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Small Riesz spaces

BY G. BUSKES
University of Mississippi, Mississippi MS 38677, *U.S.A.*
AND A. VAN ROOIJ
Katholieke Universiteit, 6525 *ED Nijmegen, The Netherlands*

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Many facts in the theory of general Riesz spaces are easily verified by thinking in terms of spaces of functions. A *proof* via this insight is said to use representation theory. In recent years a growing number of authors has successfully been trying to bypass representation theorems, judging them to be extraneous. (See, for instance, [**9**, **10**].) In spite of the positive aspects of these efforts the following can be said. Firstly, avoiding representation theory does not always make the facts transparent. Reading the more cumbersome constructions and procedures inside the Riesz space itself one feels the need for a pictorial representation with functions, and one suspects the author himself of secret heretical thoughts. Secondly, the direct method leads to repeating constructions of the same nature over and over again.

Is there a more fundamental reason for not taking the short road? An often-heard answer to this question is: the desire to avoid unnecessary use of the Axiom of Choice. Indeed, the two most widely applied representation theorems for Riesz spaces, the Kakutani Theorem for order unit spaces and the Maeda-Ogasawara Theorem, are based upon transcendental arguments.

However, as a rule of thumb, real constructions involve only a 'small' (countable) number of elements. For such constructions the full Axiom of Choice can sometimes be avoided by using representation theorems for 'small' Riesz spaces. In this paper we present such a technique.

In the context of analysis the Countable Axiom of Choice is no real burden. (Also, it seems that one can always avoid it by careful manipulation of the definitions; see [**5**].) When we started working we were quite willing to accept it. Since, however, only very little extra work turned out to be needed we have also eliminated the Countable Axiom of Choice from our proofs. At a few points this may strike the reader as strange, in particular when he finds a fact spelled out of which he has long been aware.

In the light of the above this paper may be seen as a new and unified approach to theorems that are already known. Nonetheless, our main results (Theorems 2·2, 3·6, 4·1, 4·2) are new and contain the fundamentals of much of the available knowledge on the algebraic part of Riesz space theory (in particular concerning f-algebras and orthomorphisms).

The paper has a second purpose: it also serves as the underpinning of a forthcoming article [**7**] where we investigate the logical status of some well-known statements in the theory of Riesz spaces. More precisely, we intend to show that the Boolean Prime Ideal Theorem is equivalent to the existence of maximal ideals in a Riesz space with

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order unit. This in turn will lead to a new proof of the Kakutani Representation Theorem, avoiding the Axiom of Choice in its full strength, which solves a problem that arises in reading [11], p. 168. Our method there will be to 'approximate' a Riesz space with 'small' Riesz subspaces. For that, we need a representation theorem for small Riesz spaces and a constructive proof for a generalization of Kutateladze's characterization of Riesz homomorphisms (cf. theorem 8·16 in [2]). These two results are to be found in the present paper (2·2 and 4·2). Admittedly, we could have obtained them by a shorter path, but we will get some unexpected bonuses.

This brings us to a more detailed discussion of the content. After introducing the needed terminology in §0 we prove the Representation Theorem for small Riesz spaces (Theorem 2·2). It is based on an extension procedure for Riesz homomorphisms which itself goes back to an argument by Pták [23] (and Sikorski [25]) for a theorem due to Mazur and Orlicz. (See also [26].) Theorem 2·3 is a variant of Kakutani's Representation Theorem for small Riesz spaces.

Theorem 4·1 is an extension theorem for orthomorphisms related to the Lipecki-Luxemburg-Schep theorem for Riesz homomorphisms. It strengthens theorem 8·12 of [2]. (Perhaps the theorem can be effectively used in combination with results in chapter VII of [21] and results of [28]; also note theorem 8·15 in [2].)

Finally, there is the generalization of Kutateladze's Theorem, mentioned above.

0. *Notation*

Our notation concerning the general Riesz space theory is derived from [2].

0·1. Let E be an Archimedean Riesz space. A function $p: E \to (-\infty, \infty]$ is called *sublinear* if for $x, y \in E$

$$p(x+y) \leqslant p(x)+p(y),$$

$$p(\lambda x) = \lambda p(x) \quad (\lambda \in \mathbb{R}^+)$$

with $0 . \infty = 0$. An *extended M-seminorm* is a sublinear $p: E \to [0, \infty]$ satisfying

$$p(x \vee y) = p(x) \vee p(y) \quad (x, y \in E^+),$$

$$p(|x|) = p(x) \quad (x \in E).$$

An *M-seminorm* is an extended M-seminorm that does not take the value ∞.

Let $a \in E$. The formula

$$\|x\|_a := \inf\{\lambda \in \mathbb{R}^+ : |x| \leqslant \lambda |a|\} \quad (x \in E)$$

(with $\inf \varnothing := \infty$) defines an extended M-seminorm $\| \ \|_a$ on E. For $c \in E$ and $\epsilon \in (0, \infty)$ put

$$B_a(c; \epsilon) := \{x \in E : \|x-c\|_a < \epsilon\}.$$

A subset S of E is *a-open* if for every $c \in S$ there exists an $\epsilon > 0$ with $B_a(c; \epsilon) \subset S$.

For $A \subset E$, a subset S of E is *A-open* if it is a-open for all $a \in A$. The A-open sets form a topology in E, the *A-topology*. It is clear what we will mean by terms like 'a-closed', 'A-closure' and 'a-continuous'. The E-topology is the relatively uniform topology of [18].

If D is an A-closed Riesz subspace of E that contains A, then the A-topology of D is just the relative topology induced by the A-topology of E. (If $S \subset D$ is A-open in D, then $S \cup (E \backslash D)$ is A-open in E.)

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0·2. In the sequel we will have need of the concept of uniform completeness of E. In order to avoid the Countable Axiom of Choice we employ a definition that slightly differs from the usual one (although in the presence of the Countable Axiom of Choice the two are equivalent). We call E *uniformly complete* if the following is true: if $a \in E$ and if Σ is a filter of subsets of E such that for every $\epsilon \in (0, \infty)$ there is a $c \in E$ with $B_a(c;\epsilon) \in \Sigma$, then there exists a $c \in E$ such that $B_a(c;\epsilon) \in \Sigma$ for all $\epsilon \in (0, \infty)$.

We need one more definition: a $\mathbb{Q}$-*linear sublattice* of E is a subset of E that is both a sublattice and a $\mathbb{Q}$-vector subspace.

1. *Slender Riesz spaces*

LEMMA 1·1. *Let E be an Archimedean Riesz space, A a $\mathbb{Q}$-linear sublattice of E, and $\bar{A}$ the A-closure of A in E. Then*

(i) *$\bar{A}$ is a Riesz subspace of E, contained in the Riesz ideal generated by A. Therefore the A-topology coincides with the $\bar{A}$-topology.*

(ii) *Let D be an A-closed Riesz subspace of E containing A (e.g. $D = \bar{A}$). Then the A-closure of A in D is just $\bar{A}$.*

(iii) *If E is uniformly complete, then so is $\bar{A}$ (although $\bar{A}$ may not be relatively uniformly closed in E).*

(iv) *Every Riesz homomorphism $A \to \mathbb{R}$ extends uniquely to a Riesz homomorphism $\bar{A} \to \mathbb{R}$.*

Proof. (i) If $\Phi : E \to E$ and $\alpha \in \mathbb{R}^+$ are such that

$$|\Phi(x) - \Phi(y)| \leqslant \alpha |x - y| \quad (x, y \in E),$$

then Φ is a-continuous for every a in E; hence,

$$\text{if} \quad \Phi(A) \subset \bar{A}, \quad \text{then} \quad \Phi(\bar{A}) \subset \bar{A}. \tag{$*$}$$

For example, if $x \in A$, then $x + A \subset A \subset \bar{A}$; hence $x + \bar{A} \subset \bar{A}$. Consequently, if $y \in \bar{A}$, then $A + y \subset \bar{A}$, hence $\bar{A} + y \subset \bar{A}$. Thus $\bar{A} + \bar{A} \subset \bar{A}$. If $\lambda \in \mathbb{R}$, then is is elementary that $\lambda A \subset \bar{A}$ and therefore $\lambda \bar{A} \subset \bar{A}$. We see that $\bar{A}$ is a vector subspace of E. By applying $(*)$ to $\Phi : x \mapsto |x|$ we find that $\bar{A}$ is a Riesz subspace.

The Riesz ideal generated by A is A-closed in E, hence contains $\bar{A}$.

(ii) See the concluding remark of 0·1.

(iii) Let $a \in \bar{A}$ and let Σ be a filter of subsets of $\bar{A}$ such that for every $\epsilon > 0$ there is a $c \in \bar{A}$ with $\bar{A} \cap B_a(c;\epsilon) \in \Sigma$. Let $\Sigma_E := \{S \subset E : S \text{ contains an element of } \Sigma\}$. As E is uniformly complete, there is a c in E such that $B_a(c;\epsilon) \in \Sigma_E$ for all ϵ. In particular, every $B_a(c;\epsilon)$ intersects the a-closed set $\bar{A}$, so $c \in \bar{A}$. Observing that $\bar{A} \cap B_a(c;\epsilon) \in \Sigma$ for every ϵ, we obtain the uniform completeness of $\bar{A}$.

(iv) Let $\phi : A \to \mathbb{R}$ be a Riesz homomorphism.

Let E_A be the Riesz ideal of E generated by A and define functions $\underline{\phi}, \bar{\phi}$ on E_A by

$$\underline{\phi}(x) := \sup_{\substack{u \in A \\ u \leqslant x}} \phi(u), \quad \bar{\phi}(x) := \inf_{\substack{v \in A \\ v \geqslant x}} \phi(v).$$

Both are order-preserving. $\underline{\phi} \leqslant \bar{\phi}$, whereas $\underline{\phi} = \bar{\phi} = \phi$ on A. If $x \in E_A$ and $a \in A$, then

$$\underline{\phi}(x + a) = \underline{\phi}(x) + \phi(a), \quad \bar{\phi}(x + a) = \bar{\phi}(x) + \phi(a).$$

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The set $X := \{x \in E_A : \underline{\phi}(x) = \bar{\phi}(x)\}$ contains A. We claim that it is A-closed in E. Indeed, let $a \in A$, $a \geqslant 0$; we show that $E \backslash X$ is a-open. Let $x \in E \backslash X$. We have to find an $\epsilon > 0$ such that

$$y \in E, \quad |x-y| \leqslant \epsilon a \quad \text{implies} \quad y \notin X.$$

If $x \notin E_A$ we simply take $\epsilon := 1$. Now suppose $x \in E_A$. Then $\underline{\phi}(x) < \bar{\phi}(x)$. Take $\epsilon > 0$ such that $2\epsilon\phi(a) < \bar{\phi}(x) - \underline{\phi}(x)$. If $y \in E$ and $|x-y| \leqslant \epsilon a$, then

$$\underline{\phi}(y) \leqslant \underline{\phi}(x+\epsilon a) = \underline{\phi}(x) + \epsilon\phi(a) < \bar{\phi}(x) - \epsilon\phi(a) = \bar{\phi}(x-\epsilon a) \leqslant \bar{\phi}(y),$$

so $y \notin X$.

It follows that $X \supset \bar{A}$. The rest is routine. ∎

1·2. We call an Archimedean Riesz space E *slender* if it contains a countable $\mathbb{Q}$-linear sublattice A such that E is the A-closure of A. The Riesz space c is slender, c_0 is not. (Thus a Riesz subspace of a slender Riesz space may fail to be slender.)

In the sequel we will frequently choose suitable slender Riesz subspaces of a given Riesz space. Instead of giving a construction each time we here list the main principles involved.

(i) If E is an Archimedean Riesz space and A a countable $\mathbb{Q}$-linear sublattice of E, then the A-closure of A in E is slender. (1·1(ii).)

(ii) Every countably generated Archimedean Riesz space is slender.

(iii) If D_1, D_2 are slender Riesz subspaces of a Riesz space E, then the Riesz subspace of E generated by $D_1 + D_2$ is slender. (With the Countable Axiom of Choice a similar statement is valid for infinite sequences $D_1, D_2, \ldots$.)

(iv) Every slender (and unitary) Riesz subspace of a uniformly complete Riesz space E is contained in a slender (and unitary) Riesz subspace of E that is itself uniformly complete. (1·1(iii).)

LEMMA 1·3. *Let D be a slender and unitary Riesz space and let T be an order-bounded linear map of D into an Archimedean Riesz space E. Then $T(D)$ is contained in a slender unitary Riesz subspace E_0 of E such that $T|_D$ is order-bounded as a map from D to E_0.*

Proof. Let d be a unit of D. Choose a countable $\mathbb{Q}$-linear sublattice A of D such that D is the A-closure of A in D and $d \in A$. There is an e in E^+ such that

$$x \in D, \quad |x| \leqslant d \quad \text{implies} \quad |Tx| \leqslant e. \qquad (*)$$

Let E_0 be the B-closure of B in E, where B is the smallest $\mathbb{Q}$-linear sublattice of E containing $T(A) \cup \{e\}$. Then E_0 is slender and has e as a unit. Since E_0 is e-closed in E, it follows from $(*)$ that $T^{-1}(E_0)$ is d-closed in D, so $T^{-1}(E_0) \supset D$ and $E_0 \supset T(D)$. Again by $(*)$, $T|_D$ is order-bounded as a map from D to E_0. ∎

A similar result for non-unitary Riesz spaces can be obtained with the Countable Axiom of Choice.

2. *Representation theorems*

LEMMA 2·1. *Let E be an Archimedean Riesz space and for each $i \in \mathbb{N}$ let $(x_{ij})_{j \in \mathbb{N}}$ be a decreasing sequence in E^+ whose infimum is 0. Let $e \in E$, $e > 0$. Then there exists an extended M-seminorm p on E with*

$$p(e) > 0,$$

and

$$\lim_{j \to \infty} p(x_{ij}) = 0 \quad (i \in \mathbb{N}).$$

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If E is slender, one can choose a finite-valued p with these properties.

Proof. Construct $b_1, b_2, \ldots$ in E^+ with

$$(e-b_n)^+ > 0 \quad (n\in\mathbb{N}) \tag{*}$$

as follows: $b_1 := 0$;

$$b_n := b_{n-1} + n(x_{1j_n} + x_{2j_n} + \ldots + x_{nj_n}) \quad \text{for} \quad n \geqslant 2,$$

where $$j_n := \min\{j\in\mathbb{N} : (e-b_{n-1}-n(x_{1j}+\ldots+x_{nj}))^+ > 0\}.$$

As $b_1 \leqslant b_2 \leqslant \ldots$ we have (with notation as in 0·1)

$$\| \ \|_{b_1} \geqslant \| \ \|_{b_2} \geqslant \ldots.$$

Define $$p(x) := \lim_{n\to\infty} \|x\|_{b_n} \quad (x\in E).$$

Then p is an extended M-seminorm. For all n we have $e \not\leqslant b_n$, so $p(e) \geqslant 1$. Furthermore, if $n\in\mathbb{N}$ and if $i,j\in\mathbb{N}$ are such that $i \leqslant n$ and $j \geqslant j_n$, then $b_n \geqslant nx_{ij}$, whence $p(x_{ij}) \leqslant n^{-1}$. It follows that $\lim_{j\to\infty} p(x_{ij}) = 0$ for every $i\in\mathbb{N}$.

Now suppose that E is slender. It follows from (i) of Lemma 1·1 that there exist $a_1, a_2, \ldots$ in E^+ such that for every $x\in E$ we have $|x| \leqslant a_i$ for some i. By the above, there is an extended M-seminorm p on E with $p(e) > 0$ and

$$\lim_{j\to\infty} p(x_{ij}) = 0, \quad \lim_{j\to\infty} p(j^{-1}a_i) = 0 \quad (i\in\mathbb{N}).$$

Then $p(a_i) < \infty$ for each i. Consequently p is finite-valued. ∎

By a *representation* of a Riesz space E we mean a pair $(\Omega, \hat{\ })$ where Ω is set and $x\mapsto\hat{x}$ $(x\in E)$ is an injective Riesz homomorphism of E into $\mathbb{R}^\Omega$. We call a Riesz space E *representable* if it has a representation, that is, if the Riesz homomorphisms $E\to\mathbb{R}$ separate the points of E.

THEOREM 2·2. *Every slender Riesz space is representable.*

Proof. In fact, we prove more.

(*) *Let E be a slender Riesz space and let $e\in E$, $e > 0$. Let p be an M-seminorm on E for which $p(e) > 0$. Then there exists a Riesz homomorphism $\omega : E\to\mathbb{R}$ such that $\omega \leqslant p$ and $\omega(e) > 0$.*

(The theorem follows from (*) by an application of Lemma 2·1.)

We start with a digression.

Suppose $q : E\to\mathbb{R}$ is sublinear and a $\vee$-homomorphism (i.e. $q(x\vee y) = q(x)\vee q(y)$ for all $x, y\in E$). Let $a\in E$. Then for every $x\in E$ the function

$$\lambda\mapsto q(x+\lambda a) - \lambda q(a) \quad (\lambda\in\mathbb{R}^+)$$

is decreasing and bounded from below by $-q(-x)$. Thus we can define $r : E\to\mathbb{R}$ by

$$r(x) := \lim_{\lambda\to\infty} q(x+\lambda a) - \lambda q(a) \quad (x\in E).$$

It is elementary that r is a sublinear $\vee$-homomorphism satisfying

$$-q(-x) \leqslant -r(-x) \leqslant r(x) \leqslant q(x) \quad (x\in E),$$

$$r(a) = q(a) \quad \text{and} \quad r(-a) = -q(a).$$

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Now we come to the proof of (*).

Choose a countable $\mathbb{Q}$-linear sublattice $A = \{a_1, a_2, \ldots\}$ of E such that E is the A-closure of A in E whereas $e = a_1$. The function $p_0 : x \mapsto p(x^+)$ $(x \in E)$ is a sublinear $\vee$-homomorphism. Thus the formula

$$p_n(x) := \lim_{\lambda \to \infty} p_{n-1}(x + \lambda a_n) - \lambda p_{n-1}(a_n) \quad (x \in E,\ n \in \mathbb{N})$$

determines a sequence $p_1, p_2, \ldots$ of sublinear $\vee$-homomorphisms such that for each n

$$-p_{n-1}(-x) \leqslant -p_n(-x) \leqslant p_n(x) \leqslant p_{n-1}(x) \quad (x \in E), \tag{1}$$

$$p_n(a_n) = p_{n-1}(a_n) \quad \text{and} \quad p_n(-a_n) = -p_{n-1}(a_n). \tag{2}$$

For $i \in \mathbb{N}$ we obtain from this

$$p_n(a_i) = p_{n-1}(a_i) \quad \text{and} \quad p_n(-a_i) = -p_{n-1}(a_i) \quad \text{if} \quad n \geqslant i. \tag{3}$$

By (1), $\omega(x) := \lim_{n \to \infty} p_n(x)$ exists for every x in E. Thus we have constructed a sublinear $\vee$-homomorphism $\omega : E \to \mathbb{R}$ for which (by (3))

$$\omega(e) = \omega(a_1) = p_0(a_1) = p(e^+) > 0.$$

Consider the set

$$S := \{x \in E : \omega(x) + \omega(-x) = 0\}.$$

If $S = E$, then ω is linear, hence a Riesz homomorphism, and we are done.

Now if $x, y \in E$, $a \in E$, $\epsilon \in (0, \infty)$ and $|x - y| \leqslant \epsilon |a|$, then

$$|\omega(x) - \omega(y)| \leqslant \omega(|x - y|) \leqslant \omega(\epsilon |a|) = \epsilon \omega(|a|).$$

Hence, for every $a \in E$, ω is a-continuous. In particular, S is A-closed in E. Furthermore, it follows from (3) that $S \supset A$. Then $S \supset E$. ∎

For a later application we need the following straightforward consequence of Lemma 2·1 and (*).

COROLLARY 2·3. *Let E be a slender Riesz space, and let $e \in E$, $e > 0$. For each $i \in \mathbb{N}$ let $(x_{ij})_{j \in \mathbb{N}}$ be a decreasing sequence in E^+ whose infimum is 0. Then there exists a Riesz homomorphism $\omega : E \to \mathbb{R}$ for which*

$$\omega(e) > 0 \quad \textit{and} \quad \lim_{j \to \infty} \omega(x_{ij}) = 0 \quad \textit{for each } i.$$ ∎

After Theorem 2·2 the following theorem and its proof are essentially standard.

THEOREM 2·4. *Let E be a slender Riesz space that has a weak unit e. Let Ω be the set of all Riesz homomorphisms $\omega : E \to \mathbb{R}$ with $\omega(e) = 1$. For $x \in E$ let $\hat{x} : \Omega \to \mathbb{R}$ be the function $\omega \mapsto \omega(x)$ $(\omega \in \Omega)$ and put $\hat{E} := \{\hat{x} : x \in E\}$. Endow Ω with the weakest topology that makes every $\hat{x}$ $(x \in E)$ continuous. Then*

(i) *$(\Omega, \hat{})$ is a representation of E; $\hat{e} = 1_\Omega$; Ω is metrizable; $\hat{E}$ is an order-dense Riesz subspace of $C(\Omega)$.*

(ii) *If e is a strong unit, then Ω is compact and $\hat{E}$ is uniformly dense in $C(\Omega)$; if, in addition, E is uniformly complete, then $\hat{E} = C(\Omega)$.*

Proof. For $a \in E^+$ let $\Omega_a := \{\omega \in \Omega : \omega(a) > 0\}$. The sets Ω_a $(a \in E^+)$ form a base for the topology of Ω.

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(i) If $x \in E^+$ and $\hat{x} = 0$, then for every Riesz homomorphism $\phi : E \to \mathbb{R}$ we have $\phi(x \wedge e) = \phi(x) \wedge \phi(e) = 0$. Hence $(\Omega, \hat{})$ is a representation of E.

Let $A = \{a_1, a_2, \ldots\}$ be a countable $\mathbb{Q}$-linear sublattice of E such that E is the A-closure of A. If $\phi, \omega \in \Omega$, then $\{x \in E : \phi(x) = \omega(x)\}$ is a-closed in E for every $a \in E$. It follows that the formula

$$d(\phi, \omega) := \sum_{n=1}^{\infty} 2^{-n} \wedge |\phi(a_n) - \omega(a_n)| \quad (\phi, \omega \in \Omega)$$

defines a metric d on Ω. It is clear that the d-topology is weaker than the given topology of Ω. On the other hand, the set $\{x \in E : \hat{x} \text{ is } d\text{-continuous}\}$ contains A and is a-closed in E for every $a \in E$; hence it is equal to E. Consequently the topology of Ω is just the d-topology.

To see that $\hat{E}$ is order-dense in $C(\Omega)$ (actually, we prove a bit more), we take $f \in C(\Omega)^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}$, $0 < \lambda < f(\omega)$; we find an x in E^+ with $\hat{x} \leqslant f$ and $\hat{x}(\omega) = \lambda$. Choose $a \in E^+$ with $\omega \in \Omega_a \subset \{\phi \in \Omega : f(\phi) > \lambda\}$ and $\omega(a) = \lambda$. Let $x := a \wedge \lambda e$. If $\phi \in \Omega$ and $\phi(x) > 0$, then $\phi \in \Omega_a$ and $\phi(x) \leqslant \phi(\lambda e) = \lambda < f(\phi)$. Hence $\hat{x} \leqslant f$. Moreover $\hat{x}(\omega) = \omega(a) \wedge \lambda\omega(e) = \lambda$.

(ii) Assume that e is a strong unit in E, so that $\| \ \|_e$ (see 0·1) is an M-norm. We prove that Ω is compact. Let $S \subset E^+$ be such that $\{\Omega_s : s \in S\}$ covers Ω; we show the existence of $s_1, \ldots, s_N \in S$ for which $\Omega \subset \Omega_{s_1} \cup \ldots \cup \Omega_{s_N}$.

Let J be the Riesz ideal of E generated by S. The formula

$$p(x) := \inf\{\|x - s\|_e : s \in J\}$$

produces an M-seminorm p on E. We distinguish two cases: $p(e) > 0$ and $p(e) = 0$.

If $p(e) > 0$, (*) in the proof of Theorem 2·2 yields a Riesz homomorphism $\phi : E \to \mathbb{R}$ with $\phi(e) > 0$ and $\phi \leqslant p$. Setting $\omega := \phi(e)^{-1}\phi$ we obtain $\omega \in \Omega$ and $\omega|_J = 0$. Then ω lies in no Ω_s ($s \in S$). This is a contradiction.

Therefore $p(e) = 0$. In particular, there is an s in J with $\|e - s\|_e \leqslant \frac{1}{2}$, whence $s \geqslant \frac{1}{2}e$. There exist $s_1, \ldots, s_N \in S^+$ for which $s \leqslant s_1 + \ldots + s_N$. Then $e \leqslant 2s_1 + \ldots + 2s_N$. It follows that $\Omega \subset \Omega_{s_1} \cup \ldots \cup \Omega_{s_N}$.

This proves the compactness of Ω. For the uniform denseness of $\hat{E}$ in $C(\Omega)$, observe that in our proof of (i) we have seen that every element f of $C(\Omega)^+$ is the pointwise supremum of $\{\hat{x} : x \in E^+, \hat{x} \leqslant f\}$ and apply the Dini Theorem.

The proof that $\hat{E} = C(\Omega)$ for uniformly complete E is straightforward. (It differs from the usual proof in that we have to use filters rather than sequences, wishing to avoid the Countable Axiom of Choice.) ∎

2·5. The connection between part (ii) of this theorem and the Kakutani Representation Theorem for M-spaces is obvious. Part (i) may be obtained from the Maeda-Ogasawara Theorem and a category argument.

In the sequel we use the theorem for representation of finitely generated subspaces of a Riesz space. Accordingly, we are not going to need part (i). It can be used, however, for representing finitely generated subalgebras of an f-algebra. Indeed, every finitely generated f-algebra E has a weak unit. In the language of Theorem 2·4, in addition to (i) one can prove the existence of an f in $C(\Omega)^+$ with the property

$$(xy)\hat{} = f\hat{x}\hat{y} \quad (x, y \in E).$$

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2·6. Let E be a uniformly complete Riesz space with a strong unit e. With the aid of the Axiom of Choice one can show that E is Riesz isomorphic to $C(X)$ for some compact Hausdorff space X. Without it, we can still introduce a multiplication $*$ on E rendering E an f-algebra with identity element e. Our construction runs as follows.

For $a_1, \ldots, a_n \in E$ let $[e, a_1, \ldots, a_n]$ be the smallest e-closed Riesz subspace of E that contains $\{e, a_1, \ldots, a_n\}$. Thus $[e, a_1, \ldots, a_n]$ is the A-closure of A in E, where A is the smallest $\mathbb{Q}$-linear sublattice of E containing $\{e, a_1, \ldots, a_n\}$. Let $\mathscr{D}$ be the set of all these spaces $[e, a_1, \ldots, a_n]$. Each element of $\mathscr{D}$ is slender and uniformly complete and has e as an order unit. For $D \in \mathscr{D}$ let Ω_D be the set of all Riesz homomorphisms $\omega : D \to \mathbb{R}$ with $\omega(e) = 1$. By Theorem 2·4(ii) on each D we have a unique f-algebra multiplication $*_D$ satisfying

$$\omega(x *_D y) = \omega(x)\,\omega(y) \quad (x, y \in D;\ \omega \in \Omega_D).$$

Let $x, y \in D \in \mathscr{D}$. If $F := [e, x, y]$, then $\omega|_F \in \Omega_F$ for all $\omega \in \Omega_D$, so that

$$\omega(x *_D y) = \omega(x)\,\omega(y) = \omega(x *_F y) \quad (\omega \in \Omega_D),$$

and $x *_D y = x *_F y$.

Consequently we have a unique multiplication $*$ on E with

$$x * y = x *_D y \quad (x, y \in D \in \mathscr{D}).$$

It follows easily that $*$ is an f-algebra multiplication with identity element e. We also see that every e-closed Riesz subspace of E that contains e is a subalgebra.

The argument applied above can be used to establish a functional calculus for uniformly complete Riesz spaces à la Krivine. (See [**15**], theorem 1·d·1, [**16**], chapter II, §3, and also [**4**, **20**].)

As an illustration of the technique of small Riesz spaces we prove the following result for which we refer to [**3**, **19**, **22**].

THEOREM 2·7. *Let D, E be Archimedean Riesz spaces and T a linear map $D \to E$ such that $Tx \perp Ty$ whenever $x, y \in D$ satisfy $x \perp y$, and such that if $x_1, x_2, \ldots \in D$, $x_1 \geqslant x_2 \geqslant \ldots$ and $x_n \to 0$ relatively uniformly, then $\inf_n |Tx_n| = 0$. Then T is order-bounded.*

Proof. Let $a, b \in D$ and $|a| \leqslant b$. It is enough to prove $|Ta| \leqslant |Tb|$.

Let A be the smallest $\mathbb{Q}$-linear sublattice of D containing $\{a, b\}$, B the smallest $\mathbb{Q}$-linear sublattice of E containing $T(A)$, and $\bar{B}$ the B-closure of B in E. So $\bar{B}$ is slender.

Now $\bar{A}$ has b as a unit. By Theorem 2·4(ii) we may assume that $\bar{A}$ is a Riesz subspace of $C(\Omega)$ for some compact Hausdorff space Ω and that b is the constant function $\mathbf{1}$.

For $n \in \mathbb{N}$ define c_n in A by

$$c_n := \sup\{(a - (i-1)\,2^{-n}\,\mathbf{1}) \wedge ((i+1)\,2^{-n}\,\mathbf{1} - a) : -2^n \leqslant i \leqslant 2^n,\ i \text{ odd}\}.$$

Now, a being a function $\Omega \to [-1, 1]$, it is easy to see that $c_1 \geqslant c_2 \geqslant \ldots$ and that $0 \leqslant c_n \leqslant 2^{-n}\,\mathbf{1}$ for every n, so that $c_n \to 0$ relatively uniformly. Then $\inf_{n \geqslant N} |Tc_n| = 0$ in $\bar{B}$ for every $N \in \mathbb{N}$.

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It follows from Corollary 2·3 that the Riesz homomorphisms $\phi:\bar{B}\to\mathbb{R}$ for which

$$\inf_{n\geqslant N}|\phi(Tc_n)|=0 \quad \text{for every} \quad N\in\mathbb{N} \tag{1}$$

separate the points of $\bar{B}$. Hence we are done if $|\phi(Ta)|\leqslant|\phi(T1)|$ for all such ϕ.

Take a Riesz homomorphism $\phi:\bar{B}\to\mathbb{R}$ for which (1) holds and define $\tau:A\to\mathbb{R}$ by $\tau(x):=\phi(Tx)$ $(x\in A)$. Then τ is $\mathbb{Q}$-linear $A\to\mathbb{R}$; if $x,y\in A$ and $x\perp y$, then $\tau(x)=0$ or $\tau(y)=0$; and $\liminf|\tau(c_n)|=0$. We wish to show that $|\tau(a)|\leqslant|\tau(1)|$.

If $x,y\in A$, then $(x-x\wedge y)\perp(y-x\wedge y)$, so $\tau(x)=\tau(x\wedge y)$ or $\tau(y)=\tau(x\wedge y)$. Thus

$$\tau(x\wedge y)\in\{\tau(x),\tau(y)\}.$$

As $x\vee y=x+y-x\wedge y$ we also have $\tau(x\vee y)\in\{\tau(x),\tau(y)\}$. From the definition of c_n it now follows that $\tau(c_n)$ must be one of the numbers $\tau(a)-(i-1)2^{-n}\tau(1)$ or $(i+1)2^{-n}\tau(1)-\tau(a)$. In any case

$$|\tau(a)|\leqslant|\tau(c_n)|+|\tau(1)|.$$

But $\liminf|\tau(c_n)|=0$. Hence $|\tau(a)|\leqslant|\tau(1)|$. ∎

3. *Orthomorphisms*

3·1. Let E be an Archimedean Riesz space and D a Riesz subspace of E. A linear map $T:D\to E$ is called an *orthomorphism* $D\to E$ if T is order-bounded (as a map of D into E), and if $Tx\perp y$ whenever $x\in D$, $y\in E$ and $x\perp y$. An *orthomorphism* of E is an orthomorphism $E\to E$.

The definition should be handled with care. Suppose we have D and E as above and let E itself be a Riesz subspace of some Riesz space F. Then every orthomorphism $D\to F$ whose range is contained in E is also an orthomorphism $D\to E$. (The order-boundedness will follow from 3·3(i).) But an orthomorphism $D\to E$ may fail to be an orthomorphism $D\to F$. For example, let F be the space of all functions $[0,1]\to\mathbb{R}$, $E:=C([0,1])$; let $u,e\in E$ be the functions $x\mapsto x$ and $x\mapsto 1$, respectively. Take $D:=\mathbb{R}u$ and $T(\lambda u):=\lambda e$ $(\lambda\in\mathbb{R})$.

LEMMA 3·2. *Let E be an Archimedean Riesz space and D a Riesz subspace of E, and let $T:D\to E$ be an orthomorphism. If $\omega:E\to\mathbb{R}$ is a Riesz homomorphism and $\omega|_D\neq 0$, then $\omega\circ T$ is a scalar multiple of $\omega|_D$.*

Proof. It suffices to show that $\operatorname{Ker}\omega|_D\subset\operatorname{Ker}\omega\circ T$, or even that

$$D^+\cap\operatorname{Ker}\omega\subset\operatorname{Ker}\omega\circ T.$$

Let $a\in D^+\cap\operatorname{Ker}\omega$. Choose $b\in D^+$ with $\omega(b)=1$. For all $\epsilon\in(0,\infty)$ we have

$$a-a\wedge\epsilon b\perp\epsilon b-a\wedge\epsilon b, \quad \text{so that} \quad Ta-T(a\wedge\epsilon b)\perp\epsilon b-a\wedge\epsilon b,$$

and $$\text{either} \quad \omega(Ta-T(a\wedge\epsilon b))=0 \quad \text{or} \quad \omega(\epsilon b-a\wedge\epsilon b)=0.$$

However $$\omega(\epsilon b-a\wedge\epsilon b)=\epsilon\omega(b)-\omega(a)\wedge\epsilon\omega(b)=\epsilon-0\wedge\epsilon=\epsilon>0.$$

Therefore $\omega(Ta-T(a\wedge\epsilon b))=0$, i.e. $(\omega\circ T)(a)=(\omega\circ T)(a\wedge\epsilon b)$. As T is order-bounded, so is $\omega\circ T$; then $|\omega\circ T|$ exists in the dual Riesz space of E. It follows that

$$|(\omega\circ T)(a)|=|(\omega\circ T)(a\wedge\epsilon b)|\leqslant|\omega\circ T|(\epsilon b)=\epsilon|\omega\circ T|(b).$$

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This must be true for all $\epsilon > 0$. Consequently $(\omega \circ T)(a) = 0$, and $a \in \operatorname{Ker} \omega \circ T$. ∎

The condition $\omega|_D \neq 0$ in the lemma is not redundant. (Let $E = C([0,1])$ and let u be the identity map of $[0,1]$. Take $D := \{uf : f \in E\}$, $T(uf) := f$ $(f \in E)$ and $\omega(f) := f(0)$ $(f \in E)$.)

3·3. As an application we prove some elementary properties of orthomorphisms.

LEMMA. *Let D be a Riesz subspace of an Archimedean Riesz space E and let T be an orthomorphism $D \to E$.*

(i) *If $x, y \in D$ and $|x| \leqslant |y|$, then $|Tx| \leqslant |Ty|$.*

(ii) $\operatorname{Ker} T = \{x \in D : x \perp T(D)\}$. *In particular,* $\operatorname{Ker} T$ *is a Riesz ideal in D. (It may not be a band.)*

(iii) *The map $|T| : x \mapsto |T(x^+)| - |T(x^-)|$ $(x \in D)$ is a Riesz homomorphism and an orthomorphism $D \to E$; it is the absolute value of T in $L_b(D, E)$. (Thus T is regular.) For all $x \in D$, $|Tx| = ||T|(x)| = |T|(|x|)$.*

(iv) *$|T|(D)$ is the Riesz subspace of E generated by $T(D)$.*

Proof. Let $x, y \in D$. Let D_0 be the Riesz subspace of D generated by x, y and let E_0 be the Riesz subspace of E generated by $D_0 + T(D_0)$. We will construct a representation $(\Phi, \hat{})$ of E_0 and a function $t : \Phi \to \mathbb{R}$ such that

$$(Tz)\hat{} = t\hat{z} \quad (z \in D_0).$$

Once we have that, it is straightforward to show that

(i) if $|x| \leqslant |y|$, then $|Tx| \leqslant |Ty|$;

(ii) if $Tx = 0$, then $|x| \wedge |Ty| = 0$; if $|x| \wedge |Tx| = 0$, then $Tx = 0$;

(iii) $|T|(D_0) \subset E_0$; $(|T|(z))\hat{} = |t|\,\hat{z}$ $(z \in D_0)$; $|Tx| = ||T|(x)| = |T|(|x|)$.

The lemma then follows easily.

To obtain the desired $(\Phi, \hat{})$ and t, we first observe that $T(D_0)$ is contained in a slender Riesz subspace F of E such that the restriction of T to D_0 is order-bounded as a map from D_0 to F. (See Lemma 1·3.) By 1·2(iii) we may assume $D_0 \subset F$. Let $(\Omega, \hat{})$ be a representation of F and $\Phi := \{\omega \in \Omega : \hat{z}(\omega) \neq 0 \text{ for some } z \in D_0\}$.

If $z \in F$ and $\hat{z}$ vanishes identically on Φ, then $z \perp D_0$, so $z \perp T(D_0)$ and $z \perp E_0$. Thus

$$z \mapsto \hat{z}|_\Phi \quad (z \in E_0)$$

is a representation of E_0. Moreover the restriction of T to D_0 is an orthomorphism $D_0 \to F$. Then Lemma 3·2 produces a function $t : \Phi \to \mathbb{R}$ with

$$(Tz)\hat{}|_\Phi = t\hat{z}|_\Phi \quad (z \in D_0)$$

and we are done. ∎

3·4. It is clear that many properties of orthomorphisms either are consequences of the lemma or can be proved along similar lines. Among the consequences we have the following result that we will need later on.

COROLLARY. *Let D_1, D_2 be Riesz subspaces of an Archimedean Riesz space E. Let $T_1 : D_1 \to E$ and $T_2 : D_2 \to E$ be orthomorphisms such that $T_1(D_1) \subset D_2$. Then $T_2 \circ T_1$ is an orthomorphism $D_1 \to E$.*

Proof. By (iv), $|T_1|(D_1) \subset D_2$. It follows that $T_2 \circ T_1$ is regular. The rest is obvious. ∎

3·5. Lemma 3·3 enables us to weaken the order-boundedness requirement in the definition of 'orthomorphism'.

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PROPOSITION. *Let D be a Riesz subspace of an Archimedean Riesz space E and let $T: D \to E$ be linear. Suppose that for every finitely generated Riesz subspace D_0 of E the restriction map $T|_{D_0}$ is an orthomorphism $D_0 \to E$. Then T is an orthomorphism $D \to E$.*

(Apply 3·3(i) to see that T is order-bounded.)

The range $T(E)$ of an orthomorphism $T: E \to E$ need not be a Riesz subspace of E. The following theorem implies that $T(E)$ contains a fairly large Riesz ideal of E, viz. the ideal generated by $T^2(E)$.

THEOREM 3·6. *Let E be a uniformly complete Riesz space and T an orthomorphism $E \to E$. Let J_1, J_2 be the Riesz ideals of E generated by $T(E)$ and $T^2(E)$. Then the restriction of T to J_1 is a bijection $J_1 \to J_2$ whose inverse is an orthomorphism $J_2 \to E$. The restriction of $|T|$ to J_1 is a Riesz isomorphism of J_1 onto J_2.*

Proof. It follows from 3·3(i) that $T(J_1) \subset J_2$, and from 3·3(ii) that $T|_{J_1}$ is injective. We proceed to prove that $J_2 \subset T(J_1)$.

Let $a \in J_2$. Choose $b \in E$ with $|a| \leqslant |T^2 b|$. Let D be the Riesz subspace of E generated by $\{a, b, Tb, T^2 b\}$. Then D is slender and has $e := |a| + |b| + |Tb| + |T^2 b|$ as a unit. By Lemma 1·3, $T(D)$ is contained in a slender and unitary Riesz subspace D_1 of E such that $T|_D$ is an orthomorphism $D \to D_1$. By 1·2(iii) and 1·2(iv) we may assume that $D \subset D_1$ and that D_1 is uniformly complete. Choose a unit e_1 of D_1.

Let Ω be the set of all Riesz homomorphisms $\omega: D_1 \to \mathbb{R}$ with $\omega(e_1) = 1$, provided with the hull-kernel topology (2·4). The natural representation $\hat{}$ of D_1 into $\mathbb{R}^\Omega$ is a Riesz isomorphism of D_1 onto $C(\Omega)$. From Lemma 3·2 we obtain

$$\hat{x}(Ty)\hat{} = \hat{y}(Tx)\hat{} \quad (x, y \in D).$$

Define $f: \Omega \mapsto \mathbb{R}$ by

$$f(\omega) := \begin{cases} \hat{a}(\omega)\,\hat{e}(\omega)/(Te)\hat{}(\omega) & \text{if } \omega \in \Omega, \quad (Te)\hat{}(\omega) \neq 0, \\ 0 & \text{if } \omega \in \Omega, \quad (Te)\hat{}(\omega) = 0. \end{cases}$$

We have $$|\hat{a}\hat{e}| = |a|\hat{}\,|e|\hat{} \leqslant |T^2 b|\hat{}\,|e|\hat{} = |(TTb)\hat{}\,\hat{e}| = |(Tb)\hat{}\,(Te)\hat{}\,|,$$

so that (i) $\hat{a}\hat{e} = f(Te)\hat{}$ and (ii) $|f| \leqslant |Tb|\hat{} \leqslant |Te|\hat{}$. From (ii) we infer that $f \in C(\Omega)$. Hence there is a u in D_1 with $f = \hat{u}$. By (i) and (ii) we have (i′) $\hat{a}\hat{e} = \hat{u}(Te)\hat{}$, and (ii′) $|u| \leqslant |Tb|$. In particular $u \in J_1$. We now prove that $Tu = a$.

It follows from Lemma 1·3 and 1·2(iii) that $D_1 + T(D_1)$ is contained in a slender Riesz subspace D_2 of E. Let $\phi: D_2 \to \mathbb{R}$ be a Riesz homomorphism; we are done if $\phi(Tu) = \phi(a)$. Now $\phi|_{D_1}$ is a scalar multiple of some element of Ω; then by 3·2 and (i′)

$$\phi(Tu)\,\phi(e) = \phi(u)\,\phi(Te) = \phi(a)\,\phi(e),$$

whence $\phi(Tu) = \phi(a)$ in case $\phi(e) \neq 0$. If $\phi(e) = 0$ we get the same identity from the observations that $|a| \leqslant e$ and $|Tu| \leqslant |T^2 b| \leqslant e$. (See (ii′).)

So far we have shown that T induces a bijection $J_1 \to J_2$. Let S be the inverse of this bijection. Of course, S is linear. From the above (see (ii′)) we know that if $a \in J_2$, $b \in E$ and $|a| \leqslant |T^2 b|$, then $|Sa| \leqslant |Tb|$. It follows that S is order-bounded as a map $J_2 \to E$. Finally, let $a \in J_2$, $x \in E$, $x \perp a$; we have to prove $x \perp Sa$. Let b, e, D, D_1, f, u be as above but such that $x \in D_1$. Then $\hat{x}\hat{a} = 0$, so that $\hat{x} \perp f = \hat{u}$ and $x \perp u = Sa$.

To prove the last part of the theorem, apply the above to $|T|$ instead of T, observing that the Riesz ideals of E generated by $|T|(E)$ and $|T|^2(E)$ are just J_1 and J_2, which follows from 3·3(iv). ∎

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3·7. The above theorem has various consequences for orthomorphisms and f-algebras. ([21] contains a very good bibliography.)

COROLLARY. *Let E be a uniformly complete Riesz space.*

(i) *If an orthomorphism of E is bijective, then its inverse is an orthomorphism.*

(ii) (11·1 *in* [21].) *If T is an orthomorphism of E and the Riesz ideal generated by $T(E)$ is E, then T is bijective.* ($J_1 = E$, $J_2 = T(J_1) = T(E)$; *but J_2 is an ideal, so $J_2 = E$.*) *In particular, if T is an orthomorphism of E and $T \geqslant I$ (in the ordering of $L_b(E)$), then T is a Riesz isomorphism of E onto E.*

(iii) (4·5 *in* [16], 8·6 *in* [21].) *Every orthomorphism T of E is order-continuous.* (*Let $x_\lambda \downarrow 0$. By* (ii), *$I+|T|$ is a Riesz isomorphism $E \to E$, so $(I+|T|)x_\lambda \downarrow 0$. Then $|T|x_\lambda \downarrow 0$, i.e. $|Tx_\lambda| \downarrow 0$.*)

(iv) (*cf.* 11·9 *in* [21].) *If E is an f-algebra, $a, b, c \in E$, $|a| \leqslant |b^2 c|$, then $a \in bE$.*

(v) *Let E be an f-algebra, $a, b, c \in E$, $|a| \leqslant |c|$ and $|b| \leqslant |c|$. Then $abE \subset cE$.* (*In the theorem, let T be the multiplication by c.*) *Hence, if $|a| \leqslant |c|$ and a is invertible, then so is c.* (*Take $b = a$.*)

(vi) Let E be an f-algebra. It is known (and easy to derive from Lemma 3·3) that the orthomorphisms $E \to E$ form a uniformly complete f-algebra Orth (E). For $a \in E$ let $M_a \in \mathrm{Orth}\,(E)$ be the multiplication map $x \mapsto ax$ $(x \in E)$. The set $\{M_a : a \in E\}$ is a Riesz subspace and an algebra ideal in Orth (E), but it may fail to be a Riesz ideal. However, it always contains the Riesz ideal generated by $\{M_{ab} : a, b \in E\}$, and for every $c \in E^+$ it contains $M_c \wedge I$. (The proof is only slightly less direct than the proofs of (i)–(v).)

4. *Extension theorems*

THEOREM 4·1. *Let D be a majorizing Riesz subspace of a uniformly complete Riesz space E. Let $T : D \to E$ be an orthomorphism. Then T extends uniquely to an orthomorphism $E \to E$.*

Proof. For $x \in E^+$ let E_x be the Riesz ideal generated by x.

(I) *If* S_1 and S_2 are orthomorphisms $E \to E$, both extending T, then, by Lemma 3·3 (ii), $S_1 - S_2$ vanishes identically on E. Therefore T has at most one extension to E.

(II) Suppose that for every $a \in D^+$ there is an orthomorphism $S : E_a \to E$ with $Sa = Ta$. By arguments similar to (I), S and T coincide on $D \cap E_a$, and for every $a \in D^+$ there exists precisely one orthomorphism $S_a : E_a \to E$ that extends $T|_{D \cap E_a}$. It is then easy to obtain the desired extension of T by combining all these maps S_a.

(III) Thus let $a \in D^+$. We conclude our proof by constructing an orthomorphism $S : E_a \to E$ with $Sa = Ta$.

Let $b \in D^+$, $b \geqslant a + |Ta|$ and $e := b + |Tb|$. Under a suitable multiplication $*$, E_e becomes an f-algebra with identity element e (see 2·6). If $c \in E_e^+$, then the Riesz ideal generated by the range of the orthomorphism $x \mapsto c * x$ in E_e is just E_c. It follows from 3·7 that there exists an orthomorphism $W : E_{b*b} \to E$ such that

$$W(b * y) = y \quad (y \in E_b).$$

Now $x \mapsto x * Tb - b * Tx$ is an orthomorphism $D \cap E_b \to E$ vanishing at b; then by 3·3 (ii) it vanishes at a:

$$a * Tb = b * Ta.$$

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Hence $|a*Tb| = b*|Ta| \leqslant b*b$, so $x*Tb \in E_{b*b}$ for all $x \in E_a$. Then (see 3·4) $S: x \mapsto W(x*Tb)$ is an orthomorphism $E_a \to E$ and $Sa = W(a*Tb) = W(b*Ta) = Ta$. ∎

It is tempting to think that, if D is any Riesz subspace of E (not necessarily majorizing), then every orthomorphism $D \to E$ can be extended to the Riesz ideal generated by D. For a counter-example, let $E := C([0,1])$, $D := \mathbb{R}u$ where $u(x) = x$ $(x \in [0,1])$; let Tu be the constant function 1 and consider the value of a hypothetical extension of T at g where $g \in C([0,1])$, $g(x) = x \sin x^{-1}$ $(0 < x \leqslant 1)$.

THEOREM 4·2. *Let E, F be Archimedean Riesz spaces, and suppose that F is uniformly complete. Let $T \in L^+(E,F)$ and let J be the Riesz ideal of F generated by $T(E)$. Consider the following three statements:*

(α) *T is a Riesz homomorphism;*

(β) *if $S \in L_b(E,F)$ and if for every $x \in E$, Sx lies in the band of F generated by Tx, then $S = UT$ for some orthomorphism $U: J \to F$; and*

(γ) *if $S \in L^+(E,F)$ and $T - S \in L^+(E,F)$, then $S = UT$ for some orthomorphism U of J.*

Then $(\alpha) \Rightarrow (\beta)$ and $(\beta) \Rightarrow (\gamma)$. Furthermore $(\gamma) \Rightarrow (\alpha)$ if either E has the principal projection property or F is Dedekind σ-complete.

Proof. $(\alpha) \Rightarrow (\beta)$. $T(E)$ is a Riesz subspace of F. If $x \in E$ and $Tx = 0$, then $Sx = 0$. Therefore we can define a linear map $U_0: T(E) \to F$ by

$$U_0: Tx \mapsto Sx \quad (x \in E).$$

By 4·1 we are done if U_0 is an orthomorphism, and by the given properties of S that will be true provided that U_0 is order-bounded. Let $a \in E^+$; we have to find an element $b \in F^+$ such that

$$x \in E \quad \text{and} \quad |Tx| \leqslant Ta \quad \text{imply} \quad |Sx| \leqslant b. \tag{*}$$

We know that there is a b in F^+ for which

$$y \in E \quad \text{and} \quad |y| \leqslant a \quad \text{imply} \quad |Sy| \leqslant \tfrac{1}{2}b.$$

Now let $x \in E$ and $|Tx| \leqslant Ta$. Then $T(x^+) \leqslant Ta$, so $T(x^+) = T(x^+ \wedge a)$. Hence we have $|S(x^+)| = |S(x^+ \wedge a)| \leqslant \frac{1}{2}b$. Similarly $|S(x^-)| \leqslant \frac{1}{2}b$. (*) follows.

$(\beta) \Rightarrow (\gamma)$ is trivial.

$(\gamma) \Rightarrow (\alpha)$ (under suitable assumptions on E or F). Let $a, b \in E^+$ where $a \perp b$; we wish to prove $Ta \perp Tb$. If we can find an $S \in L^+(E,F)$ with $T - S \in L^+(E,F)$, $Sa = Ta$, $Sb = 0$, and if we choose U as in (γ), then Ta and Tb lie in the range and the kernel of U, respectively, and we are done by Lemma 3·3(*ii*).

It remains to find S. If E has the principal projection property we take $S = TP$, where $P: E \to E$ is the projection onto the band generated by a. If F is σ-Dedekind complete we first define Sx for $x \in E^+$ by $Sx := \sup\{T(x \wedge na) : n \in \mathbb{N}\}$ and then extend linearly. ∎

For Dedekind complete F, Kutateladze [**13**] proves the equivalence of (α) with a statement (γ') that is very similar to (γ). The difference is that the orthomorphism U in (γ') is defined on all of F. Trivially, this condition (γ') implies (γ). The proof of the reverse implication (for Dedekind complete F) is a simple consequence of the observation that the U occurring in (γ) is majorized by the identity map $J \to F$.

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