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Energy dynamics in linear MHD with ion parallel viscosity

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Analytic results for the time dependences of the kinetic and magnetic energies of an incompressible magnetofluid threaded by a strong uniform magnetic field $\mathbf{B}_0$ are obtained. The governing equations are the linearised magnetohydrodynamic (MHD) ones, but with the conventional Laplacian dissipation replaced by ion parallel viscous effects. The behaviour is shown to depend on the relative sizes of the Alfvén frequency and the viscous and resistive dissipation rates. For many cases equipartition of the kinetic and magnetic energy holds at the (Fourier) modal level. An important exception to this behaviour occurs for two-dimensional fluctuations, that is when the wavevectors are perpendicular to $\mathbf{B}_0$.

1. Introduction

In a recent paper (Oughton 1996, hereinafter referred to as Paper I) we considered linear and nonlinear solutions of the magnetohydrodynamic (MHD) equations, where the usual Laplacian viscous term was modified to reflect the dominance of ion parallel viscous effects given a strong uniform background magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. Here we extend the analysis of the linearized equations to obtain analytic solutions for the temporal dependence of the kinetic and magnetic energies.

The motivation for and derivation of the modified incompressible equations have been discussed elsewhere (Montgomery 1992; Oughton 1996; see also Montgomery 1983). The appropriate dimensionless linearized equations are,

$$\frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} p^* + \mathbf{B}_0 \cdot \boldsymbol{\nabla}\mathbf{b} + \nu_{\mathrm{ion}} \left(-\partial_{xz}, -\partial_{yz}, 2\partial_{zz}\right) v_z, \tag{1.1}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \mathbf{B}_0 \cdot \boldsymbol{\nabla}\mathbf{v} + \eta \nabla^2 \mathbf{b}, \tag{1.2}$$

where $\mathbf{v}$ and $\mathbf{b}$ are the solenoidal velocity and magnetic field fluctuations, p^* is the pressure, η the resistivity, ν_{ion} the (kinematic) ion parallel viscosity, and $\partial_{xz} = \partial^2/\partial x \partial z$, etc. The derivation requires, amongst other restrictions (Montgomery 1992; Oughton 1996), that $B_0 \gg \bar{v}, \bar{b}$, where an overbar denotes the rms value. Note that, in contrast to the standard dissipative (linear) case, the pressure gradient is *not* zero, as can be seen by taking the divergence of (1.1).

2. Results and discussion

In this section we first introduce the polarization decompositions of $\mathbf{v}$ and $\mathbf{b}$, and

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then present results on the time dependence of the kinetic and magnetic energy components in this framework.

For linear fluctuations the two linearly independent polarizations of $\mathbf{v}$ are uncoupled. It is convenient to use such a polarization basis in what follows. Consider the Fourier transform of $\mathbf{v}(\mathbf{x})$, denoted by $\mathbf{v}(\mathbf{k})$. The solenoidal nature of the velocity field allows $\mathbf{v}(\mathbf{k})$ to be decomposed using a $\mathbf{k}$-dependent coordinate system:

$$\mathbf{v}(\mathbf{k}) = \mathbf{e}_1\psi_1 + \mathbf{e}_2\psi_2, \tag{2.1}$$

with $\mathbf{e}_1 = \mathbf{k}\times\hat{\mathbf{z}}/|\mathbf{k}\times\hat{\mathbf{z}}|$ and $\mathbf{e}_2 = \mathbf{k}\times\mathbf{e}_1/|\mathbf{k}|$. Similarly, $\mathbf{b}(\mathbf{k}) = \mathbf{e}_1 a_1 + \mathbf{e}_2 a_2$. We refer to these components as the $\mathbf{e}_1$ and $\mathbf{e}_2$ polarizations. When $\mathbf{k}$ is parallel to $\hat{\mathbf{z}} \equiv \hat{\mathbf{B}}_0$, the associated physical symmetry means that the polarizations are equivalent, and we use $\mathbf{e}_1 = \hat{\mathbf{x}}$ and $\mathbf{e}_2 = \hat{\mathbf{y}}$ in this case. As noted above, the $\mathbf{e}_1$ and $\mathbf{e}_2$ components are only coupled if nonlinear terms are present. Moreover, in the linear limit only the $\mathbf{e}_2$ component is affected by ion parallel viscous effects (Montgomery 1992; Oughton 1996).

In Paper 1 the dispersion relation for the linear system (1.1), (1.2) was obtained (see also Montgomery 1992). It was shown that the damping associated with $\nu_{\rm ion}$ can be strongly anisotropic, with propagating solutions only persisting for $\mathbf{k}$ either almost parallel or almost perpendicular to $\mathbf{B}_0$. Progress can also be made in solving for the time dependence of $E_1^v(\mathbf{k}) = \frac{1}{2}|\psi_1(\mathbf{k})|^2$, $E_1^b(\mathbf{k}) = \frac{1}{2}|a_1(\mathbf{k})|^2$, etc., with the general solution being an appropriate superposition. For example, substituting (2.1) into the Fourier transforms of (1.1) and (1.2), and taking dot products with $\mathbf{e}_2\psi_2$ and $\mathbf{e}_2 a_2$, shows that the $\mathbf{e}_2$ polarization components of the energy are governed by

$$\frac{dE_2^v}{dt} = -2\omega_A H_2 - 2\gamma E_2^v, \tag{2.2}$$

$$\frac{dE_2^b}{dt} = 2\omega_A H_2 - 2\mu E_2^b, \tag{2.3}$$

$$\frac{dH_2}{dt} = \omega_A(E_2^v - E_2^b) - (\mu+\gamma)H_2, \tag{2.4}$$

where $\mu = \eta k^2$, $\gamma = 3\nu_{\rm ion}k_z^2k_\perp^2/k^2$ and $H_2 = \frac{1}{2}{\rm Im}\{\psi_2^* a_2\}$, and $\omega_A = \mathbf{k}\cdot\mathbf{B}_0$ is the Alfvén frequency. The system governing the $\mathbf{e}_1$ energy components is obtained from (2.2)–(2.4) by replacing '2' subscripts with '1' ones, and setting $\gamma = 0$; consequently solutions to the $\mathbf{e}_1$ system can be obtained from the $\mathbf{e}_2$ solutions by using the same operations. Note that E_1^v, etc. are functions of the vector $\mathbf{k}$, as are ω_A and γ, so that anisotropic effects are to be expected (Montgomery 1992; Oughton 1996).

The quantity $H = {\rm Im}\{\psi_\alpha^*(\mathbf{k})a_\alpha(\mathbf{k})\}$ has been referred to as the (spectrum of the) 'helicity of the electric field', because it stands in the same relationship to the electric field as the magnetic helicity does to the magnetic energy (Zhou and Matthaeus 1990; Oughton *et al.* 1997). Unfortunately its physical nature is currently not well understood, but if the analogy can be legitimately pursued then the electric field helicity is a measure of the knotting and twisting of the electric field (lines). As will be seen below, H acts as a sort of 'courier' connecting the kinetic and magnetic energy reservoirs, thereby enabling the dynamic maintenance of energy equipartition.

The equations are linear constant-coefficient ODEs, and are amenable to standard methods of solution (Birkhoff and Rota 1989). Specifically, the system can be written in the vector form $d\mathbf{u}/dt = \mathbf{A}\mathbf{u}$, where $\mathbf{A}$ is the matrix of coupling coefficients, and solved by finding the eigenvalues and vectors of $\mathbf{A}$. It can be shown that

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the eigenvalues are

$$\lambda = -(\mu+\gamma), \qquad -(\mu+\gamma) \pm i[4\omega_A^2 - (\mu-\gamma)^2]^{1/2}. \tag{2.5}$$

The nature of the solutions clearly depends on the relative sizes of μ, γ and ω_A, with four cases of particular interest. In all cases initial conditions appropriate for linear fluctuations are imposed, for example, for the $\mathbf{e}_2$ components at $t = 0$, $E_2^v = E_2^b = W_2$, $H_2 = \dot{H}_2 = 0$ and $\ddot{H}_2 = 2(\mu-\gamma)\omega_A W_2$, and similarly for the $\mathbf{e}_1$ polarization. We now examine the cases in detail.

(a) Oscillations: $\Omega^2 = 4\omega_A^2 - (\mu+\gamma)^2 > 0$

This situation leads to one real and two complex-conjugate eigenvalues, so that the solutions can involve damped oscillations. The results for the $\mathbf{e}_2$ polarization are

$$E_2^{v/b}(t) = W_2[1 + N^2(1-\cos\Omega t) \pm N\sin\Omega t]e^{-(\mu+\gamma)t}, \tag{2.6}$$

$$H_2(t) = \frac{2\omega_A W_2}{\Omega} N\,(1-\cos\Omega t)\,e^{-(\mu+\gamma)t}, \tag{2.7}$$

with $N = (\mu-\gamma)/\Omega$. In (2.6) and (2.8) the upper sign in the '$\pm$' terms is associated with E_2^v and the lower with E_2^b. As expected, energy is conserved for $\mu = \gamma = 0$. The point to note is that H_2 mediates a coupling between the kinetic and magnetic components, thereby inducing decay of E_2^b, *even in the absence of resistivity.* Similarly in the $\mathbf{e}_1$ equivalent of (2.2)–(2.4), where $\gamma \to 0$, the coupling through H_1 causes E_1^v and E_1^b to decay with the same envelope, but with fluctuations about it that are in antiphase. Consequently, equipartition of the kinetic and magnetic energy is maintained, on average.

Note that when $\mu = \gamma$ the oscillations in E_2^v and E_2^b are absent and $N = 0 = H_2(t)$, so that $E_2^v(t)$ and $E_2^b(t)$ remain precisely equal for all $t > 0$.

(b) Equal eigenvalues: $4\omega_A^2 = (\mu-\gamma)^2$

The degeneracy leads to solutions whose temporal dependence is the product of a quadratic in time and exponential decay:

$$E_2^{v/b}(t) = W_2[1 \pm (\mu-\gamma)t + \tfrac{1}{2}(\mu-\gamma)^2 t^2]e^{-(\mu+\gamma)t}, \tag{2.8}$$

$$H_2(t) = \omega_A W_2\,(\mu-\gamma)\,t^2 e^{-(\mu+\gamma)t}. \tag{2.9}$$

Clearly, the modal Alfvén ratio E_2^v/E_2^b approaches unity as t becomes large compared with $|\mu-\gamma|^{-1}$. Note that if $\mu = \gamma$ then $\omega_A = 0$ and there is no coupling between the kinetic and magnetic components. Moreover, as in case (*c*), if the Alfvén frequency is zero then so is γ ($k_z \equiv 0$). In other words, we have repeated zero eigenvalues corresponding to undamped fluctuations.

(c) Two-dimensional fluctuations: $\omega_A = \mathbf{k}\cdot\mathbf{B}_0 = 0$; $\mu \neq 0$

Since $k_z = 0$ for this case, we also have $\gamma = 0$, and the kinetic energy is undamped. From (2.5), the eigenvalues are real and distinct, with one being zero. Physically, the associated fluctuations are two-dimensional in nature, the wavevectors being perpendicular to $\mathbf{B}_0$. Examination of (2.2)–(2.4) for such fluctuations shows that the equations are uncoupled and may be solved directly. Using the initial conditions described above, we obtain

$$E_2^v = W_2, \qquad E_2^b = W_2 e^{-2\mu t}, \qquad H_2 = 0. \tag{2.10}$$

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Clearly, equipartition is not maintained for two-dimensional fluctuations. Although $\gamma \sim \nu_{\rm ion}$ vanishes for strictly two-dimensional fluctuations, the higher-order viscosity coefficients (Braginskii 1965; Book 1987; Balescu 1988) are non-zero and lead to weak viscous decay.

(d) Real, distinct, negative eigenvalues: $0 < 4\omega_A^2 < (\mu - \gamma)^2$

This is a more complicated version of case (*c*), with γ non-zero. The general solution for each element of the system is a sum of three (distinct) exponentially decaying terms:

$$\frac{E_2^v(t)}{W_2} = u_0 e^{-\alpha t} + u_+ e^{-\alpha_+ t} + u_- e^{-\alpha_- t}, \tag{2.11}$$

$$\frac{E_2^b(t)}{W_2} = u_0 e^{-\alpha t} + u_- e^{-\alpha_+ t} + u_+ e^{-\alpha_- t}, \tag{2.12}$$

where $H_2(t)$ is analogous to $E_2^v(t)$ with $u_i \mapsto h_i$. The other quantities are,

$$\alpha = \mu + \gamma, \qquad \alpha_\pm = \alpha(1 \pm \delta), \qquad \delta^2 = 1 - \frac{4\omega_A^2}{(\mu - \gamma)^2},$$

$$u_0 = 1 - \frac{(\gamma - \mu)^2}{(\alpha\delta)^2}, \qquad u_\pm = \frac{(\gamma - \mu)(\gamma - \mu \pm \alpha\delta)}{2\alpha^2\delta^2},$$

$$h_\pm = -\frac{h_0}{2} = \frac{\omega_A(\mu - \gamma)}{(\alpha\delta)^2}.$$

Note that $0 < \delta < 1$.

As with case (*c*), equipartition is not maintained. Defining the modal normalized energy difference for the $\mathbf{e}_2$ components as $\sigma_{D2}(\mathbf{k}, t) = (E_2^v - E_2^b)/(E_2^v + E_2^b)$, and similarly for σ_{D1}, it can be shown that these ratios asymptotically approach constants, for example,

$$\sigma_{D2} = \frac{(u_2 - u_1)\sinh(\alpha\delta t)}{u_0 + (u_2 + u_1)\cosh(\alpha\delta t)} \tag{2.13}$$

$$\to \delta\frac{\mu + \gamma}{\mu - \gamma}, \tag{2.14}$$

for $t \gg (\alpha\delta)^{-1}$. Note that equipartition does not occur for $t \neq 0$, since setting the exact result (2.13) equal to zero implies $\gamma = -\mu$, but both quantities are strictly positive. Thus, for a given wavevector mode, either the kinetic or the magnetic energy will predominate, depending on which dissipation rate is bigger. If $\mu > \gamma \Rightarrow \eta/\nu_{\rm ion} > 3(k_\perp k_z/k^2)^2$ then an excess of kinetic energy prevails. Furthermore, because the $\mathbf{e}_1$ polarization solutions have $\gamma \to 0$, they are always associated with excess kinetic energy. This completes consideration of the four cases.

As noted above, the solutions for the $\mathbf{e}_1$ energy components can be obtained by setting $\gamma \sim \nu_{\rm ion} = 0$ in the appropriate $\mathbf{e}_2$ solution and changing all '2' subscripts to '1'. In cases (*a*)–(*c*) this produces no essential change in the solutions, since the results are qualitatively the same when $\gamma = 0$.

It follows that superposition of all $\mathbf{e}_1$ and $\mathbf{e}_2$ fluctuations of types (*a*) and (*b*) will yield a total fluctuation energy that is essentially equipartitioned between the kinetic and magnetic components. On the other hand, if such fluctuations are absent or much less common than those of types (*c*) and (*d*), the total energy is unlikely to be equipartitioned, except in special cases. Whether the kinetic or the magnetic

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energy dominates depends on both the dissipation coefficients and on the initial spectral distribution of the energy. For example, if the initial fluctuations are quasi-2D, meaning that all excited modes have wavevectors approximately perpendicular to $\mathbf{B}_0$, then the kinetic energy will dominate at (almost) all scales after a time of order $(\eta k_{\min}^2)^{-1}$.

How many modes of each type are there? Consider a cubic spatial domain and Fourier-decompose the fields using N modes in each Cartesian direction, giving $O(N^3)$ modes in all. Type (c) fluctuations require $k_z = 0$, which occurs once for every k_x, k_y pair. Thus $O(1/N)$ of the modes are strictly two-dimensional.

For type (d) fluctuations $4\omega_A^2 < (\mu - \gamma)^2$, which implies

$$\frac{k^2}{|k_z|} \gtrsim \frac{2B_0}{\eta}, \tag{2.15}$$

unless $\eta k^2/\nu_{\text{ion}}$ is $O(1)$ or less. By assumption, B_0 is large and η small, so that the right-hand side is much greater than unity. Clearly, as N increases, the number of modes that satisfy the inequality also increases. With $N = 128$, 256 and 512, the fraction of modes satisfying (2.15) is 2%, 4% and 8.6%, so that for these N the bulk of the modes are of type (a). For large enough N, however, most modes will be of type (d). In any case, the type (c) two-dimensional modes will always be a minority. Nonetheless, the two-dimensional modes can still be dynamically important or even dominant.

Figure 5(b) of Paper 1 displays the time evolution of the polarization components E_i^v and E_i^b for a three-dimensional linear simulation ($N = 32$), in which all possible wavevector modes are initially isotropically populated. The run parameters were $B_0 = 10$, $\nu_{\text{ion}} = 10$ and $\eta = 10^{-2}$, with an initial fluctuation energy of unity. Clearly the original equipartition of energy is immediately broken, with the Alfvén ratio increasing monotonically ($E^v/E^b \approx 1.25$ at $t = 4$). While this is just one example, it suggests that the linear fluctuations are becoming progressively more two-dimensional, relative to the direction of $\mathbf{B}_0$, as the other fluctuations are preferentially damped. Note that nonlinear theory, closures and simulations also indicate that fluctuations evolve towards (quasi) two-dimensionality in the presence of a uniform magnetic field (Montgomery and Turner 1981; Shebalin *et al.* 1983; Carbone and Veltri 1990; Oughton *et al.* 1994; Matthaeus *et al.* 1996; Oughton 1996).

3. Summary

We have shown that the linearized MHD equations, appropriately modified to incorporate ion parallel viscous effects, can be solved to give the time dependence of the polarization components of the kinetic and magnetic energy, E_i^v and E_i^b, where $i = 1, 2$ represents the polarization component. There are four cases of particular interest, identified by the relative sizes of the Alfvén frequency and the viscous and resistive decay rates. These cases are discussed in detail in Sec. 2. Three main types of (coupled) solutions are found: oscillations with exponentially decaying envelopes, exponential decay, and exponential decay multiplied by quadratic time dependence.

In general, E_i^v and E_i^b are coupled through the 'helicity of the electric field', H_i (Zhou and Matthaeus 1990; Oughton *et al.* 1997). This coupling tends to induce approximate equipartition of the kinetic and magnetic energy components. However,

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for two-dimensional fluctuations ($\omega_A = \mathbf{k} \cdot \mathbf{B}_0 = 0$), H_i remains zero for all t, so that the energies are decoupled and the kinetic energy is undamped.

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