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On a generalized L–H transform

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Abstract. The authors, while attempting to give a new generalization of the Laplace transform, have come across a particular function in the form of Meijer's G-function

$$G_{p,q}^{m,n}\left(\alpha x^{\lambda}\middle|\begin{matrix}a_1, a_2, \ldots, a_p\\ b_1, b_2, \ldots, b_q\end{matrix}\right),$$

which, when taken as a nucleus of the transformation $K(x)$ in

$$\phi(s) = \int_0^{\infty} K(s,t) f(t)\, dt,$$

defines a new transform. This transform besides serving as a generalization of the Laplace transform and most of its generalizations existing in the literature bears the characteristic property of generalizing the Hankel transform (and hence some of its generalizations) as well. In this paper the authors after having defined the transform have given an inversion theorem which is supported by means of two examples.

1. In a paper Erdélyi (6) has studied the integral transformation

$$\phi(s) = \int_0^{\infty} K(s,t) f(t)\, dt, \tag{1·1}$$

where $K(x)$, the nucleus of transformation, is in the form

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma(\eta+\frac{1}{2}+it)\,\Gamma(\rho+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}+it)} x^{-\frac{1}{2}-it}\, dt \tag{1·2}$$

as a generalization of the Lapace transform.

This nucleus was later developed by the same author (1) in a more general form

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma\{(\eta+\frac{1}{2}+it)/n\}\,\Gamma(\rho+\frac{1}{2}+it)}{\Gamma[\alpha+(\eta+\frac{1}{2}+it)/n]} x^{-\frac{1}{2}-it}\, dt, \tag{1·3}$$

where n is any positive number not necessarily an integer. When $n = 1$, (1·3) reduces to (1·2).

By taking $K(x)$ in a modified form as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\prod_{j=1}^{m}\Gamma(b_j+\frac{1}{2}+it)\prod_{j=1}^{n}\Gamma(1-a_j-\frac{1}{2}-it)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\frac{1}{2}-it)\prod_{=n+1}^{p}\Gamma(a_j+\frac{1}{2}+it)} x^{-\frac{1}{2}-it}\, dt \tag{1·4}$$

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and expressing it as a Meijer's G-function(4) after replacing x by αx^λ, where α is a constant, we get

$$K_1(x) = G^{m,n}_{q,q}\left(\alpha x^\lambda \middle| \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right), \tag{1·5}$$

which can be treated as a kernel of an integral transform. Consequently, we define our transform by

$$\phi(s) = \int_0^\infty G^{m,n}_{p,q}\left(\alpha s^\lambda t^\lambda \middle| \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right) f(t)\, dt, \tag{1·6}$$

where $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, $p+q < 2(m+n)$, $|\arg \alpha s^\lambda| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ and λ is a positive integer.

The transform defined by (1·6) is of a most general character in the sense that many well-known generalizations of the Laplace transform, given from time to time by different authors, can be obtained as direct consequences of (1·6) on specializing the parameters. Another special feature of the transform under consideration is that the Hankel transform and hence some of the generalizations of the Hankel transform also happen to be its particular cases.

Some of the well-known particular cases of (1·6) are listed below:

2. *Particular cases of* (1·6). (A) (1) When $m = 1 = q$, $p = 0 = n$, $\alpha = 1 = \lambda$ and $b_1 = 0$, the Laplace transform comes out as a particular case of (1·6).

(2) The Meijer transform (10) is arrived at, when the substitutions $m = 2 = q, p = 1$, $n = 0$, $a_1 = -2k_1$, $b_1 = m_1 - k_1$, $b_2 = -m_1 - k_1$ and $\alpha = 1 = \lambda$ are made in (1·6).

(3) On taking $m = 2 = q$, $p = 1$, $n = 0$, $a_1 = \nu + \sigma$, $b_1 = \sigma$, $b_2 = \nu + \sigma - 1$ and $\alpha = 1 = \lambda$ in (1·6) and using

$$G^{20}_{12}\left(x \middle| \begin{matrix} \nu \\ 0, \nu - 1 \end{matrix}\right), \tag{2·1}$$

where $\Re(\nu) > 0$, we get the integro-exponential transform

$$\phi(s) = \int_0^\infty (st)^\sigma E_\nu(st) f(t)\, dt, \tag{2·2}$$

given by Bhonsle (2).

(4) On taking $m = 2 = q, p = 1 = \alpha = \lambda, n = 0, a_1 = m_1 - k_1 + \frac{1}{2}, b_1 = 2m_1$ and $b_2 = 0$ in (1·6), we get the Varma transform (16).

(5) The generalized Laplace transform given by Mainra (8) is a particular case of (1·6), when $m = 2 = q, p = 1, n = 0, \alpha = 1 = \lambda, a_1 = -m_1 - \mu, b_1 = m_1 - \mu$ and $b_2 = -m_1 - \mu$ are taken in it.

(6) The Meijer–Laplace transform due to Bhise (1) is arrived at on taking

$$q = p+1 = m, \quad n = 0, \quad \alpha = 1 = \lambda, \quad a_j = \eta_j + \alpha_j, \quad b_j = \alpha_j \quad (j = 1, 2, \ldots, p)$$

and $b_{p+1} = \rho$ in (1·6).

(7) On taking $\alpha = 1 = \lambda$ in (1·6), we get the generalized Laplace transform given by Kapoor (7).

(B) (8) On taking $\alpha = \frac{1}{4}$, $m = 1$, $n = 0 = p$, $q = 2 = \lambda$, $b_1 = \frac{1}{4} + \frac{1}{2}\nu$ and $b_2 = \frac{1}{4} - \frac{1}{2}\nu$ in (1·6), replacing $f(t)$ by $\sqrt{2} f(t)$ and using

$$2^\mu G^{10}_{02}(x | \tfrac{1}{2}\nu + \tfrac{1}{2}\mu, \tfrac{1}{2}\mu - \tfrac{1}{2}\nu) = x^\mu J_\nu(x), \tag{2·3}$$

we obtain the well-known Hankel transform (13).

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(9) The generalization of the Hankel transform as given by Roop Narain (11), viz.

$$\phi(s) = 2^{-\nu}\int_0^\infty (st)^{\nu+\frac{1}{2}}\chi_{\nu,k_1,m_1}(\tfrac{1}{4}s^2t^2)f(t)\,dt, \tag{2·4}$$

where
$$\chi_{\nu,k_1,m_1}(x) = x^{-\nu}G^{21}_{24}\left(x\left|\begin{matrix}k_1-m_1-\frac{1}{2}, \nu-k_1+m_1+\frac{1}{2}\\ \nu, \nu+2m_1, -2m_1, 0\end{matrix}\right.\right) \tag{2·5}$$

is arrived at on taking $\alpha=\frac{1}{4}$, $m=p=2=\lambda$, $n=1$, $q=4$, $a_1=k_1-m_1-\frac{1}{2}\nu-\frac{1}{4}$, $a_2=\frac{1}{2}\nu-k_1+m_1+\frac{3}{4}$, $b_1=2m_1+\frac{1}{2}\nu+\frac{1}{4}$, $b_2=\frac{1}{2}\nu+\frac{1}{4}$, $b_3=\frac{1}{4}-\frac{1}{2}\nu-2m_1$, $b_4=\frac{1}{4}-\frac{1}{2}\nu$, writing $\sqrt{2}f(t)$ for $f(t)$ in (1·6) and making a little simplification thereafter.

(10) Another generalization of the Hankel transform similar to (2·4) given by Mehra (9) comes out as a particular case of (1·6) on putting $\alpha=\frac{1}{4}$, $\lambda=2=m=p$, $n=1$, $q=4$, $a_1=4k_1-\frac{3}{2}\nu+\frac{1}{4}$, $a_2=\frac{1}{4}-\frac{1}{2}\nu$, $b_1=k_1-m_1-\frac{1}{2}\nu+\frac{1}{4}$, $b_2=k_1+m_1-\frac{1}{2}\nu+\frac{1}{4}$, $b_3=3k_1-m_1-\frac{3}{2}\nu+\frac{1}{4}$ and $b_4=3k_1+m_1-\frac{3}{2}\nu+\frac{1}{4}$ in it, writing $\sqrt{2}f(t)$ for $f(t)$ and using ((4), p. 208).

(11) On taking $m=v, n=u, p=2u, q=2v, \alpha=\frac{1}{4}, \lambda=2, a_j=\frac{1}{2}-a_i$ $(i=1,2,\ldots,u;$ $j=u+1,u+2,\ldots,2u)$ and $b_j=\frac{1}{2}-b_i$ $(i=1,2,\ldots,v;$ $j=v+1,v+2,\ldots,2v)$ in (1·6) and replacing $f(t)$ by $\sqrt{2}f(t)$, we get the following generalization of the Hankel transform given by Roop Narain (12):

$$\phi(s) = \sqrt{2}\int_0^\infty G^{v,u}_{2u,2v}\left(\tfrac{1}{4}s^2t^2\left|\begin{matrix}a_1,a_2,\ldots,a_u,\frac{1}{2}-a_1,\frac{1}{2}-a_2,\ldots,\frac{1}{2}-a_u\\ b_1,b_2,\ldots,b_v,\frac{1}{2}-b_1,\frac{1}{2}-b_2,\ldots,\frac{1}{2}-b_v\end{matrix}\right.\right)f(t)\,dt. \tag{2·6}$$

We shall call (1·6) the generalized L–H transform and $\phi(s)$, as usual, shall be called the transform or image of $f(t)$ and $f(t)$ the original of $\phi(s)$.

Before making a systematic study of the integral transform defined in (1·6) which shall be the subject of our subsequent work, we give in this paper an inversion formula for the transform (1·6). We conclude this paper by giving two examples supporting the inversion theorem.

3. *Inversion theorem.* If $\phi(s)$ is the generalized L–H transform of $f(t)$ as given by (1·6), then

$$\tfrac{1}{2}[f(t+)+f(t-)] = \frac{\lambda}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\frac{k-1}{\lambda}\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+\frac{1-k}{\lambda}\right)}{\prod_{j=1}^{m}\Gamma\left(b_j+\frac{1-k}{\lambda}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\frac{k-1}{\lambda}\right)}(\alpha)^{(1-k)/\lambda}t^{-k}F(k)\,dk, \tag{3·1}$$

where
$$F(k) = \int_0^\infty s^{-k}\phi(s)\,ds, \tag{3·2}$$

provided that

(i) the integrals
$$\int_0^\infty x^{c-1}f(x)\,dx \quad\text{and}\quad \int_0^\infty x^{-k}\phi(x)\,dx$$
are absolutely convergent ($k=c+iT$, $-\infty<T<\infty$),

(ii) $f(x)$ is of bounded variation in the neighbourhood of the point $x=t$ $(t>0)$,

(iii) $f(t)=o(t^{\mu_1})$, $\Re(\mu_1)>0$ for small values of t

$\quad\quad = o(e^{-\mu_2 t})$, $\Re(\mu_2)>0$ $(t\to\infty)$

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and

(iv) $0 \leqslant m \leqslant q,\ 0 \leqslant n \leqslant p,\ p+q < 2(m+n),\ |\arg \alpha t^{\lambda}| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,$

$$\Re(s) > 0, \quad -\min_{1\leqslant j\leqslant m} \Re(b_j) < \Re\left(\frac{1-k}{\lambda}\right) < 1-\max_{1\leqslant j\leqslant n} \Re(a_j)$$

and the generalized L–H transform of $|f(t)|$ exists.

However, if $f(t)$ is continuous at the point $x = t$, we obtain

$$f(t) = \frac{\lambda}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \frac{\prod_{j=m+1}^{q} \Gamma\left(1-b_j+\frac{k-1}{\lambda}\right) \prod_{j=n+1}^{p} \Gamma\left(a_j+\frac{1-k}{\lambda}\right)}{\prod_{j=1}^{m} \Gamma\left(b_j+\frac{1-k}{\lambda}\right) \prod_{j=1}^{n} \Gamma\left(1-a_j+\frac{k-1}{\lambda}\right)} t^{-k}(\alpha)^{(1-k)/\lambda} F(k)\, dk. \tag{3·3}$$

Proof. Multiplying both the sides of (1·6) by s^{-k} and integrating with respect to s between the limits 0 and ∞, we get

$$\begin{aligned}\int_0^{\infty} s^{-k}\phi(s)\, ds &= \int_0^{\infty} s^{-k}\, ds \int_0^{\infty} G_{p,q}^{m,n}\left(\alpha s^{\lambda} t^{\lambda} \middle| \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right) f(t)\, dt \\ &= \int_0^{\infty} f(t)\, dt \int_0^{\infty} s^{-k} G_{p,q}^{m,n}\left(\alpha s^{\lambda} t^{\lambda} \middle| \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right) ds \\ &= \frac{1}{\lambda} \int_0^{\infty} f(t)\, dt \int_0^{\infty} x^{\{(1-k)/\lambda\}-1} G_{p,q}^{m,n}\left(\alpha t^{\lambda} x \middle| \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right) dx,\end{aligned}$$

provided that the change in the order of integration is justifiable. Evaluating the later integral on the right with the help of a known result ((5), p. 418(3)), we find that

$$\int_0^{\infty} s^{-k}\phi(s)\, ds = \frac{\alpha^{(k-1)/\lambda}}{\lambda} \frac{\prod_{j=1}^{m} \Gamma\left(b_j+\frac{1-k}{\lambda}\right) \prod_{j=1}^{n} \Gamma\left(1-a_j+\frac{k-1}{\lambda}\right)}{\prod_{j=m+1}^{q} \Gamma\left(1-b_j+\frac{k-1}{\lambda}\right) \prod_{j=n+1}^{p} \Gamma\left(a_j+\frac{1-k}{\lambda}\right)} \int_0^{\infty} t^{k-1} f(t)\, dt, \tag{3·4}$$

provided that $p+q < 2(m+n)$, $|\arg \alpha t^{\lambda}| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$ and

$$-\min_{1\leqslant j\leqslant m} \Re(b_j) < \Re\left(\frac{1-k}{\lambda}\right) < 1-\max_{1\leqslant j\leqslant n} \Re(a_j).$$

Applying Mellin's inversion formula (13) in (3·4), we obtain (3·1), provided that conditions (i) and (ii) stated above are satisfied, which completes the proof.

To justify the change in the order of integration we see that the t-integral is absolutely convergent if the generalized L–H transform of $|f(t)|$ exists; the s-integral converges absolutely if

$$p+q < 2(m+n), \quad |\arg \alpha t^{\lambda}| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$$

and

$$-\min_{1\leqslant j\leqslant m} \Re(b_j) < \Re\left(\frac{1-k}{\lambda}\right) < 1-\max_{1\leqslant j\leqslant n} \Re(a_j)$$

and the resulting integral is absolutely convergent if $t^{k-1}f(t)$ belongs to $L(0,\infty)$.

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Hence the change in the order of integration is justified by De la Vallee Poussin's Theorem ((3), p. 204).

4. *Corollaries.* (i) If we take $m = 2 = q$, $p = 1$, $n = 0$, $a_1 = m_1 - k_1 + \frac{1}{2}$, $b_1 = 2m_1$, $b_2 = 0$ and $\alpha = 1 = \lambda$ in (3·1), we get

$$\tfrac{1}{2}[f(t+)+f(t-)] = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{\Gamma(m_1-k_1-k+\frac{3}{2})}{\Gamma(1-k)\,\Gamma(2m_1-k+1)}t^{-k}F(k)\,dk, \tag{4·1}$$

where

$$F(k) = \int_0^\infty s^{-k}\,\phi(s)\,ds, \tag{4·2}$$

a result given by Saxena (14) for Varma's transform.

(ii) On taking $m = 1 = q$, $\alpha = 1 = \lambda$, $p = 0 = n$ and $b_1 = 0$, the inversion formula reduces to

$$\tfrac{1}{2}[f(t+)+f(t-)] = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{t^{-k}}{\Gamma(1-k)}\int_0^\infty s^{-k}\,\phi(s)\,ds\,dk, \tag{4·3}$$

which is a well-known result (13) in the theory of Laplace transform.

(iii) When the substitutions $\alpha = \frac{1}{4}$, $\lambda = 2 = q$, $m = 1$, $p = 0 = n$, $b_1 = \frac{1}{4}+\frac{1}{2}\nu$ and $b_2 = \frac{1}{4}-\frac{1}{2}\nu$ are made in (3·1) and $\sqrt{2}f(t)$ is written for $f(t)$, we get

$$\sqrt{2}f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}2^k\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}k)\{\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}k)\}^{-1}t^{-k}F(k)\,dk,$$

which, on further simplification, reduces to the integral equation

$$f(t) = \int_0^\infty (st)^{\frac{1}{2}}J_\nu(st)\,\phi(s)\,ds, \tag{4·4}$$

known as the Hankel transform (13).

(iv) On taking $\lambda = 1 = \alpha$, $m = p+1 = q$, $n = 0$, $a_j = \eta_j+\alpha_j$, $b_j = \alpha_j$ $(j = 1, 2, \ldots, p)$ and $b_{p+1} = \rho$ in (3·1), a result given by Bhise (1) is arrived at.

5. *Examples.* (I) Let $f(t) = e^{-\beta t}$. Then

$$\phi(s) = \int_0^\infty e^{-\beta t}\,G_{p,q}^{m,n}\left(\alpha s^\lambda t^\lambda\,\middle|\,\begin{matrix}a_1, a_2, \ldots, a_p\\ b_1, b_2, \ldots, b_q\end{matrix}\right)dt.$$

Using a known result due to Saxena (15), we find that

$$\phi(s) = \frac{(2\pi)^{\frac{1}{2}(1-\lambda)}\lambda^{\frac{1}{2}}}{\beta}\,G_{p+\lambda,q}^{m,n+\lambda}\left(\frac{\alpha s^\lambda\lambda^\lambda}{\beta^\lambda}\,\middle|\,\begin{matrix}0, \frac{1}{\lambda}, \frac{2}{\lambda}, \ldots, \frac{\lambda-1}{\lambda}, (a_p)\\ b_1, b_2, \ldots, b_q\end{matrix}\right),$$

provided that

$$p+q < 2(m+n), \quad |\arg \alpha s^\lambda| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\,\pi,$$

$$|\arg\beta| < \tfrac{1}{2}\pi, \quad \lambda\,\Re(\min b_j) > -1 > \Re(\lambda a_i - \lambda - 1),$$

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for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$; λ is a positive integer and $(a_r) \equiv a_1, a_2, \ldots, a_r$. Hence

$$F(k) = \int_0^\infty s^{-k}\,\phi(s)\,ds$$

$$= \frac{(2\pi)^{\frac{1}{2}(1-\lambda)}\lambda^{\frac{1}{2}}}{\beta}\int_0^\infty s^{-k}G^{m,\,n+\lambda}_{p+\lambda,\,q}\left(\frac{\alpha s^\lambda\lambda^\lambda}{\beta^\lambda}\middle|\begin{matrix}0, \frac{1}{\lambda}, \frac{2}{\lambda}, \ldots, \frac{\lambda-1}{\lambda}, (a_p)\\ b_1, b_2, \ldots, b_q\end{matrix}\right)ds$$

$$= \frac{(2\pi)^{\frac{1}{2}(1-\lambda)}}{\beta\lambda^{\frac{1}{2}}}\int_0^\infty x^{\{(1-k)/\lambda\}-1}G^{m,\,n+\lambda}_{p+\lambda,\,q}\left(\frac{\alpha\lambda^\lambda}{\beta^\lambda}x\middle|\begin{matrix}0, \frac{1}{\lambda}, \frac{2}{\lambda}, \ldots, \frac{\lambda-1}{\lambda}, (a_p)\\ b_1, b_2, \ldots, b_q\end{matrix}\right)dx$$

$$= (2\pi)^{\frac{1}{2}(1-\lambda)}\frac{\prod_{j=1}^{m}\Gamma\left(b_j+\frac{1-k}{\lambda}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\frac{k-1}{\lambda}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\frac{k-1}{\lambda}\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+\frac{1-k}{\lambda}\right)}(\alpha)^{(k-1)/\lambda}\left(\frac{\lambda}{\beta}\right)^{k-1}$$

$$\times\Gamma\left(\frac{k+\lambda-1}{\lambda}\right)\Gamma\left(\frac{k+\lambda-2}{\lambda}\right)\ldots\Gamma\left(\frac{k+1}{\lambda}\right)\Gamma\left(\frac{k}{\lambda}\right),$$

provided that $p+q < 2(m+n)$, $|\arg\alpha\beta^{-\lambda}s^\lambda| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\,\pi$

and $$-\min_{1\leqslant j\leqslant m}\Re(b_j) < \Re\left(\frac{1-k}{\lambda}\right) < 1-\left[\max_{1\leqslant j\leqslant n}\Re(a_j), \max_{1\leqslant h\leqslant\lambda}\left(\frac{h-1}{\lambda}\right)\right].$$

Therefore

$$f(t) = \frac{\lambda}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\frac{k-1}{\lambda}\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+\frac{1-k}{\lambda}\right)}{\prod_{j=1}^{m}\Gamma\left(b_j+\frac{1-k}{\lambda}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\frac{k-1}{\lambda}\right)}(\alpha)^{(1-k)/\lambda}\,t^{-k}F(k)\,dk$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\lambda}{\beta}\right)^{k-1}(2\pi)^{\frac{1}{2}(1-k)}\lambda^{\frac{1}{2}}\Gamma\left(\frac{k}{\lambda}\right)\Gamma\left(\frac{k+1}{\lambda}\right)\ldots\Gamma\left(\frac{k+\lambda-1}{\lambda}\right)\beta^{-1}t^{-k}\,dk.$$

Now, using the multiplication theorem of Gauss (17)

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{\frac{1}{2}(n-1)}n^{\frac{1}{2}-nz}\Gamma(nz), \tag{5·1}$$

we find that $$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\beta t)^{-k}\,\Gamma(k)\,dk$$

$$= e^{-\beta t}.$$

(II) Let $f(t) = t^\sigma e^{-\frac{1}{2}t}W_{k_1,\,m_1}(t)$.

Then (15, p. 401)

$$\phi(s) = (2\pi)^{\frac{1}{2}(1-\lambda)}\lambda^{k_1+\sigma+\frac{1}{2}}G^{m,\,n+2\lambda}_{p+2\lambda,\,q+\lambda}\left(\alpha s^\lambda\lambda^\lambda\middle|\begin{matrix}\Delta(\lambda, m_1-\sigma-\frac{1}{2}), \Delta(\lambda, -m_1-\sigma-\frac{1}{2}), (a_p)\\ (b_m), \Delta(\lambda, k_1-\sigma-1), (b_{m+1,\,q})\end{matrix}\right), \tag{5·2}$$

provided that $p+q < 2(m+n)$, λ is a positive integer,

$$|\arg\alpha s^\lambda| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\,\pi$$

and $$\Re[(\sigma+m_1+\tfrac{1}{2}), (\sigma-m_1+\tfrac{1}{2})+\lambda\min b_j] > -1 > \Re[\lambda a_i-\lambda-1],$$

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for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, where $\Delta(A, \alpha)$ stands for the set of parameters

$$\frac{\alpha}{A}, \frac{\alpha+1}{A}, \ldots, \frac{\alpha+A-1}{A} \quad \text{and} \quad (a_r) \equiv a_1, a_2, \ldots, a_r;$$

$$(a_{r,s}) \equiv a_r, a_{r+1}, \ldots, a_s.$$

Hence

$$\begin{aligned}
F(k) &= \int_0^\infty s^{-k}\phi(s)\,as \\
&= (2\pi)^{\frac{1}{2}(1-\lambda)}\lambda^{k_1+\sigma+\frac{1}{2}}\int_0^\infty s^{-k}G^{m,\,n+2\lambda}_{p+2\lambda,\,q+\lambda} \\
&\qquad -\left(\alpha s^\lambda\lambda^\lambda\middle|\begin{matrix}\Delta(\lambda, m_1-\sigma-\frac{1}{2}), \Delta(\lambda, -m_1-\sigma-\frac{1}{2}), (a_p)\\ (b_m), \Delta(\lambda, k_1-\sigma-1), (b_{m+1,\,q})\end{matrix}\right) ds \\
&= (2\pi)^{\frac{1}{2}(1-\lambda)}\lambda^{k+k_1+\sigma-\frac{3}{2}\alpha^{(k-1)/\lambda}}\frac{\prod_{j=1}^{m}\Gamma\left(b_j+\frac{1-k}{\lambda}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\frac{k-1}{\lambda}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\frac{k-1}{\lambda}\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+\frac{1-k}{\lambda}\right)} \\
&\qquad \times\frac{\prod_{h=1}^{\lambda}\Gamma\left(\frac{k+m_1+\sigma+\frac{1}{2}}{\lambda}+\frac{h-1}{\lambda}\right)\prod_{h=1}^{\lambda}\Gamma\left(\frac{k-m_1+\sigma+\frac{1}{2}}{\lambda}+\frac{h-1}{\lambda}\right)}{\prod_{h=1}^{\lambda}\Gamma\left(\frac{k+\sigma-k_1+1}{\lambda}+\frac{h-1}{\lambda}\right)} \\
&= \frac{(\alpha^{k-1})^{1/\lambda}}{\lambda}\frac{\prod_{j=1}^{m}\Gamma\left(b_j+\frac{1-k}{\lambda}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\frac{k-1}{\lambda}\right)\Gamma_*(k\pm m_1+\sigma+\frac{1}{2})}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\frac{k-1}{\lambda}\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+\frac{1-k}{\lambda}\right)\Gamma(k+\sigma-k_1+1)}
\end{aligned}$$

on using ((5), p. 418) and (5·1), where $p+q < 2(m+n)$,

$$|\arg \alpha\lambda^\lambda| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q+\tfrac{1}{2}\lambda)\pi, \quad \Gamma_*(a\pm b) \equiv \Gamma(a+b)\,\Gamma(a-b)$$

and

$$-\min_{1\leqslant j\leqslant m}\Re(b_j) < \Re\left(\frac{1-k}{\lambda}\right) < 1-\max_{1\leqslant j\leqslant n}[\Re(a_j), \Delta(\lambda, m_1-\sigma-\tfrac{1}{2}), \Delta(\lambda, -m_1-\sigma-\tfrac{1}{2})].$$

Therefore, by Mellin's inversion theorem

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\lambda(\alpha)^{(1-k)/\lambda}\frac{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\frac{k-1}{\lambda}\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+\frac{1-k}{\lambda}\right)}{\prod_{j=1}^{m}\Gamma\left(b_j+\frac{1-k}{\lambda}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\frac{k-1}{\lambda}\right)}t^{-k}F(k)\,dk \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma_*(k\pm m_1+\sigma+\frac{1}{2})}{\Gamma(k+\sigma-k_1+1)}t^{-k}\,dk \\
&= G^{20}_{12}\left(t\middle|\begin{matrix}\sigma-k_1+1\\ \sigma+m_1+\frac{1}{2}, \sigma-m_1+\frac{1}{2}\end{matrix}\right) \\
&= t^\sigma e^{-\frac{1}{2}t}W_{k_1,\,m_1}(t) = f(t).
\end{aligned}$$

Hence the inversion theorem is verified.

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Note added in proof. Within the last few days I learned, in a private communication from Dr M. Newman, that he and J. Lehner, had already established the identity (1·5) for non-negative α, β by a somewhat different method in a paper not as yet published.

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