Macroeconomic Dynamics, **24**, 2020, 403–420. Printed in the United States of America.
doi:10.1017/S1365100518000275

BANK RUNS: THE PREDEPOSIT GAME

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We analyze in some detail the full *predeposit* game in a simple, tractable, yet very rich, banking environment. How does run-risk affect the optimal deposit contract? If there is a run equilibrium in the *postdeposit* game, then the optimal contract in the *predeposit* game tolerates small-probability runs. However, this does not mean that small changes in run-risk are ignored. In some cases, the optimal contract becomes—as one would expect—strictly more conservative as the run-probability increases (until it switches to the best run-proof contract), and the equilibrium allocation is not a mere randomization over the equilibrium allocations from the *postdeposit* game. In other cases, the allocation *is* a mere randomization over the equilibria from the *postdeposit* game. In the first cases (the more intuitive cases), the incentive constraint does not bind. In the second cases, the incentive constraint does bind.

Keywords: Bank Runs, Predeposit Game, Run Probability, Sunspots

1. INTRODUCTION

Bryant (1980) and Diamond and Dybvig (1983)—hereafter DD—introduced the modern literature on panic-based bank runs. The bank deposit contract is a mechanism designed to improve the welfare of depositors facing an uncertain impulse demand (i.e., becoming impatient). Since the impulse demand itself is not directly observable, it is uninsurable in the market. The deposit contract facilitates some needed "insurance" through early and late withdrawals that are Bayesian incentive compatible (hereafter BIC), i.e., so that depositors with different liquidity needs correctly self-select their types. But the "good" equilibrium in which the depositors correctly self-select is not the unique Bayes–Nash equilibrium (hereafter BNE). DD show that there is also a "bad" equilibrium in which there is a self-fulfilling bank run. In a bank run, depositors attempt to withdraw early no matter what their liquidity needs are.

But given the two equilibria of the *postdeposit* game, consumers will not deposit in the bank if they anticipate the run: a probability-one bank-run will not be an

We thank Huberto Ennis, Chao Gu, Todd Keister, Jim Peck, the referee, and the associate editor for their helpful comments. Address correspondence to: Yu Zhang, Wang Yanan Institute for Studies in Economics (WISE), Xiamen University, Xiamen 361005, Fujian Province, China; e-mail: yuzhang2012@xmu.edu.cn.

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https://doi.org/10.1017/S1365100518000275 Published online by Cambridge University Press

equilibrium in the *predeposit* game. DD recognized this problem and offered sunspots as an answer.[1] The question is how safe a bank or banking system should be given the risk of runs.

Peck and Shell (2003)—hereafter PS—analyze a banking model in which partial suspension of convertibility is permitted. They have shown that the optimal banking contract can tolerate runs if the run probability is small.[2] PS employed a 2-depositor numerical example in calculating the optimal deposit contract as a function of the run probability. The banking contract is completely characterized by the scalar c, the withdrawal of the first depositor in line in period 1. The smaller the value of c, the more conservative is the deposit contract. The optimal c is denoted by $c^*(s)$, which is a function of the exogenous, sunspot run-probability s.[3] In the calculated example in PS, $c^*(s)$ is a step function: If the probability s is greater than some fraction s_0, the optimal contract is the best run-proof contract. If the probability s is less than the threshold probability s_0, the optimal contract $c^*(s)$ tolerates runs and is a constant.

In the present paper, we ask: Why does not the optimal contract become strictly more conservative as the run probability increases (until runs are no longer tolerated)? In other words, should not we expect $c^*(s)$ to be strictly decreasing in s until the contract switches to the best run-proof contract? If yes, which economies exhibit this property and in which economies is $c^*(s)$ a step function? Are there other (perhaps mixed) cases? To answer these questions and to keep the analysis tractable, we employ the general 2-depositor banking example from PS, but instead of relying solely on a single numerical example, we provide the global comparative statics of the optimal contract.

We analyze the game backward and start the analysis from the *postdeposit* game for an arbitrary contract c. We characterize the conditions on c which allow for a run equilibrium and a nonrun equilibrium in the *postdeposit* game. The optimal contract c^* is the BIC contract which maximizes the expected utility of the depositor in the *predeposit* game.

A contract c is BIC, if for c, there is a non-run BNE in the *postdeposit* game. Therefore, the set of contracts with nonrun as the unique BNE is a subset of the set of BIC contracts. For values of the parameters in which the depositors' actions do not exhibit strategic complementarity for any contract (hereafter non-SC parameters), the two sets are the same. For the non-SC parameters, bank runs are not relevant for the optimal contract and hence the analysis of the *predeposit* game is similar to that of the corresponding *postdeposit* game. Our main focus is on the parameter values in which the depositors' actions exhibit strategic complementarity for *some* contract c (hereafter SC parameters). For the SC parameters, the set of contracts with nonrun as the unique BNE is a *strict* subset of the set of BIC contracts. The optimal c^* changes with s unless nonrun is the unique BNE. The shape of the function $c^*(s)$ depends on further specification of the remaining parameters.

To further examine $c^*(s)$, we divide the SC parameter space into three subspaces depending on if the unconstrained efficient allocation is: (1) the unique BNE;[4]

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(2) not the unique BNE but is BIC; or (3) not even BIC. In the first case, c^* is the contract supporting the unconstrained efficient allocation and it does not change with s since the "good equilibrium" is the unique BNE.[5] In the second case, c^* balances the trade-off between the nonrun and run equilibria. As s increases, the trade-off changes continuously and the expected utility of the depositor is more dependant on the run equilibrium. Hence, the optimal contract $c^*(s)$ is continuous and strictly decreasing until it switches to the best run-proof contract. In the third case, the trade-off is constrained by the incentive-compatibility constraint (hereafter ICC). When s is small, the ICC binds and the optimal contract does not respond to changes in s since the binding ICC is independent of s. In this case, a larger run probability does not induce a more conservative contract since the binding ICC has already forced $c^*(s)$ to be more conservative than it would have been absent the ICC.

Green and Lin (2000), Andolfatto et al. (2007), and Nosal and Wallace (2009) analyze models similar to PS. A main difference between these models and PS is on the amount of information that a depositor has at the time of making his withdrawal decision. Ennis and Keister (2009) show that the PS assumptions on marginal utilities are not necessary for the qualitative results in PS. Ennis and Keister (2009) also study the Green–Lin model under a more general specification of the distribution of types across agents. See Ennis and Keister (2010) for a good survey on this part of the literature. Ennis and Keister (2006) examine how the possibility of the sunspot-driven bank run affects the investment decisions made by a bank in the Cooper and Ross (1998) model. They show that when the cost of liquidating investment early is high, an increase in the probability of a run will lead the bank to invest less. However, when liquidation costs are moderate, the level of investment is increasing in the probability of a run. Gu (2011) analyzes the sunspot-driven bank run with the noisy private sunspot signals. These three papers are similar to PS and our present paper. The main difference is that these papers are based on simple deposit contracts, while PS and our paper allow for partial or full suspension of convertibility. Bank runs are instances of coordination failure. In our banking model, it is not generally optimal to choose the contract that supports the best nonrun equilibrium if the contract also supports a run equilibrium. Ennis and Keister (2005) have analyzed similar situations but in a nonbanking setting.

The remainder of the paper is organized as follows. In Section 2, we introduce the notation and the formal model. In Section 3, we analyze the *postdeposit* game and characterize both the nonrun equilibrium and the run equilibrium in the *postdeposit* game. We also identify the SC and non-SC parameters. In Section 4, we analyze the *predeposit* game and provide comparative statics for the optimal contract for SC parameter values. We focus in Section 4 on comparative statics with respect to the impulse demand parameter. Three online appendices can be found at http://karlshell.com/wp-content/uploads/2018/04/SZ-3.30.2018-appendices.pdf. Our proofs are in Appendix A. Comparative statics with respect to parameters other than the impulse demand parameter are in Appendix B. The analyses are more complicated because these parameters do not affect the optimal contract in

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the simple multiplicative way of the impulse demand parameter. In Appendix C, we provide—for completeness—analyses for the non-SC parameter values.

2. THE ENVIRONMENT

There are two consumers and three periods: 0, 1, and 2. In period 0, each consumer is endowed with y units of the consumption good. There are no endowments in periods 1 and 2. Let c^1 and c^2 denote the withdrawals of the depositor in period 1 and 2, respectively. The impatient consumers derive utility only from period-1 consumption, while the patient consumers derive utility only from period-2 consumption. The patient consumers can store consumption goods costlessly across the two periods. The impatient and patient consumers, respectively, have utility functions $u(c^1)$ and $v(c^1 + c^2)$, where

$$u(x) = A\frac{(x)^{1-b}}{1-b} \text{ where } A > 0, \tag{1}$$

$$v(x) = \frac{(x)^{1-b}}{1-b}. \tag{2}$$

A is the intensity of the impulse demand of an impatient consumer. PS introduced this parameter in the bank runs literature. The parameter $b > 1$ is the coefficient of relative risk aversion. Consumers are identical in period 0. In period 1, each consumer becomes either impatient with probability p or patient with probability $1 - p$. Types are uncorrelated and private information. Since the number of consumers is finite, the aggregate number of patient depositors is stochastic. In period 1, each depositor also observes a sunspot variable δ distributed uniformly on [0, 1]. Besides the storage technology, there is another investment technology. Investing one unit of period 0 consumption yields $R > 1$ units if harvested in period 2 and yields 1 unit if harvested in period 1.

The sequential service constraint is part of the physical environment. A depositor visits the bank only when he makes a withdrawal. When a depositor learns his type and makes his withdrawal decision, he does not know his position in the bank queue. If more than one depositor chooses to withdraw, a depositor's position in the queue is random; in particular, positions in the queue are equally probable.

The *predeposit* game is a sequential game. In the first-stage (period 0), the bank chooses the contract c and the consumers choose whether to deposit or not. If the consumers choose to deposit,[6] they move on to the second stage (period 1) which is the *postdeposit* game and decide when to make withdrawals. As usual in sequential games, we solve it by backward induction.

3. THE POSTDEPOSIT GAME

The *postdeposit* game is a static game between two depositors with incomplete information. The incomplete information is due to the fact that one's "type" is

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TABLE 1. Pay-off matrix

	(E, E)	(L, E)
(E, E)	T_1, T_1	T_3, T_2
(L, E)	T_2, T_3	T_4, T_4

private information. The action set of each depositor is $\{E, L\}$, where E (L) stands for early (late) withdrawal. An impatient depositor always chooses E. Hence, the relevant strategy of a depositor is (E, E) or (L, E), where the first (second) element in the parenthesis stands for the action chosen by the depositor when he is patient (impatient). The pay-off matrix is in Table 1. The first symbol in each cell represents the payoff to the row player and the second symbol represents the payoff to the column player. The entries in the payoff matrix are given by

$$T_1 = (1-p)\frac{v(c)+v(2y-c)}{2} + p\frac{u(c)+u(2y-c)}{2},$$

$$T_2 = (1-p)v[(2y-c)R] + p\frac{u(c)+u(2y-c)}{2},$$

$$T_3 = (1-p)\left[(1-p)v(c) + p\frac{v(c)+v(2y-c)}{2}\right] + p\left[(1-p)u(c) + p\frac{u(c)+u(2y-c)}{2}\right],$$

$$T_4 = (1-p)\left[(1-p)v(yR) + pv[(2y-c)R]\right] + p\left[(1-p)u(c) + p\frac{u(c)+u(2y-c)}{2}\right].$$

And, T_1 is the expected utility of a depositor if both depositors choose early withdrawals regardless of their types. With probability $(1-p)$, the depositor is patient and his utility is given by the function v. And, with probability p, the depositor is impatient and his utility is given by the function u. In both cases, the depositor can be either the first or the second in the queue with the same probability (i.e., $\frac{1}{2}$). We leave the derivation of T2, T3, and T4 to the reader.

The depositors' actions exhibit strategic complementarity when a patient depositor chooses early withdrawal if and only if the other patient depositor also chooses early withdrawal. That is, the *postdeposit* game has both a run equilibrium (i.e., $T_1 > T_2$) and a nonrun equilibrium (i.e., $T_4 \geq T_3$). For a deposit contract $c \in [0, 2y]$, whether the depositors' actions exhibit strategic complementarity or not depends on the parameter values. We next characterize the conditions on the parameters that allow for strategic complementarity.

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3.1. Run Equilibrium in the Postdeposit Game

A run equilibrium in the *postdeposit* game is a BNE in which both depositors choose (E, E), which requires that $T_1 > T_2$, or equivalently, that $c \in [0, 2y]$ satisfies[7]

$$[v(c) + v(2y - c)]/2 > v[(2y - c)R]. \quad (3)$$

Whether there is a contract $c \in [0, 2y]$ satisfying inequality (3) depends on the parameters b and R. This is because, from inequality (3), the patient depositor's comparison between the early and late withdrawals depends on the following: (1) his attitude toward the risk of being the second in line while participating in a run (i.e., b), and (2) the productivity of the investment technology (i.e., R). Given R, the more risk-averse the patient depositor, the less preferable is it for him to run on the bank. Given his attitude toward risk, the more productive the investment, the less preferable is it for him to run because more resources will be left in the last period. Hence, the existence of $c \in [0, 2y]$ satisfying inequality (3) restricts the parameters b and R. We have the following lemma.

LEMMA 1. *If $b < 1 + \ln 2/\ln R$ holds, then there is a run equilibrium in the postdeposit game if and only if c satisfies the inequality*

$$c > c^{early} = 2y/[(2/R^{b-1} - 1)^{1/(b-1)} + 1]. \quad (4)$$

If $b \geq 1 + \ln 2/\ln R$ holds, then there is no run equilibrium in the postdeposit game for any $c \in [0, 2y]$.

If $b < 1 + \ln 2/\ln R$ holds, then c^{early} is the level of c beyond which a patient depositor chooses early withdrawal if he expects that the other depositor—if patient—will also choose early withdrawal.

3.2. NonRun Equilibrium in the Postdeposit Game

A nonrun equilibrium in the *postdeposit* game is defined as a BNE in which both depositors choose (L, E), which requires $T_4 \geq T_3$, or equivalently, that $c \in [0, 2y]$ satisfies

$$(1 - p)v(yR) + pv[(2y - c)R] \geq (1 - p)v(c) + p[v(c) + v(2y - c)]/2. \quad (5)$$

Inequality (5) is also the ICC.

LEMMA 2. *If $b < 1 + \ln 2/\ln R$ holds, there is a nonrun equilibrium in the postdeposit game if and only if c satisfies the inequality*

$$c \leq c^{wait}, \quad (6)$$

where c^{wait} is the level of c such that (5) holds as an equality.

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Denote the difference between the LHS and the RHS of the inequality (5) by $d(c)$, which measures a patient depositor's utility differential between early and late withdrawals when the other patient depositor chooses late withdrawal. Intuitively, as the bank promises to pay more to the first depositor demanding early withdrawal, it has two effects on a patient depositor's incentive to demand early withdrawal. On the one hand, it makes early withdrawal more attractive since less resources would be left in the bank in the last period if the other depositor demands early withdrawal. On the other hand, it also makes demanding early withdrawal more risky since the second depositor in the bank queue gets less.

If $b < 1 + \ln 2/\ln R$, the depositor is not very risk-averse and the first effect dominates. We have $d(c)$ is strictly decreasing with c and $d(c^{\text{wait}}) = 0$. Thus, c^{wait} is the level of c *below* which a patient depositor chooses late withdrawal if he expects that the other patient depositor will also choose late withdrawal. Thus, the set of BIC contracts is $[0, c^{\text{wait}}]$.

If, however, $b \geq 1 + \ln 2/\ln R$, the depositor is more risk-averse and the first effect does not always dominate. It can be shown that $d(c)$ is strictly decreasing in c up to a certain level, $c_{\min}$, and thereafter strictly increasing in c. And, $d(0) = d(2y) = \infty$.Thus, depending on whether $d(c_{\min})$ is positive or not, the set of BIC contracts is either $[0, 2y]$ or the union of two separated subsets of $[0, 2y]$.

3.3. Equilibria in the Postdeposit Game for an Arbitrary Contract

The depositors' actions exhibit strategic complementarity when a patient depositor chooses early withdrawal if and only if the other patient depositor also chooses early withdrawal. That is, there is a contract $c \in [0, 2y]$ such that the *postdeposit* game has both a run equilibrium (i.e., $T_1 > T_2$) and a nonrun equilibrium (i.e., $T_4 \geq T_3$).

From Lemmas 1 and 2, we can see that the contract, for which the depositors' actions exhibit strategic complementarity, exits if and only if the parameters are such that $b < 1 + \ln 2/\ln R$ and $c^{\text{early}} < c^{\text{wait}}$. In the proof of Lemma 3, we show that $c^{\text{early}} < c^{\text{wait}}$ requires $b < 2$. Hence, we characterize the set of parameters that permit strategic complementarity by inequality (7) in Lemma 3.

LEMMA 3. *The depositors' actions exhibit strategic complementarity if and only if the parameters satisfy*

$$b < \min\{2, 1 + \ln 2/\ln R\}. \tag{7}$$

We call parameters b and R which satisfy inequality (7) the SC parameters. For the SC parameters, c^{early} and c^{wait} are well defined. For $c \in (c^{\text{early}}, c^{\text{wait}}]$, we have $T_2 < T_1$ and $T_4 \geq T_3$. The decisions of the two depositors are strategically complementary, leading to both one run and one nonrun equilibrium. The *postdeposit* game is "stag hunt" ; For $c \in [0, c^{\text{early}}]$, we have $T_2 \geq T_1$ and $T_4 > T_3$. (L, E) is the dominant strategy for each depositor. The *postdeposit* game has a unique BNE which is also a dominant strategy equilibrium with Pareto efficiency. For

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$c \in (c^{\text{wait}}, 2y]$, we have $T_2 < T_1$ and $T_4 < T_3$. (E, E) is the dominant strategy for each depositor. The *postdeposit* game is of "prisoner's dilemma" type; running is the unique equilibrium strategy.

From PS, we know that the *predeposit* game has a run equilibrium only if the *postdeposit* game has both a nonrun equilibrium and a run equilibrium. Therefore, given the SC parameters, we know that a run-equilibrium exists in the *predeposit* game only if the optimal deposit contract belongs to $(c^{\text{early}}, c^{\text{wait}}]$. In the next section, we solve for the optimal deposit contract. Before we describe the optimal contract, we provide a numerical example of SC parameters.

Example 1

The parameters are

$$b = 1.01;\ p = 0.5;\ y = 3;\ R = 1.5.$$

These parameters will be fixed throughout the examples. We see that b and R satisfy (7). Hence, c^{early} *must be strictly smaller than* c^{wait}*. Indeed, we have* $c^{early} = 4.155955$ *and* $c^{wait} = 4.280878$*. Therefore, whenever the contract c is larger than* 4.155955 *and smaller than or equal to* 4.280878*, both a run equilibrium and a nonrun equilibrium exist in the postdeposit game.*

For completeness, we take a look at the *postdeposit* game with non-SC parameters [i.e., the values of b and R that do not satisfy (7)]. Under these values of parameters, the depositors' actions do not exhibit strategic complementarity for *any* BIC contract. According to the Revelation Principle, when we search for the optimal contract we only have to focus on c which is BIC. Therefore, for non-SC parameters, bank runs are not relevant. This would correspond to the upper dominance region in Goldstein and Pauzner (2005). We discuss the optimal contract under non-SC parameters in the online appendix. For the remainder of the paper, we focus on the economy with SC parameters.

4. THE OPTIMAL CONTRACT IN THE PREDEPOSIT GAME

We have shown that with SC parameters for any $c \in (c^{\text{early}}, c^{\text{wait}}]$, we have multiple equilibria in the *postdeposit* game. We assume—as in PS—that bank runs are sunspot-driven. Hence, whether a run equilibrium exists in the *predeposit* game depends on whether in the first stage the contract chosen by the bank (i.e., the optimal contract) belongs to the set $(c^{\text{early}}, c^{\text{wait}}]$. To characterize the optimal contract, we divide the SC parameters into three cases depending on $\widehat{c}$, the contract supporting the *unconstrained efficient allocation*. The three cases are as follows: $\widehat{c} \leq c^{\text{early}}$ (Case 1), $\widehat{c} \in (c^{\text{early}}, c^{m}]$ (Case 2), and $\widehat{c} > c^{\text{wait}}$ (Case 3). We next characterize the parameters for each case. In particular, we show in the following section that each case defines a range of values of the parameter A, the impulse intensity of the impatient consumer.

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4.1. The Impulse Parameter A and the Three Cases

The contract $\widehat{c}$ supports the *unconstrained efficient allocation*, where $\widehat{c}$ is defined by

$$\widehat{c} = \arg \max_{c \in [0,2y]} \widehat{W}(c),$$

where

$$\widehat{W}(c) = p^2[u(c)+u(2y-c)]+2p(1-p)[u(c)+v[(2y-c)R]]+2(1-p)^2 v(yR). \quad (8)$$

$\widehat{W}(c)$ is the depositor's expected utility if the types of the depositors are publicly observable *ex-post*.[8] Given the CRRA utility function, we have

$$\widehat{c} = \frac{2y}{\{p/(2-p) + 2(1-p)/[(2-p)AR^{b-1}]\}^{1/b} + 1}. \quad (9)$$

From (9), we can write $\widehat{c}$ as an increasing function of A, $\widehat{c}(A)$. As the "impulse demand" is more intense, the unconstrained efficient allocation provides for a larger first-period withdrawal. From equation (9), we have

$$\lim_{A \to 0} \widehat{c}(A) = 0$$

and

$$\lim_{A \to \infty} \widehat{c}(A) = \frac{2y}{[p/(2-p)]^{1/b} + 1}.$$

From equations (4) and (5), we know that neither c^{early} nor c^{wait} depends on A. This is intuitive: c^{early} and c^{wait} are determined solely by the patient depositor's utility, which is independent of A.

Hence, if A is sufficiently small, we have Case 1. Furthermore, if

$$\frac{2y}{[p/(2-p)]^{1/b} + 1} \leq c^{\text{early}},$$

only Case 1 obtains.

If

$$c^{\text{early}} < \frac{2y}{[p/(2-p)]^{1/b} + 1} \leq c^{\text{wait}},$$

there is a unique level of A, denoted by A^{early}, such that

$$\widehat{c}(A) = c^{\text{early}}. \quad (10)$$

Hence, if $A \leq A^{\text{early}}$, we have Case 1. If $A > A^{\text{early}}$, we have Case 2. The unconstrained efficient allocation never leads to Case 3.

If

$$c^{\text{wait}} < \frac{2y}{[p/(2-p)]^{1/b} + 1},$$

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there is a unique level of A, denoted by A^{wait}, such that

$$\widehat{c}(A) = c^{\text{wait}}. \tag{11}$$

We now have all three cases: if $A \leq A^{\text{early}}$, we are in Case 1; if $A^{\text{early}} < A \leq A^{\text{wait}}$, we are in Case 2; if $A > A^{\text{wait}}$, we are in Case 3.[9]

Example 2

In Example 1, we know that $c^{early} = 4.155955$ *and* $c^{wait} = 4.280878$. *Now, we calculate the thresholds of A for each case. It is easy to see that*

$$c^{wait} < \frac{2y}{[p/(2-p)]^{1/b} + 1}$$

for the specified parameter values. Therefore, all three cases exist. We have $A^{early} = 6.217686$ *and* $A^{wait} = 10.27799$. *Hence, if* $A \leq 6.217686$, *we are in Case 1; if* $6.217686 < A \leq 10.27799$, *we are in Case 2; if* $A > 10.27799$, *we are in Case 3.*

In the online appendix, we show how the other parameters—namely p, R, and b—affect $\widehat{c}$. Unlike A, these parameters also affect the values of c^{early} and/or c^{wait}, and $\widehat{c}$ is not monotonic in the parameters. The analysis is slightly more complicated in these cases, but once the parameter values are given, one can readily compute the values of $\widehat{c}$, c^{early}, and c^{wait} to determine which case is applicable.

4.2. The Optimal Contract

In this subsection, we focus on the parameter values of b and R satisfying condition (7). We will describe the optimal contract c^* for the three cases specified above. For an arbitrary contract $c \in [0, 2y]$, we have one or two equilibria in the *postdeposit* game depending on whether c belongs to $(c^{\text{early}}, c^{\text{wait}}]$ or not. As equation (8) shows, the depositor's expected utility in the nonrun equilibrium is $\widehat{W}(c)$. Let $W^{\text{run}}(c)$ denote the depositor's expected utility in the run equilibrium (if it exists). It is given by

$$\begin{aligned} W^{\text{run}}(c) = {} & p^2[u(c) + u(2y - c)] \\ & + p(1-p)[u(c) + v(2y - c) + v(c) + u(2y - c)] \\ & + (1-p)^2[v(c) + v(2y - c)]. \end{aligned} \tag{12}$$

When $c \leq c^{\text{early}}$, only the nonrun equilibrium exists and the depositor's ex-ante expected utility is simply $\widehat{W}(c)$. When $c^{\text{early}} < c \leq c^{\text{wait}}$, both the run equilibrium and the nonrun equilibrium exist in the *postdeposit* game. If the run probability is s, the depositor's ex-ante expected utility is

$$(1-s)\widehat{W}(c) + sW^{\text{run}}(c).$$

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When we have $c^{\text{wait}} < c \leq 2y$, only the run equilibrium exists and, therefore, the optimal contract is not in that range of c. Hence, when we consider the optimal contract, $[0, c^{\text{wait}}]$ is the relevant choice interval for c.

Let $c^*(s)$ denote the optimal contract which maximizes the depositor's ex-ante expected utility in the *predeposit* game given the run probability s.[10] We have

$$c^*(s) = \arg \max_{c \in [0, c^{\text{wait}}]} W(c; s),$$

where

$$W(c; s) = \begin{cases} \widehat{W}(c) & \text{if } c \leq c^{\text{early}}. \\ (1-s)\widehat{W}(c) + sW^{\text{run}}(c) & \text{if } c^{\text{early}} < c \leq c^{\text{wait}}. \end{cases} \quad (\mathbf{13})$$

Case 1. *The unconstrained efficient allocation is the unique BNE, i.e., we have* $\widehat{c} \leq c^{early}$.

Since the unconstrained efficient allocation is uniquely implementable, it is straightforward to see that the optimal contract for the *predeposit* game supports the unconstrained efficient allocation

$$c^*(s) = \widehat{c} \text{ for } s \in [0, 1],$$

and that a bank run does not occur in equilibrium. Other contracts cannot be optimal because they either deliver lower welfare in the nonrun equilibrium or, to make things worse, they might also support a run equilibrium. The next is a numerical example for Case 1.

Example 3

In Example 2, we have seen that as long as $A \leq A^{early} = 6.217686$, *we have Case 1 for this economy. Let A be equal to* 1. *We have* $c^*(s) = \widehat{c} = 3.004012$ *for any* $s \in [0, 1]$. *Since* $c^*(s) < c^{wait} = 4.280878$, *a bank run is not an equilibrium.*

As we have discussed in Section 4.1, when A is larger than A^{wait}, we have Case 2.

Case 2. *The unconstrained efficient allocation is BIC but not the unique BNE, i.e., we have* $c^{early} < \widehat{c} \leq c^{wait}$.

The optimal contract for the *predeposit* game cannot be $\widehat{c}$ except for the degenerate case when $s = 0$. This is because a run equilibrium exists at $\widehat{c}$. How much the optimal contract deviates from $\widehat{c}$ depends on s, which changes the trade-off between the expected utilities over the 2 *postdeposit* game equilibria. We show in Proposition 1 how the optimal contract to the *predeposit* game changes with the probability s.

PROPOSITION 1. *In Case 2, the optimal contract* $c^*(s)$ *satisfies: (1) if s is larger than the threshold probability* s_0 *(specified in the online appendix), the*

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optimal contract is run-proof, $c^*(s) = c^{early}$ and (2) if s is smaller than s_0, the optimal contract $c^*(s)$ tolerates runs and it is a strictly decreasing function of s. We have that $c^*(s) \leq \widehat{c}$ (with equality if and only if $s = 0$).

From Proposition 1, we see that in Case 2 the contract supporting the unconstrained efficient allocation also supports the run equilibrium from the *postdeposit* game. Except for the degenerate case of the run probability $s = 0$, this contract cannot be optimal since it delivers very low welfare in the run equilibrium. The optimal contract should optimize the ex-ante trade-off between the depositors' welfares in the run and the nonrun equilibria. The trade-off depends on the run probability s. For positive s, a more conservative contract, still tolerating runs or eliminating runs completely, is desirable. If s is larger than the threshold probability s_0, eliminating runs is less costly (in terms of ex-ante welfare) than tolerating runs; hence, the optimal contract is the best run-proof contract, c^{early}, in which the nonrun equilibrium is implemented uniquely. On the other hand, if s is smaller than s_0, tolerating runs is less costly. Furthermore, $c^*(s)$ will be strictly decreasing in s since, as s increases, the ex-ante welfare depends more on the welfare in the run equilibrium. The sunspot equilibrium allocation in this case is *not* a mere randomization over the *unconstrained efficient allocation* and the corresponding *run allocation*.

Example 4

In Example 2, we have seen that as long as $6.217686 < A \leq 10.277988$, we are in Case 2. Set $A = 8$. We have $s_0 = 1.382358 \times 10^{-3}$. If $s > s_0$, the optimal contract is run-proof and $c^*(s) = c^{early} = 4.155955$. If $s < s_0$, the optimal contract tolerates runs and $c^*(s)$ is strictly decreasing in s with $c^*(0) = \widehat{c} = 4.225479$. If $s = s_0$, both the run-proof contract (c^{early}) and the run-tolerating contract are optimal. Figure 1 is a plot of $c^*(s)$ for this case.

As we have discussed in Section 4.1, we shift from Case 2 to Case 3 for even larger values of A.

Case 3. *The unconstrained efficient allocation is not BIC, i.e., we have $c^{wait} < \widehat{c}$.*

In this case, the unconstrained efficient allocation is not implementable since $\widehat{c}$ cannot satisfy the ICC. Like Case 2, the optimal contract for the *predeposit* game also involves the trade-off between the two *postdeposit* game equilibria, but the trade-off is constrained by the ICC.[11] This changes how $c^*(s)$ tolerates runs. To provide sufficient incentives for the patient depositors to choose late withdrawal, the ICC requires that c is not too large. Hence, the binding ICC forces $c^*(s)$ to be more conservative than it would be without the ICC. If the ICC binds, then for small s when runs are still tolerated, c^* is independent of s.

PROPOSITION 2. *In Case 3, (1) if s is larger than the threshold probability s_1 (specified in the online appendix), we have $c^*(s) = c^{early}$ and the optimal contract is run-proof, and (2) if s is smaller than s_1, the optimal contract $c^*(s)$*

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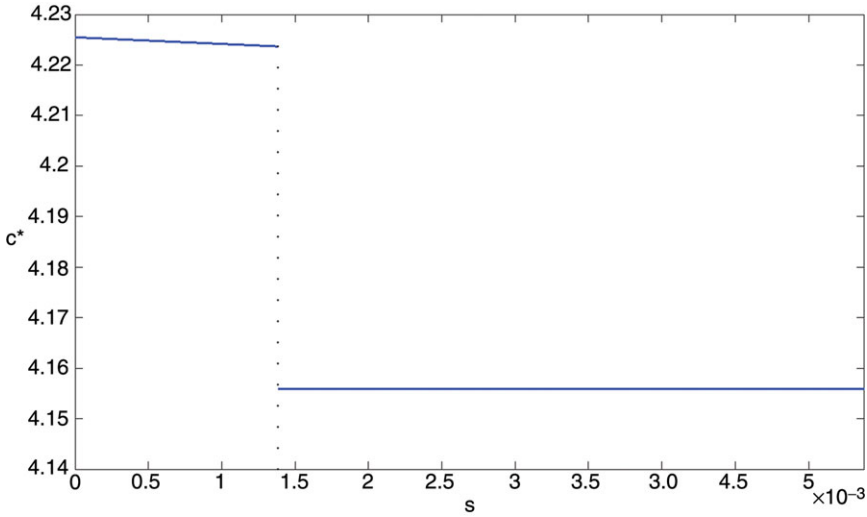


FIGURE 1. c* for $A = 8$.

tolerates runs and it is a weakly decreasing function of s. Furthermore, we have $c^*(s) = c^{wait}$ *(i.e., ICC binds) for at least part of the run tolerating range of s.*

At least when s is sufficiently small, the ICC binds. So for Case 3, when s is sufficiently small, $c^*(s)$ does not change with s since the ICC does not depend on s. The ICC might bind for merely part of the run-tolerating range of s (the first subcase) or the whole run-tolerating range of s (the second subcase). The PS numerical example belongs to the second subcase. When the ICC binds, the allocation supported by the optimal contract *is* a simple randomization over the *constrained efficient allocation*[12] and the corresponding run allocation. The *unconstrained efficient allocation* is never supported by $c^*(s)$ since it is not implementable.

We next provide two examples of the optimal contract for Case 3, which correspond to the two subcases.

Example 5

In Example 2, we have seen that as long as $A > 10.277988$*, we are in Case 3. Let A be equal to* 10.4 *and we are in the first subcase of Case 3. If s is larger than* $s_1 = 4.524181 \times 10^{-3}$*, the optimal contract is run-proof and* $c^*(s) = c^{early} = 4.155955$*. If s is smaller than* s_1*, the optimal contract* $c^*(s)$ *tolerates runs and it is a weakly decreasing function of s. To be more specific, if s is smaller than* $s_2 = 1.719643 \times 10^{-3}$*, the ICC binds and* $c^*(s) = c^{wait} = 4.280878$*; if* $s_2 \leq s < s_1$*, the ICC does not bind and* $c^*(s)$ *is strictly decreasing in s.*

Example 6

Let A be equal to 11*. We are in the second subcase of Case 3. If s is larger than* $s_1 = 5.281242 \times 10^{-3}$*, the optimal contract is run-proof and* $c^*(s) = c^{early} =$

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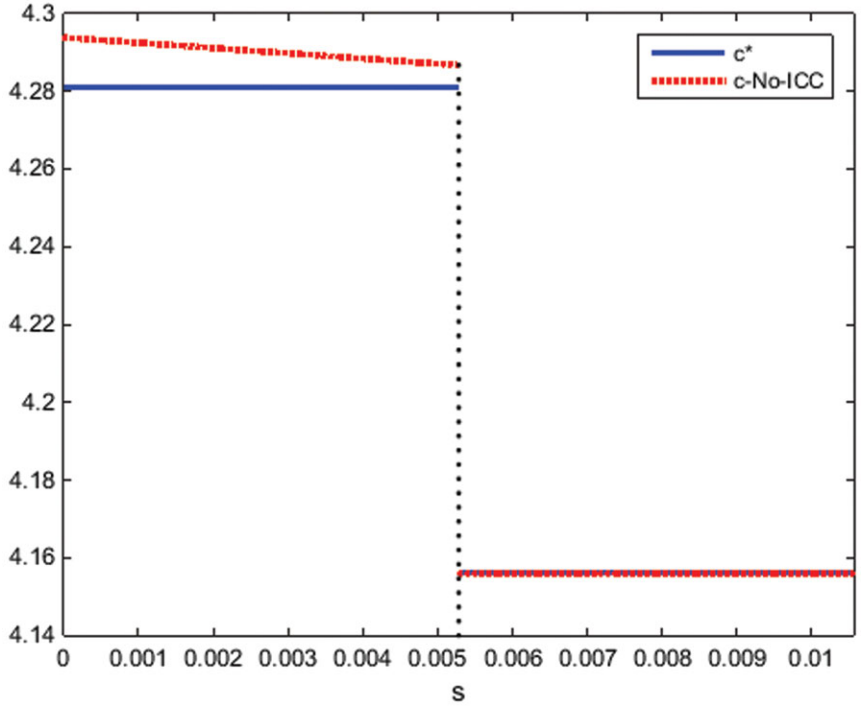


FIGURE 2. c^* and c-No-ICC for $A = 11$.

4.155955. *If* $s < s_1$, *the optimal contract tolerates runs and* $c^*(s) = c^{wait} = 4.280878$ *since ICC binds. When* $s = s_1$, *both the run-proof contract* (c^{early}) *and the run-tolerating contract* (c^{wait}) *are optimal. To show the effect of the binding ICC, we plot in Figure 2 the hypothetical optimal contract in which the ICC is removed (labelled as c-No-ICC) along with* $c^*(s)$. *The hypothetical contract is infeasible if—as we assume—types are private information.*

We plot c^* versus both s and A in Figure 3. Given the parameters we have used in previous examples, we have the following: if $A \leq A^{\text{early}} = 6.217686$, we are in Case 1 and the optimal contract does not tolerate runs and it is equal to $\widehat{c}$ the value of which depends on A. If $6.217686 < A \leq 10.277988$, we are in Case 2 and the optimal contract is strictly decreasing in s until it levels off at the best run-proof contract $c^{\text{early}} = 4.155955$. If $10.277988 < A$, we are in Case 3 and the ICC binds when s is small. The ICC may bind either in part of the run-tolerating range of s (the first subcase) or the whole run-tolerating range of s (the second subcase).

5. SUMMARY

This paper is about bank runs in the *predeposit* banking model. Understanding the *predeposit* model is essential for understanding bank runs: "Runs" in the DD

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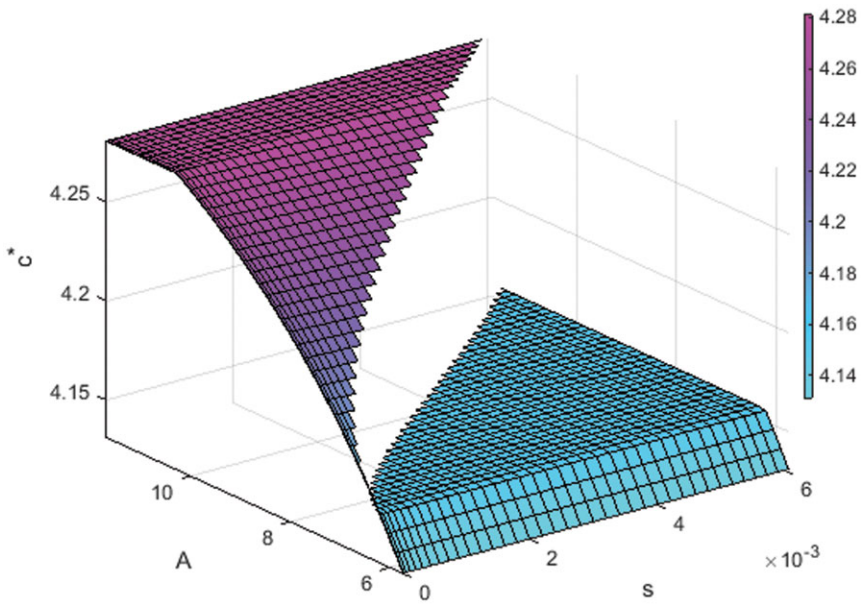


FIGURE 3. $c^*(s; A)$.

postdeposit model are not necessarily runs in the full sequential analysis, in which the bank's design of the optimal contract anticipates the withdrawal decisions of the depositors. We show that the optimal contract can depend in a nontrivial way on the probability of sunspot-driven runs.

Peck and Shell (2003) have shown that the probability of runs determines *whether or not* runs are tolerated: The optimal banking contract tolerates runs if and only if the run probability is small. One of our main contributions is the analysis of *how* the optimal banking contract responds to changes in run risk. Intuitively, changes in the run risk should not be ignored and the contract should be more conservative as run risk increases. In the numerical example in PS, however, the optimal contract does not change with the run probability for as long as the run is tolerated. We show that this (seemingly counter-intuitive) result in the PS example follows from the fact that the ICC constraint binds in the PS example. The binding ICC forces the contract to be more conservative than it would have been absent the ICC.

A lesson from daily life might be instructive. It can be risky for a walker to cross the street, but if the risks are sufficiently small, crossing might be his best option. However, this does not mean that the small risk should be ignored. We would expect that his response to a small increase in risk due to heavier traffic might lead the walker to be more cautious by looking more attentively to the left and the right. This will be the case unless the pedestrian has been constrained by a traffic law that requires him to cross the street more cautiously than he would absent the binding law. The binding law is analogous to the binding ICC constraint.

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We characterize the parameters (Cases 2 and 3) in which runs matter. In each of these cases, the optimal contract switches to being run-proof when the run probability is sufficiently large. But *how* the optimal contract tolerates runs differs between the two cases: In Case 2, the ICC does not bind since the unconstrained efficient allocation is (weakly) implementable. As a result of balancing the trade-off between the run equilibrium and nonrun equilibrium in the *postdeposit* game, the optimal contract adjusts continuously and becomes more conservative as the run probability increases. However, in Case 3—to which the PS numerical example belongs —the ICC binds for small run-probabilities. Hence, for Case 3, the optimal contract does not change with s until the ICC no longer binds.

Understanding *how* the optimal contract tolerates runs is important to banks and regulators. Optimal contracts and regulations could well be different based on the financial sector's level of fragility or pessimism as measured by the run-risk parameter s.[13] For instance, as the financial system becomes more fragile but not to the level of activating a completely run-proof mode, what should the optimal response of the financial sector be? Our result shows that if the problem of incomplete information is so severe that the ICC binds (i.e., Case 2 in our analysis), then no change should be made (as in the PS example). Otherwise, the optimal contract should become tighter and/or the regulation should be tightened.

Our paper also makes a contribution to the wider literature on sunspot equilibrium. We have provided yet other examples in which (1) not all sunspot equilibria are mere randomizations over certainty equilibria and (2) not all randomizations over certainty equilibria are sunspot equilibria. See Shell (2008) and Shell and Smith (1992).

SUPPLEMENTARY MATERIAL

To view supplementary material for this article, please visit https://doi.org/S1365100518000275.

NOTES

1. DD write on p. 410 "This could happen if the selection between the bank run equilibrium and the good equilibrium depended on some commonly observed random variable in the economy. This could be a bad earnings report, a commonly observed run at some other bank, a negative government forecast, or even *sunspots* [emphasis ours]."

2. In contrast to DD, there is also intrinsic uncertainty (as opposed to extrinsic uncertainty in the form of sunspots) in the PS model: (1) The aggregate number of impatient consumers is uncertain. It could be 0, 1, or 2. This is important because the bank will never know with certainty that a run is underway. (2) Sequential service is taken seriously (Wallace, 1988). When two depositors withdraw early, their positions in the queue are random. (3) Partial suspension of convertibility is feasible for the bank (Wallace, 1990). Typically, the first in line in period 1 will be allowed a larger withdrawal than the second in line.

3. Like other papers in the literature, PS assume that in equilibrium the bank chooses the contract that maximizes the ex-ante expected utility of depositors. This assumption can be justified when banks compete for deposits, there is no entry cost for banks, and the depositors are ex-ante identical.

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4. The *unconstrained efficient allocation* is the best allocation that can be attained when agent types (patient or impatient) are observable. In other words, the allocation maximizes the ex-ante expected utility of agents without imposing incentive compatibility, but it is still subject to the sequential service and resource feasibility constraints. See Ennis and Keister (2010). The associated contract has been called the "first-best contract."

5. Note the difference between the non-SC parameters and the parameters of the first case. With non-SC parameter values, the nonrun equilibrium and run equilibrium do not co-exist for *any* contract c. However, for the first case, the nonrun equilibrium and run equilibrium can co-exist for some contracts, but these contracts are not optimal.

6. Consumers always weakly choose to deposit in equilibrium since the contract chosen by the bank can at least mimic the autarky allocation.

7. As in other papers in the literature, we assume that a patient depositor chooses early withdrawal if he *strictly* prefers the period-1 withdrawal to the period-2 withdrawal. And, he chooses period-2 withdrawal if he *weakly* prefers to do so.

8. $\widehat{W}(c)$ is also the depositor's expected utility in the nonrun equilibrium of the *predeposit* game.

9. It is easy to see that c^{early} does not depend on p and that we have $y < c^{\text{early}} < c^{\text{wait}} < Ry$. $\lim_{p\to 1} \frac{2y}{[p/(2-p)]^{1/b}+1} = y$ and $\lim_{p\to 0} \frac{2y}{[p/(2-p)]^{1/b}+1} = 2y$. Hence, we know that for sufficiently large p, we have $\frac{2y}{[p/(2-p)]^{1/b}+1} \leq c^{\text{early}}$. If $R < 2$, we have $c^{\text{wait}} < \frac{2y}{[p/(2-p)]^{1/b}+1}$ for sufficiently small p. For intermediate values of p, we have $c^{\text{early}} < \frac{2y}{[p/(2-p)]^{1/b}+1} \leq c^{\text{wait}}$.

10. At $c^*(s; A)$, consumers must weakly prefer depositing to autarky. This is because the deposit contract can always mimic the autarky allocation by setting $c^*(s; A)$ equal to y. Hence, the participation constraint is not an issue for $c \in [0, c^{\text{wait}}]$.

11. In Case 2, the ICC does not bind. To see this, (1) we know that a contract c, which is larger than the contract supporting the *unconstrained efficient allocation*, makes the run equilibrium more devastating, and (2) it does not improve welfare at the nonrun equilibrium. Hence, to find the optimal contract, we need only to focus on contracts that are more conservative than the contract supporting the *unconstrained efficient allocation*. So, binding incentive compatibility does not occur in Case 2.

12. Ennis and Keister (2010) define the *constrained efficient allocation* as "a (contingent) consumption allocation to maximize the ex ante expected utility of agents subject to incentive compatibility, sequential service, and resource feasibility constraints." We use the same definition.

13. The parameter s might be small for an economy with a mature financial system that has recently experienced a few or no recent financial breakdowns. For an emerging country with an immature financial sector or a country with some recent, large financial breakdowns, the parameter s might be large.

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