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The critical dimension for G-measures

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Abstract. The critical dimension of an ergodic non-singular dynamical system is the asymptotic growth rate of sums of consecutive Radon–Nikodým derivatives. This has been shown to equal the average coordinate entropy for product odometers when the size of individual factors is bounded. We extend this result to G-measures with an asymptotic bound on the size of individual factors. Furthermore, unlike von Neumann–Krieger type, the critical dimension is an invariant property on the class of ergodic G-measures.

1. *Introduction*

For a conservative, non-singular ergodic dynamical system $(X, \mathcal{B}, \mu, T)$ with $\mu(X) = 1$, let $\omega_i^T(x) = ((d\mu \circ T^i)/d\mu)(x)$ denote the ith Radon–Nikodým derivative. This is written $\omega_i(x)$ when the automorphism T is apparent from the context. The asymptotic rate of growth of the sum $\sum_{i=1}^n \omega_i(x)$, called the critical dimension, was first studied in [**11**]. The critical dimension was later shown to be computable by a formula similar to the Shannon–McMillan–Breiman theorem for product measures [**5**] and Markov measures on Bratteli–Vershik diagrams [**3**].

The notion of G-measure was introduced in [**2**]. It simultaneously generalizes classical infinite product measures, classical *Riesz products*, i.e. weak* limits on the circle of the form $\prod_{k=1}^n (1 + a_k \cos(2\pi \ell(1) \cdots \ell(k)(t + \phi_k)))$ (where $\{\ell(i)\}$ is a sequence of integers, and $|a_k| < 1$ is a sequence of real numbers), and the *g-measures* of Keane [**9**], i.e. weak* limits of the form $\prod_{k=1}^n g(\ell^k x)dx$ where ℓ is a fixed integer, and g is a suitably normalized function on $\mathbb{T}$. We recall the definition of a G-measure on a product space $X = \prod_{k=1}^\infty \mathbb{Z}_{\ell(k)}$ in §3. In fact, every measure on X which is quasi-invariant for the odometer action is a G-measure for a suitable sequence of functions $\{G_n\}$; within this class, the measures for which there is a unique weak* limit are necessarily ergodic, and known as *uniquely ergodic G-measures*. All infinite product measures, Riesz products and all g-measures with g Lipschitz belong to this class, as do many other G-measures; see [**2**] for good

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behaviour results where equicontinuity is present, and [**6**] for bad behaviour where it is absent.

In this paper, we compute the critical dimension for G-measures. The structure is as follows: in §2 we define the critical dimension and *Hurewicz equivalence*. We use Hurewicz equivalence in the proofs of Lemmas 3.1 and 3.2. In §3 we show that the critical dimension for an ergodic G-measure is equal to the 'sample average'. In §4 use a law of large numbers to explicitly evaluate this 'sample average': we compute the upper and lower critical dimensions β, α of the classical Riesz product $\prod_{k=1}^{n}(1 + a_k \cos(2\pi 3^k t))$, for $|a_k| < 1$ as

$$\alpha = \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log_3\left(2\left(1 - \sqrt{1 - a_k^2}\right)a_k^{-2}\right),$$

$$\beta = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log_3\left(2\left(1 - \sqrt{1 - a_k^2}\right)a_k^{-2}\right).$$

We make the standing assumption that $(X, \mathcal{B}, \mu, T)$ is an ergodic non-singular dynamical system, where X is an infinite product of finite sets $\mathbb{Z}_{\ell(k)}$, and the odometer action on (X, μ) is conservative and of von Neumann–Krieger type III.

2. *The critical dimension and Hurewicz equivalence*

This section contains definitions and notation from [**5**, **8**], which are repeated here to establish notation.

A measure μ is non-singular for the odometer if $\mu \circ T$ is equivalent to μ; it follows that for every $i \in \mathbb{Z}$, $\mu \circ T^i \sim \mu$. Let $\omega_i(x)$ denote the Radon–Nikodým derivative $((d\mu \circ T^i)/d\mu)(x)$. Each derivative is strictly positive μ-almost everywhere. The asymptotic growth rate with respect to n is known as the critical dimension. More formally, the set

$$X_{\alpha'} = \left\{ x \in X \;\middle|\; \liminf_{n\to\infty} \frac{1}{n^{\alpha'}} \sum_{i=0}^{n-1} \omega_i(x) > 0 \right\}$$

is T-invariant and by ergodicity has measure 0 or 1. By the Hurewicz ergodic theorem, the numerator diverges almost everywhere. Define the *lower critical dimension* α as

$$\alpha = \sup\{\alpha' : \mu(X_{\alpha'}) = 1\}.$$

Similarly, the set

$$Y_{\beta'} = \left\{ x \in X \;\middle|\; \limsup_{n\to\infty} \frac{1}{n^{\beta'}} \sum_{i=0}^{n-1} \omega_i(x) = 0 \right\}$$

is T-invariant. Define the *upper critical dimension* β as

$$\beta = \inf\{\beta' : \mu(Y_{\beta'}) = 1\}.$$

The critical dimensions can also be written in the style of the abscissa of convergence

$$\alpha = \liminf_{n\to\infty} \log\left(\sum_{i=0}^{n-1} \omega_i(x)\right) \Big/ \log(n),$$

$$\beta = \limsup_{n\to\infty} \log\left(\sum_{i=0}^{n-1} \omega_i(x)\right) \Big/ \log(n).$$

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We introduce the following sufficient condition for ergodic non-singular dynamical systems to have the same critical dimensions.

Definition. Two ergodic non-singular dynamical systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are *Hurewicz equivalent* if there exists an orbit equivalence relation $\phi : X \to Y$ such that

$$\lim_{n\to\infty} \frac{1}{\log(n)} \left(\log\left(\sum_{i=1}^{n} \omega_i^S(x) \right) - \log\left(\sum_{i=1}^{n} \omega_i^T(\phi(x)) \right) \right) = 0.$$

Actually, Hurewicz equivalence implies more than equality of critical dimensions. It says that the quantities $(1/\log(n)) \log(\sum_{i=1}^{n} \omega_i^T(x))$ and $(1/\log(n)) \log(\sum_{i=1}^{n} \omega_i^S(x))$ agree on every infinite subsequence of $\mathbb{Z}^+$. This property is an important part of later proofs.

It was proven in [**11**] that conjugate non-singular dynamical systems have the same critical dimension. We extend this result to show that conjugate non-singular dynamical systems are Hurewicz equivalent.

PROPOSITION 2.1. *If two ergodic non-singular dynamical systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, T)$ are conjugate, then they are Hurewicz equivalent.*

Proof. By assumption there exists some $\phi : X \to Y$ such that $\nu \circ \phi \sim \mu$ and $\phi \circ T = S \circ \phi$. Then

$$\frac{d\nu \circ \phi}{d\mu}(T^i x)\omega_i^T(x) = \frac{d\nu \circ \phi}{d\mu}(x)\omega_i^S(\phi(x)).$$

For any $n \in \mathbb{N}$,

$$\sum_{i=0}^{n-1} \frac{d\nu \circ \phi}{d\mu}(T^i x)\omega_i^T(x) = \sum_{i=0}^{n-1} \frac{d\nu \circ \phi}{d\mu}(x)\omega_i^S(\phi(x)).$$

By the Hurewicz ergodic theorem,

$$\begin{aligned} \nu(\phi(X)) &= \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} (d\nu \circ \phi/d\mu)(T^i x)\omega_i^T(x)}{\sum_{i=0}^{n-1} \omega_i^T(x)} \\ &= \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} (d\nu \circ \phi/d\mu)(x)\omega_i^S(\phi(x))}{\sum_{i=0}^{n-1} \omega_i^T(x)} \\ &= \frac{d\nu \circ \phi}{d\mu}(x) \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega_i^S(\phi(x))}{\sum_{i=0}^{n-1} \omega_i^T(x)}. \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{\log(\nu(\phi(X)))}{\log(n)} \\ &= \lim_{n\to\infty} \frac{1}{\log(n)} \left(\log\left(\sum_{i=1}^{n} \omega_i^S(\phi(x)) \right) - \log\left(\sum_{i=1}^{n} \omega_i^T(x) \right) + \log\left(\frac{d\nu \circ \phi}{d\mu}(x) \right) \right) \\ &= \lim_{n\to\infty} \frac{1}{\log(n)} \left(\log\left(\sum_{i=1}^{n} \omega_i^S(\phi(x)) \right) - \log\left(\sum_{i=1}^{n} \omega_i^T(x) \right) \right). \end{aligned}$$

□

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This lemma allows us to construct practical examples of ergodic non-singular odometers which are Hurewicz equivalent. These were studied in [**1**, **5**].

COROLLARY 2.1. *If X is a product space, then the ergodic non-singular odometer $(X, \mathcal{B}, \mu, T)$ is conjugate to $(X, \mathcal{B}, \mu, T^{-1})$ by an involution, and hence the odometers are Hurewicz equivalent.*

Proof. Consider the map $\phi : X \to X$ given by $\phi(x)_p = \ell(p) - 1 - x_p$. This defines a conjugacy between $(X, \mathcal{B}, \mu, T)$ and $(X, \mathcal{B}, \mu, T^{-1})$ similar to an infinite 1's complement where $\mu \circ \phi \sim \mu$, $\phi^2 = 1$ and $\phi \circ T = T^{-1} \circ \phi$. The odometers are therefore Hurewicz equivalent by Proposition 2.1. □

In the context of $(X, \mathcal{B}, \mu, T)$, it is convenient to refer to $(X, \mathcal{B}, \mu, T^{-1})$ as the *reverse odometer*.

3. *G-measures*

The notion of G-measures was introduced in [**1**] in order to show that every measure on an infinite product space X which is quasi-invariant for the finite coordinate changes (or equivalently, the odometer) may be obtained from a Reisz product-like construction, which we summarize below.

For any product space $X = \prod_{n=1}^{\infty} \mathbb{Z}_{\ell(n)}$ define Γ_n as the group of finite coordinate changes $\gamma : X \to X$ such that $(\gamma x)_i = x_i$ for all $i > n$. Let $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$. For a Γ-quasi-invariant probability measure μ on X, and $n \in \mathbb{Z}^+$ define the Γ_n average of μ as

$$\mu^{(n)} = \frac{1}{|\Gamma_n|} \sum_{\gamma \in \Gamma_n} \mu \circ \gamma.$$

The measure $\mu^{(n)}$ is Γ_n invariant and equivalent to μ. Hence the Radon–Nikodým derivatives

$$G_n(x) = \frac{d\mu}{d\mu^{(n)}}(x)$$

exist. Denote the sequence $(G_n)_{n=1}^{\infty}$ by G.

The functions G_n are *normalized*, in the sense that

$$\frac{1}{|\Gamma_n|} \sum_{\gamma \in \Gamma_n} G_n(\gamma x) = 1. \tag{1}$$

They also satisfy the following compatibility condition: if $n > m$, then

$$G_n(x) G_m(\gamma x) = G_m(x) G_n(\gamma x) \quad \text{for all } \gamma \in \Gamma_m.$$

The measure μ is then said to be a quasi-invariant G-measure.

This compatibility condition implies that G_n can be written as a product of measurable functions g_n defined as

$$g_1(x) = \frac{d\mu}{d\mu^{(1)}}(x),$$

$$g_n(x) = \frac{d\mu^{(n-1)}}{d\mu^{(n)}}(x),$$

$$G_n(x) = \frac{d\mu}{d\mu^{(n)}}(x) = \prod_{i=1}^{n} g_i(x),$$

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where the functions g_n do not depend on the first n coordinates of x. That is to say, $g_n(x) = g_n(\gamma x)$ for $\gamma \in \Gamma_n$.

The group of finite coordinate changes Γ_n can be described by the powers of the odometer. Suppose that μ is a G-measure. For any $p \in \mathbb{Z}^+$ and $x \in X$ define $J^p(x)$ to be the smallest integer $n > 1$ such that

$$(T^n x)_i = \begin{cases} \ell(i) - 1 & \text{for } i \le p, \\ (T^n x)_i = x_i & \text{for } i > p. \end{cases}$$

Similarly, define $J'_p(x)$ as the unique integer n such that

$$(T^{-n} x)_i = \begin{cases} 0 & \text{for } i \le p, \\ (T^n x)_i = x_i & \text{for } i > p. \end{cases}$$

Given any integer n, $-J'_p(x) \le n \le J_p(x)$ if and only if T^n preserves x after the pth digit. For any $x \in X$, the finite coordinate change Γ_p can be represented by explicit odometer powers as $\Gamma_{p,x} = \{T^n : -J'_p(x) \le n \le J_p(x)\}$. Notice that $|\Gamma_{p,x}| = |\Gamma_p|$; however, we will continue to write this as $|\Gamma_{p,x}|$ to emphasize that we are working with particular odometer powers instead of finite coordinate changes. So for any $x \in X$, $\gamma \in \Gamma_p$ then $\gamma x = T^n x$ for some $T^n \in \Gamma_{p,x}$, and

$$\frac{d\mu \circ \gamma}{d\mu}(x) = \frac{d\mu \circ T^n}{d\mu}(x) = \prod_{i=1}^{\infty} \frac{g_i(T^n x)}{g_i(x)} = \prod_{i=1}^{p} \frac{g_i(T^n x)}{g_i(x)} = \frac{G_p(\gamma x)}{G_p(x)}.$$

By normalization,

$$\sum_{i=-J'_p(x)}^{J_p(x)} \frac{d\mu \circ T^i}{d\mu}(x) = \sum_{\gamma \in \Gamma_{p,x}} \frac{d\mu \circ \gamma}{d\mu}(x) = \sum_{\gamma \in \Gamma_{p,x}} \frac{G_p(\gamma x)}{G_p(x)} = \frac{|\Gamma_{p,x}|}{G_p(x)}. \tag{2}$$

LEMMA 3.1. *If $(X, \mathcal{B}, \mu, T)$ and $(X, \mathcal{B}, \mu, T^{-1})$ are Hurewicz equivalent, then for μ-almost every $x \in X$,*

$$\alpha \le \liminf_{p \to \infty} -\frac{1}{\log(|\Gamma_{p,x}|)} \log\left(\frac{G_p(x)}{|\Gamma_{p,x}|}\right).$$

Proof. For any $x \in X$ let

$$P_{1,x} = \{k \in \mathbb{Z}^+ : J_k(x) \ge J'_k(x)\},$$
$$P_{2,x} = \{k \in \mathbb{Z}^+ : J_k(x) \le J'_k(x)\}.$$

Note that $P_{1,x} \cup P_{2,x} = \mathbb{Z}^+$. Let $\{k_\ell\}_{\ell \in \mathbb{N}}$ be a subsequence such that

$$\liminf_{k \to \infty} -\frac{1}{\log(|\Gamma_{k,x}|)} \log\left(\frac{G_k(x)}{|\Gamma_{k,x}|}\right) = \lim_{l \to \infty} -\frac{1}{\log(|\Gamma_{k_\ell,x}|)} \log\left(\frac{G_{k_\ell}(x)}{|\Gamma_{k_\ell,x}|}\right).$$

We consider two possible cases: when $P_{1,x} \cap \{k_\ell\}_{\ell \in \mathbb{Z}^+}$ is infinite and when $P_{2,x} \cap \{k_\ell\}_{\ell \in \mathbb{Z}^+}$ is infinite. Since $P_{1,x}$ and $P_{2,x}$ cover $\mathbb{Z}^+$, at least one case will be true.

For the first case, assume that $P_{1,x} \cap \{k_\ell\}_{\ell \in \mathbb{Z}^+}$ is infinite. Enumerate the elements of this set in increasing order by p_ℓ, $\ell \in \mathbb{Z}^+$. Since $p_\ell \in P_{1,x}$,

$$|\Gamma_{p_\ell,x}| = J_{p_\ell} + J_{p'_\ell} \le 2J_{p_\ell}$$

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and

$$\begin{aligned}
\alpha &= \liminf_{n\to\infty} \frac{1}{\log(n)} \log\left(\sum_{i=1}^{n} \omega_i(x)\right) \\
&\le \liminf_{l\to\infty} \frac{1}{\log(J_{p_\ell})} \log\left(\sum_{i=1}^{J_{p_\ell}} \omega_i(x)\right) \\
&\le \liminf_{l\to\infty} \frac{1}{\log(2J_{p_\ell})} \log\left(\sum_{i=-J'_{p_\ell}}^{J_{p_\ell}} \omega_i(x)\right) \\
&\le \liminf_{l\to\infty} \frac{1}{\log(|\Gamma_{p_\ell,x}|)} \log\left(\sum_{\gamma\in\Gamma_{p_\ell,x}} \frac{d\mu\circ\gamma}{d\mu}(x)\right) \\
&= \liminf_{l\to\infty} -\frac{1}{\log(|\Gamma_{p_\ell,x}|)} \log\left(\frac{G_{p_\ell}(x)}{|\Gamma_{p_\ell,x}|}\right) \\
&= \liminf_{p\to\infty} -\frac{1}{\log(|\Gamma_{p,x}|)} \log\left(\frac{G_p(x)}{|\Gamma_{p,x}|}\right).
\end{aligned}$$

For the second case, $P_{2,x} \cap \{k_\ell\}_{\ell\in\mathbb{Z}^+}$ is infinite. Enumerate the elements of this set in increasing order by p_ℓ, $\ell \in \mathbb{Z}^+$. Since $p_\ell \in P_{2,x}$,

$$|\Gamma_{p_\ell,x}| = J_{p_\ell} + J_{p'_\ell} \le 2J'_{p_\ell}.$$

Repeating the above argument with J' and the reverse odometer T^{-1} gives

$$\liminf_{n\to\infty} \frac{1}{\log(n)} \log\left(\sum_{i=1}^{n} \omega_{-i}(x)\right) \le \liminf_{p\to\infty} \frac{1}{\log(|\Gamma_{p,x}|)} \log\left(\frac{G_p(x)}{|\Gamma_{p,x}|}\right).$$

Since $(X, \mathcal{B}, \mu, T)$ and $(X, \mathcal{B}, \mu, T^{-1})$ are Hurewicz equivalent,

$$\begin{aligned}
\alpha &= \liminf_{n\to\infty} \frac{1}{\log(n)} \log\left(\sum_{i=1}^{n} \omega_i(x)\right) \\
&= \liminf_{n\to\infty} \frac{1}{\log(n)} \log\left(\sum_{i=1}^{n} \omega_{-i}(x)\right) \\
&\le \liminf_{p\to\infty} \frac{1}{\log(|\Gamma_{p,x}|)} \log\left(\frac{G_p(x)}{|\Gamma_{p,x}|}\right).
\end{aligned}$$

This completes the proof. □

The next lemma requires a constraint on the asymptotic bound on the size of the coordinate spaces. This constraint (equation (3)) first appeared in [**10**] in the context of product measures.

LEMMA 3.2. *If* $(X, \mathcal{B}, \mu, T)$ *and* $(X, \mathcal{B}, \mu, T^{-1})$ *are Hurewicz equivalent, and*

$$\lim_{p\to\infty} \frac{\log(|\Gamma_{p,x}|)}{\log(|\Gamma_{p+1,x}|)} = 1, \tag{3}$$

then

$$\alpha \ge \liminf_{p\to\infty} -\frac{1}{\log(|\Gamma_{p,x}|)} \log\left(\frac{G_p(x)}{|\Gamma_{p,x}|}\right).$$

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Proof. For any $x \in X$ let

$$P_{1,x} = \left\{ k \in \mathbb{Z}^+ : \sum_{i=1}^{J_k} \omega_i(x) \geq \sum_{i=1}^{J'_k} \omega_{-i}(x) \right\},$$

$$P_{2,x} = \left\{ k \in \mathbb{Z}^+ : \sum_{i=1}^{J_k} \omega_i(x) \leq \sum_{i=1}^{J'_k} \omega_{-i}(x) \right\}.$$

Note that $P_{1,x} \cup P_{2,x} = \mathbb{Z}^+$ and that for any $x \in X$ and $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that

$$|\Gamma_{k,x}| \leq n < |\Gamma_{k+1,x}|.$$

Let $\{n_\ell\}_{\ell \in \mathbb{N}}$ be a subsequence along which

$$\alpha = \liminf_{n\to\infty} \frac{1}{\log(n)} \log\left(\sum_{i=1}^{n} \omega_i(x) \right) = \lim_{\ell\to\infty} \frac{1}{\log(n_\ell)} \log\left(\sum_{i=1}^{n_\ell} \omega_i(x) \right).$$

Since $(X, \mathcal{B}, \mu, T)$ and $(X, \mathcal{B}, \mu, T^{-1})$ are Hurewicz equivalent,

$$\alpha = \lim_{\ell\to\infty} \frac{1}{\log(n_\ell)} \log\left(\sum_{i=1}^{n_\ell} \omega_i(x) \right) = \lim_{\ell\to\infty} \frac{1}{\log(n_\ell)} \log\left(\sum_{i=1}^{n_\ell} \omega_{-i}(x) \right).$$

Let k_ℓ be the corresponding subsequence such that

$$|\Gamma_{k_\ell,x}| \leq n_\ell < |\Gamma_{k_\ell+1,x}|.$$

As in the proof of Lemma 3.1 we consider two possible cases: when $P_{1,x} \cap \{k_\ell\}_{\ell\in\mathbb{Z}^+}$ is infinite and when $P_{2,x} \cap \{k_\ell\}_{\ell\in\mathbb{Z}^+}$ is infinite. Since $P_{1,x}$ and $P_{2,x}$ cover $\mathbb{Z}^+$, at least one case will be true.

For the first case, assume that $P_{1,x} \cap \{k_\ell\}_{\ell\in\mathbb{Z}^+}$ is infinite. Enumerate the elements of this set in increasing order by p_ℓ, $\ell \in \mathbb{Z}^+$. Since $p_\ell \in P_{1,x}$,

$$\sum_{\gamma\in\Gamma_{p_\ell,x}} \frac{d\mu \circ \gamma}{d\mu}(x) = \sum_{i=-J'_{p_\ell}}^{J_{p_\ell}} \omega_i(x) \leq 2 \sum_{i=1}^{J_{p_\ell}} \omega_i(x) \leq 2 \sum_{i=1}^{n_\ell} \omega_i(x). \tag{4}$$

From this we deduce that

$$\begin{aligned}
\liminf_{p\to\infty} -\frac{1}{\log(|\Gamma_{p,x}|)} \log\left(\frac{G_p(x)}{|\Gamma_{p,x}|} \right) &\leq \liminf_{l\to\infty} \frac{1}{\log(|\Gamma_{p_\ell,x}|)} \log\left(\sum_{\gamma\in\Gamma_{p_\ell,x}} \frac{d\mu \circ \gamma}{d\mu}(x) \right) \\
&\leq \liminf_{l\to\infty} \frac{1}{\log(|\Gamma_{p_\ell,x}|)} \log\left(2 \sum_{i=1}^{J_{p_\ell}} \omega_i(x) \right) \\
&= \liminf_{l\to\infty} \frac{1}{\log(|\Gamma_{p_\ell+1,x}|)} \log\left(\sum_{i=1}^{J_{p_\ell}} \omega_i(x) \right) \\
&\leq \liminf_{l\to\infty} \frac{1}{\log(n_\ell)} \log\left(\sum_{i=1}^{n_\ell} \omega_i(x) \right) \\
&= \alpha.
\end{aligned}$$

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For the second case, $P_{2,x} \cap \{k_\ell\}_{\ell\in\mathbb{Z}^+}$ is infinite. Enumerate the elements of this set in increasing order by p_ℓ, $\ell \in \mathbb{Z}^+$. Since $p_\ell \in P_{2,x}$,

$$\sum_{\gamma\in\Gamma_{p_\ell,x}} \frac{d\mu\circ\gamma}{d\mu}(x) = \sum_{i=-J'_{p_\ell}}^{J_{p_\ell}} \omega_i(x) \le 2\sum_{i=1}^{J'_{p_\ell}} \omega_{-i}(x) \le 2\sum_{i=1}^{n_\ell} \omega_{-i}(x).$$

Repeating the above argument with J' and the reverse odometer T^{-1} completes the proof. □

THEOREM 3.1. *Assume that* $(X, \mathcal{B}, \mu, T)$ *and* $(X, \mathcal{B}, \mu, T^{-1})$ *are Hurewicz equivalent, and*

$$\lim_{p\to\infty} \frac{\log(|\Gamma_{p,x}|)}{\log(|\Gamma_{p+1,x}|)} = 1.$$

Then

$$\alpha = \liminf_{p\to\infty} -\frac{1}{\log(|\Gamma_{p,x}|)} \log\left(\frac{G_p(x)}{|\Gamma_{p,x}|}\right),$$
$$\beta = \limsup_{p\to\infty} -\frac{1}{\log(|\Gamma_{p,x}|)} \log\left(\frac{G_p(x)}{|\Gamma_{p,x}|}\right).$$

Proof. The first part of this theorem follows by combining Lemmas 3.1 and 3.2. The second part can be obtained by applying the same arguments for the upper critical dimension. □

Corollary 2.1 means that Theorem 3.1 applies to any product space satisfying equation (3), but not necessarily to a Bratteli–Vershik system. In [**4**], the notion of G-measure has been extended to Bratelli–Vershik systems. In future work we will extend Theorem 3.1 to that setting.

COROLLARY 3.1. *Given the product space* $(X, \mathcal{B}, \mu, T)$, $X = \prod_{n=1}^{\infty} \mathbb{Z}_{\ell(n)}$. *Let* $P^{(n)}$ *be a sequence of stochastic* $\ell(n) \times \ell(n+1)$ *matrices whose* (i, j)*th entry* $P^{(n)}_{i,j}$ *can be used to define a G-measure* $P^{(n)}_{x_n,x_{n+1}} = g_n(x_n, x_{n+1}) = g_n(x)$. *This is known as a Markov measure* [**10**]. *Assuming that*

$$\lim_{p\to\infty} \frac{\log(|\Gamma_{p,x}|)}{\log(|\Gamma_{p+1,x}|)} = 1,$$

then

$$\alpha = \liminf_{p\to\infty} -\frac{1}{\log(|\Gamma_{p,x}|)} \sum_{i=1}^{p} \log(P^{(i)}_{x_i,x_{i+1}}),$$
$$\beta = \limsup_{p\to\infty} -\frac{1}{\log(|\Gamma_{p,x}|)} \sum_{i=1}^{p} \log(P^{(i)}_{x_i,x_{i+1}}).$$

Proof. The spaces $(X, \mathcal{B}, \mu, T)$ and $(X, \mathcal{B}, \mu, T^{-1})$ are Hurewicz equivalent by Corollary 2.1. Since $G_p(x) = |\Gamma_{p,x}| \prod_{i=1}^{p} P^{(i)}_{x_i,x_{i+1}}$ defines a quasi-invariant G-measure, this is a straightforward application of Theorem 3.1. □

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The notion of unique ergodicity for G-measures was explored in [**2**, **7**], where conditions were given on the G-functions under which there is a unique G-measure. We refer to this situation as *unique ergodicity*, as the limit is then necessarily ergodic. If this condition fails, one can have bad behaviour of the set of G-measures; indeed, for any F_σ set $S \subset [0, 1]$, Dooley and Rudolph [**6**] produced examples of continuous G-functions such that there is an ergodic G-measure of type III_λ if and only if $\lambda \in S$.

Thus, in the sense of von Neumann–Krieger types, the range of possible G-measures can be quite diverse. This is not so for the critical dimension. Under certain conditions on the space X, all G-measures have the same critical dimensions.

COROLLARY 3.2. *If $X = \prod_{k=1}^{\infty} \mathbb{Z}_{\ell(k)}$ is a product space, and*

$$\lim_{p\to\infty} \frac{\log(|\Gamma_{p,x}|)}{\log(|\Gamma_{p+1,x}|)} = 1,$$

then any two quasi-invariant G-measures have the same critical dimensions.

Proof. Suppose that μ, ν are two quasi-invariant G-measures on the product space X. Then by Theorem 3.1 the upper and lower critical dimensions are both equal to

$$\alpha = \liminf_{p\to\infty} -\frac{1}{\log(|\Gamma_{p,x}|)} \log\left(\frac{G_p(x)}{|\Gamma_{p,x}|}\right),$$
$$\beta = \limsup_{p\to\infty} -\frac{1}{\log(|\Gamma_{p,x}|)} \log\left(\frac{G_p(x)}{|\Gamma_{p,x}|}\right).$$

Hence μ and ν have the same critical dimensions. □

4. *Computing the critical dimension for G-measures*

In this section we follow [**3**, **5**] and compute the critical dimension using two variants on the law of large numbers. The first lemma follows in the footsteps of [**5**].

LEMMA 4.1. *Suppose that $(X, \mathcal{B}, \mu, T)$ is a product space with $G = (G_n)$ a normalized compatible family. If there exists $k \in \mathbb{Z}^+$ such that for all n, $g_n = G_n(x)/G_{n-1}$ depends only on the next k digits of x, i.e. $g_n(x) = g_n(x_n, \ldots, x_{n+k})$, then for μ-almost every $x \in X$,*

$$\lim_{n\to\infty} \left| \frac{1}{n} \left(\sum_{i=1}^{n} \log(g_i(x)) - E(\log(g_i(x))) \right) \right| = 0.$$

Proof. Consider the random variable $X_i = \log(g_i(x)) - E(\log(g_i(x)))$. Define k sequences of random variables $(X_p^{(r)})_{p\in\mathbb{N}}$ where $X_p^{(r)} = X_{pk+r}$. For every integer $r \in [0, k)$ the random variables $X_p^{(r)}$ are independent (by assumption) and have expected value $E(X_p^{(r)}) = 0$. By the law of large numbers

$$\lim_{n\to\infty} \left| \frac{1}{n} \left(\sum_{p=1}^{n} X_p^{(r)} \right) \right| = 0.$$

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For any $n > k$,

$$0 \le \left| \frac{1}{n} \left(\sum_{i=1}^{n} \log(g_i(x)) - E(\log(g_i(x))) \right) \right|$$
$$\le \frac{\lceil n/k \rceil}{k(\lceil n/k \rceil - 1)} \sum_{r=0}^{k} \left| \frac{1}{\lceil n/k \rceil} \left(\sum_{p=1}^{\lceil n/k \rceil} X_p^{(r)} \right) \right|.$$

Taking $n \to \infty$ completes the proof. □

Example 1. Choose $\ell(p) = p$ and define a measure $\mu_p : \mathbb{Z}_{\ell(p)} \to [0, 1]$ by

$$\mu_p(i) = \begin{cases} \frac{1}{2} & \text{if } i = 0, \\ \frac{1}{2p} & \text{if } 0 < i \le p. \end{cases}$$

The product measure $\mu = \bigotimes_{p=1}^{\infty} \mu_p$ on the product space $X = \prod_{p=1}^{\infty} \mathbb{Z}_{\ell(p)}$, with odometer action T and sigma algebra $\mathcal{B}$ generated by the cylinders. The G-measure defined by $g_p(x) = p\mu_p(x)$ and

$$G_p(x) = g_1(x)g_2(x) \cdots g_p(x)$$

is quasi-invariant for the odometer $(X, \mathcal{B}, \mu, T)$. Furthermore,

$$\lim_{p\to\infty} \frac{\log(|\Gamma_{p,x}|)}{\log(|\Gamma_{p+1,x}|)} = \lim_{p\to\infty} \frac{\log(p!)}{\log((p+1)!)} = 1.$$

By Theorem 3.1,

$$\alpha = \liminf_{p\to\infty} -\frac{1}{\log(p!)} \log\left(\frac{G_p(x)}{p!}\right)$$
$$= \liminf_{p\to\infty} -\frac{1}{\log(p!)} \sum_{i=1}^{p} \log(\mu_p(x))$$
$$= \liminf_{p\to\infty} -\frac{1}{\log(p!)} \sum_{i=1}^{p} \log(\mu_p(x)).$$

This is as far as Theorem 3.1 will take us. By the law of large numbers given in Lemma 4.1,

$$\lim_{p\to\infty} \frac{1}{p} \sum_{i=1}^{p} \log(\mu_p(x)) - E(\log(\mu_p(x))) = 0.$$

Hence we compute α as

$$\alpha = \liminf_{p\to\infty} -\frac{1}{\log(p!)} \sum_{i=1}^{p} E(\log(\mu_p(x))) = \liminf_{p\to\infty} \frac{1}{\log(p!)} \sum_{i=1}^{p} 1 + \log\left(\sqrt{p}\right) = \frac{1}{2}. \quad \square$$

Remark. The example above shows how Theorem 3.1 extends the results of [**5**] from product odometers with a bounded number of edges at each level, to product odometers where the growth of the number of edges is restricted by

$$\lim_{p\to\infty} \frac{\log(|\Gamma_{p,x}|)}{\log(|\Gamma_{p+1,x}|)} = 1.$$

The same results holds for Markov odometers on a product space. □

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For the next example, we compute the critical dimension of the classical Riesz product

$$\prod_{k=1}^{n}(1 + a_k \cos(2\pi 3^k t)).$$

We begin with two lemmas. The first is an alternative form of the law of large numbers, which is attributed to [**12**].

LEMMA 4.2. *Suppose that* (X_n) *is a sequence of random variables of bounded modulus and that*

$$\sum_{N=1}^{\infty} \frac{1}{N} E(|Y_N|^2) < \infty$$

where $Y_N = (1/N) \sum_{n=1}^{N} X_n$. *Then almost surely* $Y_n \to 0$.

We need only show that this law of large numbers can be applied to the log of the Riesz product coordinate measures.

LEMMA 4.3. *Let* $X_n = \log(1 + a_n \cos(2\pi 3^n x))$. *Then*

$$\lim_{N\to\infty} \frac{1}{N}\left(\sum_{n=1}^{N} X_n - E(X_n)\right) = 0.$$

Proof. Substitute $t = 2\pi x$ and let $Y_N = (1/N) \sum_{n=1}^{N}(X_n - E(X_n))$. By Lemma 4.2, it is sufficient to prove that the sequence $(1/N)E(|Y_n|^2)$ is summable. Let $z_n = (1 - \sqrt{1 - a_n^2})/a_n$, where $|z_n| < 1$. Then

$$\begin{aligned} X_n + \log\left(\frac{2z_n}{a_n}\right) &= \log(1 + z_n^2 + z_n \cos(3^n t)) \\ &= \log(|1 + z_n e^{i3^n t}|^2) \\ &= 2\mathrm{Re}\,\mathrm{Log}\,(1 + z_n e^{i3^n t}) \\ &= 2\sum_{l=1}^{\infty} \frac{(-1)^l z_n^l}{l} \cos(l3^n t). \end{aligned}$$

From this it can be seen that

$$E(X_n) = -\log\left(\frac{2z_n}{a_n}\right).$$

Furthermore, for any integers $j \le k$, by Parseval's formula,

$$\begin{aligned} \mathrm{Cov}(X_j, X_k) &= E((X_j - E(X_j))(X_k - E(X_k))) \\ &= \frac{2}{3^{k-j}} \sum_{n=1}^{\infty} \frac{z_j^n z_k^{3^{k-j}n}}{n^2} \\ &= \frac{2}{3^{k-j}} \mathrm{Li}_2(z_j^n z_k^{3^{k-j}n}) \le \frac{2}{3^{k-j}} \end{aligned}$$

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where Li is the dilogarithm. Hence

$$\begin{aligned}\frac{1}{N}E(|Y_N|^2) &= \frac{1}{N}E(|Y_N - E(Y_N)|^2)\\ &= \frac{1}{N}\operatorname{Var}(Y_N)\\ &= \frac{2}{N}\sum_{k=1}^{N}\sum_{j=1}^{k}\operatorname{Cov}(X_j, X_k)\\ &\le \frac{4}{N}\sum_{k=1}^{N}\sum_{j=1}^{k}3^{j-k}\end{aligned}$$

is summable. The result follows as a consequence of Lemma 4.2. □

Example 2. Choose $\ell(p) = 3$ and the sequence a_p such that $|a_p| < 1$. Define $g_p(x) = 3(1 + a_p \cos(2\pi 3^p x))$ and

$$G_p(x) = g_1(x)g_2(x)\cdots g_p(x).$$

Then $X = \prod_{i=1}^{\infty}\mathbb{Z}_3$ is a product space satisfying

$$\lim_{p\to\infty}\frac{\log(|\Gamma_{p,x}|)}{\log(|\Gamma_{p+1,x}|)} = \lim_{p\to\infty}\frac{p\log(3)}{(p+1)\log(3)} = 1.$$

By Theorem 3.1 and Lemma 4.3,

$$\begin{aligned}\alpha &= \liminf_{p\to\infty} -\frac{1}{p\log(3)}\log\left(\frac{G_p(x)}{3^p}\right)\\ &= \liminf_{p\to\infty} -\frac{1}{p\log(3)}\sum_{i=1}^{p}\log(g_i(x))\\ &= \liminf_{p\to\infty} -\frac{1}{p\log(3)}\sum_{i=1}^{p}\log(1 + a_p\cos(2\pi 3^i x))\\ &= \liminf_{p\to\infty} -\frac{1}{p\log(3)}\sum_{i=1}^{p}\log\left(2\left(1-\sqrt{1-a_p^2}\right)a_p^{-2}\right).\end{aligned}$$

Similarly, the upper critical dimension can be computed as

$$\beta = \limsup_{p\to\infty} -\frac{1}{p\log(3)}\sum_{i=1}^{p}\log\left(2\left(1-\sqrt{1-a_p^2}\right)a_p^{-2}\right).$$

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REFERENCES

[**1**] G. Brown and A. H. Dooley. Ergodic measures are of weak product type. *Math. Proc. Cambridge Philos. Soc.* **98** (1985), 129–145.
[**2**] G. Brown and A. H. Dooley. Odometer actions on G-mesaures. *Ergod. Th. & Dynam. Sys.* **11** (1991), 297–307.
[**3**] A. H. Dooley and R. Hagihara. Computing the critical dimension of Bratteli–Vershik systems with multiple edges. *Ergod. Th. & Dynam. Sys.* **32** (2011), 103–117.
[**4**] A. H. Dooley and T. Hamachi. Nonsingular dynamical systems, Bratteli diagrams and Markov odometers. *Israel J. Math.* **138** (2003), 93–123.
[**5**] A. H. Dooley and G. Mortiss. On the critical dimension of product odometers. *Ergod. Th. & Dynam. Sys.* **29** (2009), 475–485.
[**6**] A. H. Dooley and D. Rudolph. Non-uniqueness in G-measures. *Ergod. Th. & Dynam. Sys.* **32** (2011), 575–586.
[**7**] A. H. Dooley and Ö. Stenflo. A criterion for uniqueness in G-measures and perfect sampling. *Math. Proc. Cambridge Philos. Soc.* **140** (2006), 545–551.
[**8**] R. H. Herman, I. F. Putnam and C. F. Skau. Ordered Bratteli diagrams, dimension groups and topological dynamics. *Internat. J. Math.* **3**(6) (1992), 827–864.
[**9**] M. Keane. Strongly mixing g-measures. *Invent. Math.* **16** (1972), 309–324.
[**10**] G. Mortiss. Average co-ordinate entropy. *J. Aust. Math. Soc.* **73** (2002), 171–186.
[**11**] G. Mortiss. An invariant for non-singular isomorphism. *Ergod. Th. & Dynam. Sys.* **23** (2003), 885–893.
[**12**] G. Rauzy. *Propriétés statistiques de suites arithmétiques*. Presses Universitaires de France, Paris, 1976.

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