Proc. Camb. Phil. Soc. (1970), **67**, 613
PCPS 67–68
Printed in Great Britain

Some formulae for Appell functions

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(*Received* 17 *June* 1968)

1. *Introduction.* The object of this paper is to obtain some generating functions for Appell functions with the help of fractional derivatives. The results are general in character and include as particular cases some of the results given earlier by Mexiner (3) and the author (4, 5, 6).

According to Erdélyi (2) the fractional derivative is defined by

$$D_x^{\lambda}[x^{\mu-1}] = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)} x^{\mu-\lambda-1}, \tag{1}$$

which holds for all values of λ except $\lambda = \mu$. We extend the definition of the Saran function F_k (8) as

$$F_k\left[\begin{matrix}(\alpha_p);\ (B_q);\ (\gamma_r);\ (\rho_k);\ (\delta_e);\ x,y,z\\ (a_m);(b_n);\ —;\ —;\ —;\end{matrix}\right]$$

$$= \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{\infty} \frac{[(\alpha_p)]_{u+w}[(B_q)]_{v+w}[(\gamma_r)]_u[(\rho_k)]_v[(\delta_e)]_w}{[(a_m)]_{u+w}[(b_n)]_{v+w}} \frac{x^u}{u!}\frac{y^v}{v!}\frac{z^w}{w}, \tag{2}$$

where p, q, r, k, e, m and n are positive integers, $p+r = m$ or $p+r-1 = m$, $q+k = n$ or $q+k-1 = n$, $p+q+e = m+n$ or $p+q+e-1 = m+n$.

2. The first result to be proved is

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1[\mu; -n, \alpha'; \gamma; \alpha, b]\, F_1[\rho; -n, \beta; \delta; c, e] t^n$$

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n(\rho)_n(t\alpha c)^n}{n!\,(\gamma)_n\,(\delta)_n(1-t)^{2n}} F_1\left[\mu+n; \alpha', \lambda+n; \gamma+n; b, \frac{t\alpha}{t-1}\right],$$

$$F_1\left[\rho+n; \beta, \lambda+n; \delta+n; e, \frac{tc}{t-1}\right], \tag{3}$$

provided

$$|c|<1, \quad |e|<1, \quad |t|<1, \quad |\arg\alpha| < \tfrac{1}{3}\pi, \quad |\arg c| < \tfrac{1}{3}\pi, \quad \pi \leqslant \arg t \leqslant \tfrac{3}{2}\pi.$$

Proof. To prove (3), we start with the elementary identity

$$(1-t)^{\lambda}(1-t+t\alpha x)^{-\lambda}(1-t+tcy)^{-\lambda}\left[1 - \frac{\alpha cxyt}{(1-t+t\alpha x)(1-t+tcy)}\right]^{-\lambda}$$

$$= [1-t(1-\alpha x)(1-cy)]^{-\lambda}, \tag{4}$$

or

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} t^n \sum_{p=0}^{n} \frac{(-n)_p}{p!} (\alpha x)^p \sum_{q=0}^{n} \frac{(-n)_q}{q!} (cy)^q = (1-t)^{-\lambda}$$

$$\times \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{(t\alpha cxy)^n}{(1-t)^{2n}} \sum_{p=0}^{\infty} \frac{(\lambda+n)_p}{p!} \left(\frac{x\alpha t}{t-1}\right)^p \sum_{q=0}^{\infty} \frac{(\lambda+n)_q}{q!} \left(\frac{yct}{t-1}\right)^q, \tag{5}$$

https://doi.org/10.1017/S030500410004593X Published online by Cambridge University Press

provided

$$|\alpha x|<1,\quad |cy|<1,\quad |t|<1,\quad \pi\leqslant \arg t\leqslant \tfrac{3}{2}\pi,\quad |\arg(\alpha x)|<\tfrac{1}{3}\pi,\quad |\arg(cy)|<\tfrac{1}{3}\pi.$$

Multiply both the sides of (5) by $x^{\mu-1}y^{\rho-1}(1-bx)^{-\alpha'}(1-ey)^{-\beta}$ and applying the operators $D_x^{\mu-\gamma}$ and $D_y^{\rho-\delta}$, we have

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}t^n\sum_{r=0}^{\infty}\sum_{p=0}^{n}\frac{(-n)^p(\alpha')_r}{p!\quad r!}\alpha^p b^r D_x^{\mu-\gamma}(x^{p+r+\mu-1})\sum_{q=0}^{n}\sum_{s=0}^{\infty}\frac{(-n)_q(\beta)_s}{q!\quad s!}c^q e^s,$$

$$D_y^{\rho-\delta}(y^{q+s+p-1})=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{(t\alpha c)^n}{(1-t)^{2n}}$$

$$\times\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\lambda+n)_p(\alpha')_r}{p!\quad r!}\left(\frac{\alpha t}{t-1}\right)^p b^r D_x^{\mu-\gamma}(x^{p+r+n+\mu-1})\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda+n)_q(\beta)_s}{q!\quad s!}\left(\frac{ct}{t-1}\right)^q e^s D_y^{\rho-\delta}(y^{q+s+n+\rho-1}),\tag{6}$$

valid for

$$R(\gamma)>R(\mu)>0,\quad R(\delta)>R(\rho)>0,\quad |\alpha x|<1,\quad |cy|<1,\quad |ey|<1,\quad |t|<1,$$
$$|\arg(\alpha x)|<\tfrac{1}{3}\pi,|\arg(cy)|<\tfrac{1}{3}\pi,\quad \pi\leqslant\arg t\leqslant\tfrac{3}{2}\pi.$$

On using (1) in (6), we obtain (3) under the conditions stated therein. This completes the proof of (3).

Particular cases. We shall mention some interesting particular cases of our general result (3)

(*a*) Taking $b=e=0$, (3) reduces to a known result due to Mexiner ((2), p. 84, equation 8).

(*b*) Taking $e=0$ and $\delta=\lambda$, (3) reduces to a known result due to Manocha and Sharma ((4), p. 74, equation 9).

(*c*) If in (3) we replace t by t/λ and let $\lambda\to\infty$, then it reduces to a known result due to Manocha and Sharma ((6), p. 472, equation 18).

(*d*) Replacing b by $tx/(t-1)$, α by x, e by $ty/(t-1)$, c by y and using the formula ((3), p. 239, equation 11).

$$F_1[\alpha;\beta,\beta';\gamma;x,x]={}_2F_1[\alpha,\beta+\beta';\gamma;x]\tag{7}$$

in (3), it reduces to

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_1\left[\mu;-n,\alpha';\gamma;x,\frac{tx}{t-1}\right]F_1\left[\rho;-n,\beta;\delta;y,\frac{ty}{t-1}\right]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n(\rho)_n(txy)^n}{n!(\gamma)_n(\delta)_n(1-t)^{2n}}{}_2F_1\left[\mu+n,\alpha+\lambda+n;\gamma+n;\frac{tx}{t-1}\right]$$
$${}_2F_1\left[\rho+n,\beta+\lambda+n;\delta+n;\frac{ty}{t-1}\right]\tag{8}$$

valid for $R(\gamma)>R(\mu)>0$, $R(\delta)>R(\rho)>0$, $|x|<1$, $|y|<1$, $|t|<1$, $|\arg x|<\tfrac{1}{3}\pi$, $|\arg y|<\tfrac{1}{3}\pi,\pi\leqslant\arg t\leqslant\tfrac{3}{2}\pi$. Further if we take $\delta=\beta+\lambda$ and $\gamma=\alpha'+\lambda$ in (8), we get an interesting formula.

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_1\left[\mu;-n,\alpha';\lambda+\alpha';x,\frac{tx}{t-1}\right]F_1\left[\rho;-n,\beta;\beta+\lambda;y,\frac{ty}{t-1}\right]t^n$$
$$=\frac{(1-t)^{\rho+\mu-\lambda}}{(1-t+tx)^{\mu}(1-t+ty)^{\rho}}\times{}_3F_2\left[\lambda,\mu,\rho;\lambda+\alpha',\lambda+\beta;\frac{txy}{(1-t+tx)(1-t+ty)}\right]\tag{9}$$

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valid for $|x|<1$, $|y|<1$, $|t|<1$, $|\arg x|<\frac{1}{3}\pi$, $|\arg y|<\frac{1}{3}\pi$. $\pi \leqslant \arg t \leqslant \frac{3}{2}\pi$, $R(\lambda+\alpha')>0$, $R(\lambda+\beta)>0$.

(*e*) Taking $\gamma=\lambda=\alpha'$, $\delta=\lambda+\beta$ and using the relation (3, p. 238, equation 1)

$$F_1[\alpha;\beta,\beta';\beta+\beta';x,y]=(1-y)^{-\alpha}F_1\left[\alpha,\beta;\beta+\beta';\frac{x-y}{1-y}\right] \tag{10}$$

in (3), we obtain

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_1[\mu;-n,\alpha';\lambda+\alpha';\alpha,b]F_1[\rho;-n,\beta;\lambda+\beta;c,e]t^n$$
$$=(1-t)^{\rho+\mu-\lambda}(1-t+t\alpha)^{-\mu}(1-t+tc)^{-\rho}$$
$$F_k\left[\begin{matrix}\mu;\rho;\alpha';\beta;\lambda;\frac{t(\alpha+b)-b}{1-t+t\alpha},\frac{t(c+e)-e}{1-t+tc},\frac{t\alpha c}{(1-t+t\alpha)(1-t+tc)}\\ \lambda+\alpha',\lambda+\beta;\end{matrix}\right] \tag{11}$$

valid for $R(\lambda+\alpha')>0$, $R(\lambda+\beta)>0$, $|x|<1$, $|y|<1$, $|t|<1$, $|\arg x|<\frac{1}{3}\pi$, $|\arg y|<\frac{1}{3}\pi$, $\pi \leqslant \arg t \leqslant \frac{3}{2}\pi$.

2.1. Proceeding in the same way and using the identity (4), we obtain the following formula also.

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_3[\alpha,-n,\beta,\gamma;\delta;x,y]F_3[\alpha_1,-n,\beta_1;\gamma_1;\delta_1;u,v]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\gamma)_n(\gamma_1)_n(tyv)^n}{n!(\delta)_n(\delta_1)_n(1-t)^{2n}}F_3\left[\alpha,\lambda+n,\beta,\gamma+n,\delta+n;x,\frac{-yt}{1-t}\right]$$
$$F_3\left[\alpha_1,\lambda+n,\beta_1,\gamma_1+n\,\delta_1+n;u,\frac{-vt}{1-t}\right], \tag{12}$$

valid for $R(\delta)>0$, $R(\delta_1)>0$, $|x|<1$, $|y|<1$, $|t|<1$, $|\arg x|<\frac{1}{3}\pi$, $|\arg y|<\frac{1}{3}\pi$, $\pi \leqslant \arg t \leqslant \frac{3}{2}\pi$.

In particular if we take $\gamma=\delta-\beta$, $\lambda=\delta-\alpha$, $\alpha=\alpha_1$, $\beta=\beta_1$, $\delta=\delta_1$ and use the formula ((3), p. 238, equation 4).

$$F_3[\alpha,\gamma-\alpha,\beta,\gamma-\beta;\gamma;x,y]=(1-x)^{\alpha+\beta-\gamma}{}_2F_1[\alpha,\beta;\gamma;x+y-xy] \tag{13}$$

in (12), we obtain after a little simplification

$$\sum_{n=0}^{\infty}\frac{(\delta-\alpha)_n}{n!}F_3[\alpha,-n,\beta,\delta-\beta;\delta;x,y]F_3[\alpha,-n,\beta,\delta-\beta;\delta;u,v]t^n$$
$$=(1-t)^{\delta-\alpha-2\beta}(1-t+ty)^{\alpha+\beta-\delta}(1-t+vt)^{\alpha+\beta-\delta}$$
$$F_k\left[\begin{matrix}-;-;\alpha,\beta;\alpha;\beta;\delta-\alpha,\delta-\beta,\delta-\beta;\frac{x-xt-yt+xyt}{1-t},\frac{u-ut-vt+uvt}{1-t},\frac{tyv}{(1-t+ty)(1-t+tv)}\\ \delta;\delta;\end{matrix}\right], \tag{14}$$

valid for $R(\delta)>0$, $|x|<1$, $|y|<1$, $|t|<1$, $|\arg x|<\frac{1}{3}\pi$, $|\arg y|<\frac{1}{3}\pi$, $\pi \leqslant \arg t \leqslant \frac{3}{2}\pi$.

2.2. On using the identities

$$\left.\begin{aligned}&(1-x)^{-\lambda}(1-y)^{-\lambda}[1-t(1-x)^{-1}(1-y)^{-1}]^{-\lambda}=(1-t)^{\lambda},\\ &(1-t-x)^{-\lambda}(1-t-y)^{-\lambda}\left[1-\frac{txy}{(1-t-x)(1-t-y)}\right]^{-\lambda},\end{aligned}\right\} \tag{15}$$

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and

$$(1-x)^{-\lambda}\left[1+\frac{t(1-y)}{1-x}\right]^{-\lambda}=(1+t)^{\lambda}(1+t-ty)^{-\lambda}(1-t-x)^{-\lambda}\left[1-\frac{txy}{(1+t-ty)(1+t-x)}\right]^{-\lambda}, \tag{16}$$

instead of (4), we obtain the following formulae in the same way as (3).

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_1[\mu;\lambda+n,\alpha;\gamma;x,y]F_1[\rho;\lambda+n,\beta;\delta;u,v]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n(\rho)_n(txu)^n}{n!(\gamma)_n(\delta)_n(1-t)^{2n}}F_1\left[\mu+n;\alpha,\lambda+n;\gamma+n;y,\frac{x}{1-t}\right]$$
$$F_1\left[\rho+n;\beta;\lambda+n;\delta+n;v,\frac{u}{1-t}\right], \tag{17}$$

valid for $R(\gamma)>R(\mu)>0$, $R(\delta)>R(\rho)>0$, $|x|<1$, $|y|<1$, $|u|<1$, $|v|<1$, $|t|<1$, $\pi\leqslant\arg x\leqslant\frac{3}{2}\pi$, $\pi\leqslant\arg t\leqslant\frac{3}{2}\pi$, $\pi\leqslant\arg u\leqslant\frac{3}{2}\pi$.

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_3[\alpha,\lambda+n,\beta,\gamma;\delta;x,y]F_3[\alpha_1,\lambda+n,\beta_1,\gamma_1;\delta_1;u,v]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\gamma)_n(\gamma_1)_n}{n!(\delta)_n(\delta_1)_n}\left[\frac{tyv}{(1-t)^2}\right]^n F_3\left[\alpha,\lambda+n,\beta,\gamma+n;\delta+n;x,\frac{y}{1-t}\right]$$
$$F_3\left[\alpha_1,\lambda+n,\beta_1,\gamma_1+n;\delta_1+n;u,\frac{v}{1-t}\right], \tag{18}$$

provided $R(\delta)>0$, $R(\delta_1)>0$, $|x|<1$, $|y|<1$, $|u|<1$, $|v|<1$, $|t|<1$, $\pi\leqslant\arg x\leqslant\frac{3}{2}\pi$, $\pi\leqslant\arg t\leqslant\frac{3}{2}\pi$, $\pi\leqslant\arg u\leqslant\frac{3}{2}\pi$.

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_1[\delta;-n,\mu;\rho;xy]F_1[\alpha;n+\lambda,\beta;\gamma;u,v]t^n=(1-t)^{-\lambda}$$
$$\sum_{n=0}^{\infty}\frac{(\lambda)_n(\delta)_n(\alpha)_n(-xtu)^n}{n!(\rho)_n(\gamma)_n(1-t)^{2n}}F_1\left[\delta+n;\lambda+n,\mu;\rho+n;\frac{xt}{1-t},y\right]F_1\left[\alpha+n;n+\lambda,\beta;\gamma+n;\frac{u}{1-t},v\right], \tag{19}$$

provided that $|x|<1$, $|v|<1$, $|t|<1$, $|u|<1$, $|\arg x|\leqslant\frac{1}{2}\pi$, $\pi\leqslant\arg u\leqslant\frac{3}{2}\pi$, $\pi\leqslant\arg t\leqslant\frac{3}{2}\pi$, $R(\rho)>R(\delta)>0$, $R(\gamma)>R(\alpha)>0$.

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_3[\alpha,-n,\beta,\gamma;\delta;x,y]F_3[\alpha_1,n+\gamma,\beta_1,\gamma_1;\delta_1;u,v]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\beta)_n(\beta_1)_n(-xut)^n}{n!(\delta)_n(\delta_1)_n(1-t)^{2n}}F_3\left[\alpha,n+\lambda,\beta+n,\gamma;\delta+n;\frac{xt}{1-t},y\right]$$
$$F_3\left[\alpha_1,n+\lambda,\beta_1+n,\gamma_1;\delta_1+n;\frac{u}{1-t},v\right], \tag{20}$$

provided that $R(\delta)>0$, $R(\delta_1)>0$, $|x|<1$, $|v|<1$, $|t|<1$, $|u|<1$, $|y|<1$, $|\arg x|\leqslant\frac{1}{2}\pi$, $\pi\leqslant\arg u\leqslant\frac{3}{2}\pi$, $\pi\leqslant\arg t\leqslant\frac{3}{2}\pi$.

2.3. In this section we consider the identity

$$[1-t(1-x-y)]^{-\lambda}=(1-t)^{-\lambda}\left[1+\frac{xt}{1-t}+\frac{yt}{1-t}\right]^{-\lambda}, \tag{21}$$

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or

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}t^n\sum_{p,q=0}^{p+q\leq n}\frac{(-n)_{p+q}x^py^q}{p!\quad q!}=(1-t)^{-\lambda}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{r+s}}{r!\quad s!}\left(\frac{-xt}{1-t}\right)^r\left(\frac{-yt}{1-t}\right)^s$$

provided that $|x|+|y|<1$, $R(t)<\frac{1}{2}$. If we multiply both sides by $x^{\alpha-1}y^{\beta-1}$ and apply the operators $D_x^{\alpha-\gamma}$ and $D_y^{\beta-\delta}$, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}t^n\sum_{p,q=0}^{p+q\leq n}\frac{(-n)_{p+q}}{p!\quad q!}D_x^{\alpha-\gamma}[x^{\alpha+p-1}]D_y^{\beta-\delta}[y^{\beta+q-1}]$$

$$=(1-t)^{-\lambda}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{r+s}}{r!\,s!}\left(\frac{t}{t-1}\right)^{r+s}D_x^{\alpha-\gamma}[x^{\alpha+r-1}]D_y^{\beta-\delta}[y^{\beta+s-1}].$$

On applying (1), we get the following formula

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_2[-n,\alpha,\gamma;\beta,\delta;x,y]t^n=(1-t)^{-\lambda}F_2\left[\lambda;\alpha,\gamma;\beta,\delta;\frac{-xt}{1-t},\frac{-yt}{1-t}\right],\tag{22}$$

provided $|x|+|y|<1$, $R(t)<\frac{1}{2}$, $R(\beta)>0$, $R(\delta)>0$. From (21), we can easily obtain the following formulae

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_4[-n,\alpha;\beta,\gamma;x,y]t^n=(1-t)^{-\lambda}\tag{23}$$

$$F_4\left[\lambda,\alpha;\beta,\gamma;\frac{-xt}{1-t},\frac{-yt}{1-t}\right],$$

provided that $\sqrt{|x|}+\sqrt{|y|}<1$, $R(t)<\frac{1}{2}$, $R(\beta)>0$, $R(\delta)>0$.

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_1[-n;\alpha,\beta;\gamma;x,y]t^n=(1-t)^{-\lambda}F_1\left[\lambda;\alpha,\beta;\gamma;\frac{-xt}{1-t},\frac{-yt}{1-t}\right],\tag{24}$$

provided that $|x|+|y|<1$, $R(t)<\frac{1}{2}$, $R(\gamma)>0$. If we take $\lambda=\gamma$, in (24), we have

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_1[-n;\alpha,\beta;\lambda;x,y]t^n=(1-t)^{-\lambda}\left(1+\frac{tx}{1-t}\right)^{-\alpha}\left(1+\frac{ty}{1-t}\right)^{-\beta},\tag{25}$$

provided that $|x|<1$, $|y|<1$, $R(t)<\frac{1}{2}$. We give below some formulae, these can be obtained by using well-known methods with the help of (22), (23), (24) and (25).

$$\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}}{n!}F_1(-m-n;\alpha,\beta;\lambda;x,y]t^n=(1-t)^{\alpha+\beta-\lambda-m}(1-t+tx)^{-\alpha}$$

$$(1-t+ty)^{-\beta}(\lambda-\alpha-\beta)_mF_1\left[-m,\alpha,\beta;1+\alpha+\beta-\lambda-m;\frac{(1-x)(1-t)}{1-t+xt},\frac{(1-y)(1-t)}{1-t+ty}\right],\tag{26}$$

provided that $|x|<1$, $|y|<1$, $|\arg x|\leqslant\frac{1}{2}\pi$, $|\arg y|\leqslant\frac{1}{2}\pi$, $R(t)<\frac{1}{2}$. In particular if we take $y=1$ and use the relations, Rainville ((7), p. 255 equation 8) and Erdélyi ((3), p, 239, equation 10) in (26), it reduces to a known result due to Manocha and Sharma ((5), p. 460, equation 7).

$$\sum_{m=0}^{n}\frac{(\lambda_1)_{n-m}(\lambda_2)_m}{m!\quad n-m!}F_1[-n-m;\alpha_1,\beta_1;\lambda_1;x,y]F_1[-m;\alpha_2,\beta_2,\lambda_2;x,y]$$

$$=\frac{(\lambda_1+\lambda_2)_n}{n!}F_1[-n;\alpha_1+\alpha_2,\beta_1+\beta_2;\lambda_1+\lambda_2;x,y],\tag{27}$$

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provided that $R(\lambda_1) > 0$, $R(\lambda_2) > 0$, $|x| < 1$, $|y| < 1$, $|\arg x| \leqslant \frac{1}{2}\pi$, $|\arg y| < \frac{1}{2}\pi$, $R(t) < \frac{1}{2}$. In particular if we take $y = 1$ and use the formulae, Raivnille ((7), p. 255, equation 8 and Erdélyi ((3), p. 239, equation 10) in (27), it reduces to a known result due to Manocha and Sharma ((5), p. 460, equation 9).

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1[-n,\alpha,\beta;\lambda;x,y] F_1[\delta,-n,\gamma;\rho;z,zw] t^n$$
$$= (1-t)^{\alpha+\beta-\lambda}(1-t+ty)^{-\beta}(1-t+xt)^{-\alpha}$$
$$F_D\left[\delta;\gamma,\lambda-\alpha-\beta.\alpha,\beta;\rho;wz,\frac{-zt}{1-t},\frac{zt(x-1)}{1-t+xt},\frac{zt(y-1)}{1-t+ty}\right], \tag{28}$$

provided that $R(\lambda) > 0, R(\rho) > 0, |x| < 1, |y| < 1, |z| < 1, |w| < 1, |\arg x| < \frac{1}{2}\pi, |\arg y| < \frac{1}{2}\pi, |\arg z| < \frac{1}{2}\pi$, $|\arg(wz)| < \frac{1}{2}\pi$, $R(t) < \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_2[-n,\alpha,\gamma;\beta,\delta;x,y] F_2[\lambda+n,\alpha_1,\gamma_1;\beta_1,\delta_1;z,w] t^n$$
$$= (1-t)^{-\lambda} F_A\left[\lambda;\alpha,\gamma,\alpha_1,\gamma_1;\beta,\delta,\beta_1,\delta_1;\frac{-xt}{1-t},\frac{-yt}{1-t},\frac{z}{1-t},\frac{w}{1-t}\right], \tag{29}$$

valid for $R(\beta) > 0$, $R(\delta) > 0$, $R(\beta_1) > 0$, $R(\delta_1) > 0$, $|x| < 1$, $|y| < 1$, $|z| < 1$, $|w| < 1$, $|\arg x| < \frac{1}{2}\pi$, $|\arg y| < \frac{1}{2}\pi$, $|\arg z| < \frac{1}{2}\pi$, $|\arg w| < \frac{1}{2}\pi$, $R(t) < \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_4[-n,\alpha;\beta,\gamma;x,y]\, {}_2F_1[\lambda+n,\alpha_1;\beta_1;z] t^n$$
$$= (1-t)^{-\lambda} F_E\left[\lambda,\lambda,\lambda;\alpha_1,\alpha,\alpha;\beta_1,\beta,\gamma;\frac{z}{1-t},\frac{-xt}{1-t},\frac{-yt}{1-t}\right], \tag{30}$$

provided that $R(\beta) > 0$, $R(\gamma) > 0$, $R(\beta_1) > 0$, $|x| < 1$, $|y| < 1$, $|z| < 1$, $|\arg x| \leqslant \frac{1}{2}\pi$, $|\arg y| < \frac{1}{2}\pi$, $|\arg z| \leqslant \frac{1}{2}\pi$, $R(t) < \frac{1}{2}$, where F_E is a function of three variables defined by Saran (8).

$$F_E[\alpha,\alpha,\alpha;\beta_1,\beta_2,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z]$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\alpha)_{m+n+p}(\beta_1)_m(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p m!\,n!\,p!} x^m y^m z^p. \tag{31}$$

For Appell functions and Lauricellas' functions, see (1).

The author is grateful to the referee for some useful suggestions for revising the paper.

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