define the Bochner space $L^\infty(D;C_w^0(\Gamma))$ for a parameter dependent function equipped with the norm $||g||_{L^\infty(D;C_w^0(\Gamma))}=\operatorname{ess\,sup}_{x\in D}(\max_{\mu\in\Gamma}w(\mu)|g(x,\mu)|)\equiv\max_{\mu\in\Gamma}w(\mu)(\operatorname{ess\,sup}_{x\in D}|g(x,\mu)|)$. We note that $L^\infty(D)$, as used in [1,10,13], is usually replaced with $C^0(D)$ for conventional interpolation [19].

At the discrete level, the physical domain D is replaced by a series of vertices $x\in V_x$ with finite cardinality $n_x=|V_x|<\infty$, for instance finite element nodes, and the parameter space Γ is represented by a sample set $\mu\in\Xi_\mu$ of finite cardinality $n_\mu=|\Xi_\mu|<\infty$. We present the weighted empirical interpolation method in the following generic Algorithm 1. We emphasize that the initial sample μ^1 is chosen such that the weighted function

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is maximized in $L^\infty(V_x; C^0_w(\Xi_\mu))$ norm:

$$\mu^1 = \arg\max_{\mu\in\Xi_\mu}\left[w(\mu)\left(\operatorname{ess\,sup}_{x\in V_x}|g(x,\mu)|\right)\right]. \tag{2.1}$$

In the course of the construction procedure, the quasi-optimal samples $\mu^{M+1}, M \geq 1$ can be chosen by greedy algorithm to minimize the weighted optimal approximation error (2.2) in the subspace W_M spanned by the "snapshots" $W_M := \text{span}\{g(\cdot,\mu^i), 1 \leq i \leq M\}$: find $\mu^{M+1} \in \Xi_\mu$ such that

$$\mu^{M+1} = \arg\max_{\mu\in\Xi_\mu}\left[w(\mu)\left(\inf_{h\in W_M}||g(\mu)-h||_{L^\infty(V_x)}\right)\right]. \tag{2.2}$$

However, the weighted L^∞ optimization problem (2.2) is expensive to solve by linear programming if $|V_x|$ and $|\Xi_\mu|$ are large. In practice, it can be efficiently replaced by a weighted L^2 optimization problem [10] or by a surrogate weighted L^∞ optimization problem (2.5) [13]. We state several properties of the wEIM in the following lemmas, whose proof is straightforward by noting the fact that the weight function $w : \Gamma \to \mathbb{R}_+$ is positive and therefore omitted here, see for instance [1, 10, 13] for details.

Lemma 2.1. *For any $M < M_{max}$, the subspace $Q_M = span\{q_m, 1 \leq m \leq M\}$ is of dimension M. Moreover, the matrix B^M formed in (2.8) is lower triangular with unity diagonal and thus invertible.*

Lemma 2.2. *For any function $h \in Q_M$, the empirical interpolation formula given by (2.3) is exact,* i.e. *$r_{M+1}(x,\mu) = 0, \forall x \in V_x, \mu \in \Xi_\mu$. In general, for any function $g \in L^\infty(D; C^0_w(\Gamma))$, we have*

$$||g - \mathcal{I}_M[g]||_{L^\infty(D)} \leq (1+\Lambda_M)\inf_{h\in Q_M}||g-h||_{L^\infty(D)} \text{ with } \Lambda_M \leq 2^M - 1. \tag{2.9}$$

3. *A PRIORI* CONVERGENCE ANALYSIS

The interpolation error obtained in (2.9) with the Lebesgue constant $\Lambda_M \leq 2^M - 1$ (see proof in [10]) by the empirical interpolation procedure is too pessimistic, far from the result for conventional interpolation error based on certain prescribed interpolation nodes (*e.g.* Chebyshev nodes) with $\Lambda_M \sim \log(M)$ [19]. An explicit *a priori* convergence rate of the weighted empirical interpolation error is not available for generic functions. In order to measure the accuracy of the approximation by wEIM, in the following theorem we compare it with the Kolmogorov N-width [17], which quantifies the optimal approximation error of a subset $\mathcal{F}$ in a Banach space $\mathcal{H}$ by any possible N dimensional subspace F_N, defined as

$$d_N(\mathcal{F},\mathcal{H}) := \inf_{F_N}\sup_{g\in\mathcal{F}}\inf_{f\in F_N}||g-f||_{\mathcal{H}}. \tag{3.1}$$

When the subset $\mathcal{F}$ becomes the same as $\mathcal{H}$, we denote $d_N(\mathcal{H}) \equiv d_N(\mathcal{F},\mathcal{H})$ for simplicity.

Theorem 3.1. *The error of wEIM can be bounded as follows*

$$||g - \mathcal{I}_M[g]||_{L^\infty(V_x)} \leq C_w(M+1)2^M d_M(L^\infty(V_x)) \tag{3.2}$$

where the constant C_w depends on the weight function w but is independent of M.

Remark 3.2. In fact, the result (3.2) is obtained in the subspace $L^\infty(V_x)$ for the constructive wEIM and can be straightforwardly extended to $L^\infty(D)$ when the vertex set V_x tends to D such that the points outside the vertex set V_x can be sufficiently well represented by the points inside. To be rigorous, we take the vertex set V_x such that for almost every $x \in D$, there exists $y \in V_x$ satisfying

$$|g(x) - g(y)| \leq ||g - \mathcal{I}_M[g]||_{L^\infty(V_x)}. \tag{3.3}$$

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Algorithm 1. A weighted empirical interpolation method.

procedure INITIALIZATION:
  Given finite vertex set $V_x \subset D$, sample set $\Xi_\mu \subset \Gamma$, weight w and function $g \in L^\infty(V_x; C_w^0(\Xi_\mu))$;
  find $\mu^1 \in \Xi_\mu$ such that $\mu^1 = \arg\max_{\mu\in\Xi_\mu} w(\mu)(\operatorname{ess\,sup}_{x\in V_x} |g(x,\mu)|)$; set $W_1 = \operatorname{span}\{g(x,\mu^1)\}$;
  find $x^1 \in V_x$ such that $x^1 = \arg\operatorname{ess\,sup}_{x\in V_x} |g(x,\mu^1)|$;
  define $r_1 = wg, q_1(x) = r_1(x,\mu^1)/r_1(x^1,\mu^1), B_{11}^1 = 1$, set $M = 1$, specify tolerance ε_{tol};
end procedure
procedure CONSTRUCTION:
  while $M < M_{max}$ & $r_M(x^M,\mu^M) > \varepsilon_{tol}$ **do**
    find $\Theta^M(\mu^M) = (\Theta_1^M(\mu^M),\dots,\Theta_M^M(\mu^M))^T$ by solving

$$\sum_{j=1}^{M} \Theta_j^M(\mu^M) q_j(x^i) = g(x^i,\mu^M) \quad 1 \le i \le M; \tag{2.3}$$

    define $r_{M+1} : D \times \Gamma \to \mathbb{R}$ as

$$r_{M+1}(x,\mu) = g(x,\mu) - \sum_{j=1}^{M} \Theta_j^M(\mu) q_j(x); \tag{2.4}$$

    find $\mu^{M+1} \in \Xi_\mu$ such that

$$\mu^{M+1} = \arg\max_{\mu\in\Xi_\mu} \left[w(\mu) \left(\operatorname*{ess\,sup}_{x\in V_x} |r_{M+1}(x,\mu)| \right) \right], \tag{2.5}$$

    find $x^{M+1} \in V_x$ such that

$$x^{M+1} = \arg\operatorname*{ess\,sup}_{x\in V_x} |r_{M+1}(x,\mu^{M+1})|; \tag{2.6}$$

    define $q_{M+1} : D \to \mathbb{R}$ as

$$q_{M+1}(x) = \frac{r_{M+1}(x,\mu^{M+1})}{r_{M+1}(x^{M+1},\mu^{M+1})}; \tag{2.7}$$

    update matrix $B^{M+1} \in \mathbb{R}^{(M+1)\times(M+1)}$ as

$$B_{ij}^{M+1} = q_j(x^i) \quad 1 \le i,j \le M+1; \tag{2.8}$$

  end while
end procedure
procedure EVALUATION:
  For $\forall \mu \in \Xi_\mu$, construct approximation (1.1) by solving (2.3), then evaluate (1.1) at $\forall x \in V_x$.
end procedure

Consequently, we have the error bound

$$\begin{aligned} ||g - \mathcal{I}_M[g]||_{L^\infty(D)} &\le ||g - \mathcal{I}_M[g]||_{L^\infty(D\setminus V_x)} + ||g - \mathcal{I}_M[g]||_{L^\infty(V_x)} \\ &\le 2||g - \mathcal{I}_M[g]||_{L^\infty(V_x)} \\ &\le C_w(M+1)2^{M+1} d_M(L^\infty(V_x)) \\ &\le C_w(M+1)2^{M+1} d_M(L^\infty(D)). \end{aligned} \tag{3.4}$$

The proof of (3.2) adopts a constructive approach inspired from that for the greedy algorithm in reduced basis approximation [3, 21]. Some preliminary results are needed, see the next two lemmas.

For simplicity, we use the shorthand notation $r_m(x) = r_m(x,\mu^m), 1 \le m \le M+1$ obtained in Algorithm 1 and define the functions $t_j(x^i) = r_i(x^j), 1 \le i,j \le M+1$ and $t_j(x^i) = 0, 1 \le j \le M+1, i > M+1$.

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Lemma 3.3. *The matrix T^{M+1} defined by $T^{M+1}_{ij} = t_j(x^i), 1 \le i, j \le M+1$ is upper triangular matrix with dominating diagonal elements,* i.e. $t_j(x^i) = 0, i > j$ *and* $|t_j(x^i)| \le |t_j(x^j)|, i \le j$.

Proof. From the result of Lemma (2.1), we know that the matrix B^{M+1} is lower triangular with unity diagonal. By the definition of $q_i, 1 \le i \le M+1$ in (2.7) and the definition of $t_j, 1 \le j \le M+1$, we have $t_j(x^i) = q_i(x^j) r_i(x^i)$, so that $t_j(x^i) = q_i(x^j) = 0, i > j$ and $|t_j(x^i)| \le |t_j(x^j)| = |r_j(x^j)|, i \le j$ due to (2.6). □

Lemma 3.4. *For any $1 \le m \le M+1$, there exists a unique $b = (b_1, \dots, b_m)^T \in \mathbb{R}^m$ such that*

$$r_m(x) e_m(x) = \sum_{j=1}^{m} b_j t_j(x) \quad \forall x \in V_x, \tag{3.5}$$

where $e_m, 1 \le m \le M+1$ are unit vectors, i.e. $e_m(x^m) = 1$ *and* $e_m(x^n) = 0$ *if* $n \ne m$. *In addition, we have $b_m = 1$ and the bound $|b_i| \le 2^{m-i-1}, 1 \le i < m$ so that $|b_1| + \cdots + |b_m| \le 2^{m-1}$.*

Proof. For any $x = x^i, i > M+1$, we have $e_m(x) = 0$ and $t_j(x) = 0$, so that both sides of the equation vanish and we only need to verify the statement for $x = x^i, 1 \le i \le M+1$, in which case the system (3.5) becomes

$$s = Tb \quad \text{with } s = (0, \dots, 0, r_m(x^m))^T. \tag{3.6}$$

Thanks to Lemma 3.3, we have that T is invertible and thus there exists a unique solution b. Moreover, the last line of the system (3.6) $r_m(x^m) = t_m(x^m) b_m$ leads to the solution $b_m = 1$ since $r_m(x^m) = t_m(x^m)$. For any other line $i, 1 \le i < m$, we have by the fact that T is an upper triangular matrix

$$0 = \sum_{j=i}^{m} b_j t_j(x^i). \tag{3.7}$$

By recalling that $|t_j(x^i)| \le |t_i(x^i)|, j > i$, this yields the following bound for $b_i, 1 \le i < m$

$$|b_i| = \left| - \sum_{j=i+1}^{m} b_j \frac{t_j(x^i)}{t_i(x^i)} \right| \le \sum_{j=i+1}^{m} |b_j|, \tag{3.8}$$

so that $|b_i| \le 2^{m-i-1}, 1 \le i < m$ and $|b_1| + \cdots + |b_m| \le 2^{m-1}$ being $b_m = 1$ and using a recursive argument. □

We are now ready to prove Theorem 3.1 using the representation of the residual in Lemma 3.4.

proof of Theorem 3.1. Suppose there exists a subspace $H_M \subset L^\infty(V_x)$ of dimension M achieving the Kolmogorov M-width as defined in (3.1), then we have a series of elements $h_j \in H_M, 1 \le j \le M+1$ such that

$$||t_j - h_j||_{L^\infty(V_x)} \le d_M(L^\infty(V_x)), 1 \le j \le M+1. \tag{3.9}$$

We define the functions

$$s_m(x) = \sum_{j=1}^{m} b_j h_j(x), 1 \le m \le M+1. \tag{3.10}$$

Since all the elements $h_j, 1 \le j \le M+1$ belong to the M dimensional subspace H_M and s_m is a linear combination of these elements for any $m = 1, \dots, M$, there exists a vector $\alpha = (\alpha_1, \dots, \alpha_{M+1})^T$ with $|\alpha_1| + \cdots + |\alpha_{M+1}| = 1$ such that

$$\sum_{m=1}^{M+1} \alpha_m s_m = 0. \tag{3.11}$$

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Thanks to the result in Lemma 3.4 together with bound (3.9) and representation (3.10) and (3.11), we obtain the following bound for every $x \in V_x$

$$\begin{aligned}\left|\sum_{m=1}^{M+1}\alpha_m r_m(x)e_m(x)\right| &= \left|\sum_{m=1}^{M+1}\alpha_m\left(r_m(x)e_m(x)-s_m(x)\right)\right| \\ &\leq \left(\sum_{m=1}^{M+1}|\alpha_m|\right)\max_{m=1,\ldots,M+1}||r_m e_m - s_m||_{L^\infty(V_x)} \\ &\leq \max_{m=1,\ldots,M+1}\left(\sum_{j=1}^{m}|b_j|\right)\max_{j=1,\ldots,m}||t_j-h_j||_{L^\infty(V_x)} \\ &\leq 2^M d_M\left(L^\infty(V_x)\right). \end{aligned} \tag{3.12}$$

Since $|\alpha_1|+\cdots+|\alpha_{M+1}|=1$, there must exists α_m such that $|\alpha_m| \geq 1/(M+1)$. Setting $x = x^m$ in (3.12), we have $|\alpha_m r_m(x^m)| \leq 2^M d_M(L^\infty(V_x))$ and thus

$$|r_m(x^m)| \leq (M+1)2^M d_M(L^\infty(V_x)). \tag{3.13}$$

By the construction of weighted empirical interpolation approximation in Algorithm 1, we have

$$\operatorname*{ess\,sup}_{x\in V_x}|r_{M+1}(x)| \leq \left|r_{M+1}\left(x^{M+1}\right)\right| \leq \left|r_M\left(x^M\right)\right| \leq \cdots \leq |r_m\left(x^m\right)|. \tag{3.14}$$

A combination of (3.13) and (3.14) leads to the following error bound

$$||g-\mathcal{I}_M[g]||_{L^\infty(V_x)} \leq \operatorname*{ess\,sup}_{x\in V_x}|r_{M+1}(x)| \leq (M+1)2^M d_M\left(L^\infty(V_x)\right). \tag{3.15}$$

□

Corollary 3.5. *Under the assumption* $d_M(L^\infty(V_x)) \leq c\mathrm{e}^{-rM}$ *being* $r > \log(2)$, *we have the following* a priori *error estimate of the wEIM: for* $\forall g \in L^\infty(V_x; C_w^0(\Xi_\mu))$

$$||g-\mathcal{I}_M[g]||_{L^\infty(V_x)} \leq c(M+1)\mathrm{e}^{-(r-\log(2))M}. \tag{3.16}$$

Remark 3.6. The result (3.16) is an improvement of that recently obtained in [13], in which r is required to satisfy $r > 2\log(2)$ and the exponential convergence rate becomes $r - 2\log(2)$. In fact, when the function g is analytic with respect to the parameter $\mu \in \mathbb{R}$, the Kolmogorov width is bounded by the exponentially decaying error from the truncation of Fourier expansion of order M of g, see [8].

Remark 3.7. The result obtained in Theorem 3.1 can not be improved in the exponential growth 2^M for *a priori* convergence analysis of general parametric functions. In fact, it can be proved that $||g-\mathcal{I}_M[g]||_{L^\infty(V_x)} \geq (1-\varepsilon)2^M d_M(L^\infty(V_x))$ for arbitrary small $\varepsilon > 0$ under certain assumptions [3].

4. Applications

In this section, we study the accuracy and efficiency of the weighted empirical interpolation method (wEIM) compared to the conventional empirical interpolation method (EIM) as well as the stochastic collocation method (SCM) for one dimensional problem and sparse grid stochastic collocation method (SG-SCM) [15] for multidimensional problem. Given a function g, we denote by g_M its approximation using M "elements" (either basis functions for wEIM and EIM, or interpolation nodes for SCM and SG-SCM) and we define the error in the following two norms

$$||g-g_M||_{L^\infty(D;C^0(\Gamma))} \quad \text{and} \quad ||\mathbb{E}[g]-\mathbb{E}[g_M]||_{L^\infty(D)}, \tag{4.1}$$

where the expectation $\mathbb{E}[g]$ is computed by Gauss quadrature formula specified when in need.

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4.1. Parametric function in one dimension – geometric Brownian motion

We consider a geometric Brownian motion S_t satisfying a stochastic ordinary differential equation $dS_t = kS_t dt + \sigma S_t dB_t$ (This is, *e.g.*, the most widely used model of stock price S_t at time t with drift k, volatility σ and standard Brown motion B_t [16]). The solution is given by $S_t = \exp(\sigma B_t + (k - \sigma^2/2)t)$. For simplicity, we set $S_0 = 1$, $\sigma = 1$ and $k = 1/2$ so that S_t can be written as $S_t = \exp(\sqrt{t}B_1)$, where B_1 is a standard Gauss random variable $B_1 \sim \mathcal{N}(0,1)$. By denoting $x \equiv t$, $\mu \equiv B_1 \in \mathbb{R}^K, K = 1$ and $g = S_t$, we seek the following affine expansion by wEIM given in Algorithm 1

$$g(x,\mu) = \exp(\sqrt{x}\mu) \approx g_M(x,\mu) = \sum_{j=1}^{M} \Theta_j(\mu) q_j(x) \text{ where } \mu \sim \mathcal{N}(0,1). \tag{4.2}$$

Moreover, we are interested in the expectation of g at time x, which can be approximated by Gauss–Hermite quadrature with abscissas and weights $(\mu_n, w_n), 1 \le n \le N$

$$\mathbb{E}_\mu[g](x) \approx \sum_{j=1}^{M} \left(\int_{-\infty}^{\infty} \Theta_j(\mu)\rho(\mu)d\mu \right) q_j(x) \approx \sum_{j=1}^{M} \left(\sum_{n=1}^{N} \Theta_j(\mu_n) w_n \right) q_j(x), \tag{4.3}$$

where ρ is standard normal density function. The advantage of (4.3) is that we do not need to compute the function g for $\mu_n, 1 \le n \le N$ at every x but only at the empirical interpolation nodes $x^m, 1 \le m \le M$, which is attributed to solving a small linear system (2.3) for $\Theta_j(\mu_n), 1 \le j \le M, 1 \le n \le N$. When the evaluation of the function itself at (x,μ) is expensive and we have a large number of points x, the wEIM can be employed for efficient computation of the statistics. We set the tolerance as $\varepsilon_{tol} = 1 \times 10^{-12}$, take 1000 equidistant points in the vertex set V_x and 1000 normal distributed samples in the sample set Ξ_μ, we also take an independent 1000 normal distributed samples to test different interpolation methods. The weight in Algorithm 1 is taken as the normalized Gauss density function $w(\mu) = \rho(\mu)/\rho(0)$. As for the evaluation of the expectation of $\mathbb{E}[g_M]$, we use 12 quadrature abscissas in (4.3), which is sufficiently accurate for this example. We examine the convergence of "EIM bound" and "wEIM bound" ($r_M(x^M)$), error by "EIM test" and "wEIM test" (error computed from test samples) and test error by stochastic collocation method "SCM test".

The convergence property of different methods is displayed in Figure 1, from which we can see that all the methods achieve exponential convergence rate and wEIM converges faster than both SCM and EIM in $L^\infty(D; C^0(\Gamma))$ norm. However, as for the expectation in $L^\infty(D)$ norm, SCM is the best and wEIM is evidently better than EIM which does not take the weight into consideration. The reason for these results is that wEIM and EIM select the samples by $L^\infty_w(\Xi_\mu)$ and $L^\infty(\Xi_\mu)$ optimization, leading to small error in $L^\infty(D; C^0(\Gamma))$ norm and relatively large error for the evaluation of expectation.

4.2. Parametric function in multidimension – Karhunen–Loève expansion

For the case of multidimensional parameters, we consider the function g truncated from Karhunen–Loève expansion of a Gaussian random field with correlation length L and eigenvalues $\lambda_n, 1 \le n \le N_t$, written as [15]

$$g(x,\mu) - g_0(x) = C \exp\left(\left(\frac{\sqrt{\pi}L}{2} \right)^{\frac{1}{2}} \mu_1(\omega) + \sum_{n=1}^{N_t} \sqrt{\lambda_n} \left(\sin(n\pi x)\mu_{2n}(\omega) + \cos(n\pi x)\mu_{2n+1}(\omega) \right) \right) \tag{4.4}$$

where $\mu_i \sim \mathcal{N}(0,1), 1 \le i \le 2N_t + 1$ are standard Gauss random variables defined in sample space $\Omega \ni \omega$. This function is widely used, *e.g.* in modelling the random property of porous medium in material science, geophysics, etc. To compare the convergence properties of different methods, we take $g_0 = 0$, $C = \exp(5)$, $N_t = 2$, $L = 1/8$ and $\lambda_1 = 0.213, \lambda_2 = 0.190$; $x \in [0,1]$ is discretized by 1000 equidistant vertices. We set tolerance $\varepsilon_{tol} = 1 \times 10^{-12}$, and use 1000 five dimensional independent normal distributed samples and another 1000 test samples. For the computation of $\mathbb{E}[g]$, we apply SG-SCM based on Gauss–Hermite quadrature with the deepest interpolation level 4 in each dimension.

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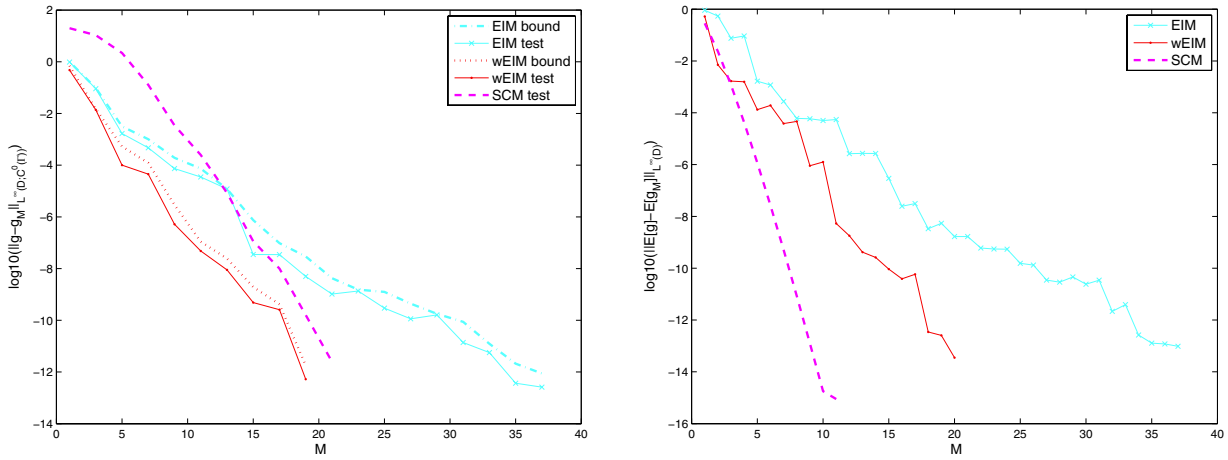


FIGURE 1. Comparison of convergence property of EIM, wEIM and SCM in different norms. Left: decreasing of the error $||g - g_M||_{L^\infty(D;C^0(\Gamma))}$; right decreasing of the error $||\mathbb{E}[g] - \mathbb{E}[g_M]||_{L^\infty(D)}$.

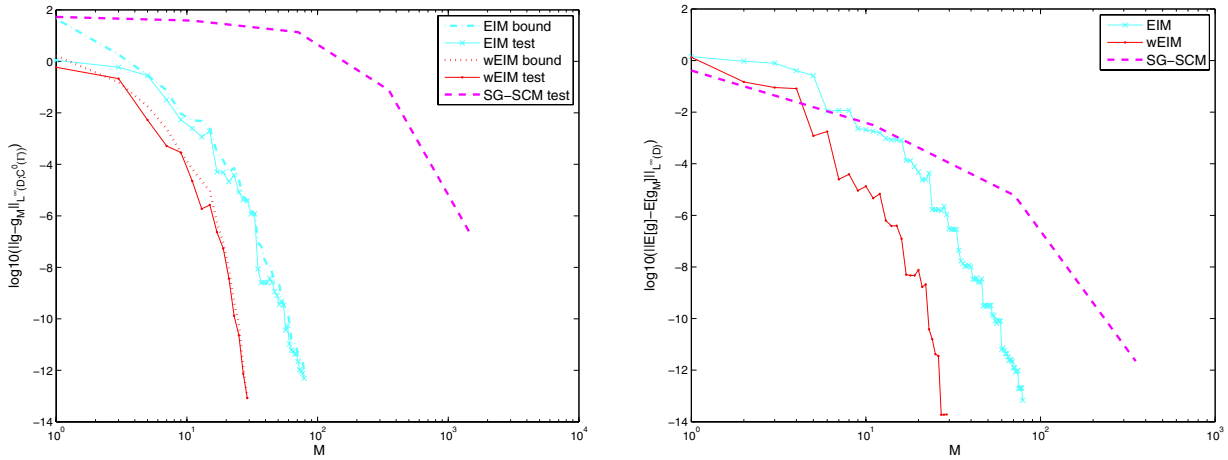


FIGURE 2. Comparison of convergence property of EIM, wEIM and SG-SCM in different norms. Left: decreasing of the error $||g - g_M||_{L^\infty(D;C^0(\Gamma))}$; right decreasing of the error $||\mathbb{E}[g] - \mathbb{E}[g_M]||_{L^\infty(D)}$.

Figure 2 depicts the convergence rate of different methods, from which we can observe that in multidimensional problems wEIM and EIM perform much better than SG-SCM in both $||g - g_M||_{L^\infty(D;C^0(\Gamma))}$ error and $||\mathbb{E}[g] - \mathbb{E}[g_M]||_{L^\infty(D)}$ error. Both wEIM and EIM achieve fast exponential convergence rate and considerably alleviate the "curse-of-dimensionality" suffered by SG-SCM. wEIM uses only 29 samples while EIM needs 80 samples and thus 80 expansion terms, which is far less efficient than the weighted type in practical applications, *e.g.*, in approximating the non-affine terms of reduced basis method.

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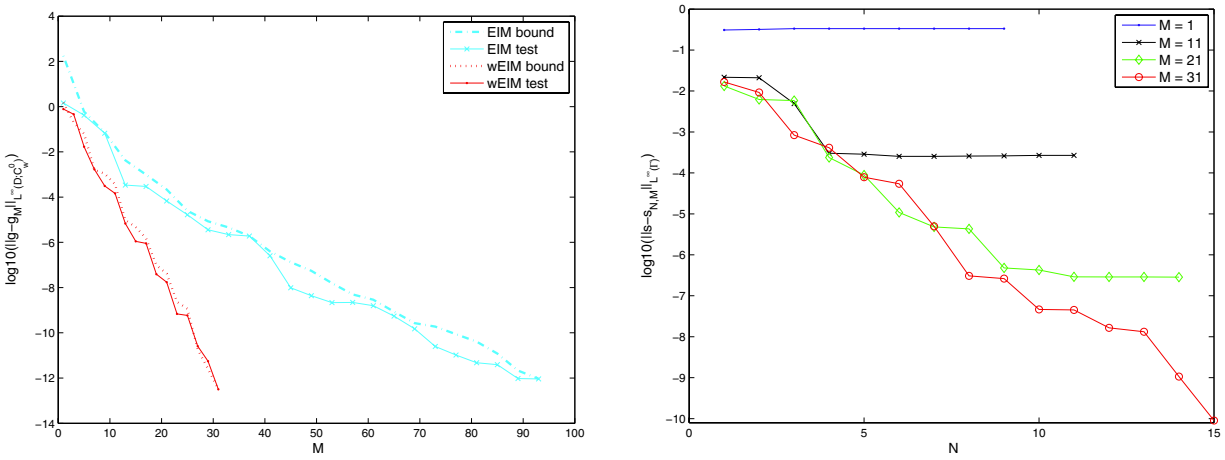


FIGURE 3. Convergence property of wEIM in reduced basis approximation. Left: decreasing of the error $||g - g_M||_{L^\infty(D;C^0(\Gamma))}$ for EIM and wEIM; right: decreasing of the error $||s - s_{N,M}||_{L^\infty(\Gamma)}$.

4.3. Parametric equation – application in non-affine reduced basis method

As mentioned before, EIM was originally developed to deal with non-affine terms in reduced basis discretization of partial differential equations in [1]. The efficiency of the reduced basis method depends critically on the number of affine terms for both offline construction and online evaluation [6,10,11,14]. Therefore, wEIM is more suitable for reduced basis approximation of non-affine parametric equation with weighted parameters.

We consider the following elliptic equation with random coefficient and homogeneous Dirichlet boundary condition: find $u : \bar{D} \times \Omega \to \mathbb{R}$ such that

$$-\nabla(g(x,\omega)\nabla u(x,\omega)) = f(x) \quad (x,\omega) \in D \times \Omega, \tag{4.5}$$

where the random coefficient $g(x,\omega)$ is a Gauss random field represented by a truncated Karhunen–Loève expansion as in (4.4). We set $D = (0,1)^2$, $f = 1$, $g_0 = 0.1$, $C = \exp(5)$, $L = 1/16$, $N_t = 5$, $\lambda_1 = 0.110, \lambda_2 = 0.107, \lambda_3 = 0.101, \lambda_4 = 0.095, \lambda_5 = 0.087$, and identify the eigenfunctions in (4.4) as $\sin(n\pi x_1)$ and $\cos(n\pi x_2)$ with $x_1, x_2 \in [0,1]$. The tolerance for weighted empirical interpolation method is taken as $\varepsilon_{tol} = 1 \times 10^{-12}$. Note that the problem has 11 independent and normal distributed random variables $\mu_K \sim \mathcal{N}(0,1), 1 \le K \le 11$ and all the random variables have relatively equivalent importance due to very close eigenvalues. Therefore, we employ isotropic sparse grid stochastic collocation method based on Gauss–Hermite quadrature [15] for the computation of statistics.

We first run wEIM and EIM with finite element vertices $|V_x| = 185$ and normal distributed samples $|\Xi_\mu| = 10000$ to build an affine expansion 1.1 for the coefficient g of problem (4.5). Another independent 1000 normal distributed samples are used to test the accuracy of the two expansions. The results are shown on the left of Figure 3, from which we can observe that wEIM is much more efficient with only 31 affine terms than EIM requiring 94 terms to achieve the same approximation accuracy in $L^\infty(D;C^0(\Gamma))$ norm.

We use the affine expansion constructed by wEIM to build a weighted reduced basis approximation (introduced in [7] for stochastic problems) with finite element discretization in physical domain D to the stochastic elliptic problem (4.5). The quantity of interest is the integral of the solution over the physical domain D, $s = \int_D u dx$, which is computed from the finite element solution. We denote $s_{N,M}$ the approximation of s based on using N reduced bases and M affine terms. The convergence of $||s - s_{N,M}||_{L^\infty(\Gamma)}$ is displayed on the right of Figure 3, which demonstrates that wEIM is efficient in the application to reduced basis method resulting in

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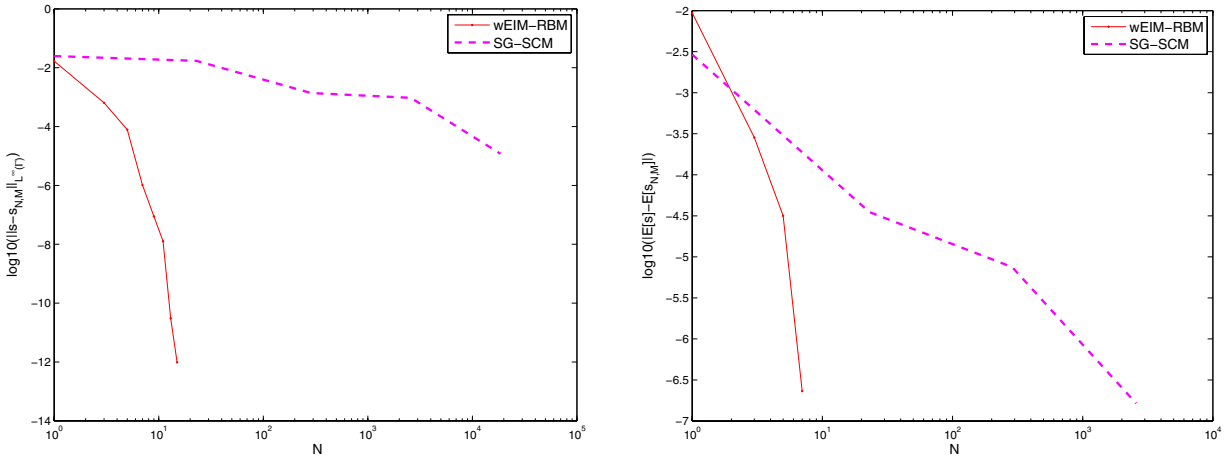


FIGURE 4. Comparison of the convergence property between methods wEIM-RBM and SG-SCM. Left: decreasing of the error $||s - s_{N,M}||_{L^\infty(\Gamma)}$; right: decreasing of the error $|\mathbb{E}[s] - \mathbb{E}[s_{N,M}]|$.

only a few elements in the reduced basis space. Moreover, we can see that the accuracy of wEIM, represented by different number of affine terms $M = 1, 11, 21, 31$, is clearly influential to the accuracy of the reduced basis approximation.

Finally, we compare the proposed approach, a combination of weighted empirical interpolation with weighted reduced basis approximation (wEIM-RBM), to one of the most efficient stochastic computational methods – SG-SCM [15] for their accuracy and efficiency. The result of this comparison, for the $||s - s_{N,M}||_{L^\infty(\Gamma)}$ norm, is depicted on the left of Figure 4, from which the "curse-of-dimensionality" of SG-SCM can be obviously observed. In contrast, wEIM-RBM effectively alleviates this computational burden, using merely 15 bases to accurately approximate the stochastic solution depending on 11 independent normal distributed random variables.

As for the approximation of expectation $\mathbb{E}[s]$, we only need to compute the quantity $s_{N,M}$ with $N = 15, M = 31$ by online evaluation of reduced basis method at the sparse Gauss quadrature abscissas and then $\mathbb{E}[s_{N,M}]$ by sparse grid Gauss quadrature formula.

The comparison of wEIM-RBM with SG-SCM on the right of Figure 4 shows that in order to achieve the same accuracy, it takes 7 bases by reduced basis approximation while 2575 collocation nodes for stochastic collocation approximation. It is worth to mention that the online evaluation of the reduced basis method is independent of the degree of freedom ($|V_x|$) of the deterministic system. Therefore, when solving the underlying deterministic system is computational demanding (with large $|V_x|$) and the dimension of the stochastic space becomes high (with more random variables), wEIM-RBM is much more efficient than SG-SCM for non-affine stochastic problems, see [6] for detailed comparison of computational cost.

5. Concluding remarks

In order to approximate parametric functions with weighted parameters, *e.g.* random variables with various probability distributions, we extended the empirical interpolation method by taking the weight into account for the construction of interpolation formula. *A priori* convergence analysis of the weighted empirical interpolation method was provided. We obtained a direct comparison of the interpolation error to the Kolmogorov N-width, which improved the result obtained recently in [13].

By the applications in approximating geometric Brownian motion in one dimension and exponential Karhunen–Loève expansion in multidimension, we demonstrated numerically the exponential convergence rate

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of the weighted empirical interpolation method and its advantage in accuracy and efficiency over the empirical interpolation method as well as over the sparse grid stochastic collocation method. We also applied the proposed method to the weighted reduced basis approximation [7] for non-affine stochastic elliptic equation and illustrated its efficiency and especially its effectiveness in alleviating the "curse-of-dimensionality" in comparison with the sparse grid stochastic collocation method.

The weighted empirical interpolation method can be straightforwardly applied to nonlinear stochastic partial differential equations with reduced basis approximation and can also be employed effectively in various fields embracing weighted parameters or random variables, *e.g.* image science, geophysics, mathematical finance, material science, bioengineering and uncertainty quantification at large.

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