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ON THE UPPER FIRST-EXIT TIMES OF COMPOUND *G/M* PROCESSES

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For a compound process with exponential jumps at renewal times, we determine, in closed form, the density of the first time an upper linear boundary is crossed. It is shown how simple formulas for the Laplace transform and the first two moments can be directly derived from this density.

1. INTRODUCTION

Let $(X_n)_{n\in\mathbb{N}}$ and $(E_n)_{n\in\mathbb{N}}$ be two independent and identically distributed (i.i.d.) sequences of positive random variables, where the X_n have the common distribution function F and the E_n are $\exp(\lambda)$ distributed. Let $X_0 = E_0 = 0$ and $N(t) = \sup\{n \in \mathbb{Z}_+ | X_0 + \cdots + X_n \le t\}$. We consider the compound G/M process

$$Y(t) = E_0 + E_1 + \cdots + E_{N(t)}, \qquad t \ge 0$$

and its upper first-exit time

$$T_U(\beta) = \inf\{t \ge 0 | Y(t) \ge \beta + t\}, \qquad \beta > 0.$$

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In [4], the following identity for the Laplace transform of $T_U(\beta)$ was derived:

$$E(e^{-\xi T_U(\beta)}) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1} e^{-\lambda\beta}}{(n-1)!} E\left(e^{-(\xi+\lambda)S_n}\left(\beta + \frac{S_n}{n}\right)(\beta + S_n)^{n-2}\right), \qquad \xi \geq 0, \tag{1.1}$$

where $S_n = X_1 + \cdots + X_n$. In this article, we present an alternative approach, leading to a direct formula for the (possibly defective) density $\psi(t|\beta)$ of $T_U(\beta)$. There seems to be no closed-form result on this density in the literature. The following theorem fills this gap.

THEOREM. *Let F have the density f on $[0,\infty)$. The nth convolution of f with itself is denoted by f^{*n}. Let*

$$h(y;t) = \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} f^{*n}(y), \qquad y, t > 0.$$

$T_U(\beta)$ has the density

$$\psi(t|\beta) = f(t)e^{-\lambda(t+\beta)} + \frac{1}{t+\beta}\int_0^t (\beta + x)h(t-x;t+\beta)f(x)\,dx, \quad 0 < t < \infty. \tag{1.2}$$

$h(\cdot;t)$ is, of course, the density of $Y(t)$ on $(0,\infty)$; formula (1.2) represents $\psi(t|\beta)$ as a weighted convolution of $h(\cdot;t+\beta)$ with f.

First-exit problems for compound processes continually attract much attention due to their importance in queuing theory, insurance mathematics, sequential analysis, and related fields. For example, $T_U(\beta)$ can be viewed as the ruin time of the surplus process of an insurance with premiums arriving continuously at unit rate, initial capital β, and exponentially distributed claims made at renewal times. In the compound Poisson case, elegant results for the corresponding boundary crossing times have recently been obtained in [6–9]; see also [5], for more references, and the monograph [1].

The proof of (1.2) is based on a reflection argument and Kendall's formula for lower skip-free Lévy processes (see [2, p. 66, Thm. 1] and the recent proof in [3]). In [10], a companion formula for the density of $T_U(\beta)$ in the compound Poisson case has been derived.

In Figure 1 a typical sample path of Y is depicted in which the first jump (at time X_1) does not cross the upper boundary.

The theorem is proved in Section 2. The explicit result for the density can be used to give an alternative proof of the Laplace transform formula (1.1) and also to derive the first moments of $T_U(\beta)$ in the case $\lambda E[X_1] < 1$. This is carried out in Section 3.

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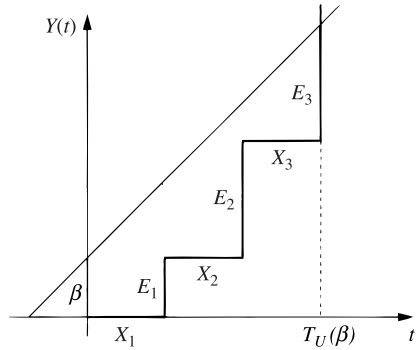


FIGURE 1. Y stopped at $T_U(\beta)$.

2. PROOF OF THE THEOREM

Conditional on X_1, we have almost surely

$$P(T_U(\beta) \geq X_1 | X_1) = 1$$

and

$$P(T_U(\beta) = X_1 | X_1) = e^{-\lambda(\beta + X_1)}.$$

Consider the compound Poisson process $Y^*(t)$, $t \geq 0$, with jump times $E_1, E_1 + E_2, E_1 + E_2 + E_3, \ldots$ and jump sizes $X_2, X_3, \ldots$. The corresponding *lower* first-exit time is denoted by $T_L^*(\beta) = \inf\{t \geq 0 | Y^*(t) = -\beta + t\}$. A little reflection shows that

$$T_U(\beta) = T_L^*(\beta + X_1) - \beta;$$

see Figure 2.

By Kendall's formula, the density of $T_L^*(\beta + X_1)$ on $(\beta + X_1, \infty)$, conditional on X_1, is

$$q(t | \beta, X_1) = \frac{\beta + X_1}{t} h(t - \beta - X_1; t), \qquad t \in (\beta + X_1, \infty).$$

Thus, the conditional density of $T_U(\beta)$, given X_1, is

$$\psi(t | \beta, X_1) = \frac{\beta + X_1}{t} h(t - X_1; t + \beta), \qquad t \in (X_1, \infty).$$

Deconditioning, we obtain (1.2).

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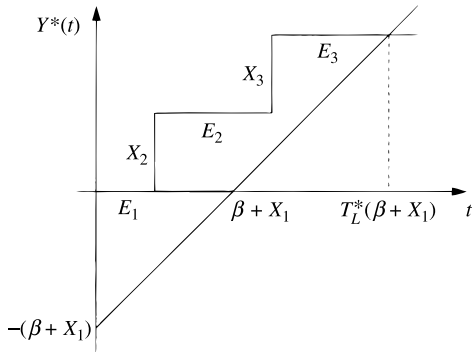


FIGURE 2. Y^* stopped at $T_L^*(\beta + X_1)$.

3. LAPLACE TRANSFORM AND MOMENTS OF $T_U(\beta)$

We now show how (1.1) can be directly deduced from our formula for the density. By (1.2), we have, for every $\xi \geq 0$,

$$E(e^{-\xi T_U(\beta)}) = \int_0^\infty e^{-\xi t} f(t) e^{-\lambda(t+\beta)}\, dt + \int_0^\infty \frac{e^{-\xi t}}{t+\beta} \left[\int_0^t (\beta + x) \sum_{n=1}^\infty e^{-\lambda(t+\beta)} \frac{(\lambda(t+\beta))^n}{n!} \times f^{*n}(t-x) f(x)\, dx \right] dt. \tag{3.1}$$

The first term on the right-hand side of (3.1) is equal to $e^{-\lambda\beta} E(e^{-(\xi+\lambda)S_1})$, which is the summand in the series (1.1) for $n = 1$. Now, let $\tilde{S}_n = X_2 + \cdots + X_{n+1}$. The second term on the right-hand side of (3.1) can be transformed as follows:

$$\begin{aligned} & e^{-\lambda\beta} \sum_{n=1}^\infty \frac{\lambda^n}{n!} \int_0^\infty (\beta + x) f(x) \left[\int_x^\infty e^{-(\lambda+\xi)t} (t+\beta)^{n-1} f^{*n}(t-x)\, dt \right] dx \\ &= e^{-\lambda\beta} \sum_{n=1}^\infty \frac{\lambda^n}{n!} \int_0^\infty (\beta + x) f(x) \left[\int_0^\infty e^{-(\lambda+\xi)(x+u)} (x+u+\beta)^{n-1} f^{*n}(u)\, du \right] dx \\ &= e^{-\lambda\beta} \sum_{n=1}^\infty \frac{\lambda^n}{n!} E((\beta + X_1) e^{-(\lambda+\xi)(X_1+\tilde{S}_n)} (X_1 + \tilde{S}_n + \beta)^{n-1}) \\ &= e^{-\lambda\beta} \sum_{n=1}^\infty \frac{\lambda^n}{n!} E\left(\left(\beta + \frac{S_{n+1}}{n+1} \right) e^{-(\lambda+\xi)S_{n+1}} (S_{n+1} + \beta)^{n-1} \right), \end{aligned} \tag{3.2}$$

where the last equality follows from symmetry. The right-hand side of (3.2) is the same as the sum on the right-hand side of (1.1) minus its first term. This yields (1.1).

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Let us now assume that $\rho = \lambda E[X_1] < 1$. This implies that $T_U(\beta)$ is almost surely finite; in fact, $E[T_U(\beta)^k] < \infty$ for all $k \in \mathbb{N}$. These moments can also be computed directly from the density (1.2). First, let $k = 1$. By (1.2),

$$E[T_U(\beta)] = e^{-\lambda\beta} \int_0^\infty t f(t) e^{-\lambda t}\, dt + \int_0^\infty \frac{t}{t+\beta} \int_0^t (\beta + x) h(t - x; t + \beta) f(x)\, dx\, dt. \tag{3.3}$$

Since $\psi(t|\beta)$ is a proper density, integrating (1.2) yields

$$\int_0^\infty \frac{\beta}{t+\beta} \int_0^t (\beta + x) h(t - x; t + \beta) f(x)\, dx\, dt = \beta(1 - e^{-\lambda\beta}\phi(\lambda)), \tag{3.4}$$

where $\phi(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt$ is the Laplace transform of X_1. By (3.2),

$$\int_0^\infty \frac{t}{t+\beta} \int_0^t (\beta + x) h(t - x; t + \beta) f(x)\, dx\, dt = \int_0^\infty \int_0^t (\beta + x) h(t - x; t + \beta) f(x)\, dx\, dt - \beta(1 - e^{-\lambda\beta}\phi(\lambda)). \tag{3.5}$$

Furthermore,

$$\int_0^\infty \int_0^t (\beta + x) h(t - x; t + \beta) f(x)\, dx\, dt = \int_0^\infty (\beta + x) f(x) \int_0^\infty h(t; t + \beta + x)\, dt\, dx. \tag{3.6}$$

Formula (3.6) of [10] gives

$$\int_0^\infty h(t; t + \beta + x)\, dt = (1 - \rho)^{-1} - e^{-\lambda(\beta+x)}. \tag{3.7}$$

Therefore, by (3.3)–(3.5),

$$\begin{aligned} &\int_0^\infty \frac{t}{t+\beta} \int_0^t (\beta + x) h(t - x; t + \beta) f(x)\, dx\, dt \\ &\quad = \int_0^\infty (\beta + x) f(x)((1 - \rho)^{-1} - e^{-\lambda(\beta+x)})\, dx - \beta(1 - e^{-\lambda\beta}\phi(\lambda)) \\ &\quad = \frac{\mu + \beta}{1 - \rho} - \beta - e^{-\lambda\beta} \int_0^\infty x f(x) e^{-\lambda x}\, dx, \end{aligned} \tag{3.8}$$

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where $\mu = E[X_1]$. By (3.1) and (3.6), we finally obtain

$$E[T_U(\beta)] = \frac{\mu + \rho\beta}{1-\rho}. \tag{3.9}$$

The second moment $E[T_U(\beta)^2]$ can be computed similarly. Start from

$$E[T_U(\beta)^2] = e^{-\lambda\beta}\int_0^\infty t^2 f(t)e^{-\lambda t}\,dt + \int_0^\infty \frac{t^2}{t+\beta}\int_0^t (\beta+x)h(t-x;t+\beta)f(x)\,dx\,dt. \tag{3.10}$$

The second integral in (3.10) can be written as

$$\begin{aligned}\int_0^\infty \frac{t^2}{t+\beta}\int_0^t (\beta+x)h(t-x;t+\beta)f(x)\,dx\,dt &= \int_0^\infty (t+\beta)\int_0^t (\beta+x)h(t-x;t+\beta)f(x)\,dx\,dt \\ &\quad - 2\beta\int_0^\infty \frac{t}{t+\beta}\int_0^t (\beta+x)h(t-x;t+\beta)f(x)\,dx\,dt \\ &\quad - \beta^2\int_0^\infty \frac{1}{t+\beta}\int_0^t (\beta+x)h(t-x;t+\beta)f(x)\,dx\,dt. \end{aligned} \tag{3.11}$$

The last two double integrals on the right-hand side of (3.11) are given by (3.8) and (3.4). The first one is equal to

$$\int_0^\infty (\beta+x)f(x)\left(\int_0^\infty (u+x+\beta)h(u;u+x+\beta)\,du\right)dx. \tag{3.12}$$

By [10], the inner integral in (3.12) is given by

$$\int_0^\infty (u+x+\beta)h(u;u+x+\beta)\,du = \frac{\beta}{(1-\rho)^2} + \frac{\lambda\mu_2}{(1-\rho)^3} - \beta e^{-\lambda(x+\beta)}, \tag{3.13}$$

where $\mu_2 = E[X_1^2]$. Combining all these calculations, we arrive at

$$E[T_U(\beta)^2] = (\beta+\mu)\left(\frac{\beta}{(1-\rho)^2} + \frac{\lambda\mu_2}{(1-\rho)^3}\right) - 2\beta\,\frac{\beta\rho+\mu}{1-\rho} - \beta^2 - \beta e^{-\lambda\beta}\int_0^\infty x f(x)e^{-\lambda x}\,dx + e^{-\lambda\beta}\int_0^\infty x^2 f(x)e^{-\lambda x}\,dx. \tag{3.14}$$

Instead of using our density formula, the moments can also be derived by standard martingale arguments as follows. Let $Z_0 = 0$, $Z_n = \sum_{j=1}^n (E_j - Z_j)$ for $n \geq 1$, and $\nu = \inf\{n \geq 1 | Z_n \geq \beta\}$. Then $T_U(\beta) = S_\nu$. Clearly, $\phi(s) = E(e^{sZ_1}) < \infty$ for

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$s < \lambda$, and $M_n = e^{sZ_n}\phi(s)^{-n}$, $n \geq 0$, is a martingale. As $E(Z_1) > 0$, the stopping rule ν is almost surely finite and has finite moments of all orders. Since the E_j are exponential, the overshoot $Z_\nu - \beta$ is $\exp(\lambda)$ distributed and independent of ν. The optional stopping theorem yields

$$1 = E(M_0) = E(M_\nu) = E(e^{sZ_\nu}\phi(s)^{-\nu}) = \frac{e^{\beta s}}{1 - (s/\lambda)} E(\phi(s)^{-\nu});$$

that is,

$$E(\phi(s)^{-\nu}) = e^{-\beta s}(1 - (s/\lambda)).$$

Now, we can immediately get $E(\nu)$ by taking the derivative of the last equation, yielding

$$-\phi'(s)E(\nu\phi(s)^{-(\nu-1)}) = -e^{-\beta s}[\beta(1 - (s/\lambda)) + (1/\lambda)],$$

and then setting $s = 0$. We obtain

$$E(\nu) = \frac{1 + \beta\lambda}{1 - \rho}.$$

Therefore, by Wald's equation,

$$E(T_U(\beta)) = E(S_\nu) = E(X_1)E(\nu) = \frac{\mu + \beta\rho}{1 - \rho}.$$

Finally, $E(T_U(\beta)^2) = E(S_\nu^2)$ can be found using the martingale $\widetilde{M}_n = (S_n - \mu n)^2 - n\,\mathrm{Var}(X_1)$.

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