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A local variational principle for the topological entropy

PIERRE-PAUL ROMAGNOLI
Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170-3 Correo 3, Santiago, Chile
(e-mail: promagno@dim.uchile.cl)

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Abstract. In this paper we introduce two notions of measure-theoretical entropy for covers and one notion of topological entropy for open covers. These notions extend the classical concept of measure-theoretical entropy of measurable partitions to measurable covers. Using these notions we prove a local variational principle for any given open cover.

1. *Introduction*

Let (X, T) be a topological dynamical system. That is, $T : X \to X$ is a continuous surjective map on a compact metric space X. Entropy is one of the most widely used notions of chaoticity that is associated to dynamical systems. Among the notions of entropy, two of the classical ones are a purely topological notion known as topological entropy and a measure-theoretical one known as measure-theoretical entropy. These entropies, though different in nature, are related by a topological variational principle [**G2**]. The topological entropy of a dynamical system is first defined for all finite open covers of X and then computed as the supremum over all these finite open covers. In a similar way the measure-theoretical entropy of a cover is defined, with respect to a T-invariant Borel probability measure, for any finite measurable partition and then computed as the supremum over all measurable partitions. The topological variational principle establishes that for any topological dynamical system the supremum over all T-invariant Borel probability measures of the measure-theoretical entropy is the topological entropy.

In the last twenty years a series of works has been published to study the connection between measure-theoretical entropy notions and topological entropy ones. This research involves proving new variational principles and giving new definitions of topological and measure-theoretical notions. For instance, topological dynamical systems have been studied in terms of their local behavior, by which we mean here the action of the dynamics over the set of covers separating two given points. An example of this is the

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notion of topological entropy pairs and entropy pairs for a measure (see [**B**], [**B-R**]). Ghys *et al* [**GLW**] give a definition of entropy for a linear operator in $L^{\infty}_{\mu}(X)$ using partitions of unity. Brin and Katok give a topological version of the Shannon–McMillan–Breiman Theorem with a local decomposition of the measure-theoretical entropy (see [**BK**]). Misiurewicz gives a definition of a topological conditional entropy in [**M**], and recently Boyle *et al* further developed this idea in [**BFF**]. There are also the conditional variational principles of Ledrappier and Walters [**LW**] and Downarowicz and Serafin [**DS**].

In the original proof of the topological variational principle for the topological entropy, one of the main issues is direct comparison between the topological entropy of an open cover with the measure-theoretical entropy of a measurable partition. However in some proofs this comparison is completely avoided, for instance by considering an alternative definition of the topological entropy that does not involve open covers (see [**DGS**]). In this paper we follow the direction given in [**BGH**], where the authors give a local variational inequality between the topological entropy of an open cover for a chosen T-invariant Borel probability measure and the measure-theoretical entropy of all partitions finer than the cover. We use the idea given in [**BGH**] to propose two extensions of the measure-theoretical entropy from partitions to covers. We define a new notion of the measure-theoretical entropy $h^{-}_{\mu}(\mathcal{U}, T)$ of an open cover $\mathcal{U}$ with respect to a measure μ that satisfies a local variational principle with respect to the classic topological entropy. We also give a formal definition of another measure-theoretical entropy $h^{+}_{\mu}(\mathcal{U}, T)$ of an open cover with respect to a measure μ that was implicit in [**BGH**]. There is a close relation between the two definitions and it remains to be seen if they are actually the same notion. If these notions turn out to coincide, then many of the properties of the measure-theoretical entropy of partitions are true for covers for the notion implicitly defined in [**BGH**]. If they do not coincide then $h^{-}_{\mu}(\mathcal{U}, T)$ is a quantity that satisfies a local variational principle for the topological entropy of open covers and extends many of the properties of measure-theoretical entropy from partitions to covers.

This paper is divided into four sections. In §2 we give the basic definitions of measure-theoretical entropy for covers and state the main theorem of this paper, Theorem 2, which is the local variational principle for $h^{-}_{\mu}(\mathcal{U}, T)$. In §3 we study the relation between our new notion and the work done in [**BGH**], and we establish the basic properties of this notion. In §4 we prove our main theorem.

2. *Basic definitions and the main theorem*

Let us introduce the basic notation and definitions used in this article. For more on measure-theoretical and topological entropy we refer to [**DGS**] and [**P**]. Let (X, T) be a topological dynamical system. This means that X is a compact metric space and $T : X \to X$ is a continuous surjective map. In this article a cover of X is a finite or countable family of Borel subsets of X, whose union is X. A cover is said to be an open cover if it consists of open sets. A partition of X is a cover of X by pairwise disjoint sets. Let $\mathcal{P}_X$ denote the set of partitions of X, $\mathcal{C}_X$ the set of covers of X and $\mathcal{C}^{o}_{X}$ the set of open covers of X. Given two covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, $\mathcal{U}$ is said to be finer than $\mathcal{V}$ ($\mathcal{U} \succeq \mathcal{V}$) if, for every $U \in \mathcal{U}$, there is $V \in \mathcal{V}$ such that $U \subseteq V$. Let $\mathcal{U} \vee \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$.

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It is clear that $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{U}$ and $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{V}$. However, $\mathcal{V} \succeq \mathcal{U}$ does not imply that $\mathcal{U} \vee \mathcal{V} = \mathcal{V}$. Given integers $M \leq N$ and $\mathcal{U} \in \mathcal{C}_X$, we use the notation $\mathcal{U}_M^N = \bigvee_{n=M}^N T^{-n}\mathcal{U}$.

The set $\mathcal{M}_T(X)$ of T-invariant Borel probability measures is a convex, compact (in the weak topology), non-empty set. For $\alpha \in \mathcal{P}_X$ and $\mu \in \mathcal{M}_T(X)$ define $H_\mu(\alpha) = -\sum_{A\in\alpha} \mu(A) \log \mu(A)$. The μ-measure-theoretical entropy of α with respect to T is defined by

$$h_\mu(\alpha, T) = \lim_{N\to\infty} \frac{1}{N} H_\mu(\alpha_0^{N-1}) = \inf_{N\in\mathbb{N}} \frac{1}{N} H_\mu(\alpha_0^{N-1}). \tag{1}$$

The measure-theoretical entropy of T with respect to μ, $h_\mu(T)$, is defined as $h_\mu(T) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(\alpha, T)$.

In the topological framework, given $\mathcal{U} \in \mathcal{C}_X$, define $\mathcal{N}(\mathcal{U})$ as the minimum among the cardinals of the subcovers of $\mathcal{U}$. Then the topological entropy of $\mathcal{U}$ with respect to T is defined as

$$h_{\text{top}}(\mathcal{U}, T) = \lim_{N\to\infty} \frac{1}{N} \log \mathcal{N}(\mathcal{U}_0^{N-1}) = \inf_{N\in\mathbb{N}} \frac{1}{N} \log \mathcal{N}(\mathcal{U}_0^{N-1}). \tag{2}$$

The topological entropy of $\mathcal{U}$ with respect to T is defined as $h_{\text{top}}(T) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_{\text{top}}(\mathcal{U}, T)$. The topological variational principle for the entropy states the following.

THEOREM 1. (Topological variational principle [**G1, G2**]) *Let (X, T) be a topological dynamical system. Then for every $\mu \in \mathcal{M}_T(X)$,*

$$h_\mu(T) \leq h_{\text{top}}(T), \quad \textit{and in fact} \quad \sup_{\mu\in\mathcal{M}_T(X)} h_\mu(T) = h_{\text{top}}(T).$$

It turns out that $\mathcal{N}(\mathcal{U}) = \inf_{\alpha\succeq\mathcal{U}} |\alpha|$, so we can compute the topological entropy of a cover $\mathcal{U}$ with respect to T as the infimum of the topological entropies of all measurable partitions finer than $\mathcal{U}$. Starting from this idea we define a new notion of μ-measure-theoretical entropy for covers that extends definition (1) to covers and allows us to give a local version (for a fixed $\mathcal{U}$) of Theorem 1. For $\mathcal{U} \in \mathcal{C}_X$ define

$$H_\mu(\mathcal{U}) = \inf_{\alpha\succeq\mathcal{U}} H_\mu(\alpha). \tag{3}$$

Lemma 8 in §3 implies that $H_\mu(\mathcal{U}_0^{N-1})$ is a subadditive function of $N \in \mathbb{N}$, and so we may define the μ^--entropy of $\mathcal{U}$ with respect to T as

$$h_\mu^-(\mathcal{U}, T) = \lim_{N\to\infty} \frac{1}{N} H_\mu(\mathcal{U}_0^{N-1}) = \inf_{N\in\mathbb{N}} \frac{1}{N} H_\mu(\mathcal{U}_0^{N-1}). \tag{4}$$

When $\mathcal{U}$ is a partition, $h_\mu^-(\mathcal{U}, T) = h_\mu(\mathcal{U}, T)$ which implies that $h_\mu(T) = \sup_{\mathcal{U}\in\mathcal{C}_X} h_\mu^-(\mathcal{U}, T)$. Moreover since, for any $\mu \in \mathcal{M}_T(X)$, $H_\mu(\mathcal{U}) \leq H(\mathcal{U})$, we have

$$\text{for every } \mathcal{U} \in \mathcal{C}_X, \text{ for every } \mu \in \mathcal{M}_T(X), \ h_\mu^-(\mathcal{U}, T) \leq h_{\text{top}}(\mathcal{U}, T). \tag{5}$$

Thus the inequality stated in Theorem 1 is obtained from equation (5) by taking the supremum over all open covers. The main theorem in this paper is the following.

THEOREM 2. (Local variational principle) *Let (X, T) be a topological dynamical system. For every $\mathcal{U} \in \mathcal{C}_X^o$ there exists $\mu \in \mathcal{M}_T(X)$ such that $h_{\text{top}}(\mathcal{U}, T) = h_\mu^-(\mathcal{U}, T)$.*

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From Theorem 2 we obtain Theorem 1 by taking the supremum over all open covers. Moreover, Theorem 2 proves that if there exists a generator $\mathcal{U} \in \mathcal{C}_X^o$ for the topological entropy so that $h_{\text{top}}(T) = h_{\text{top}}(\mathcal{U}, T)$, then the supremum in Theorem 1 is attained for some $\mu \in \mathcal{M}_T(X)$ (see [**DGS**]).

Before proving Theorem 2, in the next section we establish the connection of this work with [**BGH**].

3. *Relation with [BGH]*

In [**BGH**] the authors prove the following theorem.

THEOREM 3. *Let (X, T) be a topological dynamical system. For every $\mathcal{U} \in \mathcal{C}_X^o$ there exists $\mu \in \mathcal{M}_T(X)$ such that $h_\mu(\alpha, T) \geq h_{\text{top}}(\mathcal{U}, T)$ for every $\alpha \in \mathcal{P}_X$ finer than $\mathcal{U}$.*

Let (X, T) be a topological dynamical system. Given $\mu \in \mathcal{M}_T(X)$ and $\mathcal{U} \in \mathcal{C}_X$ define

$$h_\mu^+(\mathcal{U}, T) = \inf_{\alpha \succeq \mathcal{U}} h_\mu(\alpha, T). \tag{6}$$

A simple calculation yields

$$\begin{aligned} h_\mu^+(\mathcal{U}, T) &= \inf_{\alpha \succeq \mathcal{U}} \inf_{N \in \mathbb{N}} \frac{1}{N} H_\mu(\alpha_0^{N-1}) = \inf_{N \in \mathbb{N}} \inf_{\alpha \succeq \mathcal{U}} \frac{1}{N} H_\mu(\alpha_0^{N-1}) \\ &\geq \inf_{N \in \mathbb{N}} \inf_{\beta \succeq \mathcal{U}_0^{N-1}} \frac{1}{N} H_\mu(\beta) = h_\mu^-(\mathcal{U}, T). \end{aligned} \tag{7}$$

Theorem 3 states that when $\mathcal{U} \in \mathcal{C}_X^o$ there exists a measure $\mu \in \mathcal{M}_T(X)$ such that $h_\mu^+(\mathcal{U}, T) \geq h_{\text{top}}(\mathcal{U}, T)$. Notice that inequality (7) proves that Theorem 2 implies Theorem 3. Actually, we can strengthen the relation between these two notions. First we prove that $h_\mu^-(\mathcal{U}, T)$ extends the property of $h_\mu(\alpha, T)$ with respect to powers of T.

LEMMA 4. *Let (X, T) be a topological dynamical system and $\mathcal{U} \in \mathcal{C}_X$. Then for every $M \in \mathbb{N}$,*

$$h_\mu^-(\mathcal{U}, T) = \frac{1}{M} h_\mu^-(\mathcal{U}_0^{M-1}, T^M).$$

Proof. We have

$$\begin{aligned} \frac{1}{M} h_\mu^-(\mathcal{U}_0^{M-1}, T^M) &= \lim_{N \to \infty} \frac{1}{NM} H_\mu\left[\bigvee_{n=0}^{N-1} T^{-nM}(\mathcal{U}_0^{M-1})\right] \\ &= \lim_{N \to \infty} \frac{1}{NM} H_\mu[\mathcal{U}_0^{NM-1}] \\ &= h_\mu^-(\mathcal{U}, T). \end{aligned}$$

□

This allows us to prove the following.

LEMMA 5. *For any $\mathcal{U} \in \mathcal{C}_X$,*

$$h_\mu^-(\mathcal{U}, T) = \lim_{M \to \infty} \frac{1}{M} h_\mu^+(\mathcal{U}_0^{M-1}, T^M).$$

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Proof. From Lemma 4 and inequality (7), we get that, for every $M \in \mathbb{N}$,

$$h_\mu^-(\mathcal{U}, T) = \frac{1}{M} h_\mu^-(\mathcal{U}_0^{M-1}, T^M) \leq \frac{1}{M} h_\mu^+(\mathcal{U}_0^{M-1}, T^M).$$

On the other hand, for every $N \in \mathbb{N}$,

$$\frac{1}{NM} \inf_{\alpha \succeq \mathcal{U}_0^{M-1}} H_\mu\left(\bigvee_{n=0}^{N-1} (T^M)^{-n} \alpha \right) \leq \frac{1}{M} H_\mu(\mathcal{U}_0^{M-1}).$$

Taking the limit when N tends to infinity, we obtain

$$h_\mu^-(\mathcal{U}, T) \leq \frac{1}{M} h_\mu^+(\mathcal{U}_0^{M-1}, T^M) \leq \frac{1}{M} H_\mu(\mathcal{U}_0^{M-1}).$$

Now take the limit when M tends to infinity to conclude the argument. □

The following proposition proves that the property of $h_\mu(\alpha, T)$ with respect to measure-theoretical factors extends to $h_\mu^-(\mathcal{U}, T)$.

PROPOSITION 6. *Let (X, T) and (Y, S) be topological dynamical systems, $\mu \in \mathcal{M}_T(X)$ and $\nu \in \mathcal{M}_S(Y)$. Let $\varphi : (X, T, \mu) \to (Y, S, \nu)$ be a measure-theoretical factor map and $\mathcal{U} \in \mathcal{C}_Y$. Then*

$$h_\nu^-(\mathcal{U}, S) = h_\mu^-(\varphi^{-1}\mathcal{U}, T).$$

In order to prove Proposition 6 we need Lemma 7. The proof of this lemma is basic calculus and is left to the reader. For $\{p_k\}_{k=1}^K \subseteq (0, 1)$ with $\sum_{k=1}^K p_k \leq 1$, as usual define $H(p_1, \ldots, p_K) = -\sum_{k=1}^K p_k \log p_k$.

LEMMA 7. *Fix $K \geq 2$. Suppose that $p_1, \ldots, p_K \in (0, 1)$ with $p_1 \leq \cdots \leq p_K$ and $\sum_{k=1}^K p_k \leq 1$. Let $0 < \delta_1 < p_1$ and suppose that for each $k = 2, \ldots, K$, $\delta_k \in [0, 1 - p_k)$ and $\sum_{k=2}^K \delta_k = \delta_1$. Then*

$$H(p_1, \ldots, p_K) > H(p_1 - \delta, p_2 + \delta_2, \ldots, p_K + \delta_K).$$

Proof of Proposition 6. First we prove that

$$\text{for every } \mathcal{U} \in \mathcal{C}_Y,\ H_\mu(\varphi^{-1}\mathcal{U}) = H_\nu(\mathcal{U}). \tag{8}$$

($\leq$) Consider $\mathcal{U} \in \mathcal{C}_Y$ and $\beta \succeq \mathcal{U}$. We have that $\varphi^{-1}\beta \succeq \varphi^{-1}\mathcal{U}$ and, since $\mu \circ \varphi^{-1} = \nu$,

$$H_\mu(\varphi^{-1}\mathcal{U}) \leq \inf_{\beta \succeq \mathcal{U}} H_\mu(\varphi^{-1}\beta) = \inf_{\beta \succeq \mathcal{U}} H_\nu(\beta) = H_\nu(\mathcal{U}).$$

($\geq$) Consider $\mathcal{U} \in \mathcal{C}_Y$. For every $\alpha \succeq \varphi^{-1}\mathcal{U}$ we recursively construct a partition $\beta \succeq \mathcal{U}$ such that $H_\mu(\alpha) \geq H_\nu(\beta)$.

Let $\mathcal{U} = \{U_1, \ldots, U_M\}$ and $\alpha = \{A_1, \ldots, A_K\} \succeq \varphi^{-1}\mathcal{U}$ with $\phi \neq A_k \subseteq \varphi^{-1}U_{j_k}$, $j_k \in \{1, \ldots, M\}$, for all $k \in \{1, \ldots, K\}$. We can assume that $j_k \neq j_l$ if $k \neq l$, since the partition obtained by replacing A_k and A_l by their union is coarser than α and still finer than $\varphi^{-1}\mathcal{U}$. Notice that $\varphi^{-1}U_{j_1} \setminus \bigcup_{l=2}^K \varphi^{-1}U_{j_l} \subseteq A_1$. For each $k = 1, \ldots, K$ define

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$p_k = \mu(A_k)$. Then $\sum_{k=1}^{K} p_k = 1$ and $H_\mu(\alpha) = H(p_1, \dots, p_k)$. By exchanging indices if necessary we assume $p_1 \leq p_2 \leq \cdots \leq p_K$. Let us define

$$\delta_1 = \mu(A_1) - \nu\left(U_{j_1} \setminus \bigcup_{l=2}^{K} U_{j_l}\right) \geq 0.$$

Define $B_1^1 = U_{j_1} \setminus \bigcup_{l=2}^{K} U_{j_l}$ and $A_2^1 = A_2 \cup [\varphi^{-1}U_{j_2} \cap (A_1 \setminus \varphi^{-1}B_1^1)]$. For $k = 3, \dots, K$ let

$$A_k^1 = A_k \cup \left[\varphi^{-1}U_{j_k} \cap (A_1 \setminus \varphi^{-1}B_1^1) \cap \left(\bigcup_{l=2}^{k-1} A_l^1\right)^c\right].$$

Define $\alpha_1 = \{\varphi^{-1}B_1^1, A_2^1, \dots, A_K^1\}$. It is clear that $\alpha_1 \succeq \varphi^{-1}\mathcal{U}$, and if $\delta_1 = 0$ then $H_\mu(\alpha) = H_\mu(\alpha_1)$. If $\delta_1 > 0$, using Lemma 7 with $(p_1, \dots, p_K)$ and $\delta_k = \mu(A_k^1) - \mu(A_k)$, for every $k \in \{2, \dots, K\}$ we prove that $H_\mu(\alpha) \geq H_\mu(\alpha_1)$.

Inductively, for each $n = 2, \dots, K$ we construct

$$\alpha_n = \{\varphi^{-1}B_1^n, \dots, \varphi^{-1}B_n^n, A_{n+1}^n, \dots, A_K^n\} \succeq \varphi^{-1}\mathcal{U},$$

which satisfies $H_\mu(\alpha_{n+1}) \leq H_\mu(\alpha_n)$. Let us give the construction of α_{n+1} given α_n. For each $k \in \{1, \dots, n\}$ define $B_k^{n+1} = B_k^n$. Let $B_{n+1}^{n+1} = U_{j_{n+1}} \setminus \bigcup_{l=n+2}^{K} U_{j_l}$ and $A_{n+2}^{n+1} = A_{n+2}^n \cup [\varphi^{-1}U_{j_{n+2}} \cap (A_{n+1}^n \setminus \varphi^{-1}B_{n+1}^{n+1})]$. For each $k \in \{k+3, \dots, K\}$ define

$$A_k^{n+1} = A_k^n \cup \left[\varphi^{-1}U_{j_k} \cap (A_{n+1}^n \setminus \varphi^{-1}B_{n+1}^{n+1}) \cap \left(\bigcup_{l=2}^{k-1} A_l^{n+1}\right)^c\right].$$

As before, for each $k \in \{1, \dots, K-n\}$ define $p_k = \mu(A_{n+k}^n)$ and $\delta_k = \mu(A_{n+k}^{n+1}) - \mu(A_{n+k}^n)$. Let $\delta_1 = \mu(A_{n+1}^n) - \nu(B_{n+1}^{n+1})$. By exchanging indices if necessary we assume $p_1 \leq p_2 \leq \cdots \leq p_{K-n}$. Using Lemma 7, we prove that $H_\mu(\alpha_{n+1}) \leq H_\mu(\alpha_n)$.

We have that $H_\mu(\alpha) \geq H_\mu(\alpha_K)$ and $\alpha_K = \varphi^{-1}\beta$ with $\beta \succeq \mathcal{U}$. This proves that

$$H_\mu(\varphi^{-1}\mathcal{U}) \geq \inf_{\beta \succeq \mathcal{U}} H_\mu(\varphi^{-1}\beta) = H_\nu(\mathcal{U}).$$

Now, using property (8) for $\bigvee_{n=0}^{N-1} T^{-n}(\varphi^{-1}\mathcal{U})$ and any $N \in \mathbb{N}$, since $S \circ \varphi = \varphi \circ T$ we get

$$H_\mu\left(\bigvee_{n=0}^{N-1} T^{-n}(\varphi^{-1}\mathcal{U})\right) = H_\nu\left(\bigvee_{n=0}^{N-1} S^{-n}\mathcal{U}\right).$$

Dividing by N and taking the limit when N goes to infinity proves the result. □

It is easy to prove that many of the properties of the entropy function $H_\mu(\alpha)$ extend to $H_\mu(\mathcal{U})$, from partitions to covers.

LEMMA 8. *Let $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathcal{C}$. Then:*

(1) $H_\mu(\mathcal{U}) \geq 0$;
(2) *If $\mathcal{U} \succeq \mathcal{V}$, then $H_\mu(\mathcal{U}) \geq H_\mu(\mathcal{V})$;*
(3) *If $\mathcal{V} \succeq \mathcal{U}$, then $H_\mu(\mathcal{U} \vee \mathcal{V} \vee \mathcal{W}) = H_\mu(\mathcal{V} \vee \mathcal{W})$;*
(4) $H_\mu(\mathcal{U} \vee \mathcal{V}) \leq H_\mu(\mathcal{U}) + H_\mu(\mathcal{V})$;
(5) $H_\mu(T^{-1}\mathcal{U}) = H_\mu(\mathcal{U})$.

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The proofs of all these properties are straightforward. Notice that (5) is just formula (8) in Proposition 6 with $\varphi = T$. This proves in particular that $H_\mu(\mathcal{U}_0^{N-1})$ is subadditive.

In the topological framework, we define

$$h^+_{\text{top}}(\mathcal{U}, T) = \inf_{\alpha \succeq \mathcal{U}} h_{\text{top}}(\alpha, T). \tag{9}$$

It is clear that $h^+_\mu(\mathcal{U}, T) \leq h^+_{\text{top}}(\mathcal{U}, T) \leq \log \mathcal{N}(\mathcal{U})$. Moreover we can state a topological analog of Lemma 5.

LEMMA 9. *For every* $\mathcal{U} \in \mathcal{C}_X$,

$$h_{\text{top}}(\mathcal{U}, T) = \lim_{M \to \infty} \frac{1}{M} h^+_{\text{top}}(\mathcal{U}_0^{M-1}, T^M).$$

The proof is omitted since it follows the same lines as the proof of Lemma 5. All the properties in Lemma 8 are satisfied if the notions $h^-_\mu(\mathcal{U}, T)$ and $h^+_\mu(\mathcal{U}, T)$ coincide, and $h_{\text{top}}(\mathcal{U}, T)$ and $h^+_{\text{top}}(\mathcal{U}, T)$ coincide as well. So some very natural question are the following:

Question 10. Is it true that for every $\mu \in \mathcal{M}_T(X)$, $h^+_\mu(\mathcal{U}, T) = h^-_\mu(\mathcal{U}, T)$?

Question 11. Is it true that $h^+_{\text{top}}(\mathcal{U}, T) = h_{\text{top}}(\mathcal{U}, T)$?

We do not know the answers to these questions. Note that, by Theorem 2, if Question 10 is true then Question 11 is true. If Question 10 is true then we can improve Theorem 3 since in that case formula (5) proves that the inequality in Theorem 3 is an equality. This allows us to prove new properties for $h^+_\mu(\mathcal{U}, T)$ such as Lemma 4 and Proposition 6 that thereby simplify some calculations done in [**BGH**]. However, even if Question 10 is not true, then we still have defined $h^-_\mu(\mathcal{U}, T)$, which extends many of the properties of the entropy function from partitions to covers. Even though we are not yet able to give an answer to these questions, the local variational principle given in Theorem 2 improves Theorem 3. In the next section we give the proof of Theorem 2.

4. *Proof of the main theorem*

To prove Theorem 2 we generalize the technique used in [**BGH**]. Fix a topological dynamical system (X, T). First we give some basic definitions. Consider a finite alphabet $\mathcal{A} = \{1, \ldots, D\}$, $D \in \mathbb{N}$. Sequences $w = w_0 \ldots w_{N-1}$ in $\mathcal{A}^N$ for $N \in \mathbb{N}$ are called words of length N on $\mathcal{A}$. For any word v of length k and any word w of length N on $\mathcal{A}$ with $N, k \in \mathbb{N}$, $k \leq N$, denote by $d_v(w)$ the frequency of occurrences of v in w as

$$d_v(w) = \frac{1}{N-k+1} \big| \{ n \in \{0, \ldots, N-k\} \mid w_n \ldots w_{n+k-1} = v_0 \ldots v_{k-1} \} \big|.$$

Let $\phi(x) = -x \log x$ in $]0, 1]$ and $\phi(0) = 0$. For every word w of length N on $\mathcal{A}$ and $k \leq N$ define

$$H_k(w) = \sum_{v \in \mathcal{A}^k} \phi[d_v(w)].$$

We state (without proof) the following lemma of [**BGH**].

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LEMMA 12. **[BGH]** *For every $h > 0$, $\varepsilon > 0$, $k \in \mathbb{N}$, there exists $\tilde{N} \in \mathbb{N}$ such that for every $N \geq \tilde{N}$,*

$$\left|\{w \in \mathcal{A}^N \mid H_k(w) \leq kh\}\right| \leq e^{k(h+\varepsilon)}.$$

Let us fix $M, K \in \mathbb{N}$ and let $\{\alpha_l\}_{l=0}^{K-1}$ be a family of measurable partitions such that $\alpha_l = \{A_1^l, \ldots, A_D^l\}$. For $N \in \mathbb{N}$ and $i_0, \ldots, i_{M-1} \in \{0, \ldots, K-1\}$, define $w(\alpha_{i_0}, \ldots, \alpha_{i_{M-1}}, N, x) \in \mathcal{A}^{NM}$ as

$$w(\alpha_{i_0}, \ldots, \alpha_{i_{M-1}}, N, x)_{nM+m} = d \quad \text{if } T^{nM+m}x \in A_d^{i_m}, \tag{10}$$

for each $n \in \{0, \ldots, N-1\}$ and $m \in \{0, \ldots, M-1\}$.

This coincides with the definition given in [**BGH**] when $M = 1$ and gives an analog of Lemma 2 in [**BGH**].

LEMMA 13. *Let $\mathcal{U} \in \mathcal{C}_X$, let $M, K \in \mathbb{N}$, and let $\{\alpha_l\}_{l=0}^{K-1}$ be a family of measurable partitions of X with $\alpha_l \succeq \mathcal{U}$, each α_l having D elements. Then for every $\varepsilon > 0$, there exist $x \in X$ and $\tilde{N} \in \mathbb{N}$ such that, for any $N \geq \tilde{N}$, $k \in \{0, \ldots, KM-1\}$ and $i_0, \ldots, i_{M-1} \in \{0, \ldots, K-1\}$, one has*

$$H_k(w(\alpha_{i_0}, \ldots, \alpha_{i_{M-1}}, N, x)) \geq k(h_{\text{top}}(\mathcal{U}, T) - \varepsilon).$$

Proof. For $k \in \{0, \ldots, KM-1\}$ and $N \geq K$, put

$$\Omega(N, k) = \{w \in \mathcal{A}^{NM} \mid H_k(w) < k(h_{\text{top}}(\mathcal{U}, T) - \varepsilon)\}.$$

By Lemma 12 there exists $\tilde{N}_1 \in \mathbb{N}$ such that, for every $N \geq \tilde{N}_1$, for every $k \in \{0, \ldots, KM-1\}$,

$$|\Omega(N, k)| \leq e^{NM(h_{\text{top}}(\mathcal{U}, T) - \varepsilon/2)}.$$

For $i_0, \ldots, i_{M-1} \in \{0, \ldots, K-1\}$ denote

$$Z(k, i_0, \ldots, i_{M-1}) = \{x \in X \mid w(\alpha_{i_0}, \ldots, \alpha_{i_{M-1}}, N, x) \in \Omega(N, k)\}.$$

From definition (10), this set is the union of at most $|\Omega(N, k)|$ elements of $\bigvee_{n=0}^{N-1} \bigvee_{m=0}^{M-1} T^{-(nM+m)}\alpha_m$. Since this partition is finer than $\mathcal{U}_0^{NM-1}$, the set $Z(k, i_0, \ldots, i_{M-1})$ can be covered by less than $|\Omega(N, k)|$ elements of $\mathcal{U}_0^{NM-1}$.

Consider $\tilde{N}_2 \in \mathbb{N}$ such that for every $N \geq \tilde{N}_2$, $K^{M+1}M < e^{NM\varepsilon/2}$. For $N \geq \max\{\tilde{N}_1, \tilde{N}_2\}$,

$$\bigcup_{k=0}^{KM-1} \bigcup_{i_0, \ldots, i_{M-1} \in \{0, \ldots, K-1\}} Z(k, i_0, \ldots, i_{M-1})$$

is covered by $K^{M+1}Me^{NM(h_{\text{top}}(\mathcal{U}, T) - \varepsilon/2)}$ elements of $\mathcal{U}_0^{NM-1}$, and this is strictly smaller than $e^{NMh_{\text{top}}(\mathcal{U}, T)}$.

Since

$$h_{\text{top}}(\mathcal{U}, T) = \inf_{N \to \infty} \frac{1}{NM} \log \mathcal{N}(\mathcal{U}_0^{NM-1}),$$

every subcover of $\mathcal{U}_0^{NM-1}$ has at least $e^{NMh_{\text{top}}(\mathcal{U}, T)}$ elements, proving that

$$\bigcup_{k=0}^{KM-1} \bigcup_{i_0, \ldots, i_{M-1} \in \{0, \ldots, K-1\}} Z(k, i_0, \ldots, i_{M-1}) \neq X.$$

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Therefore the complement of

$$\bigcup_{k=0}^{KM-1} \bigcup_{i_0,\dots,i_{M-1}\in\{0,\dots,K-1\}} Z(k, i_0, \dots, i_{M-1})$$

is a non-empty set. Take any x in this set to conclude the proof. □

Proof of Theorem 2. First, assume that X is 0-dimensional, that is, there exists a fundamental basis of the topology consisting of open and closed (clopen) sets. Fix $\mathcal{U} = \{U_1, \dots, U_D\} \in \mathcal{C}_X^o$. The set of clopen partitions finer than $\mathcal{U}$ of the form $\{A_1, \dots, A_D\}$ is countable. Let $\{\alpha_l\}_{l\in\mathbb{N}}$ be an enumeration of this family.

For $M \in \mathbb{N}$, using Lemma 13, there exist an increasing sequence of integers $\{N_K\}_{K\in\mathbb{N}}$ and points $\{x_K\}_{K\in\mathbb{N}} \subseteq X$ such that for every $k \in \{0, \dots, KM-1\}$ and every $i_0, \dots, i_{M-1} \in \{0, \dots, K-1\}$,

$$H_k(w(\alpha_{i_0}, \dots, \alpha_{i_{M-1}}, N_K, x_K)) \geq k\left(h_{\text{top}}(\mathcal{U}, T) - \frac{1}{K+1}\right).$$

Define the probability measure

$$\mu_K = \frac{1}{N_K M} \sum_{n=0}^{N_K M-1} \delta_{T^n x_K}.$$

We may assume (by taking a subsequence) that there exists $\mu \in \mathcal{M}_T(X)$ such that $\{\mu_K\}_{K\in\mathbb{N}}$ converges weakly to some measure μ.

Fix $i_0, \dots, i_{M-1} \in \{0, \dots, K-1\}$ and $k = RM$ with $R \in \{0, \dots, K-1\}$. For $v \in \mathcal{A}^k$ take $E_v = \{x \in X \mid w(\alpha_{i_0}, \dots, \alpha_{i_{M-1}}, R, x) = v\}$. If E_v is non-empty then E_v is an element of $\bigvee_{r=0}^{R-1} T^{-rM} \bigvee_{m=0}^{M-1} T^{-m}\alpha_{i_m}$. Then for every K one has

$$|\mu_K(E_v) - d_v[w(\alpha_{i_0}, \dots, \alpha_{i_{M-1}}, N_K, x_K)]| \leq \frac{2k}{N_K \cdot M}.$$

Since E_v is clopen,

$$\mu(E_v) = \lim_{K\to\infty} \mu_K(E_v) = \lim_{K\to\infty} d_v[w(\alpha_{i_0}, \dots, \alpha_{i_{M-1}}, N_K, x_K)].$$

By the continuity of ϕ,

$$\phi[\mu(E_v)] = \lim_{K\to\infty} \phi(d_v[w(\alpha_{i_0}, \dots, \alpha_{i_{M-1}}, N_K, x_K)]).$$

If we sum over all $v \in \mathcal{A}^k$, we get that for every $R \in \{0, \dots, K-1\}$ and every $i_0, \dots, i_{M-1} \in \{0, \dots, K-1\}$,

$$\frac{1}{RM} H_\mu\left[\bigvee_{r=0}^{R-1} T^{-rM} \bigvee_{m=0}^{M-1} T^{-m}\alpha_{i_m}\right] = \frac{1}{RM} \lim_{K\to\infty} H_{RM}(w(\alpha_{i_0}, \dots, \alpha_{i_{M-1}}, N_K, x_K)) \geq h_{\text{top}}(\mathcal{U}, T).$$

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Letting R tend to infinity gives

$$\frac{1}{M}h_\mu\left(\bigvee_{m=0}^{M-1} T^{-m}\alpha_{i_m}, T^M\right) \geq h_{\text{top}}(\mathcal{U}, T)$$

for every $M \in \mathbb{N}$ and all $i_0, \ldots, i_{M-1}$ in $\mathbb{N}$.

Since X is 0-dimensional the family of partitions $\{\alpha_l\}_{l\in\mathbb{N}}$ is dense with respect to the distance of $L^1(\mu)$ in the set of partitions finer than $\mathcal{U}$ and having cardinal D. We conclude that

$$\text{for every } M \in \mathbb{N}, \text{ for every } \alpha \succeq \mathcal{U}_0^{M-1}, \ \frac{1}{M}h_\mu(\alpha, T^M) \geq h_{\text{top}}(\mathcal{U}, T).$$

By definition this is equivalent to $(1/M)h_\mu^+(\mathcal{U}_0^{M-1}, T^M) \geq h_{\text{top}}(\mathcal{U}, T)$. Since this is true for all $M \in \mathbb{N}$, by Lemma 5, $h_\mu^-(\mathcal{U}, T) \geq h_{\text{top}}(\mathcal{U}, T)$. By definition $h_\mu^-(\mathcal{U}, T) \leq h_{\text{top}}(\mathcal{U}, T)$ so we have the equality.

When (X, T) is not 0-dimensional we recall that (X, T) always has a 0-dimensional topological extension (see [**BGH**]). Since for every $M \in \mathbb{N}$, $(1/M)h_{\text{top}}(\mathcal{U}_0^{M-1}, T^M) = h_{\text{top}}(\mathcal{U}, T)$, using Proposition 6 we conclude the proof. □

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