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103.35 Hölder's inequality revisited

Based on the same idea provided by Razminia [1], we are going to present a new proof for Hölder's inequality.

Lemma 1: For all positive numbers a_1, a_2, b_1 and b_2, we have

$$a_1b_1 + a_2b_2 \leqslant \left(a_1^p + a_2^p\right)^{1/p}\left(b_1^q + b_2^q\right)^{1/q}, \tag{1}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ with $p, q > 1$. Moreover, the equality holds when $\dfrac{a_1^p}{b_1^q} = \dfrac{a_2^p}{b_2^q}$.

Proof: Let us define the following function

$$f(x) = \left(1 + x^p\right)^{1/p}\left(1 + y^q\right)^{1/q} - 1 - xy, \tag{2}$$

with a given y. Clearly $f(x)$ is continuous and differentiable, so the critical points of $f(x)$ can be obtained by solving the following equation

$$\frac{d}{dx}f(x) = x^{p-1}\left(1 + x^p\right)^{\frac{1}{p}-1}\left(1 + y^q\right)^{\frac{1}{q}} - y = 0. \tag{3}$$

Moving y to the right-hand side of (3), and raising both sides of the equation to the power q, we obtain

$$x^{(p-1)q}\left(1 + x^p\right)^{\left(\frac{1}{p}-1\right)q}\left(1 + y^q\right) = y^q.$$

Now, using the fact that $\frac{1}{p} - 1 = -\frac{1}{q}$ and $q(p-1) = p$, we obtain

$$x^p\left(1 + x^p\right)^{-1}\left(1 + y^q\right) = y^q \text{ or } \left(1 + y^q\right) = y^q\left(1 + x^{-p}\right),$$

from which we conclude that $1 = y^q x^{-p}$ or, equivalently, $x^p = y^q \left(x = y^{\frac{q}{p}}\right)$.

On the other hand, by taking the derivative of (3), which gives

$$\left[(p-1)x^{p-2}\left(1 + x^p\right)^{\frac{1}{p}-1} + (1-p)x^{2p-2}\left(1 + x^p\right)^{\frac{1}{p}-2}\right]\left(1 + y^q\right)^{\frac{1}{q}},$$

and after factorisation, it becomes

$$\frac{d^2}{dx^2}f(x) = (p-1)x^{p-2}\left(1 + x^p\right)^{\frac{1}{p}-2}\left(1 + y^q\right)^{\frac{1}{q}},$$

which is clearly positive and, consequently, $f(x)$ is convex. Based on

$$\left.\frac{df}{dx}\right|_{\left(x = y^{q/p}\right)} = f\left(y^{\frac{q}{p}}\right) = 0$$

and the convexity of $f(x)$, we can conclude that $f(x)$ is always non-negative.

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Now, by choosing $x = \dfrac{a_2}{a_1}$ and $y = \dfrac{b_2}{b_1}$ in (2), we get (1). It should be noted that the equality holds in (1) when $\dfrac{a_1^p}{b_1^q} = \dfrac{a_2^p}{b_2^q}$ or equivalently, $x^p = y^q$ for when $f(x) = 0$.

Theorem 2: Hölder's inequality for sums states that

$$\sum_{k=1}^{n} a_k b_k \leqslant \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}, \tag{4}$$

where all $a_k, b_k > 0$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$with $p, q > 1$. The equality holds when $\dfrac{a_k^p}{b_k^q} = \dfrac{a_j^p}{b_j^q}$ for all $k, j = 1, 2, \ldots, n$. Moreover, if $p = q = 2$, (4) becomes Cauchy's inequality.

Proof: By dividing both sides of (4) by $\max_k(a_k)$ and $\max_k(b_k)$, we have

$$\sum_{k=1}^{n} x_k y_k \leqslant \left(\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} y_k^q\right)^{\frac{1}{q}},$$

where $x_k = \dfrac{a_k}{\max_k(a_k)}$ and $y_k = \dfrac{b_k}{\max_k(b_k)}$. Hence, all x_k, y_k belong to the closed interval $[0, 1]$. Let us define the following function on $[0, 1]^{2n}$

$$g(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = \left(\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} y_k^q\right)^{\frac{1}{q}} - \sum_{k=1}^{n} x_k y_k.$$

According to Weierstrass's theorem, which says a continous function has a global minimum on a closed interval, we conclude that the function g has a global minimum on $[0, 1]^{2n}$. This minimum value occurs when $\dfrac{x_k^p}{y_k^q} = \dfrac{x_j^p}{y_j^q}$ for all $k, j = 1, 2, \ldots, n$. By use of the contradiction approach, let us assume that there is a minimum point $A = (\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n)$, where at least two $\dfrac{\alpha_k^p}{\beta_k^q}$ are unequal, and without loss of generality, say $\dfrac{\alpha_1^p}{\beta_1^q} \neq \dfrac{\alpha_2^p}{\beta_2^q}$. From Lemma 1, $\alpha_1\beta_1 + \alpha_2\beta_2 < (\alpha_1^p + \alpha_2^p)^{\frac{1}{p}}(\beta_1^q + \beta_2^q)^{\frac{1}{q}}$, and then by rearranging the terms in this inequality and adding some terms to both sides, we obtain

$$\left(\sum_{k=1}^{n} \alpha_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \beta_k^q\right)^{\frac{1}{q}} - (\alpha_1^p + \alpha_2^p)^{\frac{1}{p}}(\beta_1^q + \beta_2^q)^{\frac{1}{q}} - \sum_{k=3}^{n} \alpha_k\beta_k$$

$$< \left(\sum_{k=1}^{n} \alpha_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \beta_k^q\right)^{\frac{1}{q}} - \sum_{k=1}^{n} \alpha_k\beta_k, \tag{5}$$

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The right-hand side and left-hand side of (5) represents $g(x)$ at points A and

$$B = \left(\left(\frac{\alpha_1^p + \alpha_2^p}{2}\right)^{\frac{1}{p}}, \left(\frac{\alpha_1^p + \alpha_2^p}{2}\right)^{\frac{1}{p}}, \alpha_3, \ldots, \alpha_n, \left(\frac{\beta_1^q + \beta_2^q}{2}\right)^{\frac{1}{q}}, \left(\frac{\beta_1^q + \beta_2^q}{2}\right)^{\frac{1}{q}}, \beta_3, \ldots, \beta_n\right),$$

respectively. It should be noted that B belongs to $[0, 1]^{2n}$ (because $0 < \left(\frac{\alpha_1^p + \alpha_2^p}{2}\right)^{\frac{1}{p}}, \left(\frac{\beta_1^q + \beta_2^q}{2}\right)^{\frac{1}{q}} \leqslant 1$). We have therefore found another point, which makes $g(x)$ smaller than the global minimum, i.e. $g(A)$. It brings us to a contradiction and completes the proof.

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103.36 Three footnotes to Cartwright's proof that π is irrational

Dame Mary Cartwright's proof that π is irrational, which was famously set as an examination question at Cambridge University in 1945, may be found in many places such as [1, 2, 3, 4]. In particular, [1] traces the connection between this proof, Hermite's earlier proof and Niven's (probably more familiar) slightly later one, which is the one presented in [5, 6].

In examination question format, Cartwright's proof goes as follows.

- Define $I_n(x) = x^{2n+1}\int_{-1}^{1}(1-t^2)^n \cos xt\, dt$ for non-negative integers, n.
- Use two successive integration by parts to derive the reduction formula

  $I_n(x) = 2n(2n-1)I_{n-1}(x) - 4n(n-1)x^2 I_{n-2}(x)$ for $n \geqslant 2$, with initial conditions $I_0(x) = 2\sin x$ and $I_1(x) = 4\sin x - 4x\cos x$.
- Use induction to establish that

$$x^{2n+1}\int_{-1}^{1}(1-t^2)^n \cos xt\, dt = I_n(x) = n!\left[P_n(x)\sin x + Q_n(x)\cos x\right] \quad (1)$$

  where P_n and Q_n are polynomials of degree at most n with integer coefficients.

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