MAXIMAL FUNCTIONS WITH MITIGATING FACTORS IN THE PLANE

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1. *Introduction*

Given a smooth, compactly supported hypersurface S in $\mathbb{R}^n$ that does not pass through the origin, and denoting by tS the surface dilated by a factor $t > 0$, we can consider the averaging operator defined for functions $f \in \mathscr{S}$, the Schwartz class of functions, by

$$A_t f(x) = \int_{tS} f(x-y)\, d\sigma(y)$$

where $d\sigma$ is Lebesgue measure on S. We can now define a maximal average operator,

$$Mf(x) = \sup_{t>0} |A_t f(x)|.$$

In the case where S is S^{n-1}, the sphere of unit radius in $\mathbb{R}^n$, we are looking at Stein's spherical maximal function, as treated in his paper [**10**]. Stein proved that M is a bounded operator on the L^p spaces if and only if $p > n/(n-1)$ when $n > 3$. Subsequently, Bourgain [**2**] showed that if S is any compactly supported smooth curve with non-vanishing Gaussian curvature, then M will be bounded on L^p, if and only if $p > 2$, thus dealing with the case of the circular maximal function in the plane. (For related results, see also [**6**] and [**7**]). In the case where S is a curve whose curvature vanishes to order at most $m-2$ at a single point, Iosevich [**4**] showed that M is bounded on L^p for $p > m$, and unbounded if $p = m$. If we study curves given by $\Gamma(s) = (s, \gamma(s)+1)$, $s \in [0,1]$, for some suitably smooth γ, where $\gamma(0) = \gamma'(0) = \ldots = \gamma^{(m-1)}(0) \neq \gamma^{(m)}(0) > 0$, then we can reinterpret his results as follows. Define

$$M_k f(x) = \sup_{t>0} \left| \int_{2^{-k}}^{2^{1-k}} f(x - t\Gamma(s)) ds \right|$$

for Schwartz functions f. Iosevich proved that

$$\|M_k\|_{L^p - L^p} \leqslant c_p\, 2^{-k(1-m/p)}$$

for $p > 2$. If we note that $\kappa(s)$, the curvature of the curve $\Gamma(s)$ is approximately $2^{-k(m-2)}$ whenever $s \in [2^{-k}, 2^{1-k}]$, then we have that the operator

$$M_\sigma f(x) = \sup_{t>0} \left| \int_0^1 f(x - t\Gamma(s))\, (\kappa(s))^\sigma ds \right|$$

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is bounded on L^p for some $p > 2$, if σ is sufficiently large, since

$$\|M_\sigma\|_{L^p-L^p} \leqslant C \sum_{k \geqslant 0} 2^{-k(m-2)\sigma} \|M_k\|_{L^p-L^p}$$

$$\leqslant C \sum_{k \geqslant 0} 2^{-k((m-2)\sigma+1-m/p)}$$

which is finite so long as $\sigma > (m/p-1)(m-2)^{-1}$. If we want to choose σ independent of $m > 2$, the type of the curve, such that M_σ is bounded on L^p for some fixed $p > 2$, then clearly we can take $\sigma = 1/p$. In this paper we show that M_σ will be bounded on L^p for $p > \max\{\sigma^{-1}, 5\}$ for a class of infinitely flat, convex curves in the plane. Counterexamples will show that this is the best possible result, in that there exist flat curves for which M_σ is unbounded for $2 < p \leqslant \sigma^{-1}$.

THEOREM 1. *Let* $\Gamma(s) = (s, \gamma(s)+1)$, $s \in [0,1]$, *be a curve in* $\mathbb{R}^2$ *such that* $\gamma \in C^5[0,1]$, $\gamma(0) = \gamma'(0) = 0$, $\gamma^{(r)}(s)$ *is single-signed and monotonic for* $r = 1,2,3,4,5$, *and both* $\log \gamma''(s)$ *and* $\log \gamma'''(s)$ *are concave. Then the maximal function*

$$M_\sigma f(x) = \sup_{t>0} \left| \int_0^1 (\kappa(s))^\sigma f(x - t\Gamma(s))\, ds \right|$$

is bounded on L^p *so long as* $p > \max\{\sigma^{-1}, 2\}$.

As $|\Gamma'(s)|$ is bounded between two constants independent of everything, we have

$$0 \leqslant c\gamma''(s) \leqslant \kappa(s) \leqslant C\gamma''(s)$$

and so we may replace $\kappa(s)$ by $\gamma''(s)$ in the maximal function. Notice also that as $\gamma''(s)$ is bounded, it suffices to consider only $0 < c \leqslant \gamma''(s) \leqslant 1$. The strange conditions concerning $\log \gamma''$ and $\log \gamma'''$ are more than we require here, though weaker conditions are less tidy.

2. *Scaling*

Lemma 2, which is essentially the same fact as that used by Iosevich in [**4**], is critical in the proof of our theorem.

LEMMA 2. *Let* $\tilde{\Omega}(s) = (s+\mu, \omega(s)+\lambda)$, $s \in [0,1]$, *be a curve in* $\mathbb{R}^2$ *with* $\omega \in C^5[0,1]$, *and* $\omega(0) = \omega'(0) = 0$, $\omega'(s) \leqslant 1/2$ *and* $0 < c \leqslant \gamma''(s) \leqslant C < \infty$, *where* λ *and* μ *are constants satisfying* $|\lambda| \geqslant \max\{|\mu|, 2\}$. *Let*

$$\tilde{A}_t f(x) = \int_0^1 f(x - t\tilde{\Omega}(s))\, ds$$

$$\tilde{M} f(x) = \sup_{t>0} |A_t f(x)|$$

be the averaging operator and maximal function corresponding to $\tilde{\Omega}$, *respectively. Then* $\tilde{M}$ *is bounded on* L^p *for* $p > 2$, *with bounds bounded by* $C_p|\lambda|^{1/p}$, *where* C_p *depends only on* c, C *and the* C^5 *norm of* $\omega(s)$.

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The proof of Lemma 2 is fairly simple once one has tried to work through a proof of Bourgain's theorem in the style of [**6**]. One simply repeats this proof and studies how the translation affects the norms.

We shall need one more lemma in our proof of Theorem 1. This is a classical result, the proof of which may be found in [**11**, pp. 483–485].

LEMMA 3. *Let L be an invertible linear map from $\mathbb{R}^2$ to itself, and let $\Gamma(s)$ and $\tilde{\Gamma}(s)$ be two curves in the plane, related by $\tilde{\Gamma}(s) = L(\Gamma(s))$. Then the following two maximal functions have identical $L^p - L^p$ operator norms*:

$$Mf(x) = \left| \sup_{t>0} \int f(x - t\Gamma(s))\, ds \right|$$

$$\tilde{M}f(x) = \left| \sup_{t>0} \int f(x - t\tilde{\Gamma}(s))\, ds \right|.$$

3. *Decomposition*

Iosevich was able to decompose his operator into parts where $s \sim 2^{-k}$ because he was interested only in finite type curves. Such a decomposition is not appropriate when one considers infinitely flat curves. Here we use the other well-known alternative, the decomposition into parts where the curvature is approximately 2^{-k}; clearly these two methods are equivalent for finite type curves, and indeed, for such curves we shall obtain essentially the same estimates as Iosevich. For non-negative f, we have

$$M_\sigma f(x) \leqslant C \sum_{k \geqslant 0} 2^{-k\sigma} M_k f(x)$$

where

$$M_k f(x) = \sup_{t>0} \left| \int_{I_k} f(x - t\Gamma(s))\, ds \right|$$

and where $I_k = [s_{k+1}, s_k]$ is the interval where $2^{-k-1} \leqslant \gamma''(s) \leqslant 2^{-k}$. By scaling and a change of variable, it is easy to see that M_k will have the same $L^p - L^p$ operator norms as the operator M_k, given by

$$\tilde{M}_k f(x) = \sup_{t>0} |I_k| \left| \int_0^1 f(x - t(s + s_{k+1}|I_k|^{-1}, \eta_k^{-1}\gamma(s|I_k| + s_{k+1}) + \eta_k^{-1}))\, ds \right|$$

for any fixed $\eta_k > 0$. Suppose there existed constants $\alpha_k, \lambda_k, \mu_k$ such that

$$(s + s_{k+1}|I_k|^{-1}, \eta_k^{-1}\gamma(s|I_k| + s_{k+1}) + \eta_k^{-1}) = (s + \mu_k, \omega(s) + \lambda_k + \alpha_k(s + \mu_k)) \tag{1}$$

for some curve ω of the type dealt with in Lemma 2, and with C^5 norms bounded independent of k. By rights, ω ought to have a k subscript, but the idea is that since its C^5 norm can be bounded independent of k, it is effectively a constant curve (almost) independent of k. By Lemma 3 (with $Lv = (v_1, v_2 - \alpha_k v_1)$), we would then have that $\tilde{M}_k$ has operator norms controlled by $|I_k|$ times those of the maximal function corresponding to the curve

$$(s + \mu_k, \omega(s) + \lambda_k)$$

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which by Lemma 2 has operator norms bounded by a constant times

$$C_p |I_k| |\lambda_k|^{1/p}$$

for $p > 2$, with C_p independent of k, if $|\lambda_k| \geqslant \max\{|\mu_k|, 2\}$. The question is, how can we arrange that (1) be true? A Taylor expansion of $\gamma(s)$ about $s = s_{k+1}$ suggests writing

$$\gamma(s|I_k| + s_{k+1}) = \gamma(s_{k+1}) + s|I_k|\gamma'(s_{k+1}) + \tilde{\omega}(s)$$

where $\tilde{\omega}(0) = \tilde{\omega}'(0) = 0$ and $\tilde{\omega}$ and its derivatives inherit their monotonicity from γ. To see this, notice that

$$\tilde{\omega}'(s) = |I_k|\{\gamma'(s|I_k| + s_{k+1}) - \gamma'(s_{k+1})\}$$

which is clearly non-negative. Differentiating once more shows that

$$\tilde{\omega}''(s) = |I_k|^2 \gamma''(s|I_k| + s_{k+1})$$

which is non-negative. From this it is clear that the monotonicity of further derivatives of γ is passed on to the corresponding derivatives of $\tilde{\omega}$. Notice that for $s \in [0, 1]$, $\tilde{\omega}''(s) \sim 2^{-k}|I_k|^2$. Hence it is natural to choose

$$\omega(s) = 2^k |I_k|^{-2}\{\gamma(s|I_k| + s_{k+1}) - \gamma(s_{k+1}) - s|I_k|\gamma'(s_{k+1})\} \tag{2}$$

so that $1/2 \leqslant \omega''(s) \leqslant 1$. Clearly, ω inherits the monotonicity properties of $\tilde{\omega}$. Notice that because the second derivative of ω is small, and because $\omega(0) = \omega'(0) = 0$, we have that $\omega'(s) \leqslant s$ and $\omega(s) \leqslant s^2/2$. The point here is that although ω depends on k, we shall show that it has C^5 norms independent of k, and satisfies all the conditions of Lemma 2 with constants independent of k. We have still to check the third, fourth and fifth derivative estimates; we deal with these later. For now, let us see how we must choose η_k, μ_k and α_k. From (2) we have that

$$\eta_k^{-1}\gamma(s|I_k| + s_{k+1}) + \eta_k^{-1} = \eta_k^{-1}\{2^{-k}|I_k|^2\omega(s) + s|I_k|\gamma'(s_{k+1}) + \gamma(s_{k+1} + 1)\}$$

and so choosing $\eta_k = 2^{-k}|I_k|^2$, we see that (1) will hold so long as

$$\mu_k = s_{k+1}|I_k|^{-1}$$
$$\alpha_k = 2^k |I_k|^{-1}\gamma'(s_{k+1})$$

and

$$2^k |I_k|^{-1}\gamma'(s_{k+1}) s_{k+1}|I_k|^{-1} + \lambda_k = 2^k |I_k|^{-2}\gamma(s_{k+1}) + 2^k |I_k|^{-2},$$

this last equation being

$$\lambda_k = 2^k |I_k|^{-2}\{1 + \gamma(s_{k+1}) - s_{k+1}\gamma'(s_{k+1})\}.$$

Notice that for all but finitely many k,

$$s_{k+1}\gamma'(s_{k+1}) \leqslant 1/2$$

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so that $|\lambda_k| \geqslant 2^{k-1}|I_k|^{-2}$, for all but finitely many k. Thus, for all but finitely many k we have $|\lambda_k| \geqslant \max\{|\mu_k|, 2\}$, since

$$2^{k-1}|I_k|^{-2} \geqslant 2 \Leftrightarrow |I_k|^2 \leqslant 2^{k-2}$$

and

$$2^{k-1}|I_k|^{-2} \geqslant \frac{s_{k+1}}{|I_k|} \Leftrightarrow |I_k| \leqslant \frac{2^{k-1}}{s_{k+1}}$$

both hold for all but finitely many k. Assuming we can show that $\omega^{(r)}(s) \leqslant C$, for $r = 3, 4, 5$ and C independent of k, then we have that M_k has $L^p - L^p$ operator norms bounded by

$$C_p|\lambda_k|^{1/p}|I_k| \leqslant C_p 2^{k/p}|I_k|^{1-2/p}$$

so long as $p > 2$. This would then complete the proof of Theorem 1, as

$$\|M_\sigma\|_{L^p-L^p} \leqslant C \sum_{k \geqslant 0} 2^{-k\sigma}\|M_k\|_{L^p-L^p}$$

$$\leqslant C_p \sum_{k \geqslant 0} 2^{-k(\sigma-1/p)}|I_k|^{1-2p} < \infty$$

if $\sigma > 1/p$. We return now to those higher derivative estimates for ω. We have

$$\omega^{(r)}(s) = 2^k|I_k|^{r-2}\gamma^{(r)}(s|I_k| + s_{k+1})$$

for $r = 3, 4, 5$. Given the monotonicity of $\gamma^{(r)}$, and given that $2^{-k} = \gamma''(s_k)$, we clearly require only that

$$|I_k|^{r-2}\frac{\gamma^{(r)}(s_k)}{\gamma''(s_k)} \leqslant C \tag{3}$$

with C independent of k. This is where we shall use those conditions on $\log\gamma''$ and $\log\gamma'''$. In particular, our assumptions imply that $\gamma''(s)/\gamma'''(s)$ is monotonic, that

$$\gamma''(s)\gamma^{(4)}(s) \leqslant (\gamma'''(s))^2$$

and that

$$\gamma'''(s)\gamma^{(5)}(s) \leqslant \gamma^{(4)}(s).$$

Things are clarified further if we show that $|I_k|$ is bounded above and below by constants times $\gamma''(s_k)/\gamma'''(s_k)$. This is a consequence of our monotonicity conditions. The mean value theorem gives us the following:

$$|I_k| = (\gamma'')^{-1}(2^{-k}) - (\gamma'')^{-1}(2^{-k-1}) = 2^{-k-1}\frac{1}{\gamma'''(z)}$$

for some $z \in (s_{k+1}, s_k)$, and so

$$c\frac{\gamma''(s_k)}{\gamma'''(s_k)} \leqslant |I_k| \leqslant C\frac{\gamma''(s_{k+1})}{\gamma'''(s_{k+1})}. \tag{4}$$

As our assumptions imply that $\gamma''(s)/\gamma'''(s)$ is monotonic increasing, and as $s_{k+1} < s_k$, the right-hand side of (4) is bounded by a constant times the left-hand side giving the stated result, that $|I_k| \sim \gamma''(s_k)/\gamma'''(s_k)$. Notice that this is (3) in the case $r = 3$. Now consider the case $r = 4$; we require that

$$|I_k|^2\frac{\gamma^{(4)}(s_k)}{\gamma''(s_k)} \leqslant C.$$

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By what we have just shown, this is equivalent to

$$\frac{\gamma^{(4)}(s_k)}{\gamma''(s_k)} \leqslant C\left(\frac{\gamma'''(s_k)}{\gamma''(s_k)}\right)^2$$

which is true, as $\gamma^{(4)}(s)\,\gamma''(s) \leqslant (\gamma'''(s))^2$, by our $\log\gamma''$ concave condition. In a similar fashion, the $\log\gamma'''$ concave condition gives us (3) for $r = 5$.

4. *The case* $\sigma = 1/p$

It is easy to see that a power $\sigma < 1/p$ of the curvature will not guarantee L^p boundedness for our maximal function. For example, consider $\Gamma(s) = (s, s^m + 1)$ and the test function

$$f(x) = \eta(x)\,|x_2|^{-1/q}$$

for some $q > p$, where $\eta \in C_0^\infty$ is a smooth bump function such that $\eta(x) = 1$ if $|x| \leqslant 1/2$, and $\eta(x) = 0$ if $|x| \geqslant 1$. Notice that $f \in L^r$ for all $r < q$, and hence f is in L^p. Consider only points x that lie within distance $1/4$ of the origin, so that $|x - x_2\,\Gamma(s)| \leqslant 1/2$ for all $s \in [0, 1]$. Then we have

$$\begin{aligned} Mf(x) &\geqslant C\int_0^1 f(x - x_2\,\Gamma(s))\,s^{(m-2)\sigma}ds \\ &\geqslant C\int_0^1 |x_2|^{-1/q} s^{(m-2)\sigma - m/q}ds \end{aligned}$$

which diverges unless $(m-2)\sigma - m/q > -1$. So, if we wish to choose σ independent of $m \geqslant 2$, we clearly need $\sigma \geqslant 1/q$. As q was arbitrarily close to p, we require that $\sigma \geqslant 1/p$. It is more interesting to ask what happens when we consider $\sigma = 1/p < 1/2$. In this case we know by our earlier work that

$$\|Mf\|_{L^p} \leqslant C_p\,\|f\|_{L^p} \sum_{k \geqslant 0} |I_k|^{1-2/p}$$

so if $|I_k|$ decays sufficiently fast, then we will have that M is L^p bounded. For example, we have the following corollary.

COROLLARY 4. *Suppose* Γ *and* γ *are as in Theorem* 1, *and suppose in addition that there exists an* $\epsilon > 0$ *such that*

$$|\log\gamma''(s)|^{1+\epsilon}\left(\frac{\gamma''(s)}{\gamma'''(s)}\right)$$

is bounded. Then we have $|I_k| \leqslant Ck^{-1-\epsilon}$, *and so the maximal function of Theorem* 1 *is bounded on* L^p *if* $p = \sigma^{-1} > 2(\epsilon^{-1} + 1)$.

Notice that this applies to the family of flat curves $\gamma(s) = \exp(-s^{-N})$, for $N \geqslant 1$, with $\epsilon = 1/N$. Thus the corresponding maximal function, with power $\sigma = 1/p$ of the curvature, is bounded on L^p if $p > 2N + 2$. The proof of Corollary 4 is a matter of simple calculus, working with the function $g: \mathbb{R}^+ \longrightarrow [0, 1]$, defined implicitly by

$$\log\gamma''(g(x)) = -(x^{-1/\epsilon})\log 2.$$

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This is chosen so that

$$g(k^{-\epsilon}) = s_k$$

where s_k is, as before, defined by $\gamma''(s_k) = 2^{-k}$. We have that $|I_k| = g(k^{-\epsilon}) - g((k+1)^{-\epsilon})$, which is $(k^{-\epsilon} - (k+1)^{-\epsilon}) g'(z)$, for some $z \in (k, k+1)$. We ask that g' be uniformly bounded, this being the hypothesis of Corollary 4, and hence $|I_k| \leqslant Ck^{-\epsilon-1}$, from which the desired result follows easily. What is surprising about this result is that it is almost sharp. If one considers the test function

$$f(x) = \eta(x) \left(\frac{1}{|x_2| \, |\log |x_2||^{1+\delta}} \right)^{1/p}$$

where $\eta \in C_0^\infty$ is a smooth bump function such that $\eta(x) = 1$ if $|x| \leqslant 1/2$, and $\eta(x) = 0$ if $|x| \geqslant 3/4$, we can see that $f \in L^p$ if $\delta > 0$ and $p < \infty$. Considering now the average of this function over the curve dilated by x_2, we have

$$Mf(x) \geqslant C \int_0^1 \eta(x - x_2 \Gamma(s)) \, (\gamma''(s))^{1/p} \left(\frac{1}{|x_2 \gamma(s)| \, |\log (x_2 \gamma(s))|^{1+\delta}} \right)^{1/p} ds.$$

Now we consider only points x such that $|x - x_2 \Gamma(s)| \leqslant 1/2$, for all $s \in [0, 1]$. Notice that as we are considering Γ to be bounded as usual, it suffices to consider x in $B = \{x; |x| \leqslant c, x_2 \neq 0\}$, for sufficiently small c. For such points we have

$$Mf(x) \geqslant C |x_2|^{-1/p} \int_0^1 \left(\frac{\gamma''(s)}{\gamma(s)} \right)^{1/p} \left| \frac{1}{\log |x_2| + \log \gamma(s)} \right|^{(1+\delta)/p} ds. \tag{5}$$

If this is infinite for some small range of the x_2 near the origin, then Mf is infinite on a set of non-zero measure, and hence cannot be in any L^p space. Thus, on noting that we can arrange that $\gamma(s) < \min \{|x_2|, 1\}$ for s in a suitably small neighbourhood of the origin, we have found a necessary condition for the $\sigma = 1/p$ result to hold.

THEOREM 5. *Let M be the maximal function of Theorem* 1, *with power $\sigma = 1/p < 1/2$ of the curvature. Then a necessary condition that M be L^p bounded is that*

$$\int_0^\theta \left(\frac{\gamma''(s)}{\gamma(s)} \right)^{1/p} \left| \frac{1}{\log \gamma(s)} \right|^{(1+\delta)/p} ds$$

be finite, for any $\delta > 0$, and some $\theta > 0$. Further, if γ satisfies the additional hypothesis

$$|\log \gamma''|^{1+\epsilon} \left(\frac{\gamma''(s)}{\gamma'''(s)} \right) \geqslant C > 0$$

is a neighbourhood of $s = 0$, then M is unbounded if $p = \sigma^{-1} < 2(e^{-1} + 1)$.

(Alex Josevich has observed that the first displayed condition in Theorem 5 is also a sufficient condition. To see this, consider the same proof, but integrating over $[\theta, 1]$ instead of $[0, \theta]$.) The second part of this theorem should be compared with Corollary 4. If we consider the case $\gamma(s) = \exp(-s^{-N})$, for some $N \geqslant 1$, we see that the first part of Theorem 5 requires that

$$\int_0^\theta s^{-2(N+1)/p} s^{N(1+\delta)/p} ds$$

should be finite for some positive θ. Thus we need $p > N + 2 - N\delta$, and since $\delta > 0$ is arbitrary, we need $p \geqslant N + 2$. The second part also applies for this curve, with

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$\epsilon = 1/N$, giving that M is unbounded if $p = \sigma^{-1} < 2N+2$. Note that Corollary 4 gave us that M was bounded for $p = \sigma^{-1} > 2N+2$. This leaves open the question of what happens when $\sigma^{-1} = p = 2N+2$.

The proof of the second part of Theorem 5 relies on the following inequalities:

$$0 \leqslant \gamma(s) \leqslant \gamma''(s) \tag{6}$$

$$\frac{\gamma''(s)}{\gamma(s)} \geqslant C\left(\frac{\gamma'(s)}{\gamma(s)}\right)^2 \tag{7}$$

$$\frac{\gamma'''(s)}{\gamma''(s)} \geqslant C\frac{\gamma''(s)}{\gamma'(s)} \tag{8}$$

$$\frac{\gamma''(s)}{\gamma'(s)} \geqslant C\frac{\gamma'(s)}{\gamma(s)}. \tag{9}$$

Inequality (6) is clear from the monotonicity of γ and its derivatives. Notice that (7) and (9) are in fact the same inequality. Inequalities (8) and (9) are clearly of a similar type. To prove (8), we consider writing $(\gamma''(s))^2$ as the integral of its derivative:

$$(\gamma''(s))^2 = 2\int_0^s \gamma''(t)\,\gamma'''(t)\,dt \leqslant 2\gamma'''(s)\int_0^s \gamma''(t)\,dt$$

by the monotonicity of γ''', and hence $\gamma'' \leqslant 2\gamma'\gamma'''$, which is essentially the required inequality. Similarly, (9) follows from the monotonicity of γ''. We can now prove the second part of Theorem 5. Consider (5), and consider only the part of the integral over $[0,\theta]$, where $\theta = \gamma^{-1}(|x_2|)$, for x in the set B near the origin, as above:

$$Mf(x) \geqslant C\,|x_2|^{-1/p}\int_0^\theta \left(\frac{\gamma''(s)}{\gamma(s)}\right)^{1/p}\left|\frac{1}{\log\gamma(s)}\right|^{(1+\delta)/p} ds.$$

Using the extra hypothesis on γ, this is bounded below by

$$C\,|x_2|^{-1/p}\int_0^\theta \left(\frac{\gamma''(s)}{\gamma(s)}\right)^{1/p}\left(\frac{\gamma'''(s)}{\gamma''(s)}\right)^{1-2/p}\left|\frac{1}{\log\gamma(s)}\right|^{\alpha}\left|\frac{1}{\log\gamma''(s)}\right|^{\beta} ds$$

where $\alpha = (1+\delta)/p$ and $\beta = (1+\epsilon)(1-2/p)$. We use (6) to control the $\log\gamma''$, and (7), (8) and (9) to control the ratios of various derivatives of γ, and thus obtain

$$Mf(x) \geqslant C\,|x_2|^{-1/p}\int_0^\theta \left(\frac{\gamma'(s)}{\gamma(s)}\right)\left|\frac{1}{\log\gamma(s)}\right|^{\alpha+\beta} ds$$

which can be evaluated exactly via the change of variables $u = \gamma(s)$. Using $\theta = \gamma^{-1}(|x_2|)$, we then have

$$Mf(x) \geqslant \frac{C}{|x_2|^{1/p}\,|\log|x_2||^{\alpha+\beta-1}}$$

for all $x \in B$. Thus, Mf is not in L^p if $\alpha+\beta-1 \leqslant 1/p$, which is $p \leqslant 2(1+\epsilon^{-1})-\delta/\epsilon$. The required result follows once one notices that $\delta > 0$ was arbitrary.

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5. *Comment*

As has been observed by J. G. Bak, such estimates for maximal functions with mitigating factors give rise to Orlicz space estimates for the corresponding maximal functions with no mitigating factor. See, for example [**1**] for the higher dimensional equivalents. Roughly speaking, the Orlicz space $L^{\Phi}(\mathbb{R}^n)$ is the space of all functions f such that the integral

$$\int \Phi(|f(x)|)\,dx$$

converges. Certain restrictions have to be imposed on Φ for this definition to be made workable, but it is usual to consider $\Phi\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ to be defined as follows. Let ϕ be a non-decreasing function such that $\phi(t)$ for $t \leqslant 1$ and $\phi(t) > 0$ for $t > 1$. Then we define

$$\Phi(u) = \int_0^u \phi(t)\,dt.$$

Using Bak's interpolation lemma in [**1**], one can easily prove the following.

COROLLARY 6. *Let $\Gamma(s) = (s, \gamma(s)+1)$ be a curve of the type considered in Theorem 1, and suppose in addition that $\gamma''(s) \leqslant Cs$, $\gamma''(\epsilon)\,\gamma''(\delta) \geqslant C\gamma''(\epsilon\delta)$. If σ is sufficiently large that the maximal function*

$$M_\sigma f(x) = \sup_{t>0} \left| \int_0^1 f(x - t\Gamma(s))\,(\gamma''(s))^\sigma\,ds \right|$$

is bounded on L^p, for some $p > 2$, then the maximal function M_0 is bounded on the Orlicz space L^{Φ}, where Φ is defined as above, with ϕ given by

$$\phi(u^\theta) = \frac{u^{p-1}}{(\gamma''(u^{-1}))^{\sigma p}}$$

for $u \geqslant 2$, and $\theta < 1$ being arbitrary. The definition of $\phi(u)$ for u small may be taken so that ϕ is of the appropriate monotonicity.

Notice that the additional two hypotheses on γ are satisfied by the curve $\gamma(s) = \exp(-s^{-N})$.

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