# 12

# Beta decay and gamma decay

In this chapter we present some of the theory of $\beta$-decay and $\gamma$-decay. In both cases, a fuller treatment requires more quantum mechanics than is usually contained in an undergraduate course, but we shall see that much of the experimental phenomena can be understood qualitatively without the complete relativistic theory.

## 12.1 What must a theory of $\beta$-decay explain?

In $\beta$-decay, introduced in §3.5, the charge of a nucleus changes while $A$ remains fixed. This occurs either by the simultaneous emission of an electron and an anti-neutrino, or a positron and a neutrino, or by the capture of an atomic electron with the emission of a neutrino. The appropriate stability conditions were discussed in §4.6. Several nuclei, for example $^{64}$Cu, can decay by any of these processes (Fig. 4.5). In electron capture, the neutrino energy and the recoil energy of the nucleus are sharply defined. In the other processes, the electron (or positron) can take any energy between zero and the maximum allowed by energy conservation. Figure 12.1 shows the experimentally determined energy spectra for electron emission and for positron emission from $^{64}_{29}$Cu. It was the observation of continuous energy distributions such as these that led Pauli to infer the existence of the neutrino in 1931: given that the energy levels of a nucleus are discrete, the electron and nuclear recoil energies in the centre-of-mass frame would by energy and momentum conservation be likewise discrete, unless some third particle (the neutrino) were present to share energy and momentum. The neutrino mass can be deduced to be small since, when the electron takes its maximum energy, the energy



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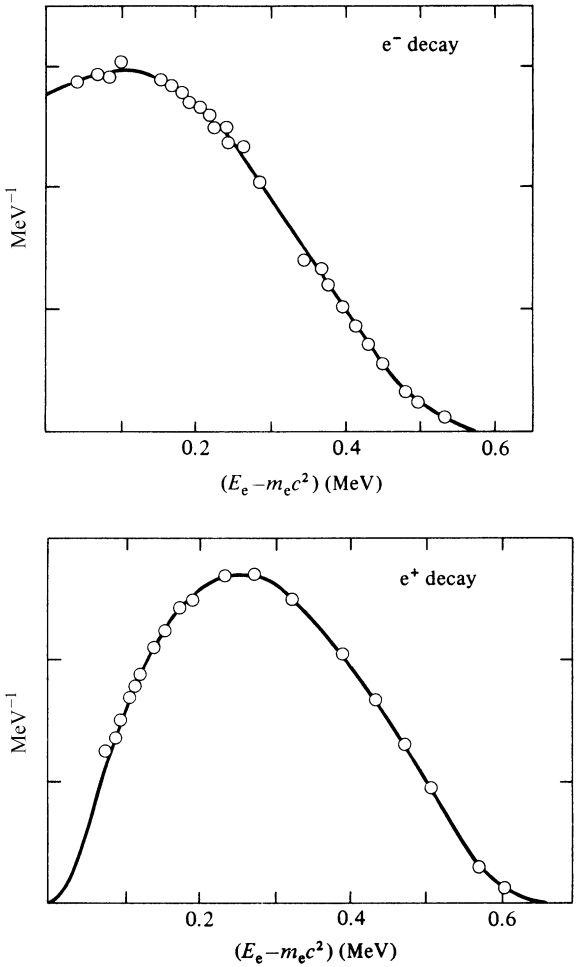



**Fig. 12.1** Electron and positron energy spectra from the $\beta$-decay of $^{64}$Cu, giving the probability distributions in energy (both normalised to unity) from a large sample of decays. The experimental points are from Langer, L. M. *et al.* (1949), *Phys. Rev.* **76**, 1725. The curves are fits to the data using equations (12.5) and (12.6).

balance to within the accuracy of present experiments is complete. Maximum electron energy corresponds to the neutrino carrying no momentum, so that the neutrino energy then would be its rest-mass energy $m_\nu c^2$. We consider recent experimental limits on $m_\nu$ in §13.2. Because the neutrino interacts so weakly with other particles, it was not

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until 1959 that its existence was more directly confirmed by the observation of the reaction

$$\bar{\nu}_e + p \rightarrow n + e^+,$$

using the high anti-neutrino flux associated with a nuclear reactor (which arises from $\beta$-decays of the neutron-rich fission fragments).

As in the case of $\gamma$-decay mean lives, $\beta$-decay mean lives span many orders of magnitude. For example, the most common isotope of indium, $^{115}_{49}$In, is $\beta$-unstable, but its mean life is $\sim 10^{14}$ years, whereas $\beta$-decay mean lives of the order of seconds or minutes are common. As with $\gamma$-decay, mean lives depend strongly on the nuclear-spin change in the decay.

The first experimental evidence for the violation of mirror symmetry at the subatomic level was found in $\beta$-decay by Wu in 1957, following a suggestion by Lee and Yang. The experiment measured the angular distributions of electrons from the decay of $^{60}$Co:

$$^{60}_{27}\text{Co} \rightarrow {}^{60}_{28}\text{Ni} + e^- + \bar{\nu}_e.$$

The $^{60}$Co nucleus, of spin $5\hbar$, has a large magnetic moment, and the nuclei in the sample were polarised by a magnetic field. The electrons from the decays were observed to be preferentially emitted in the opposite direction to the nuclear spins. Such a correlation violates the principle of mirror symmetry, since in the mirror image of this experiment, more electrons appear to be emitted in the same direction as the nuclear spin. An examination of Fig. 12.2 will make this clear. (*Any* mirror plane may

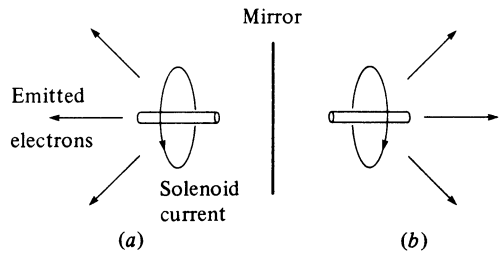



**Fig. 12.2** A schematic representation of the $^{60}$Co decay experiment in real space (*a*), and the mirror image of the experiment (*b*). In both (*a*) and (*b*) the spin of the cobalt nucleus is pointing to the right; the spin is a pseudo-vector which does not change direction under this reflection. The sample is polarised by the magnetic field produced by a current flowing in the direction indicated.

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be chosen, and will lead to the same conclusion.) Since the description of the experiment and of its mirror image differ, it follows that parity cannot be a symmetry of the weak interaction (§2.6). The breakdown of parity conservation in the decay of the muon, described in §2.6, was discovered shortly afterwards.

## 12.2 The Fermi theory of $\beta$-decay

A simple theory of $\beta$-decay was suggested by Fermi in 1934. Although this theory is incomplete (it does not allow for parity violation, for example), it is able to describe the spectra of Fig. 12.1, and gives a qualitative understanding of the range of $\beta$-decay mean lives.

To be specific, we consider in the shell model a decay in which a proton outside a doubly closed shell changes to a neutron:

$$^{17}_{9}\mathrm{F} \rightarrow {}^{17}_{8}\mathrm{O} + \mathrm{e}^{+} + \nu_{\mathrm{e}}.$$

A proton in the closed shell cannot change into a neutron since the neutron shell is also full and the Pauli principle forbids the transition. Thus the nucleons in the closed shells play no part in the decay and we can take the initial state of the system to be simply

$$\Psi_0 = \psi_{\mathrm{p}}(\mathbf{r}_{\mathrm{p}}),$$

where $\psi_{\mathrm{p}}$ is the state of the single proton in the $\mathrm{d}_{\frac{5}{2}}$ shell. The final state of the system consists of a neutron in the same shell, a positron $\mathrm{e}^{+}$, and a neutrino $\nu_{\mathrm{e}}$,

$$\Psi_{\mathrm{f}} = \psi_{\mathrm{n}}(\mathbf{r}_{\mathrm{n}})\psi_{\mathrm{e}}(\mathbf{r}_{\mathrm{e}})\psi_{\nu}(\mathbf{r}_{\nu}),$$

(in an obvious notation). Note that we are ignoring the spins of the particles involved and neglecting the recoil of the nucleus.

The transition rate from $\Psi_0$ to $\Psi_{\mathrm{f}}$ is given in perturbation theory by:

$$\text{transition rate} = \frac{2\pi}{\hbar}|H_{\mathrm{f0}}|^2 n_{\mathrm{f}}(E_0), \tag{12.1}$$

where $H_{\mathrm{f0}}$ is the matrix element linking the initial and final states, and $n_{\mathrm{f}}(E_0)$ is the density of (specified) states $\Psi_{\mathrm{f}}$ at the energy $E_0$ released in the decay. This result is obtained in Appendix D (equation (D.6)) and is often called 'Fermi's golden rule' in texts on quantum mechanics.

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We saw in Chapter 2 that the weak interaction responsible for $\beta$-decay, mediated by the heavy bosons $W^{\pm}$, is of very short range $\approx 2 \times 10^{-3}$ fm. This fact was anticipated by Fermi, who suggested that at the moment of interaction all particles were at the same point in space, so that the interaction matrix element

$$H_{\mathrm{f0}} = \int \Psi_{\mathrm{f}}^{*} H \Psi_0 \mathrm{d}^3\mathbf{r}_{\mathrm{n}} \mathrm{d}^3\mathbf{r}_{\mathrm{p}} \mathrm{d}^3\mathbf{r}_{\mathrm{e}} \mathrm{d}^3\mathbf{r}_{\mathrm{v}}$$

could be of the form

$$H_{\mathrm{f0}} = G_{\mathrm{w}} \int \psi_{\mathrm{n}}^{*}(\mathbf{r}) \psi_{\mathrm{e}}^{*}(\mathbf{r}) \psi_{\nu}^{*}(\mathbf{r}) \psi_{\mathrm{p}}(\mathbf{r}) \mathrm{d}^3\mathbf{r},$$

where the constant $G_{\mathrm{w}}$ is a measure of the strength of the weak interaction.

We may take for the neutrino a plane wave state

$$\psi_{\nu}(\mathbf{r}) = \frac{1}{V^{\frac{1}{2}}} \mathrm{e}^{\mathrm{i}\mathbf{k}_{\nu}\cdot\mathbf{r}_{\nu}},$$

with the wave-function normalised in an arbitrarily large volume $V$ for mathematical convenience. We take the positron wave-function to be also a plane wave

$$\psi_{e}(r) = \frac{1}{V^{\frac{1}{2}}} \mathrm{e}^{\mathrm{i}\mathbf{k}_{\mathrm{e}}\cdot\mathbf{r}_{\mathrm{e}}},$$

though this is only a first approximation; since a positron (or electron) is charged, its wave-function will be modified by the Coulomb field of the daughter nucleus. The matrix element $H_{\mathrm{f0}}$ becomes in this approximation

$$H_{\mathrm{f0}} = \frac{G_{\mathrm{w}}}{V} \int \psi_{\mathrm{n}}^{*}(\mathbf{r}) \psi_{\mathrm{p}}(\mathbf{r}) \mathrm{e}^{-\mathrm{i}(\mathbf{k}_{\mathrm{e}}+\mathbf{k}_{\nu})\cdot\mathbf{r}} \mathrm{d}^3\mathbf{r}.$$

The energies involved in $\beta$-decay are generally at most a few MeV, and the corresponding momenta $\hbar\mathbf{k}_{\mathrm{e}}$, $\hbar\mathbf{k}_{\nu}$ a few MeV/$c$. Hence the wave-vectors $\mathbf{k}_{\mathrm{e}}$, $\mathbf{k}_{\nu}$ are $\sim$MeV/$\hbar c \sim 10^{-2}$ fm$^{-1}$, and the exponent of the exponential in the integral is small over the range of the nuclear wave-functions. It is therefore an excellent approximation to expand the exponential to give

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$$H_{f0} = \frac{G_w}{V}\int \psi_n^*(\mathbf{r})\psi_p(\mathbf{r})d^3\mathbf{r} - \frac{iG_w}{V}(\mathbf{k}_e + \mathbf{k}_\nu)\cdot\int \psi_n^*(\mathbf{r})\mathbf{r}\psi_p(\mathbf{r})d^3\mathbf{r} + \cdots,$$

and keep only the first non-vanishing term. This argument is clearly also valid when the nuclear wave-functions involved are more complicated than in our simple example.

A decay is said to be allowed if the first term is finite. It is said to be first forbidden if the first term is zero as happens for example if the initial and final nuclear states are of opposite parity, but not the second, and so on. The diversity of $\beta$-decay rates is largely accounted for by the degree of forbiddenness of the transition and this in turn by the change in nuclear spin (as in $\gamma$-decay). In the case of indium, previously cited, the first term in the expansion not to vanish is found to be the fifth and the decay is fourfold forbidden.

We shall concentrate our discussion on allowed transitions, in which case the matrix element is

$$H_{f0} = (G_w/V)M_F, \tag{12.2}$$

where $M_F$ is the appropriate nuclear matrix element. In our example of the decay of $^{19}F$, the spatial shell model wave-functions of the proton and neutron are the same apart from Coulomb effects, and hence in this simplified theory

$$M_F = \int \psi_n^*(\mathbf{r})\psi_p(\mathbf{r})d^3\mathbf{r} \approx 1.$$

## 12.3 Electron and positron energy spectra

Consider an allowed $\beta$-decay in which the electron is emitted in a particular state of (relativistic) energy $E_e$. For simplicity we neglect the recoil energy of the daughter nucleus, which is in any case always a small correction. Then the relativistic neutrino energy $E_\nu$ is given by $E_0 = E_e + E_\nu$, where $E_0/c^2$ is the nuclear mass difference. The density of final states factor in the formula (12.1) is thus the density of neutrino states, $n_\nu(E_0 - E_e)$, at energy $(E_0 - E_e)$. There are $n_e(E_e)dE_e$ electron states with energies between $E_e$, $E_e + dE_e$, where $n_e$ is the density of

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electron states. Thus the total transition rate $\mathrm{d}R$ for decays to electron states with energies in the range $E_\mathrm{e}$, $E_\mathrm{e} + \mathrm{d}E_\mathrm{e}$, is

$$\mathrm{d}R = \frac{2\pi}{\hbar}|H_{\mathrm{f}0}|^2 n_\nu(E_0 - E_\mathrm{e}) n_\mathrm{e}(E_\mathrm{e})\mathrm{d}E_\mathrm{e}. \tag{12.3}$$

We need expressions for the densities of states $n_\nu$, $n_\mathrm{e}$. As is explained in Appendix B, neglecting spin there are $[V/(2\pi)^3]4\pi k^2 \mathrm{d}k$ plane wave states with $|\mathbf{k}|$ in the range $k$, $k + \mathrm{d}k$. For the relation between $E$ and $k$ we must use for both electrons and neutrinos the relativistic formula

$$E^2 = p^2c^2 + m^2c^4 = (\hbar k)^2c^2 + m^2c^4,$$

so that $E\,\mathrm{d}E = \hbar^2c^2k\,\mathrm{d}k$, $k = (E^2 - m^2c^4)^{\frac{1}{2}}/\hbar c$. Thus the relativistic density of states formula for a particle of mass $m$ is

$$n(E)\mathrm{d}E = \frac{V}{(2\pi)^3}\frac{4\pi}{\hbar^3c^3}(E^2 - m^2c^4)^{\frac{1}{2}}E\,\mathrm{d}E. \tag{12.4}$$

Note that $E$ is here the total energy, which includes the rest-mass energy. Substituting in equation (12.3) and using (12.2) for the matrix element of an allowed transition gives

$$\mathrm{d}R = \frac{G_\mathrm{w}^2|M_\mathrm{F}|^2}{2\pi^3\hbar^7c^6}S_0(E_\mathrm{e})\mathrm{d}E_\mathrm{e} \tag{12.5}$$

where

$$S_0(E_\mathrm{e}) = [(E_0 - E_\mathrm{e})^2 - m_\nu^2c^4]^{\frac{1}{2}}(E_0 - E_\mathrm{e})(E_\mathrm{e}^2 - m_\mathrm{e}^2c^4)^{\frac{1}{2}}E_\mathrm{e}.$$

The arbitrary normalisation volume $V$ cancels out from the final result (12.5), as we should expect. We have allowed the neutrino mass to be finite in order to discuss the direct experimental evidence for its small or zero value.

The electron (positron) energy dependence in the transition rate (12.5) comes *entirely* from the lepton density-of-state factors included in $S_0(E)$: the other factors are independent of electron energy. The formula can be improved by allowing for the interaction between the electron and the Coulomb field of the daughter nucleus of charge $Z_\mathrm{d}$. Since only the electron wave-function at the nucleus is important, $S_0(E_\mathrm{e})$ is modified to

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$$S_c(E_e) = F(Z_d, E_e) S_0(E_e), \tag{12.6}$$

where

$$F(Z_d, E_e) = \left| \frac{\psi_e(Z_d, 0)}{\psi_e(0, 0)} \right|^2,$$

and $\psi_e(Z_d, \mathbf{r})$ is the electron wave-function at energy $E_e$ in the Coulomb potential $\pm Z_d e^2/4\pi\varepsilon_0 r$. Extensive tables of $F(Z, e)$ are available for precise calculations, but a simple approximation is the non-relativistic formula

$$F(Z, E_e) = \frac{2\pi\eta}{1 - e^{-2\pi\eta}},$$

where $\eta = \pm Ze^2/(4\pi\varepsilon_0 \hbar v)$; the positive sign holds for electrons, the negative for positrons, and $v$ is the electron (positron) final velocity.

As $v \to 0$, $F(Z, E_e) \to 2\pi\eta$ for electrons. The $(1/v)$ factor makes $S_c(E_e)$ non-vanishing at the origin, where $E_e \to m_e c^2 + \frac{1}{2} m_e v^2$; the decay rate is enhanced at low energies since the Coulomb field for electrons is attractive.

For positrons at low energies $F(Z, E_e) \to 2\pi|\eta| e^{-2\pi|\eta|}$. The Coulomb field is repulsive for positrons and we can recognise the exponential as the tunnelling factor through the Coulomb barrier, which suppresses positron emission at low energies.

Figure 12.1 shows fits to the experimental statistical spectra for electron and positron emission from $^{64}$Cu. The Coulomb-corrected $S_c(E_e)$ give excellent agreement with the shapes of the experimental curves. The more detailed theory of $\beta$-theory retains this factor.

The total transition rate for a particular allowed decay is obtained by integrating the partial rate (12.5) over all electron energies, to give for the mean lifetime $\tau$ the formula

$$\frac{1}{\tau} = \frac{G_w^2 |M_F|^2 m_e^5 c^4}{2\pi^3 \hbar^7} f(Z_d, E_0),$$

where

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$$f(Z, E_0) = \left(\frac{1}{m_e c^2}\right)^5 \int_{mc^2}^{E_0} F(Z, E_e)(E_0 - E_e)^2 (E_e^2 - m_e^2 c^4)^{\frac{1}{2}} E_e \mathrm{d}E_e. \tag{12.7}$$

To obtain (12.7) we have set $m_\nu = 0$. $f(Z, E)$ is a dimensionless function for which again there are extensive tables. Some representative graphs are given in Fig. 12.3.

## 12.4 Electron capture

In an atomic environment, $\beta$-decay by electron capture always competes with positron emission, and is sometimes the only energetically allowed $\beta$-decay. To take our previous example, $^{17}_{9}\mathrm{F}$ can also decay by electron capture:

$$^{17}_{9}\mathrm{F} + \mathrm{e}^- \rightarrow {}^{17}_{8}\mathrm{O} + \nu_e.$$

The electron and proton wave-functions now constitute the initial state,

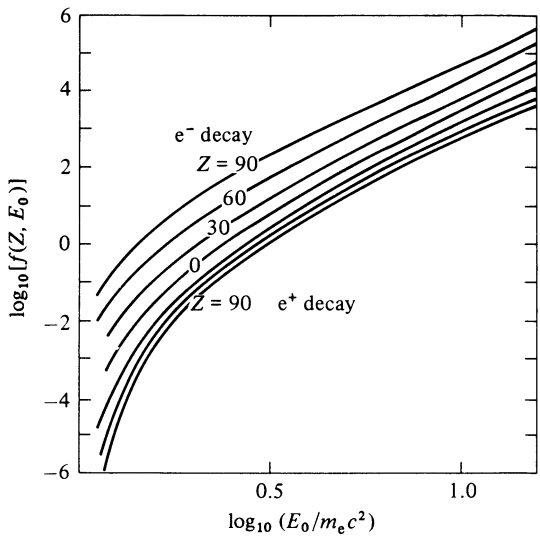



**Fig. 12.3** The function $f(Z, E_0)$. The sequence of curves is for $Z = 90$, 60, 30 and 0 for $e^-$ decay and continuing with $Z = 30$, 60, 90 for $e^+$ decay. (Formulae can be found in Feenberg, E. & Trigg, G. (1950), *Rev. Mod. Phys.* **22**, 399.)

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$$\Psi_0 = \psi_{\mathrm{p}}(\mathbf{r}_{\mathrm{p}})\psi_{\mathrm{e}}(\mathbf{r}_{\mathrm{e}}),$$

and the electron is most likely to be a K-shell electron, since K-shell wave-functions have the greatest overlap with the nucleus. To a good approximation, these wave-functions are hydrogen-like, little influenced by the outer shell atomic electrons, so we can take

$$\psi_e(\mathbf{r}) = \pi^{-\frac{1}{2}}\left(\frac{Zm_{\mathrm{e}}e^2}{4\pi\varepsilon_0\,\hbar^2}\right)^{\frac{3}{2}}\exp\left(-\frac{Zm_{\mathrm{e}}e^2 r}{4\pi\varepsilon_0\,\hbar^2}\right),$$

where $Z$ is the atomic number of the parent nucleus. The final state is

$$\Psi_{\mathrm{f}} = \psi_{\mathrm{n}}(\mathbf{r}_{\mathrm{n}})\psi_{\nu}(\mathbf{r}_{\nu}).$$

For the neutrino we again take a plane wave state $V^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ normalised in a volume $V$.

In the simple Fermi theory, we now have

$$H_{\mathrm{f}0} = G_{\mathrm{w}}\int \psi_{\mathrm{n}}^*(\mathbf{r})\psi_{\nu}^*(\mathbf{r})\psi_{\mathrm{p}}(\mathbf{r})\psi_{\mathrm{e}}(\mathbf{r})\mathrm{d}^3\mathbf{r}.$$

Assuming that the transition is allowed, this reduces to

$$H_{\mathrm{f}0} = \frac{G_{\mathrm{w}}}{V^{\frac{1}{2}}}\psi_{\mathrm{e}}(0)\int \psi_{\mathrm{n}}^*(\mathbf{r})\psi_{\mathrm{p}}(\mathbf{r})\mathrm{d}^3\mathbf{r} = \frac{G_{\mathrm{w}}}{V^{\frac{1}{2}}\pi^{\frac{1}{2}}}\left(\frac{Zm_{\mathrm{e}}e^2}{4\pi\varepsilon_0\,\hbar^2}\right)^{\frac{3}{2}}M_{\mathrm{F}},$$

since the electron and neutrino wave-functions can be treated as constant over the nuclear volume.

Neglecting the nuclear recoil, the emitted neutrino has energy $E_{\nu}$, where $E_{\nu}/c^2$ is the *atomic* mass difference $\approx (E_0/c^2 + m_{\mathrm{e}})$ (cf. (4.13)). The appropriate density of states in the formula for the transition rate is the neutrino density of states at this energy, given by equation (12.4). Setting $m_{\nu} = 0$,

$$n_{\nu}(E_{\nu}) = \frac{V}{(2\pi)^3}\frac{4\pi}{\hbar^3 c^3}E_{\nu}^2,$$

and we obtain

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$$\text{decay rate for electron capture} = \frac{2\pi}{\hbar}|H_{\mathrm{f0}}|^2 n_\nu(E_\nu) = \frac{G_{\mathrm{w}}^2 |M_{\mathrm{F}}|^2 E_\nu^2}{\pi^2 \hbar^4 c^3}\left(\frac{Z m_{\mathrm{e}} e^2}{4\pi\varepsilon_0 \hbar^2}\right)^3.$$

Consistently with our neglect of electron spin, only one K electron is included in the calculation. The ratio of the electron capture rate $R_{\mathrm{K}}$ to the positron emission rate $R_{\mathrm{e}^+}$ is independent of $G_{\mathrm{w}}$ and the nuclear matrix element, and is

$$\frac{R_{\mathrm{K}}}{R_{\mathrm{e}^+}} = 2\pi\left(\frac{E_\nu}{m_{\mathrm{e}}c^2}\right)^2\left(\frac{Z}{137}\right)^3\frac{1}{f(Z_{\mathrm{d}}, E_0)}.$$

Note that $E_\nu \approx E_0 + m_{\mathrm{e}}c^2$, and $e^2/4\pi\varepsilon_0\hbar c \approx \frac{1}{137}$. For low values of $Z$ this ratio is usually small, but at high $Z$ the $Z^3$ factor, and the increasing Coulomb barrier for positron emission which reduces $f(Z_{\mathrm{d}}, E_0)$, make electron capture the dominant process.

## 12.5 The Fermi and Gamow–Teller interactions

In the simple Fermi theory of $\beta$-decay, the interaction matrix element was written as a 'contact' interaction. For our example of $^{17}$F decay,

$$H_{\mathrm{f0}} = \int \Psi_{\mathrm{f}}^* H \Psi_{\mathrm{i}} \mathrm{d}(\text{coordinates}) = G_{\mathrm{w}} \int \psi_{\mathrm{n}}^*(\mathbf{r})\psi_{\mathrm{e}}^*(\mathbf{r})\psi_\nu^*(\mathbf{r})\psi_{\mathrm{p}}(\mathbf{r})\mathrm{d}^3\mathbf{r},$$

which we might represent diagrammatically as in Fig. 12.4. Reference to spin has been suppressed, though we know that the particles involved are all fermions with intrinsic spin quantum number $s = \frac{1}{2}$.

In the full theory of $\beta$-decay, the interaction is mediated by the charged W bosons, so that the process above is represented by Fig. 12.5. At a more fundamental level, the interaction is with a quark rather than a nucleon, as in Fig. 3.3, but phenomenologically the principal missing feature of the simple Fermi theory is the description of spin effects. We shall now describe how the results of the previous section are modified when spin is taken into account. The nucleon states in our example can still be described non-relativistically, but in general include 'spin-up' and 'spin-down' contributions (see Appendix C, §C.2):

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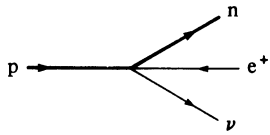



**Fig. 12.4** $\beta$-decay of a proton in a nucleus as a 'contact' interaction.

$$\psi(\mathbf{r}) = \psi_+(\mathbf{r})\begin{pmatrix}1\\0\end{pmatrix} + \psi_-(\mathbf{r})\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\psi_+\\\psi_-\end{pmatrix},$$

and hence are two-component wave-functions. The complex conjugate wave-function generalises to the adjoint row matrix $\psi^\dagger(\mathbf{r}) = (\psi_+^*, \psi_-^*)$. However, both the positron and neutrino move at relativistic speeds and for these the relativistic wave-functions must be used. Except in terms of the Dirac wave-functions, there is no simple form for the lepton part of the matrix element.

The contribution to the interaction from the Coulomb-like part of the W-field is most like the simple Fermi theory discussed in the previous sections and it is called the Fermi interaction. This part does not change the nucleon spins, and for allowed transitions the positron and neutrino angular momenta must combine to give a total lepton angular momentum of zero.

The contribution of the Fermi interaction to the interaction matrix element is

$$H_{\mathrm{f0}}^{\mathrm{F}} = G_{\mathrm{w}} \int \psi_{\mathrm{n}}^\dagger(\mathbf{r})\psi_{\mathrm{p}}(\mathbf{r})\mathrm{d}^3\mathbf{r} \times (\text{lepton part}).$$

The subtlety of the weak interaction is contained in the bracketed lepton part. This involves the neutrino and positron wave-functions eval-

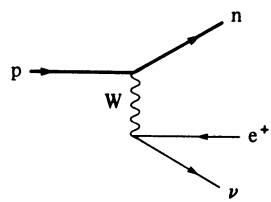



**Fig. 12.5** $\beta$-decay of a proton in a nucleus mediated by the exchange of a virtual W boson.

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uated at the nucleon coordinate **r**, as in the simple Fermi theory, but also describes the alignment of the neutrino and positron spins and the angular correlation between their directions. (The neutrino direction can be inferred by measuring the small nuclear recoil.) The lepton part is given very precisely by the Standard Model of particle physics and books on this subject should be consulted for detailed calculations. However, an experiment which only measures the electron energy spectrum and does not distinguish these correlations corresponds to an averaging over directions and spins, and then the spectrum is given exactly as in the simple theory. If only the Fermi interaction contributes to the decay, the energy spectra, decay rates, mean lives and electron capture rates are given by the previous formulae, but with the nuclear matrix element given by

$$M_{\mathrm{F}} = \int \psi_{\mathrm{n}}^{\dagger}(\mathbf{r})\psi_{\mathbf{p}}(\mathbf{r})\mathrm{d}^{3}\mathbf{r}.$$

The constant $G_{\mathrm{w}}$ is given in terms of more fundamental constants of particle physics by

$$G_{\mathrm{w}} = G_{\mathrm{F}}V_{\mathrm{ud}}.$$

Here $G_{\mathrm{F}}$ is the *Fermi constant* and $V_{\mathrm{ud}}$ is an element of the '*Kobayashi–Maskawa matrix*', which appears in the Standard Model theory of the weak interaction between leptons and quarks. $G_{\mathrm{F}}$ may be determined experimentally from the decay rate of the muon and has the value

$$G_{\mathrm{F}} = 1.166\,39(2) \times 10^{-11}(\hbar c)^{3}\ \mathrm{MeV}^{-2}. \tag{12.8}$$

From a range of nuclear data, it is found that

$$V_{\mathrm{ud}} = 0.9744(10).$$

But this is not the whole story, even for allowed transitions. The magnetic-like part of the W-field leads to a term in the transition matrix element, known as the Gamow–Teller interaction, in which the total lepton angular momentum **J** has quantum number $j = 1$, and the nuclear part of the interaction (again treated non-relativistically) contains the Pauli operator $\boldsymbol{\sigma}$ (see Appendix C). There is a term $\boldsymbol{\sigma} \cdot \mathbf{J}$ in the interaction Hamiltonian. The Gamow–Teller matrix element for our $^{17}$F example is

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$$H_{\mathrm{f0}}^{\mathrm{GT}} = g_{\mathrm{A}} G_{\mathrm{w}} \int \psi_{\mathrm{n}}^{\dagger}(\mathbf{r}) \boldsymbol{\sigma} \psi_{\mathrm{p}}(\mathbf{r}) \mathrm{d}^3\mathbf{r} \cdot (\text{lepton part}),$$

where (lepton part) is now a vector and $\cdot$ denotes a scalar product.

If we define

$$\mathbf{M}_{\mathrm{GT}} = g_{\mathrm{A}} \int \psi_{\mathrm{n}}^{\dagger}(\mathbf{r}) \boldsymbol{\sigma} \psi_{\mathrm{p}}(\mathbf{r}) \mathrm{d}^3\mathbf{r} = (M_{\mathrm{GT}}^{x}, M_{\mathrm{GT}}^{y}, M_{\mathrm{GT}}^{z}),$$

and sum over all allowed decays to the $j = 0$ and the three $j = 1$ states, the total decay rate to electrons with energies in the range $E_{\mathrm{e}}$, $E_{\mathrm{e}} + \mathrm{d}E_{\mathrm{e}}$ is given by

$$\mathrm{d}R(E_{\mathrm{e}}) = \frac{G_{\mathrm{F}}^2 V_{\mathrm{ud}}^2}{2\pi^3 \hbar^7 c^6} [|M_{\mathrm{F}}|^2 + |M_{\mathrm{GT}}^{x}|^2 + |M_{\mathrm{GT}}^{y}|^2 + |M_{\mathrm{GT}}^{z}|^2] S_{\mathrm{c}}(E_{\mathrm{e}}) \mathrm{d}E_{\mathrm{e}},$$

and the mean life is given by

$$\frac{1}{\tau} = \frac{G_{\mathrm{F}}^2 V_{\mathrm{ud}}^2 m_{\mathrm{e}}^5 c^4}{2\pi^3 \hbar^7} [|M_{\mathrm{F}}|^2 + |M_{\mathrm{GT}}^{x}|^2 + |M_{\mathrm{GT}}^{y}|^2 + |M_{\mathrm{GT}}^{z}|^2] f(Z_{\mathrm{d}}, E_0). \tag{12.9}$$

An allowed decay may be pure Fermi, pure Gamow–Teller, or a mixture of both, depending on the details of the nuclear matrix elements. Note that the electron energy spectrum is independent of these details. In general, of course, the initial and final nuclear states which enter into $M_{\mathrm{F}}$ and $\mathbf{M}_{\mathrm{GT}}$ are more complicated than those of our $^{17}$F example. $M_{\mathrm{F}}$ and $\mathbf{M}_{\mathrm{GT}}$ always vanish if the initial and final nuclear states are of opposite parity, since $\boldsymbol{\sigma}$ is an axial vector. Thus there can be no parity change in the nuclear states in an allowed transition.

The *axial coupling constant* $g_{\mathrm{A}}$ which appears in the expressions above is a parameter of the theory. It is not a fundamental particle physics parameter. We shall see in §12.6 that $g_{\mathrm{A}} \approx 1.26$.

For a Fermi transition, the change $\Delta j$ in nuclear spin must be zero. For a Gamow–Teller transition, $\Delta j = 0$ or 1, by the rules for addition of angular momentum, except that $0 \rightarrow 0$ transitions are forbidden since the matrix element of $\boldsymbol{\sigma}$ vanishes between two spherically symmetrical states.

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## 12.6 The constants $V_{ud}$ and $g_A$

In the decay

$$^{14}_{8}\mathrm{O} \rightarrow ^{14}_{7}\mathrm{N}^{*} + \mathrm{e}^{+} + \nu_{\mathrm{e}}$$

the transition occurs, with 99.7% probability, to the first excited state of the daughter nucleus, which has spin and parity $0^{+}$. The even–even nucleus $^{14}_{8}$O also has spin and parity $0^{+}$, so that from the selection rules above the transition is allowed, and pure Fermi. Also, in the nuclear shell model, the nuclei differ only in that $^{14}$O has two protons in $1p_{\frac{1}{2}}$ states outside a $^{12}_{6}$C core, and $^{14}$N* has one proton and one neutron. Thus, because of the charge independence of the strong nuclear force and the smallness of the Coulomb effects in these light nuclei, the wave-functions of the initial and final nuclear states are very similar, and $|M_{\mathrm{F}}|^2 \approx 2$ (since either of the two protons in $1p_{\frac{1}{2}}$ states can decay). The energy $E_0 = 2.32$ MeV, $Z_{\mathrm{d}} = 7$ and $f(7, 2.32) = 42.8$. The measured mean life is 102 s. Thus from the formula (12.9) for the mean life we can calculate

$$G_{\mathrm{F}} V_{\mathrm{ud}}/(\hbar c)^3 = 1.16 \times 10^{-11}\ \mathrm{MeV}^{-2}.$$

From (12.8) we see immediately that $V_{\mathrm{ud}}$ is close to unity. This mean life measurement gave one of the first estimates of the particle physics parameter $V_{\mathrm{ud}}$.

The constant $g_{\mathrm{A}}$ is most directly determined from the lifetime of a free neutron, since there are then no uncertainities in the computation of nuclear wave-functions. Indeed, if we neglect recoil, the spatial parts of the initial neutron and final proton wave-functions are the same. Suppose the spin state of the neutron is $|+\frac{1}{2}\rangle$. If the proton spin state is $|+\frac{1}{2}\rangle$ then, using the properties of the $\boldsymbol{\sigma}$ matrices (Appendix C), $M_{\mathrm{F}} = 1$ and $\mathbf{M}_{\mathrm{GT}} = (g_{\mathrm{A}} G_{\mathrm{w}})(0, 0, 1)$. If the proton spin state is $|-\frac{1}{2}\rangle$, $M_{\mathrm{F}} = 0$ and $\mathbf{M}_{\mathrm{GT}} = (g_{\mathrm{A}} G_{\mathrm{w}})(1, i, 0)$. The neutron–proton mass difference gives $E_0 = 1.29$ MeV and $f(1, 1.29) = 1.6$. The total decay rate to all possible spin states is therefore, from equations (12.6) and (12.8), given by

$$\frac{1}{\tau} = \frac{1.6[1 + 3g_{\mathrm{A}}^2] G_{\mathrm{F}}^2 V_{\mathrm{ud}}^2 m^5 c^4}{2\pi^3 \hbar^7}.$$

The measured mean life of the neutron is 887 s, which yields $g_{\mathrm{A}} = 1.3$, using this formula. However, there are 'radiative corrections'

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to our simple expression and the present accepted value of $g_A$ is a little lower,

$$g_A = 1.26.$$

## 12.7 Electron polarisation

The lepton part of the interaction matrix element leads to angular correlations between the various spins and momenta of the four particles involved in a $\beta$-decay. These correlations can be detected in suitable experiments, as for example the spin-polarised $^{60}$Co experiment discussed in §12.1; the observed angular distribution of electrons in this experiment is in accord with the theory.

The non-parity conserving nature of the weak interaction is most clearly seen in the lepton states. All neutrinos are 'left-handed' and all anti-neutrinos 'right-handed'. The theory also predicts that in $\beta$-decay left-handed electrons are produced more copiously than right-handed electrons, whereas positrons produced in $\beta$-decay are predominantly right-handed. More precisely, the probability of an electron emitted with velocity $v$ being in a left-handed state (with intrinsic spin **s** anti-parallel to momentum **p**) is

$$P_L = \frac{1}{2}\left(1 + \frac{v}{c}\right),$$

and the probability of its being emitted in a right-handed state (with **s** parallel to **p**) is

$$P_R = \frac{1}{2}\left(1 - \frac{v}{c}\right).$$

Hence

$$P = \frac{P_R - P_L}{P_R + P_L} = -\frac{v}{c}$$

(and for positrons $P = +v/c$).

Figure 12.6 shows experimental measurements of $P$, the 'longitudinal polarisation', plotted against $v/c$ for a variety of $\beta$-decays. The complete polarisation of the neutrino and anti-neutrino can be regarded as a generalisation of this result, since $v = c$ for massless particles.

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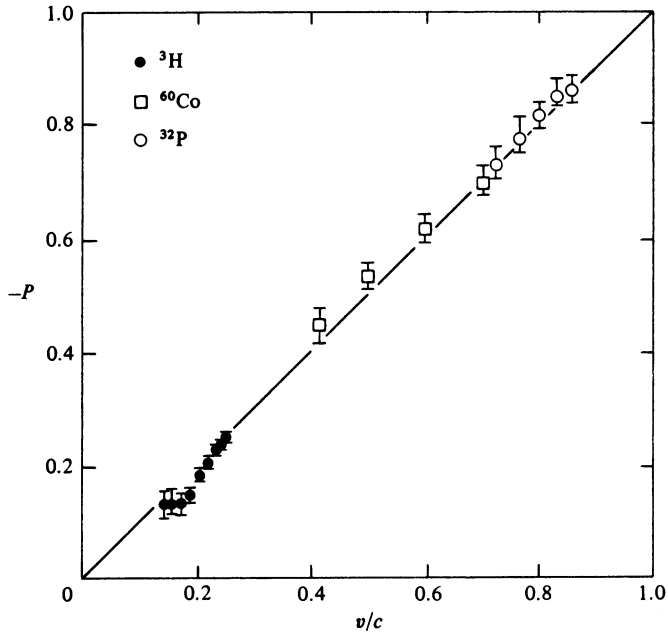



**Fig. 12.6** Measured degree of longitudinal polarisation $P$ for allowed $e^-$ decays. (Data from Koks, F. W. J. & van Klinken, J. (1976), *Nucl. Phys.* **A272**, 61.)

## 12.8 Theory of $\gamma$-decay

In $\gamma$-decay, a nucleus in an excited state falls to a lower state with the emission of a photon (§7.3). The electromagnetic interaction which governs this process is very well understood theoretically, but a full discussion requires the quantised equations of the electromagnetic field, rather than the classical Maxwell equations, and is beyond the scope of this book. However, we can understand the main features of $\gamma$-decay, and in particular the great range of $\gamma$-decay lifetimes described in §7.3, using semi-classical arguments to write down an approximate expression for the interaction energy between a nucleus and a photon.

We again enclose our system in a large volume $V$. Consider the plane electromagnetic wave

$$\mathbf{E} = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t), \quad \mathbf{B} = \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t).$$

The standard Maxwell theory tells us that in such a wave $|\mathbf{B}| = |\mathbf{E}|/c$, and the energy is divided equally between the electric and magnetic fields and is given in total by

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$$\varepsilon_0 \int \mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \mathrm{d}^3\mathbf{r} = \tfrac{1}{2}\varepsilon_0 \mathbf{E}_0^2 V,$$

since the cosine squared averages to $\frac{1}{2}$ over the volume $V$. If we identify this wave with a single photon of wave-vector $\mathbf{k}$ and energy $\hbar\omega$ we must therefore set

$$\tfrac{1}{2}\varepsilon_0 \mathbf{E}_0^2 V = \hbar\omega, \quad \text{or} \quad |\mathbf{E}_0| = (2\hbar\omega/\varepsilon_0 V)^{\frac{1}{2}}.$$

In a typical $\gamma$-decay, $\hbar\omega$ is at most a few MeV, so that $|\mathbf{k}| = \hbar\omega/\hbar c \sim (1\ \text{MeV})/(197\ \text{MeV fm}) \sim 10^{-2}\,\text{fm}^{-1}$. Hence to a good approximation we can neglect the change in $(\mathbf{k} \cdot \mathbf{r})$ over the dimensions of the nucleus (~fm), which we can take to be centred at $\mathbf{r} = 0$. The electric field over the nucleus is then

$$\mathbf{E} = \mathbf{E}_0 \cos\omega t = \tfrac{1}{2}\mathbf{E}_0(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}),$$

and the potential energy of the nucleus in such a field is given classically by

$$-e \sum_{\text{protons}} \mathbf{E} \cdot \mathbf{r}_\mathrm{p} = -\frac{e}{2} \sum_{\text{protons}} \mathbf{E}_0 \cdot \mathbf{r}_\mathrm{p}(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}).$$

In a $\gamma$-decay, we start with a state in which there is no photon present, and end with a state in which there is one photon present and the nucleus is in a lower energy state. As in our discussion of $\beta$-decay, we shall neglect the small nuclear recoil energy. It is clear from the derivation of the result (D.6) in Appendix D that only the term with $\mathrm{e}^{\mathrm{i}\omega t}$ in the interaction can contribute to this transition, so that the matrix element to be employed in the formula for the decay rate is

$$-\frac{e}{2}\int \Psi_\mathrm{f}^* \left( \sum_{\text{protons}} \mathbf{E}_0 \cdot \mathbf{r}_\mathrm{p} \right) \Psi_0 \mathrm{d}(\text{coordinates}) = -\frac{e}{2}\mathbf{E}_0 \cdot \mathbf{R}_{\mathrm{f}0},$$

where $\Psi_0$, $\Psi_\mathrm{f}$ are the initial and final nuclear states and

$$\mathbf{R}_{\mathrm{f}\mathbf{0}} = \int \Psi_\mathrm{f}^* \left( \sum_{\text{protons}} \mathbf{r}_\mathrm{p} \right) \Psi_0 \mathrm{d}(\text{coordinates}). \tag{12.10}$$

If $\mathbf{R}_{\mathrm{f}0}$ is non-vanishing the transition is said to be *electric dipole* (E1).

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Let us assume $\mathbf{R}_{f0}$ is non-vanishing and real. The treatment is easily extended to the case when $\mathbf{R}_{f0}$ is a complex vector $\mathbf{R}_1 + i\mathbf{R}_2$ since

$$|\mathbf{E}_0 \cdot (\mathbf{R}_1 + i\mathbf{R}_2)|^2 = |\mathbf{E}_0 \cdot \mathbf{R}_1|^2 + |\mathbf{E}_0 \cdot \mathbf{R}_2|^2.$$

For a given direction of photon emission, there are two independent photon states with polarisations which we can take as in the plane defined by $\mathbf{R}_{f0}$ and $\mathbf{k}$, and perpendicular to this plane (Fig. 12.7). For the latter $\mathbf{E}_0 \cdot \mathbf{R}_{f0} = 0$, so that the transition probability to this state vanishes.

If $\theta$ is the angle between the direction $\mathbf{k}$ and $\mathbf{R}_{f0}$, for the state with polarisation in the plane we have

$$|\mathbf{E}_0 \cdot \mathbf{R}_{f0}| = |\mathbf{E}_0||\mathbf{R}_{f0}| \sin\theta$$

since $\mathbf{E}_0$ is perpendicular to $\mathbf{k}$. The density of states at energy $E_\gamma = \hbar\omega$, for photons emitted in a solid angle $d\Omega = \sin\theta\, d\theta\, d\phi$, is

$$\frac{V}{(2\pi)^3} k^2 \frac{dk}{dE_\gamma} d\Omega = \frac{V}{(2\pi)^3} \frac{\omega^2}{\hbar c^3} \sin\theta\, d\theta\, d\phi,$$

since $E_\gamma = \hbar\omega = \hbar c k$. Hence the 'Fermi golden rule' formula gives the transition rate

$$\frac{2\pi}{\hbar} \frac{e^2}{4} |\mathbf{E}_0|^2 |\mathbf{R}_{f0}|^2 \frac{V}{(2\pi)^3} \frac{\omega^2}{\hbar c^3} \sin^2\theta\, d\Omega. \tag{12.11}$$

There is a characteristic $\sin^2\theta$ angular distribution of the emitted photons. Such angular distributions can be observed experimentally, if for example the nuclei in a sample are oriented in the same direction by a strong magnetic field.

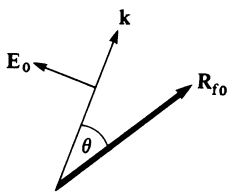



**Fig. 12.7** Direction of emission $\mathbf{k}$ and polarisation vector $\mathbf{E}_0$ for an allowed electric dipole transition.

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The mean life is obtained by integrating the expression (12.11) over all directions in space. Using $|\mathbf{E}_0| = (2\hbar\omega/\varepsilon_0 V)^{\frac{1}{2}}$ we obtain

$$\begin{aligned}\frac{1}{\tau_{E1}} &= \frac{4}{3}\left(\frac{e^2}{4\pi\varepsilon_0}\right)\frac{\omega^3}{\hbar c^3}|\mathbf{R}_{\mathrm{f0}}|^2 \\ &= \left(\frac{E_\gamma}{1\ \mathrm{MeV}}\right)^3\left(\frac{|\mathbf{R}_{\mathrm{f0}}|}{1\ \mathrm{fm}}\right)^2 \quad 0.38\times 10^{15}\ \mathrm{s}^{-1},\end{aligned} \tag{12.12}$$

where the last form indicates the order of magnitude to be expected for the mean lives.

From equation (12.10), we see that $\Psi_0$ and $\Psi_\mathrm{f}$ must be of opposite parity for an electric dipole transition to take place, since if they have the same parity the integral vanishes. It can also be shown from the angular part of the integration, using the properties of spherical harmonics, that the change $\Delta j$ in the nuclear spin quantum number for an electric dipole transition must be $\Delta j = 0$ or $\Delta j = \pm 1$, except that $0 \to 0$ transitions are forbidden (see §7.3). An estimate of the magnitude of $|\mathbf{R}_{\mathrm{f0}}|$ requires a knowledge of the nuclear wave-functions. Even in the simple shell model such calculations are not easy.

The nucleus also couples to the magnetic field of the photon, and at a similar level of approximation the interaction with the magnetic field $\mathbf{B}$ is given classically by $-\boldsymbol{\mu}\cdot\mathbf{B}_0\cos\omega t$, where $\boldsymbol{\mu}$ is the total magnetic moment of the nucleus. The magnetic moment operator is given in the simple shell model by equation (5.24),

$$\boldsymbol{\mu} = \sum_{\text{nucleons}} \mu_\mathrm{N}[g_L\mathbf{L} + g_\mathrm{s}\mathbf{s}]/\hbar,$$

where $\mu_\mathrm{N} = e\hbar/2m_\mathrm{p}$ is the nuclear magneton. The transition rate induced by this interaction will be of the same form as (2.11), with $e\mathbf{E}_0\cdot\mathbf{R}_{\mathrm{f0}}$ replaced by $\mathbf{B}_0\cdot\mathbf{M}_{\mathrm{f0}}$ where

$$\mathbf{M}_{\mathrm{f0}} = \int \Psi_\mathrm{f}^*\boldsymbol{\mu}\Psi_0 \mathrm{d(coordinates)}. \tag{12.13}$$

If $\mathbf{M}_{\mathrm{f0}}$ is non-vanishing, the transition is said to be *magnetic dipole* (M1). Since $|\mathbf{B}_0| = |\mathbf{E}_0|/c$, the mean life is given by

$$\frac{1}{\tau_{\mathrm{M1}}} = \frac{4}{3}\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{\omega^3}{\hbar^2 c^5}|\mathbf{M}_{\mathrm{f0}}|^2. \tag{12.14}$$

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From (12.13), $\mathbf{M}_{f0}$ is non-vanishing only if $\Psi_0$ and $\Psi_f$ have the same parity, since $\boldsymbol{\mu}$ is a pseudo-vector (§2.6). Electric and magnetic dipoles transitions are therefore mutually exclusive. The angular momentum selection rules, $\Delta j = 0, \pm 1 (0 \to 0$ forbidden) are the same as for electric dipole transitions.

The ratio of mean lives for magnetic and electric dipole transitions at the same energy is

$$\frac{\tau_{M1}}{\tau_{E1}} = \frac{e^2 c^2 |\mathbf{R}|^2}{|\mathbf{M}|^2}.$$

If we take $|\mathbf{R}| \sim$ nuclear radius $\sim A^{\frac{1}{3}}$ fm, and $M \sim e\hbar/m_p$, we obtain

$$\frac{\tau_{M1}}{\tau_{E1}} \sim \frac{(m_p c^2)^2 (A^{\frac{2}{3}} \text{ fm}^2)}{(\hbar c)^2} \sim 20 A^{\frac{2}{3}}.$$

Thus the mean lives for magnetic dipole transitions are generally longer than those of electric dipole transitions at the same energy by a considerable factor, though this estimate is of course very crude.

If both $\mathbf{R}_{f0}$ and $\mathbf{M}_{f0}$ vanish, as is not uncommon, then we can no longer neglect the variation in the photon field over the dimensions of the nucleus. The expansion of $\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$ in powers of $(\mathbf{k}\cdot\mathbf{r})$ gives matrix elements for higher-order electric and magnetic transitions. Each power of $(\mathbf{k}\cdot\mathbf{r})$ introduces an additional factor of $-1$ in the parity selection rule, and an additional unit of orbital angular momentum in the $\Delta j$ selection rule so that, for example, for electric quadrupole transitions there is no change in parity and $\Delta j = \pm 2, \pm 1, 0$, except that $0 \to 0$ and $\frac{1}{2} \to \frac{1}{2}$ transitions are both forbidden, by conservation of angular momentum. To each type of transition there corresponds a characteristic angular dependence and polarisation of emitted $\gamma$-rays.

Each power of $(\mathbf{k}\cdot\mathbf{r})$ reduces the order of magnitude of the matrix element by a factor $\sim (kR)$, where $R$ is the nuclear radius, and hence increases the lifetime by a factor of $(kR)^{-2}$. For a 1 MeV photon and $A \sim 50$, $(kR)^{-2} = 0.24 \times 10^4$. The curves of Fig. 7.6 have been drawn using only a more sophisticated version of this argument, but they are nevertheless a useful guide to the interpretation of experimental lifetimes.

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## 12.9 Internal conversion

A nucleus in an excited state can also decay electromagnetically by 'internal conversion'. In this process, an atomic electron in a state $\phi_0(\mathbf{r}_e)$ takes up the energy released in the decay and is excited to a state $\phi_f(\mathbf{r}_e)$ which must be initially empty. If the energy release is greater than the binding energy of the electron, as is usually the case, the electron is ejected from the atom and the state $\phi_f(\mathbf{r}_e)$ may be approximated by a plane wave.

Thus the initial state is of the form

$$\Psi_0 = \psi_0^{\text{nuc}}\phi_0(\mathbf{r}_e)$$

and the final state of the form

$$\Psi_f = \psi_f^{\text{nuc}}\phi_f(\mathbf{r}_e).$$

The main contribution to the interaction energy between the electron and the nucleus is the Coulomb energy

$$\sum_{\text{protons}} \frac{-e^2}{4\pi\varepsilon_0|\mathbf{r}_p - \mathbf{r}_e|}$$

and the corresponding matrix element for the transition is

$$H_{f0} = \int \psi_f^{\text{nuc}^*}\phi_f^* \sum_{\text{protons}} \frac{-e^2}{4\pi\varepsilon_0|\mathbf{r}_p - \mathbf{r}_e|} \times \psi_0^{\text{nuc}}\phi_0 \mathrm{d}^3\mathbf{r}_e \mathrm{d}(\text{nuclear coordinates}).$$

We shall not pursue the evaluation of this matrix element, but note that it can be non-vanishing for $0 \rightarrow 0$ transitions between nuclear states of the same parity.

The process of internal conversion always competes with $\gamma$-decay with similar nuclear matrix elements appearing. As in the case of K-capture in $\beta$-decay, there is a factor $Z^3$ in the transition rate arising from the normalisation of the initial state electron wave-function. Thus internal conversion becomes increasingly significant in the electromagnetic decays of the heavier elements. The *internal conversion coefficient* is defined as the ratio of the rate of internal conversion to the rate of $\gamma$-emission, for a given type of electromagnetic transition. Extensive tables of these coefficients can be found in the literature.

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## Problems

12.1 The product of a $\beta$-decay half life $T_{\frac{1}{2}}$ (§2.4) and the number $f(Z_d, E_0)$ is the '$fT_{\frac{1}{2}}$' value. From equations (12.7) and (12.9) the $fT_{\frac{1}{2}}$ value gives a direct empirical determination of the nuclear matrix element $|M_F|^2 + |\mathbf{M}_{GT}|^2$. Calculate the $fT_{\frac{1}{2}}$ value for the decay ${}^{31}_{16}\mathrm{S} \rightarrow {}^{31}_{15}\mathrm{P} + \mathrm{e}^+ + \nu$, for which $T_{\frac{1}{2}} = 2.60$ s, $E_0 = 4.94$ MeV and $f(15, 4.94) = 1830$. In the simple shell model, this decay involves a $2s_{\frac{1}{2}}$ proton changing to a $2s_{\frac{1}{2}}$ neutron. Compare this $fT_{\frac{1}{2}}$ value with that of a free neutron. Why do the two values differ?

12.2 The cross-section for the reaction

$$\bar{\nu}_e + p \rightarrow n + e^+$$

is given in perturbation theory by

$$\sigma = \frac{2\pi}{\hbar} \frac{1}{(\text{neutrino flux})} |H_{f0}|^2 n(E_e),$$

where $n(E_e)$ is the relativistic density of states (equation (12.4)). Show that

$$\sigma \approx \frac{G_w^2}{\pi(\hbar c)^4}[1 + 3g_A^2](cp_e)E_e.$$

Calculate this cross-section for a 2 MeV anti-neutrino.

12.3 If an electric dipole (E1) decay mean life is known, then equation (12.12) can be used to calculate the corresponding dipole matrix element $|\mathbf{R}_{f0}|$. The first excited state of $^{11}$Be decays to the ground state through an E1 transition. The mean life is $1.79 \times 10^{-13}$ s and the photon energy is 0.32 MeV. Calculate $|R_{f0}|$.

An example of an electric dipole transition in atomic physics is the decay of the 2p excited state of the hydrogen atom, for which the mean life is $1.6 \times 10^{-9}$ s and the photon energy is 10.2 eV. Calculate $|\mathbf{R}_{f0}|$ and compare it with the nuclear matrix element.

12.4 The nucleus ${}^{108}_{47}\mathrm{Ag}$, which has spin and parity $1^+$, is $\beta$-unstable with a mean life of 3.4 minutes. It has an excited state at 109 keV excitation energy, spin and parity $6^+$, which is an isomeric state with a mean life of 180 years. Explain how an excited state of a nucleus can be more stable than the ground state.

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