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RESEARCH ARTICLE

Hjorth's Reflection Argument

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Abstract

In [7], Hjorth, assuming AD + ZF + DC, showed that there is no sequence of length ω_2 consisting of distinct $\mathbf{\Sigma}^1_2$-sets. We show that the same theory implies that for $n \geq 0$, there is no sequence of length $\boldsymbol{\delta}^1_{2n+2}$ consisting of distinct $\mathbf{\Sigma}^1_{2n+2}$ sets. The theorem settles Question 30.21 of [16], which was also conjectured by Kechris in [17] (see Conjecture in Chapter 4 of [17] and the last paragraph of Chapter 4 of [17]).

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1. Introduction

A central theme in descriptive set theory is the study of the complexity of various natural processes in terms of their ordinal lengths. In this line of thought, it is often shown that some ordinal is unreachable via processes of certain complexity. For example, there is no analytic well-founded relation of length ω_1, and so ω_1 is inaccessible with respect to analytic surjections with domain $\mathbb{R}$.

One way to study the complexity of a definability class $\mathbf{\Gamma}$ is to seek definable ways of assigning sets from $\mathbf{\Gamma}$ to ordinals in such a way that no two ordinals are assigned to the same set. For example, if α is a countable ordinal then we can assign to α the set $A_\alpha \subseteq \mathbb{R}$ consisting of those reals that code α in some natural way. Each A_α is Borel and clearly the assignment $\alpha \mapsto A_\alpha$ is definable. However, a remarkable theorem of Harrington (see [5, Theorem 4.5]) says that such an assignment cannot exist if we further demand that each set comes from a specific Borel class.

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Below AD is the Axiom of Determinacy, ZF are the axioms of the Zermelo-Fraenkel set theory (which does not include the Axiom of Choice) and DC is the axiom of dependent choice.

Theorem 1.1 (Harrington). *Assume* AD + ZF + DC. *If $\beta < \omega_1$ then there is no injection $f : \omega_1 \to \mathbf{\Pi}^0_\beta$.*

Notice that the content of Harrington's theorem isn't that $|\mathbf{\Pi}^0_\beta| < \omega_1$, which is in fact not true as $\mathbf{\Pi}^0_\beta$ has continuum many distinct sets. The content of Harrington's theorem is that if we fix $\beta < \omega_1$ and devise an algorithm that picks a new set from the Borel class $\mathbf{\Pi}^0_\beta$ then at some stage less than ω_1 our algorithm will stop outputting anything[1] .

Harrington's theorem was recently used by Marks and Day to prove the decomposability conjecture (see [4]).

Definition 1.2. Suppose $\mathbf{\Gamma} \subseteq \wp(\mathbb{R})$. We say κ is $\mathbf{\Gamma}$-reachable if there is an injection $f : \kappa \to \mathbf{\Gamma}$, and that κ is $\mathbf{\Gamma}$-unreachable if it is not $\mathbf{\Gamma}$-reachable.

Let Θ be the least ordinal that is not a surjective image of $\mathbb{R}$. Then in $L(\mathbb{R})$, Θ is $\wp(\mathbb{R})$-reachable. Indeed, in $L(\mathbb{R})$, if $\alpha < \Theta$ then there is a pre-well-ordering (pwo[2]) $\leq^*$ of $\mathbb{R}$ that is ordinal definable and has length α. We can then let $\leq^*_\alpha$ be the least ordinal definable pwo of $\mathbb{R}$ that has length α. Thus we have assigned an ordinal definable pwo of length α to each α in an ordinal definable manner.

The above construction has a well-known Harrington-type analogue. Assume AD. If $\beta < \Theta$ and $\mathbf{\Gamma}$ consists of those sets of reals whose Wadge rank[3] is $\leq \beta$ then Θ is $\mathbf{\Gamma}$-unreachable. This is because Wadge's lemma[4] implies that there is a surjection $f : \mathbb{R} \to \mathbf{\Gamma}$, so if Θ was $\mathbf{\Gamma}$-reachable then we could find a surjection $g : \mathbb{R} \to \Theta$.

Perhaps the most natural way of showing that κ is $\mathbf{\Gamma}$-reachable is to find a surjection $f : \mathbb{R} \to \kappa$ such that for each $\alpha < \kappa$, $A_\alpha = \{x \in \mathbb{R} : f(x) < \alpha\} \in \mathbf{\Gamma}$. In this case, the sets $(A_\alpha : \alpha < \kappa)$ form a strictly $\subset$-increasing sequence. Thus, the fact that κ is not $\mathbf{\Gamma}$-reachable via a strictly $\subset$-increasing sequence implies that κ is inaccessible with respect to $\mathbf{\Gamma}$-surjections.

An equivalence relation on $\mathbb{R}$ is called *thin* if it does not have a perfect set of inequivalent elements. Assuming AD holds in $L(\mathbb{R})$, or in fact just AD^+, Woodin (see [2], [3] and [6, Theorem 0.3]), generalizing Harrington's earlier result on Π^1_2-equivalence relations, showed that if $E \subseteq \mathbb{R}^2$ is a thin equivalence relation then the set $\{[x]_E : x \in \mathbb{R}\}$ is well-orderable. Thus, assuming AD^+, if κ is $\mathbf{\Gamma}$-unreachable then any thin equivalence relation $E \in \wp(\mathbb{R}^2) \cap \mathbf{\Gamma}$ has $< \kappa$ many equivalence classes.

Below $\boldsymbol{\delta}^1_n$ is the supremum of the lengths of $\mathbf{\Delta}^1_n$ pre-well-orderings of $\mathbb{R}$. In a seminal work, Jackson, building on an early work of Kechris, Kunen, Martin, Moschovakis and Solovay computed $\boldsymbol{\delta}^1_n$. Before Jackson's work, it was known that assuming $\mathsf{AD}^{L(\mathbb{R})}$, $\delta^1_1 = \aleph_1$, $\delta^1_2 = \aleph_2^{L(\mathbb{R})}$, $\delta^1_3 = \aleph^{L(\mathbb{R})}_{\omega+1}$, $\delta^1_4 = \aleph^{L(\mathbb{R})}_{\omega+2}$, for every n, $\delta^1_{2n+2} = ((\delta^1_{2n+1})^+)^{L(\mathbb{R})}$, for every n, δ^1_{2n+1} is itself a successor cardinal of $L(\mathbb{R})$ and $\mathbf{\Sigma}^1_{2n+2}$ is exactly the collection of δ^1_{2n+1}-Suslin sets[5]. In [12], Jackson computed the remaining δ^1_n's and in particular, showed that $\delta^1_5 = \aleph^{L(\mathbb{R})}_{\omega^{\omega^\omega}+1}$.

In [17], Kechris proved a partial generalization of Harrington's theorem[6].

Theorem 1.3 (Kechris). *Assume* ZF + AD + DC. *Then δ^1_{2n+2} is $\mathbf{\Delta}^1_{2n+1}$-unreachable.*

Moreover, in [17], Kechris showed that there is no $f : \boldsymbol{\delta}^1_{2n+2} \to \mathbf{\Sigma}^1_{2n+2}$ such that for all $\alpha, \beta < \delta^1_{2n+2}$, if $\alpha \neq \beta$ then $f(\alpha) \cap f(\beta) = \emptyset$. In Chapter 4 of [17], Kechris conjectured that in fact a general form of Harrington-type theorem is true for projective pointclasses.

Conjecture 1.4 (Kechris 1st Conjecture). *Assume* ZF + AD + DC. *Then $\boldsymbol{\delta}^1_{2n+2}$ is $\mathbf{\Delta}^1_{2n+2}$-unreachable.*

[1] Harrington's Theorem can also be viewed as an effective cardinality theorem. This type of question has been investigated in [9] and [1].

[2] $\leq^*$ is a pwo if $\leq^*$ is a well-founded relation such that for every $x, y \in \mathrm{dom}(\leq^*)$, either $x \leq^* y$ or $y \leq^* x$.

[3] E.g., [16, Theorem 29.16] or [13, Chapter 2.3].

[4] See [16, Lemma 29.15].

[5] See [13, Theorem 2.18, Chapter 3].

[6] As pointed out by the referee, the theorems and conjectures stated below do not need DC. See, for example, [18]. However, the original versions of these results assumed DC, and so we also assume it.

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In [11], Jackson proved Kechris' 1st Conjecture by establishing the following remarkable theorem (see [11, Corollary 4.5]).

Theorem 1.5 (Jackson). *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$. *Then* δ^1_{2n+2} *is* $\mathbf{\Delta}^1_{2n+2}$*-unreachable.*

Jackson's proof used his computation of the projective ordinals. In particular, that $\boldsymbol{\delta}^1_{2n+1}$ has the strong partition property. In the last paragraph of [17, Chapter 4], Kechris made the following stronger conjecture.

Conjecture 1.6 (Kechris 2nd Conjecture). *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$. *Then* $\boldsymbol{\delta}^1_{2n+2}$ *is* $\mathbf{\Sigma}^1_{2n+2}$*-unreachable.*

This was partially resolved by Jackson and Martin who showed the following (see the Theorem on Page 84 of [11]).

Theorem 1.7 (Jackson-Martin). *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$. *Then there is no strictly* $\subset$*-increasing or* $\subset$*-decreasing sequence* $(A_\alpha : \alpha < \boldsymbol{\delta}^1_{2n+2}) \subseteq \mathbf{\Sigma}^1_{2n+2}$.

Chuang then generalized Theorem 1.7 by showing that in fact there are no strictly $\subset$-increasing or $\subset$-decreasing sequences of $\mathbf{\Gamma}$ sets of length $\delta^+_{\mathbf{\Gamma}}$ provided $\mathbf{\Gamma}$ is closed under $\forall^{\mathbb{R}}$, $\wedge$, $\vee$ and that $\mathbf{\Gamma}$ has the pre-well-ordering property (see [13, Theorem 3.5]). Here $\delta_{\mathbf{\Gamma}}$ is the supremum of the lengths of $\mathbf{\Delta}_{\mathbf{\Gamma}}$-pwos[7].

Kechris' 2nd Conjecture also appears in Kanamori's book where it appears as [16, Question 30.21]. In [7], Hjorth verified Kechris' 2nd Conjecture for $n = 0$ using techniques from inner model theory. In this paper, we prove Kechris' 2nd Conjecture.

Theorem 1.8 (Hjorth-S.). *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$. *Then* $\boldsymbol{\delta}^1_{2n+2}$ *is* $\mathbf{\Sigma}^1_{2n+2}$*-unreachable.*

The following is an immediate corollary of Theorem 1.8 and Woodin's result mentioned above[8]. The case $n = 0$ is due to Hjorth ([7]).

Corollary 1.9. *Assume* $\mathsf{ZFC} + \mathsf{AD}^{L(\mathbb{R})}$. *If* E *is a thin* $\mathbf{\Pi}^1_{2n+2}$*-equivalence relation then it has* $\leq \boldsymbol{\delta}^1_{2n+1}$*-many equivalence classes. In particular, any thin* $\mathbf{\Pi}^1_2$ *equivalence relation has* $\leq \omega_1$*-equivalence classes and any thin* $\mathbf{\Pi}^1_4$ *equivalence relation has* $\leq \aleph^{L(\mathbb{R})}_{\omega+1}$*-equivalence classes.*

Our proof of Theorem 1.8 uses inner model theory and directly builds on [7] and [29]. From [7], we will mainly use the *reflection argument*[9] used by Hjorth which appears on page 104 of [7]. We state it as Lemma 4.7. According to Page 95 of [7], Hjorth's reflection argument is inspired by Woodin's unpublished proof of the pre-well-ordering property for Π^1_3. We strongly believe that it can have many other applications.

The reason that Theorem 1.8 has been open since [29] is that the proof in [7] uses the well-known Kechris-Martin theorem (see (i) and (ii) on page 105 of [7], see [19] for the Kechris-Martin theorem). It has been quite challenging to extend Kechris-Martin result in a way that could be useful to us. However, as it turns out, the use of Kechris-Martin theorem can be removed from [7], and this is our main new idea (see Section 4.5). Clearly Theorem 1.5 is a corollary of Theorem 1.8, and so the inner model proof of Theorem 1.5 avoids the sophisticated machinery developed by Jackson in [12], though it uses inner model theory.

The main technical ingredient of our argument is the directed systems of mice. This is the system that Steel used in his calculation of $(\mathrm{HOD}|\Theta)^{L(\mathbb{R})}$ ([38]) and Woodin used in his calculation of the $\mathrm{HOD}^{L(\mathbb{R})}$. The theory of these directed systems of mice has appeared in [40]. We will use the material developed in [40, Chapter 6]. [26] has a nice introduction to the subject.

We expect that our methods will generalize and settle the following conjecture. AD^+ is an extension of AD introduced by Woodin ([21]).

Conjecture 1.10. *Assume* AD^+. *Suppose* κ *is a regular Suslin cardinal and* $\mathbf{\Gamma}$ *is the pointclass of* κ*-Suslin sets. Then* κ^+ *is* $\mathbf{\Gamma}$*-unreachable.*

[7]Here, $\mathbf{\Delta}_{\mathbf{\Gamma}}$ consists of all those sets of reals A such that both A and $\mathbb{R} - A$ belong to Γ.
[8]Notice that κ is Γ-unreachable if and only if κ is $\check{\Gamma}$-unreachable.
[9]Hjorth didn't call it a reflection argument.

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Conjecture 1.10 is a global conjecture like those made by Jackson (see [13, Conjecture 6.4], the conjectures in [10] and [41, Problem 19]) though perhaps given the result of this paper Conjecture 1.10 is somewhat easier than those made by Jackson. Such global conjectures test our understanding of projective sets. It is one of the deepest mysteries of descriptive set theory that, assuming AD, the complete theory of analytic and co-analytic sets doesn't immediately generalize to projective hierarchy. Perhaps the most well-known example of this phenomenon is that Δ^1_3 to (Π^1_3, Σ^1_3) is not the same as Δ^1_1 to (Π^1_1, Σ^1_1) (see [20]). From a current point of view, it seems that the function $x \mapsto x^\#$ provides singularly magical coding of subsets of ω_1, and that coding – which was used by Martin to establish the strong partition property for ω_1 (see [16, Theorem 28.12]) and by Kechris-Martin to establish their celebrated Kechris-Martin theorem (see [19]) – doesn't yet have a proper inner model theoretic generalization to higher levels of the projective hierarchy and beyond. Our current understanding is based on Jackson's deep analysis of measures (see [13]). The fact that global conjectures such as Conjecture 1.10 and Jackson's conjectures are still open seems to suggest that our current understanding of the projective hierarchy, just like it was with our understanding of (Π^1_1, Σ^1_1), may not be the final one, as whatever methods we discover to settle these global conjectures, the projective case will have to be the special case of these conjectures, and so these yet-to-be-discovered ideas will come with new insight into the projective hierarchy.

2. Remarks, notations and terminology

Experts can skip this and the following sections and directly go to Section 4. The results in Section 3 are not fundamentally new and go back to [37]. However, [37] doesn't state them in the exact form that we need. Below we make some remarks, and set up our notation and terminology.

Review 2.1. Basic concepts from inner model theory:

The reader unfamiliar with basic concepts of inner model theory might find it helpful to consult [39]. Also, the introduction of [26] is accessible and introduces many of the concepts that we need. Extenders were treated both in [14] and [16].

1. Suppose $x \in \mathsf{HC}$[10] is such that there is a wellordering of x in $L_1[x]$ which can be defined without parameters over $L_0[x]$[11]. We say $(\mathcal{M}, \Phi)$ is an x-mouse pair if $\mathcal{M}$ is an x-premouse and Φ is an (ω_1, ω_1)-iteration strategy for $\mathcal{M}$ (e.g., see [39, Definition 2.19, 3.9 and 4.4][12])[13]. We say $(\mathcal{M}, \Phi)$ is a countable mouse pair if $\mathcal{M}$ is countable. We will often say that $\mathcal{M}$ is a premouse or $(\mathcal{M}, \Phi)$ is a mouse pair without mentioning the x.
2. Suppose $\mathcal{M} = \mathcal{J}^{\vec{E}}_\beta$ is a premouse and $\alpha \leq \mathsf{Ord} \cap \mathcal{M}$. We let $\vec{E}^{\mathcal{M}}$ be the extender sequence of $\mathcal{M}$. Because we will allow padded iterations, we let $\mathrm{dom}(\vec{E})^{\mathcal{M}} = \beta$ and for those γ such that $\mathcal{M}$ doesn't have an extender indexed at γ, we set $\vec{E}^{\mathcal{M}}(\gamma) = \emptyset$. We then let $\mathcal{M}|\alpha = (\mathcal{J}^{\vec{E}\restriction\omega\alpha}_{\omega\alpha}, \vec{E} \restriction \omega\alpha, \in)$ and $\mathcal{M}||\alpha = (\mathcal{J}^{\vec{E}\restriction\omega\alpha}_{\omega\alpha}, \vec{E} \restriction \omega\alpha, \vec{E}(\omega\alpha), \in)$.
3. Suppose $\mathcal{M}$ is a premouse. We say η is a cutpoint of $\mathcal{M}$ if for all $E \in \vec{E}^{\mathcal{M}}$ with the property that $\mathrm{crit}(E) < \eta$, $\mathrm{lh}(E) \leq \eta$.
4. Under AD, as ω_1 is measurable, ω_1-iterability[14] is what is needed to prove the Comparison Theorem for countable mouse pairs (see [39, Theorem 3.11]).
5. For a definition of an iteration[15] and an iteration strategy see [39, Definition 3.3 and 3.9]. Given an iteration $\mathcal{T}$ of a premouse $\mathcal{M}$, we write $\mathcal{T} = ((\mathcal{M}_\alpha : \alpha < \mathrm{lh}(\mathcal{T})), (E_\alpha : \alpha + 1 < \mathrm{lh}(\mathcal{T})), \mathcal{D}, T)$ where

[10] HC is the set of hereditarily countable sets.
[11] Let $L_0[x]$ be the the transitive closure of x.
[12] Here we only consider $(\omega, \omega_1, \omega_1)$-iteration strategies, and so we drop 'ω' for convenience.
[13] An x-premouse is defined similarly to a premouse except one requires that $\mathcal{J}^{\mathcal{M}}_0$ is the transitive closure of x
[14] This concept is defined in [39, Definition 3.9]. Here again we ignore the integer k of that definition as we take it to be ω. If κ is a measurable cardinal then κ-iterability implies $\kappa + 1$-iterability.
[15] Following Jensen, we will use iteration for iteration trees.

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(a) $E_\alpha \in \vec{E}^{\mathcal{M}_\alpha}$ is the extender picked from $\mathcal{M}_\alpha$,
(b) $\mathcal{D}$ is the set of those α where a drop occurs, and
(c) T is the tree order of $\mathcal{T}$.
We allow padded iterations, and so it is possible that $E_\alpha = \emptyset$.

6. Suppose $(\mathcal{M}, \Phi)$ is a mouse pair and $\mathcal{N}$ is a Φ-iterate of $\mathcal{M}$ via iteration $\mathcal{T}$. We then let $\Phi_{\mathcal{N},\mathcal{T}}$ be the strategy of $\mathcal{N}$ induced by the pair $(\Phi, \mathcal{T})$. More precisely, $\Phi_{\mathcal{N},\mathcal{T}}(\mathcal{U}) = \Phi(\mathcal{T}^\frown\mathcal{U})$.
7. Continuing with $(\mathcal{M}, \Phi)$, $\mathcal{N}$ and $\mathcal{T}$ as above, we say $\mathcal{N}$ is a complete Φ-iterate if the iteration embedding $\pi^{\mathcal{T}}$ is defined, which happens if and only if there is no drop on the main branch of $\mathcal{T}$ (see the paragraph after [39, Definition 3.3]).
8. We say $\mathcal{M}$ is an almost knowledgeable mouse if $\mathcal{M}$ has a unique (ω_1, ω_1)-iteration strategy Φ such that whenever $\mathcal{T}$ is an iteration of $\mathcal{M}$ via Φ and $\mathcal{N}$ is the last model of $\mathcal{T}$ then
   (a) $\Phi_{\mathcal{N},\mathcal{T}}$ is independent of $\mathcal{T}$, and[16]
   (b) if $\mathcal{N}$ is a complete iterate of $\mathcal{M}$ then $\pi^{\mathcal{T}}$ is independent of $\mathcal{T}$[17] .
9. We say $\mathcal{M}$ is knowledgeable if letting Φ be the unique (ω_1, ω_1)-iteration strategy of $\mathcal{M}$, whenever $\mathcal{N}$ is a complete iterate of $\mathcal{M}$ via Φ, $\mathcal{N}$ is almost knowledgeable.
10. If $\mathcal{M}$ is knowledgeable then we let $\Phi_{\mathcal{M}}$ be its unique (ω_1, ω_1)-iteration strategy and for each Φ-iterate $\mathcal{N}$ of $\mathcal{M}$, we let $\Phi_{\mathcal{N}} = \Phi_{\mathcal{N},\mathcal{T}}$ where $\mathcal{T}$ is some iteration of $\mathcal{M}$ via Φ with last model $\mathcal{N}$. If $\mathcal{M}$ is knowledgeable and $\mathcal{N}$ is a $\Phi_{\mathcal{M}}$-iterate of $\mathcal{M}$ then we say that $\mathcal{N}$ is an iterate of $\mathcal{M}$. If $\mathcal{N}$ is a complete $\Phi_{\mathcal{M}}$-iterate of $\mathcal{M}$ then we say that $\mathcal{N}$ is a complete iterate of $\mathcal{M}$.
11. Suppose $\mathcal{M}$ is a knowledgeable mouse and $\mathcal{N}$ is a normal iterate[18] of $\mathcal{M}$. Then we let $\mathcal{T}_{\mathcal{M},\mathcal{N}}$ be the unique normal iteration of $\mathcal{M}$ according to $\mathcal{M}$'s unique iteration strategy whose last model is $\mathcal{N}$[19]. In general, if $\mathcal{N}$ is an iterate of $\mathcal{M}$ then $\mathcal{M}$-to-$\mathcal{N}$ iteration may not be unique[20]. If $\mathcal{N}$ is a complete iterate of $\mathcal{M}$, we let $\pi_{\mathcal{M},\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ be the iteration embedding.
12. Suppose $\mathcal{M}$ is an x-premouse and $\mathcal{T}$ is an iteration of $\mathcal{M}$ of limit length. We let $\delta(\mathcal{T}) = \sup\{\mathrm{lh}(E^{\mathcal{T}}_\alpha) : \alpha < \mathrm{lh}(\mathcal{T})\}$ and $\mathsf{cop}(\mathcal{T}) = \cup_{\alpha<\mathrm{lh}(\mathcal{T})}\mathcal{M}^{\mathcal{T}}_\alpha|\mathrm{lh}(E^{\mathcal{T}}_\alpha)$[21]. For more on these objects see [39, Definition 6.9].
13. Suppose $x \in \mathbb{R}$. $\mathcal{M}_n(x)$ is the minimal class size x-mouse with n Woodin cardinals. $\mathcal{M}^{\#}_n(x)$ is the minimal active[22] x-mouse with n Woodin cardinals. We say $\mathcal{M}^{\#}_n(x)$ exists if there is an $\omega_1 + 1$-iterable active x-premouse with n Woodin cardinals. Assuming AD, $\mathcal{M}^{\#}_n(x)$ exists (e.g., see [25] or [31, Sublemma 3.2]). For each n, $\mathcal{M}^{\#}_n(x)$ is knowledgeable (assuming it exist). If every real has a sharp and $\mathcal{M}_n(x)$ exists then $\mathcal{M}_n(x)$ is knowledgeable. For the proof of these and relevant results see [40, Chapter 2 and 3] and especially [40, Theorem 3.23].
14. Suppose $\mathcal{M}$ is an x-premouse, $\mathcal{T}$ is an iteration of $\mathcal{M}$ of limit length and b is a branch. We say $\mathcal{Q}(b, \mathcal{T})$ exists if there is α such that $\mathcal{M}^{\mathcal{T}}_b||\alpha \vDash$ '$\delta(\mathcal{T})$ is a Woodin cardinal' but $\mathcal{J}_1[\mathcal{M}^{\mathcal{T}}_b||\alpha] \vDash$ '$\delta(\mathcal{T})$ is not a Woodin cardinal'. If $\mathcal{Q}(b, \mathcal{T})$ exists we let it be $\mathcal{M}^{\mathcal{T}}_b||\alpha$ where α is the largest such that $\mathcal{M}^{\mathcal{T}}_b||\alpha \vDash$ '$\delta(\mathcal{T})$ is a Woodin cardinal'.
15. $\mathcal{Q}(b, \mathcal{T})$ defined above is one of the most used objects in inner model theory. The reader may want to consult [39, Definition 6.11] and [37, Definition 2.11]. The reason that $\mathcal{Q}(b, \mathcal{T})$ is important is that it uniquely identifies b. More precisely, if $c \neq b$ is another cofinal branch of $\mathcal{T}$ such that $\mathcal{Q}(c, \mathcal{T})$ exists then $\mathcal{Q}(c, \mathcal{T}) \neq \mathcal{Q}(b, \mathcal{T})$.

[16] Such strategies are usually called positional, see, for example, [30, Chapter 2.6].
[17] Such strategies are usually called commuting, see, for example, [30, Chapter 2.6].
[18] We say $\mathcal{N}$ is a *normal iterate* of $\mathcal{M}$ if there is a normal iteration tree $\mathcal{T}$ on $\mathcal{M}$ whose last model is $\mathcal{N}$. Normal iteration trees are exactly the iteration trees defined in [39] on page 23. In such trees, the lengths of the extenders increase and each extender is applied to the earliest possible model appearing in the tree. Normal iteration trees are the trees that appear in the comparison process.
[19] $\mathcal{T}_{\mathcal{M},\mathcal{N}}$ is unique because it is the iteration of $\mathcal{M}$ that is build via the comparison process, see [39, Chapter 3.2].
[20] [35] establishes the following remarkable theorem. Suppose that $(\mathcal{M}, \Phi)$ is a mouse pair, $\mathcal{M}$ is sound and projects to ω and Φ is the unique iteration strategy of $\mathcal{M}$. Then Φ has full normalization, that is, every Φ-iterate of $\mathcal{M}$ can be obtained as a normal Φ-iterate of $\mathcal{M}$.
[21] $\mathsf{cop}(\mathcal{T})$ is usually denoted by $\mathcal{M}(\mathcal{T})$. However, $\mathcal{M}$ gets overused in inner model theory, so we decided to change the notation.
[22] This just means that it has a last extender predicate indexed at the ordinal height of the mouse.

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16. Suppose $\mathcal{M}$ is an x-premouse and $\mathcal{T}$ is an iteration of $\mathcal{M}$. We say $\mathcal{T}$ is below δ if for every $\alpha < \mathrm{lh}(\mathcal{T})$ either $\pi^{\mathcal{T}}_{0,\alpha}$ is not defined or $E^{\mathcal{T}}_\alpha \in \vec{E}^{\mathcal{M}^{\mathcal{T}}_\alpha | \pi^{\mathcal{T}}_{0,\alpha}(\delta)}$. We say $\mathcal{T}$ is above ν if for every $\alpha < \mathrm{lh}(\mathcal{T})$, $\mathrm{crit}(E^{\mathcal{T}}_\alpha) \geq \nu$.
17. We remark that when we say that 'κ is a measurable cardinal in a premouse $\mathcal{M}$' or 'κ is a strong cardinal in a premouse $\mathcal{M}$' or say other similar expressions we tacitly assume that these large cardinal properties are witnessed by the extenders on the extender sequence of $\mathcal{M}$. See [36] for results showing that such a restriction is unnecessary.

Review 2.2. The directed system:

The theory of directed systems, by now, has a long history. It originates in Steel's seminal paper [38]. Since then it has been used to establish a number of striking applications of inner model theory to descriptive set theory. To learn more about the subject the interested reader may consult [39, Chapter 8], [40, Chapter 6], [24], [26], [28], [30], [32] and many other sources.

1. Suppose $\mathcal{P}$ is a knowledgeable mouse. Let $\mathcal{I}_{\mathcal{P}}$ be the set of complete iterates $\mathcal{N}$ of $\mathcal{P}$ such that $\mathcal{P}$-to-$\mathcal{N}$ iteration has a countable length.
2. Suppose that either there is some $\nu < \omega_1$ such that $\mathcal{P} = L[\mathcal{P}|\nu]$ or $\mathcal{P}$ itself is countable. Then comparison implies that if $\mathcal{R}, \mathcal{S} \in \mathcal{I}_{\mathcal{P}}$ then there is $\mathcal{W} \in \mathcal{I}_{\mathcal{P}}$ such that $\mathcal{W}$ is a complete iterate of both $\mathcal{R}$ and $\mathcal{S}$. Define $\leq_{\mathcal{P}}$ on $\mathcal{I}_{\mathcal{P}}$ by setting $\mathcal{R} \leq_{\mathcal{P}} \mathcal{S}$ if and only if $\mathcal{S}$ is an iterate of $\mathcal{R}$. Assuming $\mathcal{P}$ is knowledgeable, we then get a directed system $\mathcal{F}_{\mathcal{P}}$ whose models consist of the models in $\mathcal{I}_{\mathcal{P}}$, whose directed order is $\leq_{\mathcal{P}}$ and whose embeddings are the iteration embeddings $\pi_{\mathcal{R},\mathcal{S}}$.
3. Assuming $\mathcal{P}$ is as above, we let $\mathcal{M}_\infty(\mathcal{P})$ be the direct limit of $\mathcal{F}_{\mathcal{P}}$ and given $\mathcal{N} \in \mathcal{I}_{\mathcal{P}}$, we let $\pi_{\mathcal{N},\infty} : \mathcal{N} \to \mathcal{M}_\infty(\mathcal{P})$ be the iteration embedding according to $\Phi_{\mathcal{N}}$. Comparison Theorem (see [39, Theorem 3.11]) implies that $\mathcal{M}_\infty(\mathcal{P})$ is well-founded (see, e.g., the remark after [39, Definition 8.15])).

Review 2.3. The extender algebra:

The extender algebra, which was discovered by Woodin, is the magic tool of inner model theory. The reader may consult [39, Chapter 7.2].

1. Suppose $\mathcal{P}$ is a premouse, δ is a Woodin cardinal of $\mathcal{P}$ and $\nu < \delta$.

   Definition 2.4. We say that $\mathcal{E}$ is **weakly appropriate** at δ if $\mathcal{E}$ is a set consisting of extenders $E \in \vec{E}^{\mathcal{P}|\delta}$ such that

   (a) $\nu(E)$[23] is an inaccessible cardinal in $\mathcal{P}$,
   (b) $\mathcal{E}$ witnesses that δ is a Woodin cardinal[24].

   If in addition

   (c) for each $E \in \mathcal{E}$, $\pi_E(\mathcal{E}) \cap (\mathcal{P}|\nu(E)) = \mathcal{E} \cap (\mathcal{P}|\nu(E))$,

   then we say that $\mathcal{E}$ is **appropriate**.

   When δ is clear from the context we will omit the expression 'at δ'. Suppose now that $\mathcal{E}$ is weakly appropriate. We then let $\mathsf{Ea}^{\mathcal{P}}_{\delta,\nu,\mathcal{E}}$ be the extender algebra of $\mathcal{P}$ defined using extenders $E \in \mathcal{E}$ such that $\mathrm{crit}(E) > \nu$. If $\nu = 0$ then we omit it from our notation. If $\mathcal{E}$ consists of all extenders or if its role is irrelevant then we omit it from the notation.

   $\mathsf{Ea}^{\mathcal{P}}_{\delta,\nu,\mathcal{E}}$ is the basic extender algebra for adding a real: thus, it has only countably many predicate symbols. The conditions in $\mathsf{Ea}^{\mathcal{P}}_{\delta,\nu,\mathcal{E}}$ are formulas, and so given $g \subseteq \mathsf{Ea}^{\mathcal{P}}_{\delta,\nu,\mathcal{E}}$ and $r \in \mathsf{Ea}^{\mathcal{P}}_{\delta,\nu,\mathcal{E}}$, we will often write $g \vDash r$ instead of $r \in g$.

   Instead of making our notation endlessly complicated, we will abuse our terminology and notation in the following way. Suppose $a = L[b]$ where for some α, $b \subseteq \alpha$. Then we will say that a is generic

[23] Recall from [39, Definition 2.2] that $\nu(E)$ is the supremum of the generators of E, the natural length of E.
[24] In the sense that for every $A \subseteq \delta$ there is an $E \in \mathcal{E}$ such that $A \cap \nu(E) = \pi_E(A) \cap \nu(E)$.

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for $\mathsf{Ea}^{\mathcal{P}}_{\delta,\nu,\mathcal{E}}$ to mean that if α is the least such that there is $b \in a$ with the property that $b \subseteq \alpha$ and $a = L[b]$, then b is generic for the extender algebra at δ that uses α many generators and extenders who critical points are strictly greater than ν.

A celebrated theorem of Woodin says that $\mathsf{Ea}^{\mathcal{P}}_{\delta,\nu,\mathcal{E}}$ has the δ-c.c. condition (assuming only that $\mathcal{E}$ is weakly appropriate, see [39, Chapter 7.2]).

2. Suppose $(\mathcal{P}, \Sigma)$ is a mouse pair, δ is a Woodin cardinal of $\mathcal{P}$, $\nu < \delta$, $\mathcal{E}$ is an appropriate[25] set of extenders and $(x_1, ..., x_k) \in \mathbb{R}^k$. We say $\mathcal{T}$ is the $(\vec{x}, \delta, \nu, \mathcal{E})$-genericity iteration of $\mathcal{P}$ if $\mathcal{T}$ is according to Σ and for each $\alpha < lh(\mathcal{T})$, setting $\mathcal{M}_\alpha =_{def} \mathcal{M}^{\mathcal{T}}_\alpha$ and $E_\alpha =_{def} E^{\mathcal{T}}_\alpha \in \vec{E}^{\mathcal{M}_\alpha}$, $\mathrm{lh}(E_\alpha)$ is the least $\gamma \in \mathrm{dom}(\vec{E}^{\mathcal{M}_\alpha})$ such that setting $E = \vec{E}^{\mathcal{M}_\alpha}(\gamma)$, the following clauses hold:
   (a) $\mathrm{crit}(E) > \nu$ and $\mathrm{lh}(E) < \pi^{\mathcal{T}}_{0,\alpha}(\delta)$,
   (b) $E \in \pi^{\mathcal{T}}_{0,\alpha}(\mathcal{E})$ (and hence, E measures all subsets of $\mathrm{crit}(E)$ in $\mathcal{M}_\alpha$),
   (c) for some $i \leq k$, x_i doesn't satisfy an axiom of $\pi_{\mathcal{P},\mathcal{M}_\alpha}(\mathsf{Ea}^{\mathcal{P}}_{\delta,\nu,\mathcal{E}})$ that is generated by E. More precisely, $x_i \not\Vdash A_{E,\vec{\phi}}$ where $\vec{\phi} \in \mathcal{M}_\alpha | (\mathrm{crit}(E)^+)^{\mathcal{M}_\alpha}$ and $A_{E,\vec{\phi}}$ is the axiom $\bigvee \vec{\phi} \leftrightarrow \bigvee \pi^{\mathcal{M}_\alpha}_E(\vec{\phi}) \restriction \nu(E)$, and
   (d) for all $\gamma' < \gamma$, if $\gamma' \in \mathrm{dom}(\vec{E}^{\mathcal{M}_\alpha})$ then $E^{\mathcal{M}_\alpha}_{\gamma'}$ does not satisfy clauses (a)-(c) above.
3. Suppose $\phi(v_0, ..., v_k)$ is a Σ^1_{2n+2} formula, $\mathcal{M}$ is a premouse, $\delta_0 < ... < \delta_{2n-1}$ are Woodin cardinals of $\mathcal{M}$, $\kappa < \delta_0$, $d = (\kappa, \delta_0, ..., \delta_{2n-1})$ and $\vec{a} \in [\mathbb{R}^{\mathcal{M}}]^{<\omega}$. By induction, we define $\phi_{\mathcal{M},d}$ and the meaning of $\mathcal{M} \vDash \phi_{\mathcal{M},d}[\vec{a}]$. If $n = 0$ then $\phi_{\mathcal{M},d} = \phi$ and $\mathcal{M} \vDash \phi_{\mathcal{M},d}[\vec{a}]$ if and only if $\mathcal{M} \vDash \phi[\vec{a}]$. Next let ψ be a Σ^1_{2n} formula such that $\phi(v_0, ..., v_k) \leftrightarrow \exists u_0 \forall u_1 \psi(u_0, u_1, v_0, ..., v_k)$. We then write $\mathcal{M} \vDash \phi_{\mathcal{M},d}[\vec{a}]$ if and only if there is $p \in \mathsf{Ea}_{\delta_0,\kappa}$ such that if g is the name of the generic for $\mathsf{Ea}_{\delta_0,\kappa}$ then p forces that there is $x \in \mathbb{R}$ such that every $q \in \mathsf{Ea}^{\mathcal{M}[g]}_{\delta_1,\delta_0}$ forces that for all $y \in \mathbb{R}$, $\psi_{\mathcal{M},d'}[x, y, \vec{a}]$ where $d' = (\delta_1, \delta_2, ..., \delta_{2n-1})$.[26] Similarly we can define $\phi_{\mathcal{M},d}$ for a Π^1_{2n+3} formula ϕ and $d = (\kappa, \delta_0, \delta_1, ..., \delta_{2n})$ (assuming that $\delta_0, \delta_1, ..., \delta_{2n}$ are Woodin cardinals of $\mathcal{M}$).
4. We will need the following basic applications of the extender algebra.

Proposition 2.5. *Suppose $x \in \mathbb{R}$, $\mathcal{M}$ is countable $\omega_1 + 1$-iterable mouse over x, $\mathcal{M} \vDash \mathsf{ZFC}$, $\mathcal{M}$ has $2n$ Woodin cardinals, and $\phi(\vec{v})$ is a Σ^1_{2n+2} formula. Let $\delta_0 < \delta_1 < ... < \delta_{2n-1}$ be the Woodin cardinals of $\mathcal{M}$ and let $\kappa < \delta_0$ be any ordinal. Set $d = (\kappa, \delta_0, ..., \delta_{2n-1})$ and suppose that for some $\vec{a} \in \mathcal{M} \cap \mathbb{R}^k$, $\mathcal{M} \vDash \phi_{\mathcal{M},d}[\vec{a}]$. Then $\phi[\vec{a}]$.*

Proof. We give the proof of the prototypical case of $n = 1$. Suppose ϕ is $\exists u_0 \forall u_1 \psi[u_0, u_1, \vec{v}]$ where ψ is Σ^1_2. Let $p \in \mathsf{Ea}^{\mathcal{M}}_{\delta_0,\kappa}$ be a condition witnessing $\phi_{\mathcal{M},d}[\vec{a}]$. Let $g \subseteq \mathsf{Ea}^{\mathcal{M}}_{\delta_0,\kappa}$ be $\mathcal{M}$-generic such that $p \in g$ and $g \in V$. Let then b_0 be a real in $\mathcal{M}[g]$ such that every $q \in \mathsf{Ea}^{\mathcal{M}[g]}_{\delta_1,\delta_0}$ forces that if u is a real then $\psi[b_0, u, \vec{a}]$.

We now want to see that for all $b_1 \in \mathbb{R}$, $\psi[b_0, b_1, \vec{a}]$ holds. Fix $b_1 \in \mathbb{R}$ and an $\omega_1 + 1$-strategy Σ for $\mathcal{M}$. Let $\mathcal{N}$ be a complete iterate of $\mathcal{M}$ via iteration $\mathcal{T}$ that is above δ_0 and is such that b_1 is generic over $\mathcal{N}$ for $\pi^{\mathcal{T}}(\mathsf{Ea}^{\mathcal{M}[g]}_{\delta_1,\delta_0})$. Because $\mathrm{crit}(\pi^{\mathcal{T}}) > \delta_0$, we have that in $\mathcal{N}[g]$, every $q \in \pi^{\mathcal{T}}(\mathsf{Ea}^{\mathcal{M}[g]}_{\delta_1,\delta_0})$ forces that if u is a real then $\psi[b_0, u, \vec{a}]$. Therefore, we have that $\mathcal{N}[g][b_1] \vDash \psi[b_0, b_1, \vec{a}]$. But then it follows from the upward absoluteness of Σ^1_2 formulas that $\psi[b_0, b_1, \vec{a}]$. □

The same proof also gives the following.

Proposition 2.6. *Suppose $x \in \mathbb{R}$, $\mathcal{M}$ is countable $\omega_1 + 1$-iterable mouse over x, $\mathcal{M} \vDash \mathsf{ZFC}$, $\mathcal{M}$ has $2n + 1$ Woodin cardinals, and $\phi(\vec{v})$ is a Π^1_{2n+3} formula. Let $\delta_0 < \delta_1 < ... < \delta_{2n}$ be the Woodin cardinals of $\mathcal{M}$ and let $\kappa < \delta_0$ be any ordinal. Set $d = (\kappa, \delta_0, ..., \delta_{2n})$ and suppose that for some $\vec{a} \in \mathcal{M} \cap \mathbb{R}^k$, $\mathcal{M} \vDash \phi_{\mathcal{M},d}[\vec{a}]$. Then $\phi[\vec{a}]$.*

5. In [7], Hjorth showed that the product of the extender algebra with itself is still δ-c.c. More precisely, suppose M is a transitive model of ZFC, δ is a Woodin cardinal of M, $\mathcal{E}$ is a weakly appropriate set of extenders and $\nu, \nu' < \delta$. Then $\mathsf{Ea}^{M}_{\delta,\nu,\mathcal{E}} \times \mathsf{Ea}^{M}_{\delta,\nu',\mathcal{E}}$ is δ-c.c. For the proof see [7, Lemma 1.2].

[25] Here, we need appropriateness to ensure that the resulting iteration is normal.
[26] We believe that the reader can extract the exact formula $\phi_{\mathcal{M},d}$ from this equivalence.

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6. Suppose M, δ and $\nu, \nu' < \delta$ are as above. We then let ea be the name for the generic object for $\mathsf{Ea}^M_{\delta,\nu}$ and let $(\mathsf{ea}_1, \mathsf{ea}_2, ..., \mathsf{ea}_n)$ be the sequence of reals ea codes. We let $(\mathsf{ea}^l, \mathsf{ea}^r)$ be the name for the generic of $\mathsf{Ea}^M_{\delta,\nu} \times \mathsf{Ea}^M_{\delta,\nu'}$. We then have that $(\mathsf{ea}^l_1, ..., \mathsf{ea}^l_n)$ and $(\mathsf{ea}^r_1, ..., \mathsf{ea}^r_m)$ are the finite sequences of reals coded by ea^l and ea^r.

Review 2.7. Backgrounded constructions and S-constructions:

S-construction is a method of translating the mouse structure to a similar structure over some inner model. The details of such a construction first appeared in [33, Lemma 1.5] where it was called P-constructions. In [29], the author renamed them S-constructions[27]. In this paper, we will need a special instance of S-constructions. The reader is advised to review the notion of *fully backgrounded constructions*[28] as defined in [23, Chapter 11].

1. Suppose $\mathcal{P}$ is a premouse, δ is a $\mathcal{P}$-cardinal and $z \in \mathcal{P}|\delta$. We then let $\mathsf{Le}(z)$ be the last model[29] of the fully backgrounded construction of $\mathcal{P}|\delta$ over z as defined in [23, Chapter 11]. In this construction, all extenders used have critical points $> \eta$ where η is the least such that $z \in \mathcal{P}|\eta$. If $z = \emptyset$ then we omit it from our notation. We will only consider such fully backgrounded constructions over z that can be easily coded as a subset of ordinals. Typical examples of z's that we will consider are reals and premice.

   If $\mathcal{P}$ is $\omega_1 + 1$-iterable then $\mathsf{Ord} \cap \mathsf{Le}(z) = \delta$, and if $\alpha = \mathsf{Ord} \cap \mathcal{P}$ and δ is Woodin in $\mathcal{P}$ then $L_\alpha[\mathsf{Le}(z)] \vDash$ 'δ is a Woodin cardinal' (see [23, Chapter 11]).

   Below, in clauses 2-6, $\mathcal{P}$, δ and z are as above, δ is a Woodin cardinal of $\mathcal{P}$ and $u \in \mathbb{R} \cap \mathcal{P}$.
2. Suppose the z-premouse $\mathcal{M} \subseteq \mathcal{P}$ is such that $\mathsf{Le}(z) \trianglelefteq \mathcal{M}$ and for every $\alpha < \mathsf{Ord} \cap \mathcal{M}$, $\mathcal{M}||\alpha \in \mathcal{P}$. Then for any $\nu < \delta$, u is generic over $\mathsf{Ea}^{\mathcal{M}}_{\delta,\nu}$. This is because every extender $E \in \vec{E}^{\mathsf{Le}(z)}$ such that $\nu(E)$ is inaccessible in $\mathsf{Le}(z)$ is backgrounded by an extender $F \in \vec{E}^{\mathcal{P}}$ such that there is a factor map $\tau : Ult(\mathsf{Le}(z), E) \to \pi^{\mathcal{P}}_F(\mathsf{Le}(z))$ with the property that $\mathrm{crit}(\tau) \geq \nu(E)$. Fixing now $\vec{\phi} \in \mathsf{Le}(z)|(\mathrm{crit}(E)^+)^{\mathsf{Le}(z)}$ such that $u \vDash \bigvee \pi^{\mathsf{Le}(z)}_E(\vec{\phi}) \restriction \nu(E)$, we have that $u \vDash \bigvee \pi^{\mathcal{P}}_F(\vec{\phi})$. But now because $u \in \mathcal{P}$, we have that $u \vDash \bigvee \vec{\phi}$. Thus, $u \vDash A_{E,\vee\phi}$.
3. We say $\mathcal{P}$ is translatable if for every $z \in \mathcal{P} \cap \mathbb{R}$, $\mathsf{Ord} \cap \mathsf{Le}(z) = \delta$[30] .
4. Given a translatable $\mathcal{P}$ and $z \in \mathcal{P} \cap \mathbb{R}$, we let $\mathsf{StrLe}(\mathcal{P}, z)$ be the result of the S-construction over $\mathsf{Le}(z)$ that translates the extenders of $\mathcal{P}$ with critical points $> \delta$ into extenders over $\mathsf{Le}(z)$.
5. It is shown in [33, Lemma 1.5] that if $E \in \vec{E}^{\mathcal{P}}$ is such that $\mathrm{crit}(E) > \delta$ then $E \cap \mathsf{StrLe}(\mathcal{P}, z) \in \vec{E}^{\mathsf{StrLe}(\mathcal{P},z)}$.
6. It follows that every Woodin cardinal of $\mathcal{P}$ greater than δ is a Woodin cardinal of $\mathsf{StrLe}(\mathcal{P}, z)$, and also if $\mathcal{P}$ is ω_1-iterable then $\mathsf{StrLe}(\mathcal{P}, z)$ is ω_1-iterable.
7. Suppose $x, z \in \mathbb{R}$, $\mathcal{N}$ is a complete iterate of $\mathcal{M}^{\#}_n(z)$ such that $\mathcal{T}_{\mathcal{M}^{\#}_n(z),\mathcal{N}}$ is below the least Woodin cardinal of $\mathcal{M}^{\#}_n(z)$, and some real recursive in x codes $\mathcal{N}$. Set $\mathcal{P} = \mathsf{StrLe}(\mathcal{M}_n(x), z)$. Then $\mathcal{P}$ is a complete iterate of $\mathcal{N}$. The proof proceeds as follows. First it is shown that there is a normal iteration $\mathcal{T}$ of $\mathcal{N}$ which is below the least Woodin cardinal of $\mathcal{N}$ and if $\mathcal{P}'$ is the last model of $\mathcal{T}$ then $(\mathcal{N}(z))^{\mathcal{M}_n(x)} = \mathcal{P}'|\delta$ where δ is the least Woodin cardinal of $\mathcal{P}'$ (and also $\mathcal{M}_n(x)$). To establish this result one uses the stationarity of the backgrounded constructions which says that in the comparison of $\mathcal{P}$ and $\mathcal{P}'$, the iteration of $\mathcal{P}$ is trivial. One then shows that $\mathcal{P}' = \mathcal{P}$, and here, the important fact is that $\mathcal{M}_n(x)|\delta$ is generic over $\mathcal{P}$. This in particular implies that $\mathcal{P}[\mathcal{M}_n(x)|\delta] = \mathcal{M}_n(x)$. One then concludes that every set in $\mathcal{P}$ is definable from a finite sequence $s \in \delta^{<\omega}$ and a finite sequence of indiscernibles for $\mathcal{M}_n(x)$. Since $\mathcal{P}'|\delta = \mathcal{P}|\delta$ and $\mathcal{P}'$ has the same property (being a complete

[27] 'S' stands for Steel.

[28] The difference between full backgrounded constructions and K^c constructions defined in [39] as well as in many other places is that in the fully backgrounded constructions the background extenders are assumed to be total (i.e., they measure all subsets of their critical points) while in K^c constructions this requirement is relaxed. Thus, the fully backgrounded constructions can inherit large cardinals only if there are already large cardinals in V while K^c constructions can inherit large cardinals even if there are no large cardinals in V. For example, see [15].

[29] We will often say that it is the output of the fully backgrounded construction.

[30] The fully backgrounded constructions may fail to reach δ.

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iterate of $\mathcal{N}$ below its least Woodin cardinal), it follows that $\mathcal{P} = \mathcal{P}'$. The details of what we have said have appeared in a number of places. The reader may find it useful to consult [24, Lemma 3.20], [27, Definition 1.1], [32], [30, Lemma 2.11], [34, Lemma 3.23], [33, Lemma 1.3] and the discussion after [33, Lemma 1.4].

The following objects will be used in clauses 8-10. Suppose $\mathcal{P}$ is a premouse, δ is a Woodin cardinal of $\mathcal{P}$ and $a \in \mathcal{P}|\delta$. Let $\mathcal{N} = (\mathsf{Le}(a))^{\mathcal{P}|\delta}$ and suppose $\mathsf{Ord} \cap \mathcal{N} = \delta$.[31]

8. Suppose κ is a measurable cardinal of $\mathcal{N}$ as witnessed by the extenders on the sequence of $\mathcal{N}$. Then κ is a measurable cardinal in $\mathcal{P}$. This is essentially because in the fully backgrounded construction all extenders used for backgrounding purposes are total extenders.
9. Similarly, if κ is a strong cardinal of $\mathcal{N}$ then κ is a $< \delta$-strong cardinal in $\mathcal{P}$.
10. Suppose κ is a $< \delta$-strong cardinal in $\mathcal{P}$. Then

Lemma 2.8. $\mathcal{N}|\kappa = (\mathsf{Le}(a))^{\mathcal{P}|\kappa}$.

Proof. To see this, suppose not. Set $\mathcal{N}' = (\mathsf{Le}(a))^{\mathcal{P}|\kappa}$ and let ξ be the least such that the ξth model of the fully backgrounded construction of $\mathcal{P}|\delta$ over a projects across κ. Let $\mathcal{Q}$ be the ξth model of the fully backgrounded construction of $\mathcal{P}$ over a and let $\nu < \delta$ be such that $\mathcal{Q}$ is constructed by the fully backgrounded construction of $\mathcal{P}|\nu$ over a. Let $\mathcal{Q}'$ be the core of $\mathcal{Q}$. Since $\rho_\omega(\mathcal{Q}) < \kappa$, we must have that $\mathcal{Q}' \in \mathcal{P}|\kappa$. We now have that $\mathcal{Q}'$ is not constructed by the fully backgrounded construction of $\mathcal{P}|\kappa$ over a[32]. However, if $E \in \vec{E}^{\mathcal{P}|\delta}$ is any extender with $lh(E) > \nu$, then in $Ult(\mathcal{P}, E)$, $\pi_E(\mathcal{Q}') = \mathcal{Q}'$ is constructed by the fully backgrounded construction of $Ult(\mathcal{P}, E)|\pi_E(\kappa)$ over a. □

Definition 2.9. We say κ is an fb-cut in $\mathcal{P}$ if letting δ_0 be the least Woodin cardinal of $\mathcal{P}$, $\kappa < \delta_0$, κ is a $\mathcal{P}$-cardinal and

$$\mathsf{Le}^{\mathcal{P}|\delta_0}|\kappa = \mathsf{Le}^{\mathcal{P}|\kappa}.$$

We say κ is a weak fb-cut if whenever $\mathcal{Q}$ is a mouse that appears in the fully backgrounded construction of $\mathcal{P}$ over $\mathsf{Le}^{\mathcal{P}|\kappa}$ (and hence uses extenders with critical points $> \kappa$), $\rho_\omega(\mathcal{Q}) \geq \kappa$[33] .

11. (**Universality**) Suppose $\mathcal{P}$ is a translatable premouse (see clause 3 above), δ is its least Woodin cardinal and $a \in \mathcal{P}|\delta$. Suppose $\mathcal{Q}$ is a $\delta + 1$-iterable a-premouse in $\mathcal{P}$. Then either $\mathsf{Le}(a)$ has a superstrong cardinal or $\mathcal{Q} \trianglelefteq \mathsf{Le}(a)$. Moreover, if $\mathcal{N}$ is some fully backgrounded construction of $\mathcal{P}|\delta$ such that $a \in \mathcal{N}$ and $\mathsf{Ord} \cap \mathsf{Le}(a)^{\mathcal{N}} = \delta$ then either $\mathsf{Le}(a)^{\mathcal{N}}$ has a superstrong cardinal or $\mathcal{Q} \trianglelefteq \mathsf{Le}(a)^{\mathcal{N}}$. These results are due to Steel and are consequences of universality of the fully backgrounded constructions (e.g., see [30, Lemma 2.12 and 2.13]).
12. Let $\mathcal{P} = \mathcal{M}_n(x)$ where $n > 0$ and $x \in \mathbb{R}$, and let δ be the least Woodin cardinal of $\mathcal{P}$. Let $\eta < \delta$. Then $\mathcal{P}|\eta$ is $\delta + 1$-iterable inside $\mathcal{P}$. This is because if $\mathcal{T}$ is a correct iteration of $\mathcal{P}|\eta$ of length $\leq \delta$ then $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_{n-1}(\mathsf{cop}(\mathcal{T}))$ implying that the correct branch of $\mathcal{T}$ is in $\mathcal{P}$.
13. We will need the following lemma. We continue with the $\mathcal{P}$ and δ of clause 12, but the results we state are more general.

Lemma 2.10. *Suppose κ is an fb-cut. Then κ is a weak fb-cut.*

Proof. If $\mathcal{Q}$ is an $\mathsf{Le}^{\mathcal{P}|\kappa}$-premouse which is constructed by the fully backgrounded construction of $\mathcal{P}|\delta$ done over $\mathsf{Le}^{\mathcal{P}|\kappa}$ then by universality $\mathcal{Q} \trianglelefteq \mathsf{Le}^{\mathcal{P}|\delta}$ (see clause 11 above). Since κ is an fb-cut, we have that $\rho_\omega(\mathcal{Q}) \geq \kappa$. Hence, κ is a weak fb-cut. □

14. We will need the following lemma. We continue with the $\mathcal{P}$ and δ as in clause 12.

Lemma 2.11. *Let $E \in \vec{E}^{\mathcal{P}|\delta}$ be a total extender such that $\nu(E)$ is an inaccessible cardinal of $\mathcal{P}$. Then for any $\tau < \nu(E)$, τ is a weak fb-cut in $\mathcal{P}$ if and only if τ is a weak fb-cut in $Ult(\mathcal{P}, E)$.*

[31] In this construction all extenders used have critical points greater than η where η is least such that $a \in \mathcal{P}|\eta$.

[32] Because the construction reaches δ we cannot construct the same model twice at different stages as otherwise the construction will be looping between these two stages.

[33] κ must be a cutpoint in such a $\mathcal{Q}$.

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Proof. Assume first that τ is an inaccessible weak fb-cut in $\mathcal{P}$. Let $\mathcal{K} = \mathsf{Le}^{\mathcal{P}|\tau}$. Suppose $\mathcal{Q}$ is a $\mathcal{K}$-premouse constructed by the fully backgrounded construction of $Ult(\mathcal{P}, E)|\delta$ done over $\mathcal{K}$ and $\rho_\omega(\mathcal{Q}) = \tau$[34]. Since $\mathcal{Q}$ is $\delta+1$-iterable, universality implies that $\mathcal{Q}$ is constructed by the fully backgrounded construction of $\mathcal{P}|\delta$ done over $\mathcal{K}$. Hence, considering $\mathcal{Q}$ as a premouse, $\rho_\omega(\mathcal{Q}) \geq \tau$. Thus, τ is a weak fb-cut in $Ult(\mathcal{P}, E)$.

Conversely, suppose τ is an inaccessible weak fb-cut in $Ult(\mathcal{P}, E)$. Again, let $\mathcal{K} = \mathsf{Le}^{\mathcal{P}|\tau}$. Suppose $\mathcal{Q}$ is a $\mathcal{K}$-premouse constructed by the fully backgrounded construction of $\mathcal{P}|\delta$ done over $\mathcal{K}$ and $\rho_\omega(\mathcal{Q}) = \tau$. Then $\mathcal{Q}$ is $\delta+1$-iterable in $Ult(\mathcal{P}, E)$[35]. Hence, again universality implies that, considering $\mathcal{Q}$ as just a premouse, $\rho_\omega(\mathcal{Q}) \geq \tau$. □

15. Cond is the following statement in the language of premice. $\dot{\mathcal{V}}$ is used for the universe.

Cond : Suppose δ is the least Woodin cardinal, $\kappa < \delta$ is the least $< \delta$-strong cardinal (as witnessed by the extender sequence), $\gamma > \delta$ is an inaccessible cardinal, ϕ is a formula and $\dot{\mathcal{V}}|\gamma \vDash \phi[\delta, \vec{a}]$ where $\vec{a} \in [\dot{\mathcal{V}}|\delta]^{<\omega}$. There is then $\alpha < \beta < \kappa$ and $\vec{b} \in [\dot{\mathcal{V}}|\alpha]^{<\omega}$ such that
(a) $\dot{\mathcal{V}}|\beta \vDash$ 'α is the least Woodin cardinal',
(b) $\vec{b} \in [\dot{\mathcal{V}}|\alpha]^{<\omega}$,
(c) $\dot{\mathcal{V}}|\beta \vDash \phi[\alpha, \vec{b}]$, and
(d) α is an inaccessible cardinal of $\dot{\mathcal{V}}$,
(e) α is an fb-cut of $\dot{\mathcal{V}}$.

We will need the following lemma. $\mathcal{P}$ and δ are as in clause 12 above. Thus, $\mathcal{P} = \mathcal{M}_n(x)$ where $n > 0$, $x \in \mathbb{R}$ and δ is the least Woodin cardinal of $\mathcal{P}$.

Lemma 2.12. $\mathcal{P} \vDash$ Cond.

Proof. Let $\mathcal{N} = \mathsf{Le}^{\mathcal{P}|\delta}$. Let δ be the least Woodin cardinal of $\mathcal{P}$. Suppose γ, ϕ and $a \in \mathcal{P}|\delta$ are such that $\gamma > \delta$ is inaccessible and $\mathcal{P}|\gamma \vDash \phi[\delta, a]$. We can find $\mathcal{Q} \trianglelefteq \mathcal{P}|\delta$ and an elementary $\pi : \mathcal{Q} \to \mathcal{P}|\gamma$ such that if $\tau = \pi^{-1}(\delta)$ then τ is an inaccessible cardinal of $\mathcal{P}$, $\pi \restriction \tau = id$ and $a \in \mathcal{P}|\tau$. We then have that τ is an fb-cut of $\mathcal{P}$. However, we do not know that $\tau < \kappa$ where κ is the least $< \delta$-strong cardinal of $\mathcal{P}$.We now want to show that there is $\mathcal{R} \trianglelefteq \mathcal{P}|\kappa$ such that
(a) $\mathcal{R}$ has a Woodin cardinal and if ν is its least Woodin cardinal then ν is an inaccessible cardinal of $\mathcal{P}|\kappa$,
(b) $\mathcal{R}|\nu = \mathcal{P}|\nu$ and ν is an fb-cut,
(c) for some $b \in \mathcal{R}|\nu$, $\mathcal{R} \vDash \phi[\nu, b]$.

To get such an $\mathcal{R} \trianglelefteq \mathcal{P}|\lambda$, let $E \in \vec{E}^{\mathcal{P}}$ be such that $\mathcal{Q} \trianglelefteq \mathcal{P}|\mathrm{lh}(E)$, $\mathrm{crit}(E) = \kappa$ and τ is a cutpoint in $Ult(\mathcal{P}, E)$. Then in $Ult(\mathcal{P}, E)$, $\mathcal{Q}$ has the properties that we look for except that we do not know that τ is an fb-cut in $Ult(\mathcal{P}, E)$. We now prove that in fact τ is an fb-cut in $Ult(\mathcal{P}, E)$.

It follows from Lemma 2.11 that τ is a weak fb-cut in $Ult(\mathcal{P}, E)$. To see that τ is an fb-cut notice that the fully backgrounded construction of $Ult(\mathcal{P}, E)$ never adds extenders overlapping τ, and so in fact this construction is the fully backgrounded construction of $Ult(\mathcal{P}, E)$ done over $\mathcal{N}|\tau$. □

16. We will need the following two extender sequences.

Definition 2.13. Suppose $\mathcal{S}$ is a translatable premouse (see clause 3 above) and τ is its least Woodin cardinal. Let $\mathcal{N} = \mathsf{Le}^{\mathcal{S}|\tau}$. We then let $\mathcal{E}^{\mathcal{S}}_{le}$ be the set of all extenders $E \in \vec{E}^{\mathcal{S}|\tau}$ such that
(a) $\nu(E)$ is an inaccessible cardinal of $\mathcal{S}$,
(b) $\pi_E(\mathcal{N})|\nu(E) = \mathcal{N}|\nu(E)$,
(c) $\mathrm{crit}(E)$ is an fb-cut.

We let $\mathcal{E}^{\mathcal{S}}_{sm}$ be the set of $E \in \vec{E}^{\mathcal{S}|\tau}$ such that $\nu(E)$ is a measurable cardinal of $\mathcal{S}$ and $\mathrm{crit}(E)$ is a $< \tau$-strong cardinal of $\mathcal{S}$.

[34]Recall that if $\mathcal{Q}$ is a $\mathcal{K}$-premouse then we put all elements of $\mathcal{K}$ in all fine structural hulls. Also, our $\mathcal{Q}$ can also be considered as a premouse.

[35]This is because if $\mathcal{T}$ is a correct iteration of $\mathcal{Q}$ above τ then $Ult(\mathcal{P}, E)$ can find the correct branch of $\mathcal{T}$ as the function $\mathcal{T} \mapsto (\wp(\delta(\mathcal{T})) \cap \mathcal{M}_{n-1}(\mathrm{cop}(\mathcal{T})))$ is definable over $Ult(\mathcal{P}, E)|(\delta^+)^{\mathcal{P}}$.

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Lemma 2.14. *Continuing with $\mathcal{S}$ and τ as above, $\mathcal{E}^{\mathcal{S}}_{ms}$ is weakly appropriate and $\mathcal{E}^{\mathcal{S}}_{le}$ is appropriate (see Definition 2.4 and Definition 2.13).*

Proof. It is easy to show that $\mathcal{E}^{\mathcal{S}}_{sm}$ is weakly appropriate. Suppose then $E \in \mathcal{E}^{\mathcal{S}}_{le}$. Let $\mathcal{N}$ be the fully backgrounded construction of $\mathcal{S}|\tau$ and set $\mathcal{E} = \mathcal{E}^{\mathcal{S}}_{le}$. We want to see that $\pi_E(\mathcal{E}) \cap (\mathcal{S}|\nu(E)) = \mathcal{E} \cap (\mathcal{S}|\nu(E))$. We have that $\pi_E(\mathcal{N})$ is the fully backgrounded construction of $Ult(\mathcal{S}, E)|\tau$ and $\pi_E(\mathcal{N})|\nu(E) = \mathcal{N}|\nu(E)$.

Suppose now that $F \in \pi_E(\mathcal{E}) \cap (\mathcal{S}|\nu(E))$. It follows that

(1) $\nu(F)$ is an inaccessible cardinal in $Ult(\mathcal{S}, E)$,
(2) $\pi_F^{Ult(\mathcal{S},E)}(\pi_E(\mathcal{N}))|\nu(F) = \pi_E(\mathcal{N})|\nu(F)$,
(3) $\mathrm{crit}(F)$ is an fb-cut in $Ult(\mathcal{S}, E)$.

We want to see that

(4) $\nu(F)$ is an inaccessible cardinal in $\mathcal{S}$,
(5) $\pi_F(\mathcal{N})|\nu(F) = \mathcal{N}|\nu(F)$,
(6) $\mathrm{crit}(F)$ is an fb-cut in $\mathcal{S}$.

(4) is easy as $\nu(E) > \nu(F)$ and $\nu(E)$ is inaccessible in both $\mathcal{S}$ and $Ult(\mathcal{S}, E)$. To see (5) notice that because

$$\pi_F \restriction (\mathcal{S}|(\mathrm{crit}(F)^+)^{\mathcal{S}}) = \pi_F^{Ult(\mathcal{S},F)} \restriction (Ult(\mathcal{S}, F)|(\mathrm{crit}(F)^+)^{\mathcal{S}})$$

and because of (2) we have that $\mathcal{N}|\nu(F) = \pi_F(\mathcal{N})|\nu(F)$.

We now want to show (6) and we have that $\mathrm{crit}(F)$ is an fb-cut in $Ult(\mathcal{S}, F)$. It follows that $\pi_E(\mathcal{N})|\mathrm{crit}(F)$ is the fully backgrounded construction of $Ult(\mathcal{S}, F)|\mathrm{crit}(F)$. But $\pi_E(\mathcal{N})|\mathrm{crit}(F) = \mathcal{N}|\mathrm{crit}(F)$ and $Ult(\mathcal{S}, F)|\mathrm{crit}(F) = \mathcal{S}|\mathrm{crit}(F)$. Therefore,

$$\mathcal{N}|\mathrm{crit}(F) \text{ is the fully backgrounded construction of } \mathcal{S}|\mathrm{crit}(F).$$

The proof that $\mathcal{E} \cap (\mathcal{S}|\nu(E)) \subseteq \pi_E(\mathcal{E}) \cap (\mathcal{S}|\nu(E))$ is very similar. □

The same proof can be used to show that the following also holds.

Lemma 2.15. *Continuing with $\mathcal{S}$ and τ as above, if $E \in \mathcal{E}^{\mathcal{S}}_{le}$ then for all $\xi < \nu(E)$, $\mathcal{S} \vDash$ 'ξ is an fb-cut' if and only if $Ult(\mathcal{S}, E) \vDash$ 'ξ is an fb-cut'.*

Proof. Let $\mathcal{N}$ be as in the above proof. We have that $\pi_E(\mathcal{N})|\nu(E) = \mathcal{N}|\nu(E)$. Set $\mathcal{K} = \mathsf{Le}^{\mathcal{S}|\xi}$. We have that $\mathcal{K} = \mathsf{Le}^{Ult(\mathcal{S},E)|\xi}$. Thus, the following equivalence holds.

$$\begin{aligned}\mathcal{S} \vDash \text{'}\xi \text{ is an } fb\text{-cut'} &\leftrightarrow \mathcal{K} = \mathcal{N}|\xi \\ &\leftrightarrow \mathcal{K} = \pi_E(\mathcal{N})|\xi \\ &\leftrightarrow Ult(\mathcal{S}, E) \vDash \text{'}\xi \text{ is an } fb\text{-cut'}.\end{aligned}$$

□

The following corollary can now easily be proven by induction on the length of the iteration.

Corollary 2.16. *Continuing with $\mathcal{S}$ and τ as above, suppose $\mathcal{T}$ is a normal nondropping (i.e., $\mathcal{D}^{\mathcal{T}} = \emptyset$) iteration of $\mathcal{S}$ such that for every $\alpha < \mathrm{lh}(\mathcal{T})$, $E^{\mathcal{T}}_\alpha \in \pi^{\mathcal{T}}_{0,\alpha}(\mathcal{E}^{\mathcal{S}}_{le})$. Then for any $\alpha < \beta < \mathrm{lh}(\mathcal{T})$, $\mathcal{M}^{\mathcal{T}}_\beta \vDash$ '$\mathrm{crit}(E^{\mathcal{T}}_\alpha)$ is an fb-cut'.*

17. We will need the following lemma.

Lemma 2.17. *Suppose $\mathcal{S}$ is a premouse and τ is its least Woodin cardinal. Let $\mathcal{N} = \mathsf{Le}^{\mathcal{S}|\tau}$ and assume that $\mathsf{Ord} \cap \mathcal{N} = \tau$. Suppose $\alpha < \beta$ are such that*
(a) *α is an fb-cut in $\mathcal{S}$ and*
(b) *$\mathcal{S}|\beta \vDash \mathsf{ZFC} +$ 'α is a Woodin cardinal'.*

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Suppose t is a real which satisfies all the axioms of $\mathsf{Ea}^{\mathcal{S}}_{\tau,\mathcal{E}^{\mathcal{S}}_{le}}$ that are generated by the extenders in $\mathcal{S}|\alpha$. Let $\mathcal{K} = \mathsf{StrLe}(\mathcal{S}|\beta)$. Then t is generic for $\mathsf{Ea}^{\mathcal{K}}_{\alpha,\mathcal{E}^{\mathcal{K}}_{sm}}$.

Proof. Suppose $E \in \vec{E}^{\mathcal{K}|\alpha}$ such that $\nu(E)$ is a measurable cardinal of $\mathcal{K}$ and $\mathrm{crit}(E)$ is a $< \alpha$-strong cardinal of $\mathcal{K}$. Let F be the background extender of E. It follows that $E = F \cap \mathcal{K}$. Moreover, since $\mathrm{crit}(E)$ is a strong cardinal in $\mathcal{K}|\alpha$, $\mathrm{crit}(E)$ is an fb-cut in $\mathcal{S}|\alpha$, and since α itself is an fb-cut, $\mathrm{crit}(E)$ is an fb-cut in $\mathcal{S}$. Also, F, since it coheres $\mathcal{N}$, is in $\mathcal{E}^{\mathcal{S}}_{le}$. Therefore, since there is a factor map $k : \pi^{\mathcal{N}}_{E}(\mathcal{N}|(\mathrm{crit}(E)^+)^{\mathcal{N}}) \to \pi^{\mathcal{S}}_{F}(\mathcal{N}|(\mathrm{crit}(E)^+)^{\mathcal{N}})$ with $\mathrm{crit}(k) \geq \nu(E)$ and since $\pi^{\mathcal{N}}_{E}(\mathcal{N}|(\mathrm{crit}(E)^+)^{\mathcal{N}})|\nu(E) = \mathcal{N}|\nu(E)$, any axiom generated by E in $\mathcal{K}$ is satisfied by z as it is also an axiom generated by F (see clause 2 of Review 2.7). □

Review 2.18. Review of [29]:

Unless otherwise specified, we assume $\mathsf{AD}^{L(\mathbb{R})}$. The material reviewed below appears in [29] and in [40]. Other treatments of similar concepts appear in [32] and [24].

1. $(u_i : i \in \mathsf{Ord})$ is the sequence of uniform indiscernibles for reals (assuming they exist).[36] Set $s_0 = \emptyset$ and for $m \geq 1$, $s_m = (u_0, ..., u_{m-1})$
2. Fix $n \in \omega$ and $x \in \mathbb{R}$ and suppose $\mathcal{P}$ is a complete iterate of $\mathcal{M}_n(x)$. For $i \leq n$, $\delta^{\mathcal{P}}_i$ is the $i+1$st Woodin of $\mathcal{P}$. Let

$$\gamma^{\mathcal{P}}_m = \sup(Hull_1^{\mathcal{P}|u_m}(s_m) \cap \delta^{\mathcal{P}}_0).$$

Then $\sup_{m<\omega} \gamma^{\mathcal{P}}_m = \delta^{\mathcal{P}}_0$. Given two pairs $(\mathcal{P}, \alpha)$ and $(\mathcal{Q}, \beta)$ such that $\mathcal{P}$ and $\mathcal{Q}$ are complete iterates of $\mathcal{M}_n(x)$, $\alpha < \gamma^{\mathcal{P}}_m$ and $\beta < \gamma^{\mathcal{Q}}_m$, we write $(\mathcal{P}, \alpha) \leq^n_m (\mathcal{Q}, \beta)$ if and only if letting $\mathcal{R}$ be the common iterate of $\mathcal{P}$ and $\mathcal{Q}$, $\pi_{\mathcal{P},\mathcal{R}}(\alpha) \leq \pi_{\mathcal{Q},\mathcal{R}}(\beta)$. Clearly, $\leq^n_m$ depends on x, but not mentioning x makes the notation simpler[37]. We then have that there is a formula ϕ such that for all $(\mathcal{P}, \alpha)$ and $(\mathcal{Q}, \beta)$, $(\mathcal{P}, \alpha) \leq^n_m (\mathcal{Q}, \beta)$ if and only if

$$\mathcal{M}_{n-1}(\mathcal{M}_n(x)^{\#}, (\mathcal{P}, \alpha), (\mathcal{Q}, \beta)) \vDash \phi[\mathcal{M}_n(x)^{\#}, (\mathcal{P}, \alpha), (\mathcal{Q}, \beta), s_m].$$

The formula ϕ essentially records what was said above. It is the conjunction of the following statements:
(a) $\mathcal{P}$ and $\mathcal{Q}$ are complete iterates of $\mathcal{M}_n(x)$.
(b) There is $\mathcal{R}$, which is a common iterate of $\mathcal{P}$ and $\mathcal{Q}$ and $\pi_{\mathcal{P},\mathcal{R}}(\alpha) \leq \pi_{\mathcal{Q},\mathcal{R}}(\beta)$.
Explaining the above equivalence is beyond the scope of this paper. The reader can consult [29], [40], [32] and [24]. Essentially, the following happens.Suppose $\mathcal{P}$ is a complete iterate of $\mathcal{M}_n(x)$. Then, there is an iteration tree $\mathcal{T} \in \mathcal{M}_{n-1}(\mathcal{M}_n(x)^{\#}, (\mathcal{P}, \alpha), (\mathcal{Q}, \beta))$ such that $\mathcal{T}$ is an iteration tree on $\mathcal{M}_n(x)^{\#}$ according to its unique $(\omega_1 + 1)$-iteration strategy and such that either $\mathcal{T}$ has a last model which is $\mathcal{P}$ or $\mathcal{P} = L(\mathsf{cop}(\mathcal{T}))$ and $\delta(\mathcal{T})$ is the largest Woodin cardinal of $\mathcal{P}$ (such iteration trees are called *maximal*). This is the fact used to identify complete iterates of $\mathcal{M}_n(x)^{\#}$ inside models of the form $\mathcal{M}_{n-1}(\mathcal{M}_n(x)^{\#}, d)$.Suppose next that there is $\mathcal{R}$ such that $\pi_{\mathcal{P},\mathcal{R}}(\alpha) \leq \pi_{\mathcal{Q},\mathcal{R}}(\beta)$. We can assume that $\mathcal{R}$ is obtained via the usual comparison procedure of [39], which means that $\mathcal{R} \in \mathcal{M}_{n-1}(\mathcal{M}_n(x)^{\#}, (\mathcal{P}, \alpha), (\mathcal{Q}, \beta))$. One then shows that

$$\mathcal{M}_{n-1}(\mathcal{M}_n(x)^{\#}, (\mathcal{P}, \alpha), (\mathcal{Q}, \beta))$$

can correctly compute $\pi_{\mathcal{P},\mathcal{R}} \restriction \gamma^{\mathcal{P}}_m$ and $\pi_{\mathcal{Q},\mathcal{R}} \restriction \gamma^{\mathcal{Q}}_m$, even though it cannot in general compute $\pi_{\mathcal{P},\mathcal{R}}$ and $\pi_{\mathcal{Q},\mathcal{R}}$.

[36] Assume that every real has a sharp, and for each $x \in \mathbb{R}$, let $\boldsymbol{C}_x$ be the class of x-indiscernibles. Thus, $\boldsymbol{C}_x$ is a class of indiscernibles for $L[x]$. We then say that u is a uniform indiscernible if $u \in \cap_{x\in\mathbb{R}} \boldsymbol{C}_x$. Thus, the class of uniform indiscernibles is a club and contains all uncountable cardinals.
[37] But $\leq^n_m$ does not depend on the choice of indiscernibles.

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3. For $x \in \mathbb{R}$ and $n \in \omega$, set $\gamma^{2n+1}_{m,x,\infty} = \pi_{\mathcal{M}_{2n+1}(x),\infty}(\gamma_m^{\mathcal{M}_{2n+1}(x)})$.
4. For $x \in \mathbb{R}$ and $n \in \omega$, set $b_{2n+1,m} = \sup_{x\in\mathbb{R}} \gamma^{2n+1}_{m,x,\infty}$.
5. Let κ^1_{2n+1} be the predecessor of δ^1_{2n+1} (e.g., see [13, Theorem 2.18]). It follows from [29] that for each n, $\sup_{m\in\omega} b_{2n+1,m} = \kappa^1_{2n+3}$. In fact, for each $x \in \mathbb{R}$, $\sup_{m<\omega} \gamma^{2n+1}_{m,x,\infty} = \kappa^1_{2n+3}$. Moreover, for each $n, m \in \omega$, $b_{2n+1,m}$ is a cardinal and $b_{2n+1,0} > \delta^1_{2n+1}$. Thus, $b_{2n+1,0} \geq \delta^1_{2n+2}$. For the proofs of these results see [29, Theorem 4.1, Corollary 5.23, Lemma 6.1].
6. It is conjectured in [29] that for all n, $b_{2n+1,0} = \delta^1_{2n+2}$. For $n = 0$ this is shown in [8]. The case $n \geq 1$ is still open.

Review 2.19. γ-stability:

The main technical fact from [29] that we will need in this paper appears in the bottom of page 760 of [29]. It claims that for each $x \in \mathbb{R}$, each $n \in \omega$ and for each $\gamma < \kappa^1_{2n+3}$, there is a γ-stable complete iterate $\mathcal{P}$ of $\mathcal{M}_{2n+1}(x)$. Because our situation is just a little bit different we outline how to obtain γ-stable iterates. The reader may find it useful as well.

γ-stable iterates:

1. Fix $n \in \omega$ and $x \in \mathbb{R}$. Let $\mathcal{M} = \mathcal{M}_{2n+1}(x)$ and suppose that $\mathcal{M}^{\#}_{2n+1} \in \mathcal{M}$[38]. Suppose γ is an ordinal such that for some m_0, $\gamma < \gamma^{2n+1}_{m_0,\infty}$. Let $\mathcal{P}$ be a complete iterate of $\mathcal{M}$.
2. Let ν be the least inaccessible of $\mathcal{P}$ above $\delta^{\mathcal{P}}_0$ and let $\mathcal{H}^{\mathcal{P}}$ be the direct limit of all iterates of $\mathcal{M}_{2n+1}$ that are in $\mathcal{P}|\nu$[39]. The construction of such limits has been carried out in [29], [24], [32]. For example see [29, Section 5.1].
3. The results of [29] and other similar calculations done, for example, in [40] show that $\mathcal{H}^{\mathcal{P}}$ is a complete iterate of $\mathcal{M}_{2n+1}$.
4. We say $\mathcal{P}$ is locally γ-stable if $\gamma \in \mathrm{rge}(\pi_{\mathcal{H}^{\mathcal{P}},\infty})$.
5. We say $\mathcal{P}$ is γ-stable if $\mathcal{P}$ is locally γ-stable and if $\xi = \pi^{-1}_{\mathcal{H}^{\mathcal{P}},\infty}(\gamma)$ then whenever $\mathcal{Q}$ is a complete iterate of $\mathcal{P}$,

$$\pi_{\mathcal{H}^{\mathcal{Q}},\infty}(\pi_{\mathcal{P},\mathcal{Q}}(\xi)) = \gamma.$$

6. The argument on page 760 of [29] can be used to show the following lemma.

Lemma 2.20. *There is a complete γ-stable iterate $\mathcal{P}$ of $\mathcal{M}_{2n+1}(x)$.*

Proof. (Outline) Towards a contradiction assume not. The key observation is that whenever $\mathcal{P}$ is a complete iterate of $\mathcal{M}_{2n+1}(x)$ and $\mathcal{Q}$ is a complete iterate of $\mathcal{P}$, the Dodd-Jensen argument (see [39, Chapter 4.2]) implies that for all $\xi \in \mathcal{H}^{\mathcal{P}}$, $\pi_{\mathcal{H}^{\mathcal{P}},\infty}(\xi) \leq \pi_{\mathcal{H}^{\mathcal{Q}},\infty}(\pi_{\mathcal{P},\mathcal{Q}}(\xi))$.

Suppose now that $\mathcal{R}$ is a complete iterate of $\mathcal{M}_{2n+1}$ such that $\gamma \in \mathrm{rge}(\pi_{\mathcal{R},\infty})$. We set $\gamma_{\mathcal{R}} = \pi^{-1}_{\mathcal{R},\infty}(\gamma)$.

Let now $\mathcal{P}$ be a compete iterate of $\mathcal{M}_{2n+1}(x)$ and $\mathcal{Q}$ be a complete iterate of $\mathcal{P}$. What we have observed implies that if $\gamma_{\mathcal{H}^{\mathcal{P}}}$ and $\gamma_{\mathcal{H}^{\mathcal{Q}}}$ are defined then $\pi_{\mathcal{P},\mathcal{Q}}(\gamma_{\mathcal{H}^{\mathcal{P}}}) \geq \gamma_{\mathcal{H}^{\mathcal{Q}}}$. Moreover, if $\mathcal{Q}$ witnesses that $\mathcal{P}$ is not γ-stable then we in fact have that $\pi_{\mathcal{P},\mathcal{Q}}(\gamma_{\mathcal{H}^{\mathcal{P}}}) > \gamma_{\mathcal{H}^{\mathcal{Q}}}$.

Observe further that given any $(\mathcal{P}, \mathcal{Q})$ we can iterate $\mathcal{Q}$ to obtain a complete iterate $\mathcal{R}$ of $\mathcal{Q}$ such that $\gamma_{\mathcal{H}^{\mathcal{R}}}$ is defined. To do this, it is enough to fix some complete iterate $\mathcal{S}$ of $\mathcal{M}_{2n+1}$ for which $\gamma_{\mathcal{S}}$ is defined and iterate $\mathcal{Q}$ to obtain a complete iterate $\mathcal{R}$ of $\mathcal{Q}$ such that $\mathcal{S}$ is generic for $\mathsf{Ea}^{\mathcal{R}}_{\delta^{\mathcal{R}}_0}$.[40] It then follows that $\mathcal{S}$ is a point in the directed system of $\mathcal{R}[\mathcal{S}]$ that converges to $\mathcal{H}^{\mathcal{R}}$, and so $\gamma_{\mathcal{H}^{\mathcal{R}}}$ is defined.

We now get that our assumption that there is no γ-stable complete iterate of $\mathcal{M}$ implies that there is a sequence $(\mathcal{P}_i : i < \omega)$ such that (a) for each i, $\mathcal{P}_{i+1}$ is a complete iterate of $\mathcal{P}_i$, (b) for each i,

[38] Any reader who is familiar with the nuts and bolts of inner model theory can see that this condition is not necessary.

[39] We say $\mathcal{Q}$ is an iterate of $\mathcal{M}_{2n+1}$ in $\mathcal{P}|\nu$ if $\mathcal{Q}$ is a complete iterate of $\mathcal{M}_{2n+1}$ and letting δ be the largest Woodin cardinal of $\mathcal{Q}$, $\mathcal{Q}|\delta \in \mathcal{P}|\nu$.

[40] According to our conventions, this means that the extender algebra over which $\mathcal{S}$ is generic has λ many generators where λ is the largest Woodin cardinal of $\mathcal{S}$.

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$\gamma_{\mathcal{H}^{\mathcal{P}_i}}$ is defined (c) $\mathcal{P}_0$ is a complete iterate of $\mathcal{M}$ and finally (d) for each i,

$$\pi_{\mathcal{P}_i,\mathcal{P}_{i+1}}(\gamma_{\mathcal{H}^{\mathcal{P}_i}}) > \gamma_{\mathcal{H}^{\mathcal{P}_{i+1}}}.$$

We thus have that the direct limit of $(\mathcal{P}_i : i < \omega)$ is ill-founded, which is clearly a contradiction. □

7. Suppose now that $\mathcal{P}$ is γ-stable, $\nu < \delta_0^{\mathcal{P}}$ and $p \in \mathsf{Ea}^{\mathcal{P}}_{\delta_0^{\mathcal{P}},\nu}$. Suppose $m \leq k$ are natural numbers (so $m \geq 1$). We say p is (γ, k, m)-good if p forces that when the generic is decoded as a k-tuple $(u_1, ..., u_k)$ (via one of the standard ways of decoding a real into tuples) u_m codes a complete iterate $\mathcal{R}$ of $\mathcal{M}_{2n+1}$ with the property that $\mathcal{T}_{\mathcal{M}_{2n+1},\mathcal{R}}$ is below the least Woodin cardinal of $\mathcal{M}_{2n+1}$ and $\gamma_{\mathcal{H}^{\mathcal{P}}} \in \mathrm{rge}(\pi_{\mathcal{R},\mathcal{H}^{\mathcal{P}}})$.
8. It is not immediately clear that the definition of (γ, k, m)-good condition is first order over $\mathcal{P}$. This follows from the fact that the directed system can be internalized to $\mathcal{P}[g]$ where $g \subseteq Coll(\omega, \delta_0^{\mathcal{P}})$ is $\mathcal{P}$-generic. To do this, one uses the concept of s_n-iterability and the details of this were carried out in [29]. For example, see [29, Definition 5.4] and [29, Chapter 5.1]. The basic ideas that are used in internalizing directed systems are due to Woodin who carried out such internalization in $L[x]$ (see [40]).
9. Because the statement 'p is (γ, k, m)-good' is first order over $\mathcal{P}$, there is a maximal antichain of (γ, k, m)-good conditions. As $\mathsf{Ea}^{\mathcal{P}}_{\delta_0^{\mathcal{P}},\nu}$ has $\delta_0^{\mathcal{P}}$-cc, we have a condition p such that p is the disjunction of the conditions of some maximal antichain consisting of (γ, k, m)-good conditions, and as a consequence, p is (γ, k, m)-good and if $g \subseteq \mathsf{Ea}^{\mathcal{P}}_{\delta_0^{\mathcal{P}},\nu}$ is generic such that g is a tuple $(g_1, g_2, ..., g_k)$ and some $r \in g$ is (γ, k, m)-good then $p \in g$. We can then let $p^{\mathcal{P}}_{\gamma,\nu,k,m}$ be the $\mathcal{P}$-least (γ, k, m)-good condition with the property that any other (γ, k, m)-good condition is compatible with it. We call $p^{\mathcal{P}}_{\gamma,\nu,k,m}$ the (γ, ν, k, m)-master condition.

3. Π^1_n-iterability

We will use the concept of Π^1_n-iterability introduced in [37, Definition 1.4]. Following [37, Definition], we say that a premouse $\mathcal{M}$ is n-small if for every $\alpha \in \mathrm{dom}(\vec{E})$,[41] $\mathcal{M}|\alpha \vDash$ 'there does not exist n Woodin cardinals'. Thus, $\mathcal{M}_1^{\#}$ is 2-small and $\mathcal{M}_1$ is 1-small while $\mathcal{M}_2^{\#}$ is 3-small and $\mathcal{M}_2$ is 2-small. Notice however that all proper initial segments of $\mathcal{M}_1^{\#}$ are 1-small. We then say that $\mathcal{M}$ is properly n-small if $\mathcal{M}$ is $n+1$-small and all of its proper initial segments are n-small.

We will use Π^1_{2n+2}-iterability in conjunction with properly $2n+1$-small premice as well as $2n+1$-small premice. There is a lot to review here, but the details are technical and we suggest that the reader consult [37]. The following is a list of facts that we will need.

1. We say $\mathcal{M}$ is $\mathcal{M}_n$-like premouse over x if $\mathcal{M}$ is n-small premouse over x with n Woodin cardinals and such that $\mathsf{Ord} \subseteq \mathcal{M}$. Similarly, we say $\mathcal{M}$ is $\mathcal{M}_n^{\#}$-like premouse over x (or just x-premouse) if $\mathcal{M}$ is active, sound, properly n-small premouse over x with n Woodin cardinals.
2. Given two premice $\mathcal{M}$ and $\mathcal{M}'$ we say $(\mathcal{T}, \mathcal{T}')$ is a coiteration of $(\mathcal{M}, \mathcal{M}')$ if the extenders of $\mathcal{T}$ and $\mathcal{T}'$ are chosen to be the least ones causing disagreement. More precisely, $\mathrm{lh}(\mathcal{T}) = \mathrm{lh}(\mathcal{T}')$ and for every $\alpha < \mathrm{lh}(\mathcal{T})$, $\mathrm{lh}(E^{\mathcal{T}}_\alpha)$ is the least β such that $\mathcal{M}^{\mathcal{T}}_\alpha|\beta = \mathcal{M}^{\mathcal{T}'}_\alpha|\beta$ but $\mathcal{M}^{\mathcal{T}}_\alpha||\beta \neq \mathcal{M}^{\mathcal{T}'}_\alpha||\beta$, and similarly for $E^{\mathcal{T}'}_\alpha$. We then must have that $\mathrm{lh}(E^{\mathcal{T}}_\alpha) = \mathrm{lh}(E^{\mathcal{T}'}_\alpha)$. It is possible that only one of $E^{\mathcal{T}}_\alpha$ and $E^{\mathcal{T}'}_\alpha$ is defined, in which case we allow padding[42].
3. We say that the coiteration $(\mathcal{T}, \mathcal{T}')$ of $(\mathcal{M}, \mathcal{M}')$ is successful if $\mathrm{lh}(\mathcal{T})$ is a successor ordinal and if $\mathcal{N}$ is the last model of $\mathcal{T}$ and $\mathcal{N}'$ is the last model of $\mathcal{T}'$ then either $\mathcal{N} \trianglelefteq \mathcal{N}'$ or $\mathcal{N}' \trianglelefteq \mathcal{N}$.
4. Suppose $\mathcal{M}$ is a premouse, $\mathcal{T}$ is an iteration of $\mathcal{M}$ of limit length, $\alpha < \omega_1$ and b is a maximal branch of $\mathcal{T}$. We say $\mathcal{N}$ is (α, b)-relevant if either $\mathcal{N} = \mathcal{M}^{\mathcal{T}}_b$ or for some $\mathcal{P} \trianglelefteq \mathcal{M}^{\mathcal{T}}_b$ and for some

[41] Recall that $\vec{E}$ is a partial function with domain some ordinal γ. $\vec{E}(\alpha)$ is the extender indexed at α.
[42] We didn't have to abuse our terminology this way if we didn't insist on having $\mathrm{lh}(\mathcal{T}) = \mathrm{lh}(\mathcal{T}')$, but this simplifies the notation.

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$E \in \vec{E}^{\mathcal{P}}$, $\mathcal{N}$ is the αth linear iterate of $\mathcal{P}$ via E. We say that b is α-good if whenever $\mathcal{N}$ is (α, b)-relevant, either $\mathcal{N}$ is well-founded or α is in the well-founded part of $\mathcal{N}$. We say $\mathcal{M}$ is Π^1_2-iterable if for every iteration $\mathcal{T}$ of $\mathcal{M}$ of countable limit length and for every ordinal α there is an α-good maximal branch b of $\mathcal{T}$.

5. Recall $\mathcal{G}(\mathcal{M}, 0, 2n+1)$, which is the iteration game introduced on page 83 of [37]. $\mathcal{G}(\mathcal{M}, 0, 1)$ defines Π^1_2-iterability. In $\mathcal{G}(\mathcal{M}, 0, 1)$, Player I plays a pair $(\mathcal{T}, \beta)$ such that $\mathcal{T}$ is an iteration of $\mathcal{M}$ and $\beta < \omega_1$. Player II must either accept $\mathcal{T}$ or play a β-good branch of $\mathcal{T}$. If $\mathrm{lh}(\mathcal{T})$ is a limit ordinal or $\mathcal{T}$ has an ill-founded last model then Player II is not allowed to accept $\mathcal{T}$. If Player II doesn't violate any of the rules of the game then Player II wins the game. We thus have that $\mathcal{M}$ is Π^1_2-iterable if and only if Player II has a winning strategy in $\mathcal{G}(\mathcal{M}, 0, 1)$.
6. Suppose $\mathcal{M}$ is as above. We say that $\mathcal{M}$ is Π^1_{2n+2}-iterable if player *II* has a winning strategy in $\mathcal{G}(\mathcal{M}, 0, 2n+1)$. The game has $2n+1$ many rounds. We describe the game for $n > 0$. We modify the game very slightly and present it as a game that has only two rounds. We start by assuming that $\mathcal{M}$ is a δ-mouse in the sense of [37, Definition 1.3][43] .
   (a) In the first round, Player I plays a pair $(\mathcal{T}_0, x_0)$ such that $\mathcal{T}_0$ is an iteration of $\mathcal{M}$ that is above $\delta + 1$[44] and $x_0 \in \mathbb{R}$. Player II must either accept $\mathcal{T}_0$ or play a maximal well-founded branch b_0 of $\mathcal{T}_0$. If $\mathrm{lh}(\mathcal{T}_0)$ is a limit ordinal or $\mathcal{T}_0$ has an ill-founded last model then Player II is not allowed to accept $\mathcal{T}_0$. If Player II accepts $\mathcal{T}_0$ then we set $\delta_1 = \sup\{\nu(E^{\mathcal{T}_0}_\alpha) : \alpha + 1 < \mathrm{lh}(\mathcal{T}_0)\}$ and $\mathcal{N}_1 = \mathcal{M}^{\mathcal{T}_0}_{\mathrm{lh}(\mathcal{T}_0)-1}$. If Player II plays b then we set $\delta_1 = \delta(\mathcal{T}_0)$ and $\mathcal{N}_1 = \mathcal{M}^{\mathcal{T}_0}_{b_0}$.
   (b) The second round is played on $\mathcal{N}_1$, which is now a δ_1-mouse. In this round, Player I plays an iteration $\mathcal{T}_1$ of $\mathcal{N}_1$ above δ_1 such that for some Π^1_{2n}-iterable $\mathcal{M}^{\#}_{2n-1}$-like $(\mathcal{T}_0, b_0, x_0)$-premouse $\mathcal{M}', \mathcal{T}_1 \in \mathcal{M}'|\omega_1^{\mathcal{M}'}$[45]. Once again Player II must either accept $\mathcal{T}_1$ or play a maximal well-founded branch b_1 of $\mathcal{T}_1$. If $\mathrm{lh}(\mathcal{T}_1)$ is a limit ordinal or $\mathcal{T}_1$ has an ill-founded last model then Player II is not allowed to accept $\mathcal{T}_1$. If Player II accepts $\mathcal{T}_1$ then we set $\delta_2 = \sup\{\nu(E^{\mathcal{T}_1}_\alpha) : \alpha + 1 < \mathrm{lh}(\mathcal{T}_1)\}$ and $\mathcal{N}_2 = \mathcal{M}^{\mathcal{T}_1}_{\mathrm{lh}(\mathcal{T}_1)-1}$. If Player II plays b_1 then we set $\delta_2 = \delta(\mathcal{T}_1)$ and $\mathcal{N}_2 = \mathcal{M}^{\mathcal{T}_1}_{b_1}$.

   Now, Player II wins if Player II doesn't violate any of the rules of the game and $\mathcal{N}_2$ is Π^1_{2n}-iterable as a δ_2-mouse.
7. For $n > 1$, the statement 'the real x codes a Π^1_n-iterable premouse' is Π^1_n.
8. Assuming that Π^1_{2n}-iterability is a Π^1_{2n}-condition, it is not hard to see that Π^1_{2n+2}-iterability is a Π^1_{2n+2}-condition.[46]
9. Suppose $x \in \mathbb{R}$, $n \in \omega$ and $\mathcal{M}$ is $\mathcal{M}^{\#}_{2n+1}$-like Π^1_{2n+2}-iterable x-premouse. Let δ be the least Woodin cardinal of $\mathcal{M}$ and suppose $\mathcal{T}$ is an iteration of $\mathcal{M}$ that is below δ. We say $\mathcal{T}$ is correct if $\mathrm{lh}(\mathcal{T}) \leq \omega_1$ and for each limit $\alpha < \mathrm{lh}(\mathcal{T})$ if $b_\alpha = [0, \alpha)_{\mathcal{T}}$ then $\mathcal{Q}(b_\alpha, \mathcal{T} \restriction \alpha)$ exists and $\mathcal{Q}(b_\alpha, \mathcal{T} \restriction \alpha) \trianglelefteq \mathcal{M}_{2n}(\mathsf{cop}(\mathcal{T} \restriction \alpha))$.
10. [37] shows that for $x \in \mathbb{R}$, $\mathcal{M}^{\#}_n(x)$ is the unique Π^1_{n+2}-iterable, active, properly n-small, sound premouse over x. For example see clause 3 of [37, Lemma 2.2], and also the remark after [37, Corollary 4.11].
11. It follows from [37] that if $\mathcal{M}$ is $2n$-small, Π^1_{2n+2}-iterable, sound δ-mouse (in the sense of [37, Definition 1.3]) over a real x then $\mathcal{M}$ is ω_1-iterable above δ. For example, see [37, Lemma 3.3].
12. [37, Corollary 4.7] implies that for all $n \in \omega$ and $x \in \mathbb{R}$, $\mathcal{M}_{2n}(x)$ is Σ^1_{2n+2}-correct. More precisely, if $\vec{a} \in \mathbb{R}^{<\omega} \cap \mathcal{M}_{2n}(x)$ and ϕ is a Σ^1_{2n+2} formula then $\mathcal{M}_{2n}(x) \vDash \phi[\vec{a}]$ if and only if $V \vDash \phi[\vec{a}]$.
13. The following is a consequence of clause 12 above and the proof of Lemma 2.5.

Proposition 3.1. *Suppose $x \in \mathbb{R}$ and $\mathcal{M}$ is a complete iterate of $\mathcal{M}_{2n}(x)$ such that $\mathcal{T}_{\mathcal{M}_{2n}(x),\mathcal{M}}$ has a countable length. Let $\exists u \phi(\vec{v})$ be a Σ^1_{2n+2} formula. Let $\delta_0 < \delta_1 < \ldots < \delta_{2n-1}$ be the Woodin*

[43] δ-mouse is also sometimes called δ-sound.
[44] For each $\alpha < \mathrm{lh}(\mathcal{T})$, $\mathrm{crit}(E_\alpha) > \delta$.
[45] The requirement that $\mathcal{T}_1 \in \mathcal{M}'|\omega_1^{\mathcal{M}'}$ is similar to the requirement that $\mathcal{T}_1 \in \Delta^{\mathsf{HC}}_{2n}(\mathcal{T}_0, b_0, x_0)$ used in [37]. It all comes down to the fact that for $\vec{s} \in \mathbb{R}^{<\omega}$, $\exists u \in \mathcal{Q}_{2n+1}(\vec{s})$ is $\Pi^1_{2n+1}(\vec{s})$ uniformly in $\vec{s}$. For this and similar results one can consult [20] or [37, Corollary 4.9].
[46] The proof appears in [37].

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cardinals of $\mathcal{M}$. Set $d = (\delta_0, ..., \delta_{2n-1})$. Suppose $\vec{a} \in \mathcal{M} \cap \mathbb{R}^k$ is such that $V \models \exists u \phi[\vec{a}]$. There is then $b \in \mathbb{R} \cap \mathcal{M}_{2n}(x)$ such that $\mathcal{M} \models \phi_{\mathcal{M},d}[b, \vec{a}]$.

The following are consequences of [37] that we will need. The proofs of these facts are implicitly present in [37], but [37] doesn't isolate these facts.

Lemma 3.2. *Suppose $\mathcal{M}$ is Π^1_2-iterable and $\mathcal{M}^{\#}_1$-like, and suppose $\mathcal{T}$ is a correct iteration of $\mathcal{M}$ of countable limit length. Let β be such that for each limit $\alpha < \mathrm{lh}(\mathcal{T})$, if $b_\alpha = [0, \alpha)_{\mathcal{T}}$ then $\mathcal{Q}(b_\alpha, \mathcal{T} \restriction \alpha)$ exists and $\mathcal{Q}(b_\alpha, \mathcal{T} \restriction \alpha) \triangleleft \mathcal{J}_\beta(\mathsf{cop}(\mathcal{T} \restriction \alpha))$. Suppose I plays $(\mathcal{T}, \beta)$ in $\mathcal{G}(\mathcal{M}, 0, 1)$ and II responds, using her winning strategy in $\mathcal{G}(\mathcal{M}, 0, 1)$, with a branch b. Then b is a cofinal branch of $\mathcal{T}$.*

Proof. Towards a contradiction suppose that b is not a cofinal branch, and set $\alpha = \sup(b)$.

Suppose for a moment that $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ exists. We claim that then $\mathcal{Q}(b, \mathcal{T} \restriction \alpha) \trianglelefteq \mathcal{J}(\mathsf{cop}(\mathcal{T} \restriction \alpha))$. To see this, suppose not. Because $\mathcal{J}_\beta(\mathsf{cop}(\mathcal{T} \restriction \alpha)) \models$ '$\delta(\mathcal{T} \restriction \alpha)$ is not a Woodin cardinal', we have that β is not in the well-founded part of $\mathcal{M}_b^{\mathcal{T} \restriction \alpha}$. This implies that $\mathcal{M}_b^{\mathcal{T} \restriction \alpha}$ is well-founded and has ordinal height less than β. But since $\mathcal{Q}(b, \mathcal{T} \restriction \alpha) \not\trianglelefteq \mathcal{J}(\mathsf{cop}(\mathcal{T} \restriction \alpha))$, $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ must have an active level above $\delta(\mathcal{T} \restriction \alpha)$. Let $E \in \vec{E}^{\mathcal{Q}(b, \mathcal{T} \restriction \alpha)}$ be such that $\mathrm{crit}(E) > \delta(\mathcal{T})$. Let $\mathcal{N}$ be the βth iterate of $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ via E. It follows that β is in the well-founded part of $\mathcal{N}$, and so $\mathcal{N}|\beta \models$ '$\delta(\mathcal{T} \restriction \alpha)$ is not a Woodin cardinal'. Hence, $\mathcal{Q}(b, \mathcal{T} \restriction \alpha) \trianglelefteq \mathcal{N}|\beta$, which is a contradiction as no mouse is an initial segment of its own iterate.

Now, because the existence of $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ implies that $\mathcal{Q}(b, \mathcal{T} \restriction \alpha) \trianglelefteq \mathcal{J}_\beta(\mathsf{cop}(\mathcal{T} \restriction \alpha))$, we must have that if $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ exists then $\mathcal{Q}(b, \mathcal{T} \restriction \alpha) = \mathcal{Q}(b_\alpha, \mathcal{T} \restriction \alpha)$ which then implies that $b = b_\alpha$. Thus, because $b \neq b_\alpha$, $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ does not exist.

It now immediately follows that β cannot be in the well-founded part of $\mathcal{M}_b^{\mathcal{T} \restriction \alpha}$. Thus, $\mathcal{M}_b^{\mathcal{T} \restriction \alpha}$ must be well-founded and $\sup(\mathsf{Ord} \cap \mathcal{M}_b^{\mathcal{T} \restriction \alpha}) < \beta$. Because $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ does not exist, $\pi_b^{\mathcal{T} \restriction \alpha}$ is defined. If now $\mathcal{N}$ is the βth iterate of $\mathcal{M}_b^{\mathcal{T} \restriction \alpha}$ via the last extender of $\mathcal{M}_b^{\mathcal{T} \restriction \alpha}$, $\mathcal{N} \models$ '$\delta(\mathcal{T} \restriction \alpha)$ is not a Woodin cardinal'. This contradicts the fact that $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ does not exist. □

Proposition 3.3. *Assume V is closed under the sharp function $x \mapsto x^{\#}$. Suppose $\mathcal{M}$ is Π^1_2-iterable and $\mathcal{M}^{\#}_1$-like, and suppose $\mathcal{T}$ is a correct iteration of $\mathcal{M}$ of countable limit length. Let β be such that for each $\alpha < \mathrm{lh}(\mathcal{T})$, if $b_\alpha = [0, \alpha)_{\mathcal{T}}$ then $\mathcal{Q}(b_\alpha, \mathcal{T} \restriction \alpha)$ exists and $\mathcal{Q}(b_\alpha, \mathcal{T} \restriction \alpha) \triangleleft \mathcal{J}_\beta(\mathsf{cop}(\mathcal{T}))$. Suppose that there is an α such that $\mathcal{J}_\alpha(\mathsf{cop}(\mathcal{T})) \models$ '$\delta(\mathcal{T})$ is a Woodin cardinal' and $\mathcal{J}_{\alpha+1}(\mathsf{cop}(\mathcal{T})) \models$ '$\delta(\mathcal{T})$ is not a Woodin cardinal', and finally, suppose I plays $(\mathcal{T}, \max(\alpha + 1, \beta))$ in $\mathcal{G}(\mathcal{M}, 0, 1)$ and II responds with a branch b (according to her winning strategy). Then b is the unique cofinal well-founded branch of $\mathcal{T}$ such that $\mathcal{J}_\alpha(\mathsf{cop}(\mathcal{T})) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$.*

Proof. We have that Lemma 3.2 implies that b is cofinal. Also, there can be at most one such well-founded cofinal branch. Towards a contradiction assume that either b is ill-founded or $\mathcal{J}_\alpha(\mathsf{cop}(\mathcal{T})) \not\trianglelefteq \mathcal{M}_b^{\mathcal{T}}$. Suppose for a moment that b is well-founded. Then we must have that $\mathcal{J}_\alpha(\mathsf{cop}(\mathcal{T})) \not\trianglelefteq \mathcal{M}_b^{\mathcal{T}}$, which implies that $\pi_b^{\mathcal{T}}$ is defined. But now if $\mathcal{N}$ is the $\max(\alpha + 1, \beta)$th iterate of $\mathcal{M}_b^{\mathcal{T}}$ via its last extender then $\mathcal{J}_{\alpha+1}(\mathsf{cop}(\mathcal{T})) \trianglelefteq \mathcal{N}$. Hence, $\mathcal{N} \models$ '$\delta(\mathcal{T})$ is not a Woodin cardinal' implying that $\mathcal{J}_{\alpha+1}(\mathsf{cop}(\mathcal{T})) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$.

We thus have that b is an ill-founded branch, and this implies that $\max(\alpha+1, \beta)$ is in the well-founded part of $\mathcal{M}_b^{\mathcal{T}}$, which then implies that $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{Q}(b, \mathcal{T}) = \mathcal{J}_\alpha(\mathsf{cop}(\mathcal{T}))$. Thus, we have that

(*) there is a cofinal branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T})$ exists, $\mathcal{Q}(b, \mathcal{T}) = \mathcal{J}_\alpha(\mathsf{cop}(\mathcal{T}))$ and this branch is ill-founded.

Let $\mathcal{M}'$ be the result of iterating the last extender of $\mathcal{M}$ out of the universe and let $\gamma > \max(\alpha, \beta)$ be such that the ill-foundedness of b can be witnessed by functions in $\mathcal{M}'|\gamma$[47]. We can now express (*) by the following formula:

(**) there are reals x, y and z such that

[47]Notice that after straighforward reorganization, $\mathcal{T}$ can be viewed as an iteration $\mathcal{T}^*$ of $\mathcal{M}'$. In the presence of sharps, the illfoundedness of b as a branch of $\mathcal{T}$ is equivalent to the illfoundedness of b as a branch of $\mathcal{T}^*$.

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1. x codes an iteration $\mathcal{U}$ of $\mathcal{M}'|\gamma$
2. y codes a well-founded model $\mathcal{N} = \mathcal{J}_\alpha(\mathsf{cop}(\mathcal{U}))$ such that $J_1[\mathcal{N}] \vDash$ '$\delta(\mathcal{U})$ is not a Woodin cardinal',
3. z codes a cofinal ill-founded branch c of $\mathcal{U}$ such that $\mathcal{N} \trianglelefteq \mathcal{M}_c^{\mathcal{U}}$.

(**) is a Σ^1_2 statement in any code of $(\mathcal{M}'|\gamma, \alpha)$, and so we get such a $\mathcal{U}$ in $\mathcal{M}[g]$ where $g \subseteq Coll(\omega, \beta)$ is generic. But then we can use the proof of [40, Corollary 4.17] to get a contradiction[48]. □

Proposition 3.4. *Suppose $n \in \omega$ and $\mathcal{M}$ is $\mathcal{M}^\#_{2n+1}$-like Π^1_{2n+2}-iterable x-premouse. Let Σ be a strategy for player II in $\mathcal{G}(\mathcal{M}, 0, 2n+1)$. Let δ be the least Woodin cardinal of $\mathcal{M}$ and suppose $\mathcal{T}$ is a correct iteration of $\mathcal{M}$ below δ such that $\mathrm{lh}(\mathcal{T})$ is a limit ordinal. For a limit $\alpha < \mathrm{lh}(\mathcal{T})$ let $b_\alpha = [0, \alpha)_{\mathcal{T}}$. Let y_0 be a real that codes the sequence $(\mathcal{Q}(b_\alpha, \mathcal{T} \restriction \alpha) : \alpha < \mathrm{lh}(\mathcal{T}) \wedge \alpha \in Lim)$ and $\mathcal{M}^\#_{2n}(\mathsf{cop}(\mathcal{T}))$. Finally set $b = \Sigma(\mathcal{T}, y)$ where y is any real that is Turing above y_0 and also set $\mathcal{N} = \mathcal{M}_{2n}(\mathsf{cop}(\mathcal{T}))$. Then the following hold.*

1. *b is a cofinal branch.*
2. *Suppose further that $\mathcal{N} \vDash$ '$\delta(\mathcal{T})$ is not a Woodin cardinal'. Then $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{N}$.*
3. *Suppose that $\mathcal{N} \vDash$ '$\delta(\mathcal{T})$ is a Woodin cardinal'. Then $\mathcal{N}|(\delta(\mathcal{T})^+)^{\mathcal{N}} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$.*

Proof. Set $\alpha = \sup(b)$. We do the proof for the prototypical case $n = 1$.

Lemma 3.5. *b is cofinal.*

Proof. Towards a contradiction suppose that $\alpha < \mathrm{lh}(\mathcal{T})$.

Case 1. $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ exists.

Set $\mathcal{R} = \mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ and $\mathcal{S} = \mathcal{Q}(b_\alpha, \mathcal{T} \restriction \alpha)$. Note that $\delta(\mathcal{T} \restriction \alpha)$ is a Woodin cardinal in both $\mathcal{R}$ and $\mathcal{S}$. We claim that $\mathcal{R} = \mathcal{S}$. To see this, we compare them inside $\mathcal{M}_1(\mathcal{R}, \mathcal{S})$. If there is a successful coiteration of $\mathcal{R}$ and $\mathcal{S}$ then we indeed have that $\mathcal{R} = \mathcal{S}$ (e.g., see [[37], Lemma 1.11]), and so any attempt to coiterate them is doomed to failure.

We attempt to coiterate $(\mathcal{R}, \mathcal{S})$ in $\mathcal{M}_1(\mathcal{R}, \mathcal{S})$ by building a coiteration $(\mathcal{U}, \mathcal{W})$ in which the branches are picked as follows. Suppose λ is a limit ordinal and we have defined $\mathcal{U} \restriction \lambda$ and $\mathcal{W} \restriction \lambda$. We want to describe our procedure for picking a branch of $\mathcal{U} \restriction \lambda$ and $\mathcal{W} \restriction \lambda$. As $\mathcal{S}$ is $\omega_1 + 1$-iterable inside $\mathcal{M}_1(\mathcal{R}, \mathcal{S})$, we have a branch d of $\mathcal{W} \restriction \lambda$ that is according to the unique strategy of $\mathcal{S}$. Also, we must have that $\mathcal{Q}(d, \mathcal{W} \restriction \lambda)$ exists as $\mathcal{S}$ is a $\delta(\mathcal{T} \restriction \alpha)$-mouse (in the sense of [[37], Definition 1.3])[49]. We now seek a branch c of $\mathcal{U}$ such that $\mathcal{Q}(c, \mathcal{U} \restriction \lambda)$ exists and $\mathcal{Q}(c, \mathcal{U} \restriction \lambda) = \mathcal{Q}(d, \mathcal{W} \restriction \lambda)$. If there is such a branch then we choose it and continue the coiteration. Otherwise we stop the coiteration. As we cannot successfully coiterate $\mathcal{R}$ and $\mathcal{S}$, we must end up with a coiteration $(\mathcal{U}, \mathcal{W})$ such that $\mathrm{lh}(\mathcal{U}) = \mathrm{lh}(\mathcal{W})$ is a limit ordinal $< \omega_1^{\mathcal{M}_1(\mathcal{R}, \mathcal{S})}$[50] and if d is the branch of $\mathcal{W}$ according to $\mathcal{S}$'s unique strategy then there is no branch c of $\mathcal{U}$ such that $\mathcal{Q}(c, \mathcal{U})$ exists and $\mathcal{Q}(c, \mathcal{U}) = \mathcal{Q}(d, \mathcal{W})$.

We are now in the lucky situation that $\mathcal{U}$ is an allowed move for I in the second round of $\mathcal{G}(\mathcal{M}, 0, 3)$. Indeed, some real recursive in y codes $\mathcal{S}$, and therefore, $\mathcal{U} \in \mathcal{M}_1(y, \mathcal{T}, b)|\omega_1^{\mathcal{M}_1(y, \mathcal{T}, b)}$[51]. Let then I play $\mathcal{U}$ and II play a branch c such that $\mathcal{M}_c^{\mathcal{U}}$ is well-founded. Once again, we only know that c is maximal. But now $\mathcal{M}_c^{\mathcal{U}}$ is Π^1_2-iterable above $\delta(\mathcal{U})$. Let $\gamma = \sup(c)$, $\nu = \delta(\mathcal{U} \restriction \gamma)$ and set $\mathcal{R}' = \mathcal{Q}(c, \mathcal{U} \restriction \gamma)$ if it exists and otherwise set $\mathcal{R}' = \mathcal{M}_c^{\mathcal{U} \restriction \gamma}$. If $\gamma < \mathrm{lh}(\mathcal{U})$ we set $\mathcal{S}' = \mathcal{Q}(c', \mathcal{U} \restriction \gamma)$ where $c' = [0, \gamma)_{\mathcal{U}}$. Otherwise let $\mathcal{S}' = \mathcal{Q}(d, \mathcal{W})$. In either of the cases, we have that $\mathcal{S}'$ is ω_1-iterable, 1-small, $\delta(\mathcal{U})$-mouse

[48] Here is a little bit more for those who are familiar with such proofs. We fix θ larger than γ and let $\pi : \mathcal{P} \to \mathcal{M}'|\theta$ be a countable elementary hull of $\mathcal{M}'|\theta$ inside $\mathcal{M}'$ such that $(\gamma, \alpha) \in \mathrm{rge}(\pi)$. Let then $g \in \mathcal{M}'$ be $\mathcal{P}$-generic for $Coll(\omega, \pi^{-1}(\gamma))$. Let $(\mathcal{T}', \mathcal{N}') \in \mathcal{P}[g]$ be a pair satisfying (**). Using Martin-Steel realizability theorem (see [22, Theorem 4.3]) we can show that the branch c' of clause 3 of (**) is the π-realizable branch of $\mathcal{T}'$ and hence, it is well-founded. The fact that $c' \in \mathcal{P}[g]$ follows from absoluteness and the fact that $\mathcal{N}'$ uniquely identifies c'.

[49] Or just $\delta(\mathcal{T} \restriction \alpha)$-sound.

[50] One wrinkle here is to show that $\mathrm{lh}(\mathcal{U}) < \omega_1^{\mathcal{M}_1(\mathcal{R}, \mathcal{S})}$. This follows from the fact that we always have a branch on the $\mathcal{S}$-side and so if the coiteration lasts $\omega_1^{\mathcal{M}_1(\mathcal{R}, \mathcal{S})}$ steps then we could prove that $\mathcal{U}$ has a branch by a standard reflection argument.

[51] Recall that for any $u \in \mathbb{R}$ and any Π^1_2-iterable $\mathcal{M}_1$-like $\mathcal{M}$, $\mathcal{M}_1(u)|\omega_1^{\mathcal{M}(u)} \trianglelefteq \mathcal{M}$. See [[37], Lemma 2.2].

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and neither $\mathcal{R}' \trianglelefteq \mathcal{S}'$ nor $\mathcal{S}' \trianglelefteq \mathcal{R}'$. Moreover, ν is a Woodin cardinal in both $\mathcal{R}'$ and $\mathcal{S}'$. Thus, $\mathcal{R}'$ and $\mathcal{S}'$ have at least 2 Woodin cardinals, $\delta(\mathcal{T} \restriction \alpha)$ and ν.

Because $\mathcal{R}'$ is Π^1_2-iterable above ν and $\mathcal{S}'$ is iterable, if $\mathcal{R}' = \mathcal{Q}(c, \mathcal{U} \restriction \gamma)$ then clause 3 of [[37], Lemma 2.2] implies that $\mathcal{R}' = \mathcal{S}'$, which is a contradiction. Thus, $\mathcal{R}' = \mathcal{M}_c^{\mathcal{U}\restriction\gamma}$ and $\mathcal{Q}(c, \mathcal{U} \restriction \gamma)$ doesn't exist. It follows that $\pi^{\mathcal{T}^\frown\mathcal{U}}$ is defined, and hence, $\mathcal{R}'$ is not 1-small. But now [[37], Lemma 3.1] implies that $\mathcal{S}' \trianglelefteq \mathcal{R}'$. Hence, after all, $\delta(\mathcal{U})$ is not a Woodin cardinal in $\mathcal{R}'$, contradiction.

Case 2. $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ doesn't exist.

Set now $\mathcal{R} = \mathcal{M}_b^{\mathcal{T}\restriction\alpha}$ and $\mathcal{S} = \mathcal{Q}(b_\alpha, \mathcal{T} \restriction \alpha)$. Let ν be the second least Woodin cardinal of $\mathcal{R}$. We now coiterate $\mathcal{R}_0 = \mathcal{R}|(\nu^+)^{\mathcal{R}}$ and $\mathcal{S}$ inside $\mathcal{M}_1(\mathcal{R}_0, \mathcal{S})$ using the same procedure as before, that is, if $(\mathcal{U}_0, \mathcal{W})$ is the coiteration that we build then at each limit stage $\lambda < \mathrm{lh}(\mathcal{U}_0) = \mathrm{lh}(\mathcal{W})$, $d = [0, \lambda)_{\mathcal{W}}$ is the branch of $\mathcal{W} \restriction \lambda$ according to the unique strategy of $\mathcal{S}$ as a $\delta(\mathcal{T} \restriction \alpha)$-mouse, and $c = [0, \lambda)_{\mathcal{U}_0}$ is the unique well-founded cofinal branch of $\mathcal{U} \restriction \lambda$ such that $\mathcal{Q}(c, \mathcal{U}_0 \restriction \lambda)$ exists and $\mathcal{Q}(c, \mathcal{U}_0 \restriction \lambda) = \mathcal{Q}(d, \mathcal{W} \restriction \lambda)$.

Assume for a moment that the coiteration is successful. Thus $\mathcal{U}_0$ and $\mathcal{W}$ have last models, say $\mathcal{R}'_0$ and $\mathcal{S}'$. If $\mathcal{S}' \trianglelefteq \mathcal{R}'_0$ then we in fact have that $\mathcal{S} \triangleleft \mathcal{R}_0$ and so $\delta(\mathcal{T} \restriction \alpha)$ is not a Woodin cardinal in $\mathcal{R}$. Thus, it must be the case that $\mathcal{R}'_0 \trianglelefteq \mathcal{S}'$. This means that $\pi^{\mathcal{U}_0}$ exists. Let now $\mathcal{U}$ be the copy of $\mathcal{U}_0$ on $\mathcal{R}$ using the identity map. The extenders of $\mathcal{U}$ and the tree structure of $\mathcal{U}$ are the same as the extenders of $\mathcal{U}_0$ and the tree structure of $\mathcal{U}_0$. Let $\mathcal{R}'$ be the last model of $\mathcal{U}$.

Now $\mathcal{U}$ again is a valid move for I in $\mathcal{G}(\mathcal{M}, 0, 3)$, and so we let I play it. II has to now either accept $\mathcal{U}$ or play a maximal branch of $\mathcal{U}$. If II plays a maximal non-cofinal branch then we have two different branches c_0, c_1 of some $\mathcal{U} \restriction \lambda$ such that both $\mathcal{Q}(c_0, \mathcal{U} \restriction \lambda)$ and $\mathcal{Q}(c_1, \mathcal{U} \restriction \lambda)$ exist, one of them is $\omega_1 + 1$ iterable and the other is Π^1_2-iterable. It then follows from clause 3 of [[37], Lemma 2.2] that $\mathcal{Q}(c_0, \mathcal{U} \restriction \lambda) = \mathcal{Q}(c_1, \mathcal{U} \restriction \lambda)$ implying that $c_0 = c_1$, contradiction. Thus, II must accept it. But now we are in the second scenario of this repetitive argument, namely if ν' is the second Woodin cardinal of $\mathcal{R}'$ then $\mathcal{R}'$ as a ν'-mouse is Π^1_2-iterable and not 1-small while $\mathcal{S}'$ as ν'-mouse is $\omega_1 + 1$-iterable and is 1-small. Thus, [[37], Lemma 3.1] implies that $\mathcal{S}' \trianglelefteq \mathcal{R}'$, which in fact implies that $\mathcal{S} \trianglelefteq \mathcal{R}$ and hence, $\delta(\mathcal{T} \restriction \alpha)$ is not a Woodin cardinal in $\mathcal{R}$.

We thus have that the coiteration $(\mathcal{U}_0, \mathcal{W})$ of $(\mathcal{R}_0, \mathcal{S})$ that we described above is not successful, which can only happen if we fail to find a branch c of $\mathcal{U}_0$ such that $\mathcal{Q}(c, \mathcal{U}_0)$ exists and $\mathcal{Q}(c, \mathcal{U}_0) = \mathcal{Q}(d, \mathcal{W})$, where once again d is the unique branch of $\mathcal{W}$ that is according to the unique strategy of $\mathcal{S}$. We again let $\mathcal{U}$ be the copy of $\mathcal{U}_0$ onto $\mathcal{R}$ via identity and let I play it in the second round of $\mathcal{G}(\mathcal{M}, 0, 3)$. Clause 3 of [[37], Lemma 2.2] implies that II must play a cofinal branch c of $\mathcal{U}$ such that $\mathcal{M}_c^{\mathcal{U}}$ is well-founded. Clause 3 of [[37], Lemma 2.2] implies that $\mathcal{Q}(c, \mathcal{U})$ cannot exists while [[37], Lemma 3.1] implies that $\mathcal{Q}(d, \mathcal{W}) \trianglelefteq \mathcal{M}_c^{\mathcal{U}}$. □

We thus have that b is a cofinal branch. Assume then $\mathcal{N} \vDash$ '$\delta(\mathcal{T})$ is not a Woodin cardinal'. Let $\mathcal{S} \trianglelefteq \mathcal{N}$ be the longest such that $\mathcal{S} \vDash$ '$\delta(\mathcal{T})$ is a Woodin cardinal'. Assume first that $\mathcal{Q}(b, \mathcal{T})$ exists and set $\mathcal{R} = \mathcal{Q}(b, \mathcal{T})$. We want to see that $\mathcal{R} = \mathcal{S}$, and this can be achieved by repeating the proof of Case 1 of the Lemma above. Assume then $\mathcal{Q}(b, \mathcal{T})$ doesn't exist. We then let $\mathcal{R} = \mathcal{M}_b^{\mathcal{T}}$ and argue as in Case 2 of the Lemma above to conclude that $\mathcal{S} \trianglelefteq \mathcal{R}$, contradicting the fact that $\mathcal{Q}(b, \mathcal{T})$ doesn't exist.

Assume next that $\mathcal{N} \vDash$ '$\delta(\mathcal{T})$ is a Woodin cardinal'. Let $\mathcal{S} \trianglelefteq \mathcal{N}$ be such that $\mathcal{S}$ is a $\delta(\mathcal{T})$-mouse. We want to see that $\mathcal{S} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$. Assume first that $\mathcal{Q}(b, \mathcal{T})$ exists and set $\mathcal{R} = \mathcal{Q}(b, \mathcal{T})$. We then compare $\mathcal{R}$ and $\mathcal{S}$ as in Case 1 of the Lemma and conclude that $\mathcal{S} \trianglelefteq \mathcal{R}$ as $\mathcal{R}$ witnesses that $\delta(\mathcal{T})$ is not a Woodin cardinal while $\mathcal{S}$ does not. Suppose then $\mathcal{Q}(b, \mathcal{T})$ doesn't exist and set $\mathcal{R} = \mathcal{M}_b^{\mathcal{T}}$. We then compare $\mathcal{R}$ with $\mathcal{S}$ as in Case 2 of the Lemma above. Just like in that proof, the conclusion once again is that $\mathcal{S} \trianglelefteq \mathcal{R}$. □

4. The proof of Theorem 1.8

After setting up some notation, we will present the main ideas of the proof in Section 4.1.

Fix n. Below we prove Theorem 1.8 for n. As [7] proves Theorem 1.8 for $n = 0$, we might just as well assume that $n \geq 1$. As was already mentioned in clause 5 of Review 2.18, we have that

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$$\sup_{m<\omega} \gamma^{2n+1}_{m,\infty} = \kappa^1_{2n+3}.$$

Fix then $m < \omega$ such that $\gamma^{2n+1}_{m,\infty} \geq \delta^1_{2n+2}$. We then let $(\mathcal{S}_0, \xi_0)$ be such that

1. $\mathcal{S}_0$ is a complete iterate of $\mathcal{M}^{\#}_{2n+1}$ such that $\mathcal{T}_{\mathcal{M}^{\#}_{2n+1},\mathcal{S}_0}$ is below the least Woodin cardinal of $\mathcal{M}^{\#}_{2n+1}$,
2. $\xi_0 \leq \gamma^{\mathcal{S}_0}_m$,
3. $\pi_{\mathcal{S}_0,\infty}(\xi_0) = \delta^1_{2n+2}$.

Let x_0 be a real coding $(\mathcal{S}_0, \xi_0)$ and let Code be the set of reals y such that y codes a pair $(\mathcal{R}_y, \tau_y)$ with a property that

1. $\mathcal{R}_y$ is a complete iterate of $\mathcal{S}_0$ such that $\mathcal{T}_{\mathcal{S}_0,\mathcal{R}_y}$ is below the least Woodin cardinal of $\mathcal{S}_0$,
2. $\tau_y < \pi_{\mathcal{S}_0,\mathcal{R}_y}(\xi_0)$.

For $x, y \in$ Code set $x \leq^* y$ if and only if letting $\mathcal{Q}$ be the common complete iterate of $\mathcal{R}_x$ and $\mathcal{R}_y$, $\pi_{\mathcal{R}_x,\mathcal{Q}}(\tau_x) \leq \pi_{\mathcal{R}_y,\mathcal{Q}}(\tau_y)$. We have that there is a formula ϕ such that for all $(x, y) \in \mathbb{R}^2$,

$$x \leq^* y \leftrightarrow \mathcal{M}_{2n}(x, y, x_0) \vDash \phi[x, y, s_m].$$

It follows that $\leq^*$ is $\Delta^1_{2n+3}(x_0)$ (see clause 10 of Section 3). For $x \in \mathrm{dom}(\leq^*)$, let $\gamma_x = \pi_{\mathcal{R}_x,\infty}(\tau_x)$. We have that the length of $\leq^*$ is δ^1_{2n+2}, and for each ordinal $\alpha < \delta^1_{2n+2}$, there is $x \in \mathrm{dom}(\leq^*)$ such that $\gamma_x = \alpha$.

Towards a contradiction, suppose $(A_\alpha : \alpha < \delta^1_{2n+2})$ is a sequence consisting of distinct $\mathbf{\Sigma}^1_{2n+2}$-sets. Let $U \subseteq \mathbb{R}^2$ be a universal Σ^1_{2n+2} formula[52] and let for $\alpha < \delta^1_{2n+2}$, $B_\alpha = \{y : \{z : U(y, z)\} = A_\alpha\}$. Using the Coding Lemma (see [13, Theorem 2.12]) we can find a real z Turing above x_0 and a $\Sigma^1_{2n+3}(z)$ set $D^* \subseteq \mathrm{dom}(\leq^*) \times \mathbb{R}$ such that

1. for each $x \in \mathrm{dom}(\leq^*)$, $D^*_x \neq \emptyset$,
2. for each $(x, y) \in D^*$, $y \in B_{\gamma_x}$ (therefore, $U_y = A_{\gamma_x}$).

Let $D \in \Pi^1_{2n+2}(z)$ be such that $D \subseteq \mathbb{R}^3$ and $(x, y) \in D^* \leftrightarrow \exists u (x, y, u) \in D$. To get that $\mathrm{dom}(D^*) = \mathrm{dom}(\leq^*)$ we use the fact that $\Sigma^1_{2n+3}(z)$ is closed under $\exists^{\mathbb{R}}$. For details, see the discussion above [13, Lemma 2.13].

We now set $\mathcal{M} =_{def} \mathcal{M}_{2n+1}(z)$. Given a complete iterate $\mathcal{N}$ of $\mathcal{M}$, let $\delta_{\mathcal{N}}$ be the least Woodin cardinal of $\mathcal{N}$ and $\kappa_{\mathcal{N}}$ be the least $< \delta_{\mathcal{N}}$-strong cardinal of $\mathcal{N}$.

If $x \in \mathbb{R}$ codes a countable premouse then we let $\mathcal{R}_x$ be this premouse and let C' be the set of reals that code a countable Π^1_{2n+2}-iterable $\mathcal{M}^{\#}_{2n+1}$-like premouse over some real. For $x \in \mathsf{C}'$ we let ν_x be the least Woodin of $\mathcal{R}_x$ and μ_x be the least $< \nu_x$-strong of $\mathcal{R}_x$. Because $\mathcal{R}_x$ may not be iterable, $\mathcal{R}_x$ may not satisfy condensation (see [39, Theorem 5.1]). Because for $u \in \mathbb{R}$, $\mathcal{M}^{\#}_{2n+1}(u)$ is iterable it does have condensation, and in particular, it follows from Lemma 2.12 that $\mathcal{M}_{2n+1}(u) \vDash$ Cond. Let now C be the set of $x \in \mathsf{C}'$ such that

1. $\mathcal{R}_x \vDash$ Cond,
2. $\mathcal{R}_x$ is translatable (see clause 3 of Review 2.7),
3. for all $\eta < \delta$, $\mathcal{R}_x|\eta$ is $\eta + 1$-iterable in $\mathcal{R}_x$ (see clause 11 of Review 2.7).

It follows from clause 1, 3 and 11 of Review 2.7 that if x codes $\mathcal{M}^{\#}_{2n+1}(u)$ then $x \in \mathsf{C}$.

We now work towards applying Hjorth's reflection argument to produce a code of each A_γ below $\kappa_{\mathcal{M}}$. Let $w \subseteq \omega$ be a $\Pi^1_{2n+3}(z)$ real that is not $\Sigma^1_{2n+3}(z)$. Let ψ be a Σ^1_{2n+2} formula such that

$$n \in w \leftrightarrow \forall z' \psi[n, z, z']$$

[52] I.e., for each $A \in \mathbf{\Sigma}^1_{2n}$ there is $y \in \mathbb{R}$ such that $A = \{u \in \mathbb{R} : U(y, u)\} =_{def} U_y$ and moreover, if $y' \in \mathbb{R}$ and $A \in \Sigma^1_{2n}(y')$ then there is a real y recursive in y' such that $A = U_y$. $U(y, u)$ essentially says that 'if ϕ is the Σ^1_{2n}-formula whose Gödel number is $y(0)$ then $\phi[y', u]$ holds where $y'(n) = y(n + 1)$'.

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4.1. The main ideas of the proof

Before we go on, we give an outline of the proof that follows. The main idea of the proof is that each $A_\alpha = U_{w_0}$ for some w_0 that can be obtained 'below $\kappa_{\mathcal{M}}$'. We say w_0 is a U code for A_α. This means that for each $\alpha < \delta^1_{2n+2}$ we would like to find some complete iterate $\mathcal{N}$ of $\mathcal{M}$, $\xi < \kappa_{\mathcal{N}}$ and a $g \subseteq Coll(\omega, \xi)$ such that there is a U-code for A_α in $\mathcal{N}[g]$.

Notice that, using genericity iterations, for each $\alpha < \delta^1_{2n+2}$, we can find a complete iterate $\mathcal{N}$ of $\mathcal{M}$ such that A_α has a U-code that is $\mathcal{N}$-generic for the extender algebra at $\delta_{\mathcal{N}}$. Because $\mathcal{N} \vDash \mathsf{Cond}$, this allows us to obtain a $\xi < \kappa_{\mathcal{N}}$ such that A_α has a U-code that is generic over the extender algebra of $\mathcal{N}|\xi$. To make this work, we need to make sure that the U-code of A_α can be identified in a first order way over $\mathcal{N}$, and this is achieved by the formula θ_0 introduced below.

However, the above is not enough. We need to find $\mathcal{N}$ as above which is α-stable in the sense that for any complete iterate $\mathcal{Q}$ of $\mathcal{N}$ there is such a w_0 in $\mathcal{Q}^{Coll(\omega, \pi_{\mathcal{N},\mathcal{Q}}(\xi))}$. Such a uniformity is not easy to achieve, and here, we use our reals w and w' introduced below. w is a $\Pi^1_{2n+3}(z)$ real which is not $\Sigma^1_{2n+3}(z)$ while w' is a $\Sigma^1_{2n+3}(z)$ subset of w.

The $\Pi^1_{2n+2}(z)$-formula θ is the key. w' is the real defined by $\exists \vec{y} \theta(x, \vec{y})$. The formulas are defined in a way that assures the following. If $k \in w'$ then the least initial segment of $\mathcal{M}$ that 'proves' that $k \in w$ is below the least initial segment of $\mathcal{M}$ that has a U-code for A_α. Thus, if $k \in w - w'$ then for any α (roughly speaking), the least initial segment of $\mathcal{M}$ that has a U-code for A_α is below the least initial segment that proves that $k \in w$. This means that if we pick ζ to be the least such that $\mathcal{M}|\zeta$ proves that $k \in w$ then each A_α will have a U-code below ζ.

Above by 'the least initial segment of $\mathcal{M}$ that proves $k \in w$' we mean that the formula '$k \in w$' is first order over $\mathcal{M}$ as explained in Proposition 2.6, and therefore, because $\mathcal{M} \vDash \mathsf{Cond}$, we have some $\xi < \kappa_{\mathcal{M}}$ such that '$k \in w$' is first order over $\mathcal{M}|\xi$. The least such ξ, which we call ζ, determines the least level of $\mathcal{M}$ that proves that $k \in w$.

Because of Cond, we can assume that ζ is a cutpoint of $\mathcal{M}$. We then finish the proof by showing that if $\mathcal{Q} = \mathcal{M}|\zeta$ then (roughly) each A_α has a U-code in a generic extension of a complete iterate of $\mathcal{Q}$. But the set of complete iterates of $\mathcal{Q}$ is too simple (this follows from calculations presented in Lemma 4.5), and one could not define δ^1_{2n+2} many distinct $\mathbf{\Sigma}^1_{2n+2}$ sets whose U-codes can be obtained in generic extensions of complete iterates of $\mathcal{Q}$. This then gives a contradiction as $(A_\alpha : \alpha < \delta^1_{2n+2})$ consists of distinct $\mathbf{\Sigma}^1_{2n+2}$ sets whose U-codes can be obtained in generic extensions of complete iterates of $\mathcal{Q}$.

4.2. The formula θ

We will need the following lemma. Recall that U is our fixed universal Σ^1_{2n+2} set.

Lemma 4.1. *Suppose $\mathcal{N}$ is a $\mathcal{M}^\#_{2n+1}$-like Π^1_{2n+2}-iterable z-premouse, $(\nu_0, \nu_1, ..., \nu_{2n})$ are the Woodin cardinals of $\mathcal{N}$ enumerated in increasing order, $\mathcal{E}$ is a weakly appropriate set of extenders (relative to ν_0), $\nu < \nu_0$ and suppose $(w_0, (w_1, w_2, w_3))$ is a sequence of reals such that*

1. *$D(w_1, w_2, w_3)$ and $U(w_2, w_0)$,*
2. *w_0 and (w_1, w_2, w_3) are generic over $\mathcal{N}$ for $\mathsf{Ea}^{\mathcal{N}}_{\nu_0,\nu,\mathcal{E}}$.*

There is then a condition $(p, q) \in \mathsf{Ea}^{\mathcal{N}}_{\nu_0,\nu,\mathcal{E}} \times \mathsf{Ea}^{\mathcal{N}}_{\nu_0,\nu,\mathcal{E}}$ such that

1. *$w_0 \vDash p$ and $(w_1, w_2, w_3) \vDash q$,*
2. *$\mathcal{N} \vDash (p, q) \Vdash$ '$D(\mathsf{ea}^r_1, \mathsf{ea}^r_2, \mathsf{ea}^r_3) \wedge U(\mathsf{ea}^r_2, \mathsf{ea}^l)$'.*

Proof. First of all, notice that $\mathcal{N} = \mathcal{M}^\#_{2n}(\mathcal{N}|\nu_0)$ (see clause 11 of Section 3). Suppose w_1 codes a pair $(\mathcal{R}, \xi)$. Working in $\mathcal{N}[w_1, w_2, w_3]$, let $\gamma = \pi_{\mathcal{R},\mathcal{H}^{\mathcal{N}}}(\xi)$. We then have that there is a condition $q_0 \in \mathsf{Ea}^{\mathcal{N}}_{\nu_0,\nu,\mathcal{E}}$ such that $(w_1, w_2, w_3) \vDash q_0$ and $q_0 \Vdash$ 'if $(\mathcal{S}, \beta)$ is the pair coded by ea_1 then $\pi_{\mathcal{S},\mathcal{H}}(\beta) = \check{\gamma}$'[53] and also, $q_0 \Vdash D(\mathsf{ea}_1, \mathsf{ea}_2, \mathsf{ea}_3)$. Let now (g_1, g_2, g_3) be $\mathcal{N}[w_0]$-generic for $\mathsf{Ea}^{\mathcal{N}}_{\nu_0,\nu,\mathcal{E}}$

[53]Here we use the fact that the directed system can be internalized to $\mathcal{N}$. See clause 8 of Review 2.19.

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such that $(g_1, g_2, g_3) \vDash q_0$. Because $\mathcal{N} = \mathcal{M}^{\#}_{2n}(\mathcal{N}|\nu_0)$ (implying that $\mathcal{N}$ is Σ^1_{2n+2}-correct), it follows that $g_1 \in \mathsf{Code}$ and that $w_0 \in A_{\gamma_{g_1}}$,[54] ,[55]. Then we must have that $\mathcal{N}[w_0][(g_1, g_2, g_3)] \vDash U(g_2, w_0)$. The desired (p, q) can now be found using the forcing theorem. □

The formula θ_0:

Let $\theta_0(u, w_0, w_1, w_2, w_3)$ be the $\Pi^1_{2n+2}(z)$-formula that is the conjunction of the following clauses:

1. $D(w_1, w_2, w_3)$[56] ,
2. $u \in \mathsf{C}$ and $\mathcal{R}_u$ is Π^1_{2n+2}-iterable,
3. $(z, w_0, w_1, w_2, w_3) \in \mathcal{R}_u \cap \mathbb{R}^5$,
4. $R_u \vDash U(w_2, w_0)$.

At this point the reader may find it useful to review Proposition 3.1, clause 3 and 6 of Review 2.3, and clause 2 of Review 2.7.

Let U' be a Π^1_{2n+1} set such that $U(\vec{a}) \leftrightarrow \exists b U'(b, \vec{a})$.

Suppose now that $\theta_0(u, w_0, w_1, w_2, w_3)$ holds. Let $\mathcal{K} = \mathsf{StrLe}(\mathcal{R}_u, z)$ and $\mathcal{E} = \mathcal{E}^{\mathcal{K}}_{sm}$ (see Definition 2.13). It follows from Lemma 4.1 that there is a condition $(p, q) \in \mathsf{Ea}^{\mathcal{K}}_{\nu_u, \mathcal{E}} \times \mathsf{Ea}^{\mathcal{K}}_{\nu_u, \mathcal{E}}$ such that

1. $w_0 \vDash p$,
2. $(w_1, w_2, w_3) \vDash q$, and
3. $(p, q) \Vdash \exists b(U'_{\mathcal{K}[(\mathsf{ea}^l, \mathsf{ea}^r)], d}(b, \mathsf{ea}^r_2, \mathsf{ea}^l))$ where letting $(\nu_u, \xi_1, ..., \xi_{2n})$ be the Woodin cardinals of $\mathcal{K}$ enumerated in increasing order, $d = (\xi_1, \xi_2, ..., \xi_{2n})$.[57]

Clause 3 above is a consequence of Proposition 3.1 and the fact that because $\mathcal{R}_u$ is Π^1_{2n+2}-iterable, it is $\omega_1 + 1$-iterable above ν_u (see clause 11 of Section 3 and recall that under AD, ω_1-iterability implies $\omega_1 + 1$-iterability). Notice that to verify U' we skip the extender algebra $\mathsf{Ea}^{\mathcal{K}}_{\xi_1, \nu_u}$. However, ξ_1 is not a lazy Woodin cardinal, it has a noble role of making sure that the witness b can be found inside $\mathcal{K}[(\mathsf{ea}^l, \mathsf{ea}^r)]$. The fact that we have skipped ξ_1 will be used in the proof of Lemma 4.2.

Because $\mathcal{R}_u \vDash \mathsf{Cond}$, we have that if $\theta_0(u, w_0, w_1, w_2, w_3)$ holds then in fact there are $\alpha < \beta < \nu_u$ such that

1. $\mathcal{R}_u|\beta \vDash \mathsf{ZFC}$ + 'there are $2n + 1$ many Woodin cardinals',
2. $\mathcal{R}_u|\beta \vDash$ 'α is the least Woodin cardinal',
3. α is an fb-cut of $\mathcal{R}_u$ (see Definition 2.9, in particular, α is an $\mathcal{R}_u$-cardinal),
4. setting $\mathcal{K} = \mathsf{StrLe}(\mathcal{R}_u|\beta, z)$ and $\mathcal{E} = \mathcal{E}^{\mathcal{K}}_{sm}$, there is a condition $(p, q) \in \mathsf{Ea}^{\mathcal{K}}_{\alpha, \mathcal{E}} \times \mathsf{Ea}^{\mathcal{K}}_{\alpha, \mathcal{E}}$ such that
   (a) $w_0 \vDash p$,
   (b) $(w_1, w_2, w_3) \vDash q$, and
   (c) $(p, q) \Vdash \exists b(U'_{\mathcal{K}[(\mathsf{ea}^l, \mathsf{ea}^r)], d}(\mathsf{ea}^r_2, \mathsf{ea}^l))$ where letting $(\alpha, \xi_1, ..., \xi_{2n})$ be the Woodin cardinals of $\mathcal{K}$ enumerated in increasing order, $d = (\xi_1, \xi_2, ..., \xi_{2n})$.

If $s = (u, w_0, w_1, w_2, w_3)$ then we say that (α, β) witnesses θ_0-reflection for s if (α, β) satisfies clause 1-4 above. Assuming $\theta_0(u, w_0, w_1, w_2, w_3)$ holds with $s = (u, w_0, w_1, w_2, w_3)$, we let (α_s, β_s) be the

[54] Notice that because $w_1 \in \mathsf{Code}$ we really do have that $\mathcal{R}$ is a complete iterate of $\mathcal{M}^{\#}_{2n+1}$. Because $\mathcal{H}^{\mathcal{N}}$ is a complete iterate of $\mathcal{R}$, this implies $\pi_{\mathcal{H}^{\mathcal{N}}, \infty}(\gamma) = \gamma_{w_1}$. Because $D(g_1, g_2, g_3)$ holds (by absoluteness, see Review 2.3), we have that if $(\mathcal{R}', \xi')$ is the pair coded by g_1 then $\mathcal{R}'$ is a complete iterate of $\mathcal{M}^{\#}_{2n+1}$ and because $\pi_{\mathcal{R}', \mathcal{H}^{\mathcal{N}}}(\xi') = \gamma$, we have that $\gamma_{g_1} = \gamma_{w_1}$. Thus, $w_0 \in A_{\gamma_{g_1}}$.

[55] The fact that $\mathcal{H}^{\mathcal{N}}$ is a complete iterate of $\mathcal{R}$ follows from generic comparisons (this point also appeared in Lemma 2.20). Recall that $\mathcal{H}^{\mathcal{N}}$ is the direct limit of all complete iterates of $\mathcal{M}^{\#}_{2n+1}$ that are in $\mathcal{N}|\nu$ where ν is the least inaccessible cardinal of $\mathcal{N}$ above the least Woodin cardinal of $\mathcal{N}$. Just like in the case of HOD analysis of $L[x][G]$ presented in [40], this direct limit can be shown to coincide with the direct limit of all iterates of $\mathcal{M}^{\#}_{2n+1}$ that appear in $N[H]$ where $H \subseteq Coll(\omega, < \nu)$ generic over $\mathcal{N}$. These points are explained in detail in various publications such as [29], [24], [32] and [40].

[56] Notice that z is implicitly build into D.

[57] See item 3 of 2.3. According to the terminology defined there, we have that $U'_{\mathcal{K}[(\mathsf{ea}^l, \mathsf{ea}^r)], d}(b, \mathsf{ea}^r_2, \mathsf{ea}^l)$ is an instances of $\phi_{\mathcal{M}, d}$ with ϕ the Π^1_{2n+1} formula defining U' and $\mathcal{M} = \mathcal{K}[(\mathsf{ea}^l, \mathsf{ea}^r)]$.

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lexicographically least witnessing θ_0-reflection for s. Because $\mathcal{R}_u \vDash \mathsf{Cond}$ we have that $(\alpha_s, \beta_s) \in \mu_u^2$. Notice that (α_s, β_s) is definable from (z, w_0, w_1, w_2, w_3) over $\mathcal{R}_u$ in the sense that there is a formula θ' such that (α_s, β_s) is the unique pair (α, β) such that

$$\mathcal{R}_u \vDash \theta'[(\alpha, \beta), z, w_0, w_1, w_2, w_3].$$

The formula θ:

Let ψ' be a Π^1_{2n+1} such that $\psi(\ldots) \leftrightarrow \exists v \psi'(\ldots, v)$ (ψ is the formula used to define w, see just before Section 4.2). Let $\theta(k, u, w_0, w_1, w_2, w_3)$ be the $\Pi^1_{2n+2}(z)$-formula that is the conjunction of the following clauses:

1. $k \in \omega$,
2. $\theta_0(u, w_0, w_1, w_2, w_3)$,
3. letting $s = (u, w_0, w_1, w_2, w_3)$, in $\mathcal{R}_u$,[58] there are ordinals $\alpha' < \beta' < \alpha_s$ such that
   (a) $\mathcal{R}_u|\beta' \vDash$ "ZFC + 'there are $2n+1$ many Woodin cardinals' + 'α' is the least Woodin cardinal',
   (b) α' is an fb-cut in $\mathcal{R}_u|\alpha_s$ (and hence, in $\mathcal{R}_u$), and
   setting $\mathcal{K} = \mathsf{StrLe}(\mathcal{R}_u|\beta')$, $\mathcal{E} = \mathcal{E}_{sm}^{\mathcal{K}|\alpha'}$ and for $i \in [1, 2n]$, letting ξ_i be the $i+1$st Woodin cardinal of $\mathcal{K}$ and $d = (\xi_1, \xi_2, \ldots, \xi_{2n})$ the following holds:
   (c) If $p \in \mathsf{Ea}^{\mathcal{K}}_{\alpha', \mathcal{E}}$ then $\mathcal{K} \vDash p \Vdash \exists v(\psi'_{\mathcal{K}[\mathsf{ea}], d}[k, z, \mathsf{ea}, v])$.

4.3. *The real w'*

Set

$$k \in w' \leftrightarrow \exists u, w_0, w_1, w_2, w_3(\theta[k, u, z, w_0, w_1, w_2, w_3]).$$

Clearly w' is $\Sigma^1_{2n+3}(z)$. The proof of the next lemma is similar to the proof of Claim 2 that appears in the proof of [7, Theorem 3.3].

Lemma 4.2. $w' \subseteq w$.

Proof. Suppose $k \in w'$. Let Σ be a winning strategy for II in $\mathcal{G}(\mathcal{R}_u, 0, 2n+1)$. We want to argue that $k \in w$. Fix $s = (u, w_0, w_1, w_2, w_3)$ such that $\theta[k, u, z, w_0, w_1, w_2, w_3]$ holds. We want to show that for each z', $\psi[k, z, z']$ holds. Fix then such a z'.

We have that $\mathcal{R}_u$ is a properly $2n+1$-small Π^1_{2n+2}-iterable premouse such that $(z, w_0, w_1, w_2, w_3) \in \mathcal{R}_u \cap \mathbb{R}^5$. We now want to use Π^1_{2n+2}-iterability of $\mathcal{R}_u$ to make z' generic over the image of $\mathsf{Ea}^{\mathcal{R}_u}_{\nu_u, \mathcal{E}^{\mathcal{R}_u}_{le}}$[59]. We iterate $\mathcal{R}_u$ producing iteration $\mathcal{T}$ of $\mathcal{R}_u$ by using the rules of $(z', \nu_u, 0)$-genericity iteration to pick extenders (see clause 2 of Review 2.3) and by picking branches so that the iteration stays correct. More precisely, if $\gamma < \mathrm{lh}(\mathcal{T})$ and $c = [0, \gamma)_{\mathcal{T}}$ then $\mathcal{Q}(c, \mathcal{T} \restriction \gamma)$ exists and $\mathcal{Q}(c, \mathcal{T} \restriction \gamma) \trianglelefteq \mathcal{M}_{2n}(\mathsf{cop}(\mathcal{T} \restriction \gamma))$. Thus the resulting iteration $\mathcal{T}$ of $\mathcal{R}_u$ is correct, has no drops and is below ν_u[60]. There are two possibilities: either

(Pos1) $\mathcal{T}$ has a last model $\mathcal{R}'$ and z' is generic over $\mathcal{R}'$ for $\mathsf{Ea}^{\mathcal{R}'}_{\mu, \mathcal{E}^{\mathcal{R}'}_{le}}$ where $\mu = \pi^{\mathcal{T}}(\nu_u)$ or

(Pos2) $\mathcal{T}$ does not have a last model and $\mathrm{lh}(\mathcal{T})$ is a limit ordinal.

Sublemma 4.3. There is $\xi < \mathrm{lh}(\mathcal{T})$ and $\alpha' < \beta'$ such that the following conditions hold:

1. The generators of $\mathcal{T} \restriction \xi$ are contained in α'.
2. $\mathcal{M}^{\mathcal{T}}_{\xi}|\beta' \vDash \mathsf{ZFC}$ + 'there are $2n+1$ many Woodin cardinals' + 'α' is the least Woodin cardinal'.
3. Setting $\mathcal{K} = \mathsf{StrLe}(\mathcal{M}^{\mathcal{T}}_{\xi}|\beta')$, $\mathcal{E} = \mathcal{E}_{sm}^{\mathcal{K}|\alpha'}$ and for $i \in [1, 2n]$, letting ξ_i be the $i+1$st Woodin cardinal of $\mathcal{K}$ and $d = (\xi_1, \xi_2, \ldots, \xi_{2n})$ the following holds:

[58] Here we need to add 'if (α_s, β_s) is such that $\mathcal{R}_u \vDash \theta'[(\alpha, \beta), z, w_0, w_1, w_2, w_3]$ then'.
[59] Recall our notation: ν_u is the least Woodin cardinal of $\mathcal{R}_u$.
[60] Recall our setup, ν_u is the least Woodin cardinal of $\mathcal{R}_u$.

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(a) If $p \in \mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}}$ then $\mathcal{K} \vDash p \Vdash \exists v(\psi'_{\mathcal{R}_u[\mathsf{ea}],d}[k,z,\mathsf{ea},v])$.
(b) z' satisfies all the axioms of $\mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}}$.
(c) If player I plays $(\mathcal{T} \restriction \xi+1, y)$ where $y \in \mathbb{R}$ is any real coding $\mathcal{T} \restriction \xi+1$ then $\Sigma(\mathcal{T} \restriction \xi+1, y) = \mathsf{accept}$ and the iterations of $\mathcal{K}[z']$ that are above α' are legal moves for player I in the second round of $\mathcal{G}(\mathcal{R}_u, 0, 2n+1)$.

Proof. **Suppose first that (Pos1) holds.** Set $(\alpha,\beta) = \pi^{\mathcal{T}}(\alpha_s,\beta_s)$. We have that $\alpha < \beta < \mu$ and α is an fb-cut in $\mathcal{R}'$. Because $k \in w'$ we have $\alpha' < \beta' < \alpha$ such that

(A1) $\mathcal{R}'|\beta' \vDash \mathsf{ZFC}$ + 'there are $2n+1$ many Woodin cardinals' +'α' is the least Woodin cardinal',
(B1) α' is an fb-cut in $\mathcal{R}'$, and

setting $\mathcal{K} = \mathsf{StrLe}(\mathcal{R}'|\beta')$, $\mathcal{E} = \mathcal{E}^{\mathcal{K}|\alpha'}_{sm}$ and for $i \in [1,2n]$, letting ξ_i be the $i+1$st Woodin cardinal of $\mathcal{K}$ and $d = (\xi_1,\xi_2,...,\xi_{2n})$ the following holds:

(C1) If $p \in \mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}}$ then $\mathcal{K} \vDash p \Vdash \exists v(\psi'_{\mathcal{R}_u[\mathsf{ea}],d}[k,z,\mathsf{ea},v])$.

We also have that if $\mathcal{E}' = \mathcal{E}^{\mathcal{R}'|\alpha}_{le}$ then z' satisfies all the axioms of $\mathsf{Ea}^{\mathcal{R}'|\alpha}_{\alpha,\mathcal{E}'}$, and hence,

(D1) z' satisfies all the axioms of $\mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}}$.

To see (D1), we use Lemma 2.17 and the fact that both α and α' are fb-cuts of $\mathcal{R}'$.
Let $\xi < \mathrm{lh}(\mathcal{T})$ be the least such that $\mathcal{M}^{\mathcal{T}}_{\xi}|\alpha' = \mathcal{R}'|\alpha'$. It follows that

(E1) the generators of $\mathcal{T} \restriction \xi$ are contained in α' and
(F1) if y is a real coding $\mathcal{T} \restriction \xi+1$ then $\Sigma(\mathcal{T} \restriction \xi+1, y) = \mathsf{accept}$ (see Proposition 3.4).

(E1) and (F1) then imply the following.

(G1) If player I plays $(\mathcal{T} \restriction \xi+1, y)$ where $y \in \mathbb{R}$ is any real coding $\mathcal{T} \restriction \xi+1$, $\Sigma(\mathcal{T} \restriction \xi+1, y) = \mathsf{accept}$ and iterations of $\mathcal{K}[z']$ that are above α' are legal moves for player I in the second round of $\mathcal{G}(\mathcal{R}_u, 0, 2n+1)$.

Because $\mathcal{M}^{\mathcal{T}}_{\xi}|\beta' = \mathcal{R}'|\beta'$ we have that (A1)-(G1) imply clauses (1)-(3) of Sublemma 4.3. Indeed, (1) is a consequence of (E1), (2) is a consequence of (A1), (3a) is a consequence of (C1), (3b) is a consequence of (D1) and (3c) is a consequence of (G1).

Suppose next that (Pos2) holds. Set $\mathcal{N} = \mathcal{M}_{2n}(\mathsf{cop}(\mathcal{T}))$ and $\kappa = \delta(\mathcal{T})$.

Claim. Let $y \in \mathbb{R}$ code the pair $(\mathcal{T}, \mathcal{N}|(\kappa^+)^{\mathcal{N}})$ and set $b = \Sigma(\mathcal{T}, y)$. Then

1. $\mathcal{N} \vDash$ 'κ is a Woodin cardinal' and $\mathcal{N}|(\kappa^+)^{\mathcal{N}} \trianglelefteq \mathcal{M}^{\mathcal{T}}_b$ and
2. κ is an fb-cut of $\mathcal{M}^{\mathcal{T}}_b$.

Proof. Clause 1 is a consequence of Proposition 3.4. To see clause 2 let $\mathcal{X}$ be the fully backgrounded construction of $\mathsf{cop}(\mathcal{T}) = \mathcal{M}^{\mathcal{T}}_b|\kappa$. Suppose there is $\mathcal{Y}$ which extends $\mathcal{X}$, is constructed by the fully backgrounded construction of $\mathcal{M}^{\mathcal{T}}_b|\pi^{\mathcal{T}}_b(\nu_u)$ and $\rho_\omega(\mathcal{Y}) < \kappa$. Let F be an extender used in b such that $\mathrm{crit}(F) > \rho_\omega(\mathcal{Y})$[61]. Let ξ be such that $E^{\mathcal{T}}_\xi = F$. Then $\mathrm{crit}(F)$ is an fb-cut of $\mathcal{M}^{\mathcal{T}}_{\xi+1}$ and $\mathrm{crit}(\pi^{\mathcal{T}}_{\xi+1,b}) > \mathrm{crit}(F)$ implying that in fact $\mathrm{crit}(F)$ is an fb-cut in $\mathcal{M}^{\mathcal{T}}_b$. Hence, $\rho_\omega(\mathcal{Y}) \geq \mathrm{crit}(F)$, contradiction. □

Using condensation (applied in $\mathcal{N}$) and Lemma 4.1, we can find $\beta \in (\kappa, (\kappa^+)^{\mathcal{N}})$ such that

1. $\mathcal{N}|\beta \vDash \mathsf{ZFC}$ + 'there are $2n+1$ many Woodin cardinals',
2. setting $\mathcal{K} = \mathsf{StrLe}(\mathcal{N}|\beta, z)$ and $\mathcal{E} = \mathcal{E}^{\mathcal{K}}_{sm}$, there is a condition $(p,q) \in \mathsf{Ea}^{\mathcal{K}}_{\kappa,\mathcal{E}} \times \mathsf{Ea}^{\mathcal{K}}_{\kappa,\mathcal{E}}$ such that
   (a) $w_0 \vDash p$,
   (b) $(w_1,w_2,w_3) \vDash q$, and
   (c) $(p,q) \Vdash \exists b(U'_{\mathcal{K}[(\mathsf{ea}^l,\mathsf{ea}^r)],d}(\mathsf{ea}^r_2, \mathsf{ea}^l))$ where letting $(\kappa,\xi_1,...,\xi_{2n})$ be the Woodin cardinals of $\mathcal{K}$ enumerated in increasing order, $d = (\xi_1,\xi_2,...,\xi_{2n})$.

[61] It might help to review Corollary 2.16

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Let now $y_0 \in \mathbb{R}$ code the pair $(\mathcal{T}, \mathcal{N}|(\kappa^+)^{\mathcal{N}})$ and set $b = \Sigma(\mathcal{T}, y_0)$. We then have that $\mathcal{N}|\beta \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$ (see clause 1 of the Claim). Notice that because κ is an fb-cut of $\mathcal{M}_b^{\mathcal{T}}$ (see clause 2 of the Claim) and because $\mathcal{N}|\beta \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$, the above clauses imply that $\pi_b^{\mathcal{T}}(\alpha_s) \leq \kappa$. Because $k \in w'$, we must have $\alpha' < \beta' < \pi_b^{\mathcal{T}}(\alpha_s) \leq \kappa$ such that

(A2) $\mathcal{M}_b^{\mathcal{T}}|\beta' \vDash$ "ZFC + 'there are $2n+1$ many Woodin cardinals' +'α' is the least Woodin cardinal',
(B2) α' is an fb-cut in $\mathcal{M}_b^{\mathcal{T}}$, and

setting $\mathcal{K} = \mathsf{StrLe}(\mathcal{M}_b^{\mathcal{T}}|\beta')$, $\mathcal{E} = \mathcal{E}_{sm}^{\mathcal{K}|\alpha'}$ and for $i \in [1, 2n]$, letting ξ_i be the $i+1$st Woodin cardinal of $\mathcal{K}$ and $d = (\xi_1, \xi_2, ..., \xi_{2n})$ the following holds:

(C2) If $p \in \mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}}$ then $\mathcal{K} \vDash p \Vdash \exists v(\psi'_{\mathcal{R}_u[\mathsf{ea}],d}[k, z, \mathsf{ea}, v])$.

Let then $\mathcal{K} = \mathsf{StrLe}(\mathcal{M}_b^{\mathcal{T}}|\beta')$. Because

(a) z' satisfies all the axioms of $\mathsf{Ea}^{\mathsf{cop}(\mathcal{T})}_{\delta(\mathcal{T}),\mathcal{E}'}$ where $\mathcal{E}' = \mathcal{E}_{le}^{\mathsf{cop}(\mathcal{T})}$, and
(b) α' is an fb-cut of $\mathcal{M}_b^{\mathcal{T}}$

(D2) z' satisfies all the axioms of $\mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}}$.

Let $\xi < \mathrm{lh}(\mathcal{T})$ be the least such that $\mathcal{M}_\xi^{\mathcal{T}}|\alpha' = \mathcal{M}_b^{\mathcal{T}}|\alpha'$ (notice that because $\alpha' < \delta(\mathcal{T})$, we really do have such a $\xi < \mathrm{lh}(\mathcal{T})$). It follows that

(E2) the generators of $\mathcal{T} \restriction \xi$ are contained in α'.

(F2) if y is a real coding $\mathcal{T} \restriction \xi + 1$ then $\Sigma(\mathcal{T} \restriction \xi + 1, y) = \mathsf{accept}$ (see Proposition 3.4).

(E2) and (F2) then imply the following.

(G2) If player I plays $(\mathcal{T} \restriction \xi{+}1, y)$ where $y \in \mathbb{R}$ is any real coding $\mathcal{T} \restriction \xi{+}1$, $\Sigma(\mathcal{T} \restriction \xi{+}1, y) = \mathsf{accept}$ and iterations of $\mathcal{K}[z']$ that are above α' are legal moves for player I in the second round of $\mathcal{G}(\mathcal{R}_u, 0, 2n{+}1)$.

As in the case of (Pos1), (A2)-(G2) imply that (1)-(3) of Sublemma 4.3 hold. □

Let then ξ, $d = (\xi_1, \xi_2, ..., \xi_{2n})$ and $\mathcal{K}$ be as in the conclusion of Sublemma 4.3. We now want to conclude that $\psi[k, z, z']$ holds. Let $v \in \mathcal{K}[z']$ be such that $\mathcal{K}[z'] \vDash \psi'_{\mathcal{K}[z'],d}[k, z, z', v]$. It is then enough to show that $\psi'[k, z, z', v]$ holds, and this will be established by Sublemma 4.4.

Sublemma 4.4. $\psi'[k, z, z', v]$ holds.

Proof. $\psi'[k, z, z', v]$ is a Π^1_{2n+1}-formula, so we can find a Σ^1_{2n} formula ψ'' such that $\psi'[k, z, z', v] \leftrightarrow \forall t \psi''[k, z, z', v, t]$. Fix $t \in \mathbb{R}$. We now want to argue that $\psi''[k, z, z', v, t]$ holds. Let I's first move in $\mathcal{G}(\mathcal{R}, 0, 2n+1)$ be $(\mathcal{T} \restriction \xi + 1, t')$ where t' is a real coding t and $\mathcal{T} \restriction \xi + 1$. As discussed above, II must accept $\mathcal{T} \restriction \xi + 1$.

Notice now that $\mathcal{K}$ as a $\mathcal{K}|\xi_1$-mouse is $2n-1$-small. Working inside $\mathcal{M}_{2n-1}(t')$, let $\mathcal{U}$ be an iteration of $\mathcal{K}$ such that

1. $\mathcal{U}$ is a (t, ξ_2, ξ_1)-genericity iteration,
2. for each limit ordinal $\lambda < \mathrm{lh}(\mathcal{U}) - 1$, if $c = [0, \lambda)_{\mathcal{U}}$ then $\mathcal{Q}(c, \mathcal{U})$ exists and is $\mathcal{Q}(c, \mathcal{U} \restriction \lambda) \trianglelefteq \mathcal{M}_{2n-2}(\mathsf{cop}(\mathcal{U} \restriction \lambda))$.

As always, we have two possibilities: either
(1) $\mathcal{U}$ has a last model $\mathcal{K}_1$ such that t is generic over $\mathcal{K}_1$ for $\mathsf{Ea}^{\mathcal{K}_1}_{\pi^{\mathcal{U}}(\xi_2),\xi_1}$ or
(2) $\mathcal{U}$ is of limit length and there is no branch c of $\mathcal{U}$ such that $\mathcal{Q}(c, \mathcal{U}) \trianglelefteq \mathcal{M}_{2n-2}(\mathsf{cop}(\mathcal{U}))$[62].

We now let I play $\mathcal{U}$ in the second round of $\mathcal{G}(\mathcal{R}, 0, 2n+1)$. If II accepts it then let $\mathcal{K}_1$ be the last model of $\mathcal{U}$. If II plays a maximal well-founded branch c then let $\mathcal{K}_1 = \mathcal{M}_c^{\mathcal{U}\restriction \sup(c)}$[63].

[62]Here we should say that 'there is no branch $c \in \mathcal{M}_{2n-1}(t')$ such that...' but this is not necessary as if there was such a branch then it had to be in $\mathcal{M}_{2n-1}(t')$.
[63]While we do not need this, it can be argued, using the minimality of $\mathcal{K}$, that II cannot play a branch.

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Suppose for a moment that II plays a branch c. Set $\iota = \sup(c)$. We want to argue that $\mathcal{Q}(c, \mathcal{U} \restriction \iota)$ doesn't exists. If it does then it is $2n-2$-small $\delta(\mathcal{U} \restriction \iota)$-mouse which is Π^1_{2n}-iterable above $\delta(\mathcal{U} \restriction \iota)$. It follows from clause 11 of Review 3 that $\mathcal{Q}(c, \mathcal{U} \restriction \iota)$ is iterable and hence, $\mathcal{Q}(c, \mathcal{U} \restriction \iota) \trianglelefteq \mathcal{M}_{2n-2}(\mathsf{cop}(\mathcal{U} \restriction \iota))$, which is a contradiction.

Let now $j : \mathcal{K} \to \mathcal{K}_1$ be the iteration map given by $\pi^{\mathcal{U}}$ or $\pi^{\mathcal{U}\restriction\iota}_c$ depending on which case was used to define $\mathcal{K}_1$. In both of these cases we have that t is generic over $\mathcal{K}_1$ for $\mathsf{Ea}^{\mathcal{K}_1}_{j(\xi_2),\xi_1}$. Let $d' = (j(\xi_2), j(\xi_3), ..., j(\xi_{2n}))$. We have that $\mathcal{K}_1[z', t] \vDash \psi''_{\mathcal{K}_1[z',t],d'}[k, z, z', v, t]$. Moreover, $\mathcal{K}_1$ above $j(\xi_2)$ is $2n-2$-small and Π^1_{2n}-iterable. It follows from clause 11 of Review 3 that $\mathcal{K}_1$ is iterable above $j(\xi_2)$ and hence, applying Proposition 2.5 to $\mathcal{K}_1[z', t]$, we get that $\psi''[k, z, z', v, t]$ holds. □

□

4.4. Hjorth's reflection

Recall that w is $\Pi^1_{2n+3}(z)$ but not $\Sigma^1_{2n+3}(z)$, and since w' is $\Sigma^1_{2n+3}(z)$, we have that $w - w'$ is not empty. Fix now $k \in w - w'$ and let $\alpha < \beta < \kappa_{\mathcal{M}}$ be the least such that

1. $\mathcal{M}|\beta \vDash$ ZFC+'there are $2n+1$ Woodin cardinals',
2. $\mathcal{M}|\beta \vDash$ 'α is a Woodin cardinal',
3. α is both an inaccessible fb-cut of $\mathcal{M}$ and a cutpoint of $\mathcal{M}$,
4. letting $(\alpha, \tau_1, ..., \tau_{2n})$ be the Woodin cardinals of $\mathcal{M}|\beta$ enumerated in the increasing order, $d = (\tau_1, ..., \tau_{2n})$ and $\mathcal{K} = \mathsf{StrLe}(\mathcal{M}|\beta)$, whenever $q \in \mathsf{Ea}^{\mathcal{K}}_{\alpha,sm}$, $\mathcal{K} \vDash q \Vdash \exists v \psi'_{\mathcal{M}|\beta,d}[k, z, \mathsf{ea}, v]$.

Notice that to get $\beta < \kappa_{\mathcal{M}}$ we are using that $\mathcal{M} \vDash$ Cond. Also, notice that if $\mathcal{N}$ is a complete iterate of $\mathcal{M}$ then setting $\pi_{\mathcal{M},\mathcal{N}}(\alpha, \beta) = (\iota_{\mathcal{N}}, \zeta_{\mathcal{N}})$, we have that $(\iota_{\mathcal{N}}, \zeta_{\mathcal{N}})$ has the same definition over $\mathcal{N}$ as (α, β) over $\mathcal{M}$. Set $\mathcal{Q} = \mathcal{M}|(\zeta^+_{\mathcal{M}})^{\mathcal{M}}$[64]. Given a complete iterate $\mathcal{N}$ of $\mathcal{M}$ we let $\mathcal{Q}_{\mathcal{N}} = \pi_{\mathcal{M},\mathcal{N}}(\mathcal{Q})$. Recall that we set $B_\gamma = \{y : U_y = A_\gamma\}$. To implement Hjorth's reflection we will need the following general lemma which is based on [37].

Lemma 4.5. *Suppose $\mathcal{R}$ is a complete iterate of $\mathcal{M}_{2n+1}$ such that $\mathrm{lh}(\mathcal{T}_{\mathcal{M}_{2n+1},\mathcal{R}}) < \omega_1$. Let $\alpha \leq \kappa_{\mathcal{R}}$ be a cutpoint of $\mathcal{R}$ and set $\mathcal{S} = \mathcal{R}|(\alpha^+)^{\mathcal{R}}$. Let $x \in \mathbb{R}$ code $\mathcal{S}$. Then the following statements hold:*

1. *The statement that 'y codes a complete iterate of S' is $\Sigma^1_{2n+2}(x)$.*
2. *The statement that 'y codes a pair $(\mathcal{N}, \xi)$, y' codes a pair $(\mathcal{N}', \xi')$, $\mathcal{N}$ and $\mathcal{N}'$ are complete iterates of $\mathcal{S}$ and if $\mathcal{N}''$ is the common complete iterate of $\mathcal{N}$ and $\mathcal{N}'$ then $\pi_{\mathcal{N},\mathcal{N}''}(\xi) \leq \pi_{\mathcal{N}',\mathcal{N}''}(\xi')$' is $\Sigma^1_{2n+2}(x)$.*

Proof. To say that 'y codes a complete iterate of $\mathcal{S}$' it is enough to say the following:

1. y codes a premouse $\mathcal{N}$,
2. there is a real u such that u codes an iteration $\mathcal{T}$ of $\mathcal{S}$ such that
   (a) $\mathcal{T}$ has a last model $\mathcal{N}$,
   (b) $\pi^{\mathcal{T}}$ is defined,
   (c) for every limit $\alpha < \mathrm{lh}(\mathcal{T})$, letting $b = [0, \alpha)_{\mathcal{T}}$, $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ exists and $\mathcal{Q}(b, \mathcal{T} \restriction \alpha) \trianglelefteq \mathcal{M}_{2n}(\mathsf{cop}(\mathcal{T} \restriction \alpha))$.

The complexity of the statement comes from clause 2.3, and [[37], Corollary 4.9] implies that it is $\Sigma^1_{2n+2}(u)$. The reason is that for any $v \in \mathbb{R}$, $\mathbb{R} \cap \mathcal{M}_{2n+2}(v)$ is the largest countable $\Sigma^1_{2n+2}(v)$ set[65]. It is then not hard to see that 'y codes a complete iterate of $\mathcal{S}$' is indeed $\Sigma^1_{2n+2}(x)$. A very similar calculation shows that the statement in clause 2 is also $\Sigma^1_{2n+2}(x)$. □

Recall that Lemma 2.20 says that for every $\gamma < \delta^1_{2n+2}$ there is a γ-stable complete iterate $\mathcal{N}$ of $\mathcal{M}$. Before we state and prove Hjorth's Reflection argument, we will need the following lemma.

[64] Notice that $(\iota^+_{\mathcal{M}})^{\mathcal{M}} = (\zeta^+_{\mathcal{M}})^{\mathcal{M}}$.

[65] Another way of seeing this is just that if for every $x \in \mathbb{R}$, $\mathcal{M}_{2n}(x)^{\#}$ exists then if $\mathcal{W}$ is a $2n$-small Π^1_{2n+1}-iterable δ-mouse (in the sense of [37,]) then $\mathcal{W}$ is $\omega_1 + 1$-iterable. This can be shown by appealing to [[37], Lemma 3.3].

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Lemma 4.6. *There is a complete γ-stable iterate $\mathcal{N}$ of $\mathcal{M}$ such that if $\mathcal{P} = \mathsf{StrLe}(\mathcal{N})$ then $\mathcal{P}$ is a γ-stable complete iterate of $\mathcal{M}$.*

Proof. Let $\mathcal{N}'$ be any γ-stable complete iterate of $\mathcal{M}$ and set $\mathcal{R} = \mathcal{M}_{2n+1}(\mathcal{M}, \mathcal{N}')$. Let $\mathcal{N} = \mathsf{StreLe}(\mathcal{R})$ and let $\mathcal{P} = \mathsf{StreLe}(\mathcal{N})$. Then both $\mathcal{N}$ and $\mathcal{P}$ are complete iterates of both $\mathcal{M}$ and $\mathcal{N}'$. Therefore, both $\mathcal{N}$ and $\mathcal{P}$ are γ-stable. □

Lemma 4.7 (Hjorth's Reflection). *Suppose $\gamma < \delta^1_{2n+2}$ and $\mathcal{N}$ is a γ-stable complete iterate of $\mathcal{M}$ such that $\mathsf{StrLe}(\mathcal{N})$ is also a complete γ-stable iterate of $\mathcal{M}$. Then whenever $g \subseteq Coll(\omega, \mathcal{Q}_{\mathcal{N}})$ is $\mathcal{N}$-generic, $B_\gamma \cap \mathcal{N}[g] \neq \emptyset$.*

Proof. Set $p = p^{\mathsf{StrLe}(\mathcal{N})}_{\gamma,0,3,1}$ (see clause 9 of Review 2.19, here we define $p^{\mathsf{StrLe}(\mathcal{N})}_{\gamma,0,3,1}$ relative to $\mathcal{E}^{\mathsf{StrLe}(\mathcal{N})}_{sm}$). Let $\mathcal{A} \in \mathcal{N}$ consist of quadruples (α', β', q, r) such that

1. $\alpha' \leq \iota_{\mathcal{N}}$, $\mathcal{N}|\beta' \vDash \mathsf{ZFC} +$ 'α' is the least Woodin cardinal'+'there are $2n+1$ many Woodin cardinals',
2. α' is an fb-cut of $\mathcal{N}$, and
   letting $\mathcal{K} = \mathsf{StrLe}(\mathcal{N}|\beta')$ and $\mathcal{E} = \mathcal{E}^{\mathcal{K}}_{sm}$, there is $(q, r) \in \mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}} \times \mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}}$,
3. r is compatible with p,
4. $(q, r) \Vdash \exists b(U'_{\mathcal{K}[\mathsf{ea}^l,\mathsf{ea}^r],d}(\mathsf{ea}^r_2, \mathsf{ea}^l, b))$ where letting $(\alpha', \xi_1, ...\xi_{2n})$ be the Woodin cardinals of $\mathcal{K}$ enumerated in increasing order, $d = (\xi_1, \xi_2, ..., \xi_{2n})$.

Sublemma 4.8. Suppose $x \in \mathbb{R}$. Then the following are equivalent.

1. $x \in A_\gamma$.
2. There is a complete iterate $\mathcal{S}$ of $\mathcal{N}$ such that $\mathcal{T}_{\mathcal{N},\mathcal{S}}$ is below $\iota_{\mathcal{N}}$ and for some $(\alpha', \beta', q, r) \in \pi_{\mathcal{N},\mathcal{S}}(\mathcal{A})$, $x \vDash q$ and letting $\mathcal{K} = \mathsf{StrLe}(\mathcal{S}|\beta')$ and $\mathcal{E} = \mathcal{E}^{\mathcal{K}}_{sm}$, x is generic over $\mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}}$.

Proof. (**Clause 1 implies Clause 2:**) Suppose $x \in A_\gamma$. Let $x_1 \in \mathsf{Code}$ be such that $\gamma_{x_1} = \gamma$, $x_2 \in B_\gamma$ and x_3 be such that $D(x_1, x_2, x_3)$. Let $\mathcal{P} = \mathcal{M}_{2n+1}(z, x, (x_1, x_2, x_3))$. Notice that if u is a real coding $\mathcal{P}$ then we in fact have that $\theta_0(u, x, x_1, x_2, x_3)$ holds.

We now compare $\mathcal{P}$ and $\mathcal{N}$ to obtain $\mathcal{P}'$ and $\mathcal{N}'$ such that

(a) the least Woodin cardinals of $\mathcal{P}'$ and $\mathcal{N}'$ coincide,
(b) if κ is the least Woodin cardinal of $\mathcal{P}'$ and $\mathcal{N}'$ then the fully backgrounded construction of $\mathcal{P}|\kappa$ coincides with the fully backgrounded construction of $\mathcal{N}'|\kappa$[66]

Let $i : \mathcal{P} \to \mathcal{P}'$ be the iteration embedding.

Claim. $i(\alpha_s) \leq \iota_{\mathcal{N}'}$.

Proof. Suppose that $i(\alpha_s) > \iota_{\mathcal{N}'}$. Because the fully backgrounded constructions of $\mathcal{P}'$ and $\mathcal{N}'$ are the same and because $\iota_{\mathcal{N}'}$ is an fb-cut of $\mathcal{N}'$, we have that $\iota_{\mathcal{N}'}$ is an fb-cut of $\mathcal{P}'$. It now follows that $\theta(k, u, x, x_1, x_2, x_3)$ holds, and therefore, $k \in w'$. □

Now, set $\mathcal{K} = \mathsf{StrLe}(\mathcal{N}'|i(\beta_s)) = \mathsf{StrLe}(\mathcal{P}'|i(\beta_s))$ and $\mathcal{E} = \mathcal{E}^{\mathcal{K}|i(\alpha_s)}_{sm}$. Let $(q, r) \in \mathsf{Ea}^{\mathcal{K}}_{i(\alpha_s),\mathcal{E}} \times \mathsf{Ea}^{\mathcal{K}}_{i(\alpha_s),\mathcal{E}}$ be such that

1. $x \vDash q$,
2. $(x_1, x_2, x_3) \vDash r$,
3. $(q, r) \Vdash \exists b(U'_{\mathcal{K}[\mathsf{ea}^l,\mathsf{ea}^r],d}(\mathsf{ea}^r_2, \mathsf{ea}^l, b))$ where letting $(\alpha_s, \xi_1, ..., \xi_{2n})$ be the Woodin cardinals of $\mathcal{K}$ enumerated in increasing order, $d = (\xi_1, \xi_2, ..., \xi_{2n})$.

[66] The reason that the $\mathcal{P}'$ and $\mathcal{N}'$ have the same Woodin cardinal is that if $\mathcal{Q}$ is some $\mathcal{M}_{2n+1}$-like premouse over some set a then the least Woodin cardinal of $\mathcal{Q}$ is the least Woodin cardinal of the fully backgrounded construction. If now, for example, the least Woodin cardinal η of $\mathcal{P}'$ is strictly smaller than the least Woodin cardinal ν of $\mathcal{N}'$ then because $\mathcal{M}_{2n}(\mathsf{Le}(\mathcal{P}'|\eta) \vDash$ 'η is a Woodin cardinal' we have that η must be a Woodin cardinal of $\mathsf{Le}^{\mathcal{N}'|\nu}$, contradiction.

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Assuming that $\pi_{\mathcal{N},\mathcal{N}'}(p)$ is compatible with r, we get that $(i(\alpha_s), i(\beta_s), q, r) \in \pi_{\mathcal{N},\mathcal{N}'}(\mathcal{A})$. To finish we need to see that in fact $\pi_{\mathcal{N},\mathcal{N}'}(p)$ is compatible with r. Notice now that because $\gamma_{x_1} = \gamma$, $(x_1, x_2, x_3) \vDash \pi_{\mathcal{N},\mathcal{N}'}(p)$[67]. Because $(x_1, x_2, x_3) \vDash r$, we must have that r is compatible with $\pi_{\mathcal{N},\mathcal{N}'}(p)$. Thus, $(i(\alpha_s), i(\beta_s), q, r) \in \pi_{\mathcal{N},\mathcal{N}'}(\mathcal{A})$.

Set now $\mathcal{T} = \mathcal{T}_{\mathcal{N},\mathcal{N}'}$ and $\tau = \mathrm{lh}(\mathcal{T}) - 1$. Because $\iota_{\mathcal{N}}$ is a cutpoint of $\mathcal{N}$, we can find $\xi < \mathrm{lh}(\mathcal{T})$ such that

1. $\mathcal{M}^{\mathcal{T}}_{\xi}|\iota_{\mathcal{N}'} = \mathcal{N}'|\iota_{\mathcal{N}'}$,
2. the generators of $\mathcal{T} \restriction \xi$ are contained in $\iota_{\mathcal{N}'}$,
3. $\xi \in [0, \tau]_{\mathcal{T}}$,
4. all the extenders used in $[0, \tau)_{\mathcal{T}}$ after stage ξ have critical points $> \iota_{\mathcal{N}}$.

Set $\mathcal{S} = \mathcal{M}^{\mathcal{T}}_{\xi}$. We then have that $\pi_{\mathcal{N},\mathcal{N}'}(\mathcal{A}) = \pi_{\mathcal{N},\mathcal{S}}(\mathcal{A})$, and the pair $(\mathcal{T} \restriction \xi + 1, (i(\alpha_s), i(\beta_s), q, r))$ witnesses clause 2 of Lemma 4.8.

(**Clause 2 implies Clause 1.**) Conversely, suppose $\mathcal{S}$ is a complete iterate of $\mathcal{N}$ such that $\mathcal{T}_{\mathcal{N},\mathcal{S}}$ is based on $\mathcal{Q}_{\mathcal{N}}$ and for some $(\alpha', \beta', q, r) \in \pi_{\mathcal{N},\mathcal{S}}(\mathcal{A})$, $x \vDash q$ and setting $\mathcal{K} = \mathsf{StrLe}(\mathcal{S}|\beta')$ and $\mathcal{E} = \mathcal{E}^{\mathcal{K}}_{sm}, x$ is generic over $\mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}}$. We now further iterate $\mathcal{S}$ above α' to get $\mathcal{S}'$ such that x is generic over $\mathsf{Ea}^{\mathcal{S}'}_{\delta_{\mathcal{S}'},\alpha'}$[68]. We then have that x is actually generic over $\mathcal{P} =_{def} \mathsf{StrLe}(\mathcal{S}')$ (see Clause 2 of Review 2.7). It follows that there is a $\mathcal{P}[x]$-generic (x_1, x_2, x_3) such that $(x_1, x_2, x_3) \vDash r \wedge \pi_{\mathcal{N},\mathcal{S}'}(p)$. Therefore, we must have that $\gamma_{x_1} = \gamma$ and $x_2 \in B_\gamma$. If we now show that $U(x_2, x)$ holds then we will get that $x \in A_\gamma$.

To show that $U(x_2, x)$ holds it is enough to show that (x_1, x_2, x_3) is generic over $\mathcal{K}[x]$ for $\mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}}$. Assuming this, we have that since

$$(q, r) \Vdash \exists b(U'_{\mathcal{K}[\mathsf{ea}^l, \mathsf{ea}^r], d}(\mathsf{ea}^r_2, \mathsf{ea}^l, b))$$

and $\mathcal{K}[x, (x_1, x_2, x_3)]$ is Σ^1_{2n+2} correct, in fact $\exists b U'(x_2, x, b)$ holds in V. Therefore, $U(x_2, x)$ holds.

The fact that (x_1, x_2, x_3) is generic over $\mathcal{K}[x]$ for $\mathsf{Ea}^{\mathcal{K}}_{\alpha',\mathcal{E}}$ follows from the Claim in the proof of [7, Theorem 2.2]. To apply that Claim, set g to be (x_1, x_2, x_3), ψ_0 to be $r \wedge \pi_{\mathcal{N},\mathcal{S}}(p)$ and y to be x. Here we use the fact that α' is an fb-cut of $\mathcal{S}'|\iota_{\mathcal{S}'}$ and $\iota_{\mathcal{S}'}$ is an fb-cut of $\mathcal{S}'$, implying that in fact α' is an fb-cut of $\mathcal{S}'$. Thus, $\mathcal{K}|\alpha' \triangleleft \mathcal{P}$ and σ' is a cardinal of $\mathcal{P}$. □

To finish the proof, notice that if $g \subseteq Coll(\omega, (\iota^+_{\mathcal{N}})^{\mathcal{N}})$ is generic over $\mathcal{N}$, $y \in \mathcal{N}[g]$ codes $\mathcal{Q}_{\mathcal{N}}$ and σ is the formula displayed in clause 2 of Sublemma 4.8 then in fact $A_\gamma \in \Sigma^1_{2n+2}(y)$ as witnessed by σ (here we use clause 1 of Lemma 4.5). Therefore, we can find y' that is Turing reducible to y and such that $u \in A_\gamma \leftrightarrow U(y', u)$. Hence, we have that $y' \in B_\gamma \cap \mathcal{N}[g]$. □

4.5. Removing the use of Kechris-Martin

Recall that $\mathcal{Q} = \mathcal{M}|(\iota^+_{\mathcal{M}})^{\mathcal{M}}$ (where $\iota_{\mathcal{M}}, \zeta_{\mathcal{M}}$ were introduced in the beginning of Section 4.4) and if $\mathcal{S}$ is a complete iterate of $\mathcal{M}$ then let $\mathcal{Q}_{\mathcal{S}} = \pi_{\mathcal{M},\mathcal{S}}(\mathcal{Q})$. We have that $\mathcal{Q} \triangleleft \mathcal{M}|\kappa_{\mathcal{M}}$ and for each γ there is a complete iterate $\mathcal{S}$ of $\mathcal{M}$ such that $\mathcal{T}_{\mathcal{M},\mathcal{S}}$ is below $\iota_{\mathcal{M}}$ and if $g \subseteq Coll(\omega, \pi_{\mathcal{M},\mathcal{S}}(\mathcal{Q}))$ is generic over $\mathcal{S}$ then $\mathcal{S}[g] \cap B_\gamma \neq \emptyset$. Fix some recursive enumeration $(\phi_e : e \in \omega)$ of recursive functions.

Fix $\gamma < \delta^1_{2n+2}$ and let $\mathcal{S}$ be a complete γ-stable iterate of $\mathcal{M}$. Let $g \subseteq Coll(\omega, \mathcal{Q}_{\mathcal{S}})$ be $\mathcal{S}$-generic, $a_g = \{(i, j) \in \omega^2 : g(i) \in g(j)\}$ and e be such that if $v = \phi_e(a_g)$ then $U_v = A_\gamma$. Let $\dot{\mathbb{P}}$ be a $Coll(\omega, \mathcal{Q}_{\mathcal{S}})$-name for $\mathsf{Ea}^{\mathcal{S}[g]}_{\delta_{\mathcal{S}}}$ and let $\dot{s} \in \mathcal{S}$ be a $Coll(\omega, \mathcal{Q}_{\mathcal{S}})$-name for $p^{\mathcal{S}[g]}_{\gamma,0,3,1}$, the $(\gamma, 0, 3, 1)$-master condition in $\mathcal{S}[g]$ (see clause 9 of Review 2.19). It follows from clause 12 of Section 3 that

(*) if $(\dot{s}, \dot{q}) \in \dot{\mathbb{P}} \times \dot{\mathbb{P}}$ then $\mathcal{S}[g] \vDash (\dot{s}_g, \dot{q}_g) \Vdash$ '$D(\mathsf{ea}^r_1, \mathsf{ea}^r_2, \mathsf{ea}^r_3) \rightarrow (U_v = U_{\mathsf{ea}^r_2})$'.

[67] We have that (x_1, x_2, x_3) is generic over $\mathsf{StrLe}(\mathcal{P}')$ and since $\mathsf{StrLe}(\mathcal{P}') = \mathsf{StrLe}(\mathcal{N}')$, we have that $(x_1, x_2, x_3) \vDash \pi_{\mathcal{N},\mathcal{N}'}(p)$.

[68] This is the extender algebra that uses extenders with critical points $> \alpha'$.

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Let $\dot{a}$ be $Coll(\omega, \mathcal{Q}_{\mathcal{S}})$-name for a_g. It follows from (*) that there is a condition $p \in Coll(\omega, \mathcal{Q}_{\mathcal{S}})$ and $e \in \omega$ such that

(**) whenever $\dot{q} \in \dot{\mathbb{P}}$, $\mathcal{S} \vDash (p, \dot{s}, \dot{q}) \Vdash$ '$D(\mathsf{ea}_1^r, \mathsf{ea}_2^r, \mathsf{ea}_3^r) \rightarrow (U_{\phi_e(\dot{a})} = U_{\mathsf{ea}_2^r})$'.

Let $(p_\gamma^{\mathcal{S}}, e_\gamma^{\mathcal{S}})$ be the lexicographically least pair (p, e) satisfying (**).

Suppose now that $\mathcal{S}'$ is a complete iterate of $\mathcal{S}$. Then because $\mathcal{S}$ is γ-stable we have that $\pi_{\mathcal{S},\mathcal{S}'}(p_\gamma^{\mathcal{S}}, e_\gamma^{\mathcal{S}}) = (p_\gamma^{\mathcal{S}'}, e_\gamma^{\mathcal{S}'})$. Set then $(p_{\gamma,\infty}, e_{\gamma,\infty}) = \pi_{\mathcal{S},\infty}(p_\gamma^{\mathcal{S}}, e_\gamma^{\mathcal{S}})$.

We now claim that if $\gamma \neq \gamma'$ then $(p_{\gamma,\infty}, e_{\gamma,\infty}) \neq (p_{\gamma',\infty}, e_{\gamma',\infty})$. Suppose to the contrary that $(p_{\gamma,\infty}, e_{\gamma,\infty}) = (p_{\gamma',\infty}, e_{\gamma',\infty})$. Let $\mathcal{S}$ be a complete iterate of $\mathcal{M}$ which is both γ and γ' stable. It is enough to show that $(p_\gamma^{\mathcal{S}}, e_\gamma^{\mathcal{S}}) \neq (p_{\gamma'}^{\mathcal{S}}, e_{\gamma'}^{\mathcal{S}})$. Suppose then that $(p_\gamma^{\mathcal{S}}, e_\gamma^{\mathcal{S}}) = (p_{\gamma'}^{\mathcal{S}}, e_{\gamma'}^{\mathcal{S}}) =_{def} (p, e)$. Let $g \subseteq Coll(\omega, \mathcal{Q}_{\mathcal{S}})$ be $\mathcal{S}$-generic such that $p \in g$ and let $\nu = \phi_e(a_g)$. Let $s \in \mathsf{Ea}_{\delta_{\mathcal{S}}}^{\mathcal{S}[g]}$ be the $(\gamma, 0, 3, 1)$-master condition and let $s' \in \mathsf{Ea}_{\delta_{\mathcal{S}}}^{\mathcal{S}[g]}$ be the $(\gamma', 0, 3, 1)$-master condition.

Because $A_\gamma \neq A_{\gamma'}$, we can, without losing generality, assume that there is $x \in A_\gamma - A_{\gamma'}$. Fix such an x.

We now have that

(***) there is a condition $(r, r') \in \mathsf{Ea}_{\delta_{\mathcal{S}}}^{\mathcal{S}[g]} \times \mathsf{Ea}_{\delta_{\mathcal{S}}}^{\mathcal{S}[g]}$ such that (r, r') extends (s, s') and

$$\mathcal{S}[g] \vDash (r, r') \Vdash D(\mathsf{ea}_1^l, \mathsf{ea}_2^l, \mathsf{ea}_3^l) \wedge D(\mathsf{ea}_1^r, \mathsf{ea}_2^r, \mathsf{ea}_3^r).$$

(***) can be shown by iterating $\mathcal{S}[g]$ to make two triples (y_1, y_2, y_3) and (y_1', y_2', y_3') generic where the triples are chosen so that

1. $D(y_1, y_2, y_3)$ and $D(y_1', y_2', y_3')$,
2. $\gamma_{y_1} = \gamma$ and $\gamma_{y_1'} = \gamma'$.

Then elementarity would imply (***). Fix such a condition (r, r').

Suppose $\mathcal{R}$ is a complete iterate of $\mathcal{S}[g]$ such that x is generic for $\mathsf{Ea}_{\delta_{\mathcal{R}}}^{\mathcal{R}[g]}$ over $\mathcal{R}[g]$. Let (x_1, x_2, x_3) be any $\mathcal{R}[g, x]$-generic for $\mathsf{Ea}_{\delta_{\mathcal{R}}}^{\mathcal{R}[g]}$ with the property that $(x_1, x_2, x_3) \vDash \pi_{\mathcal{S}[g],\mathcal{R}[g]}(r)$ and let (x_1', x_2', x_3') be any $\mathcal{R}[g, x]$-generic for $\mathsf{Ea}_{\delta_{\mathcal{R}}}^{\mathcal{R}[g]}$ with the property that $(x_1', x_2', x_3') \vDash \pi_{\mathcal{S}[g],\mathcal{R}[g]}(r')$. It follows from (***) that

1. $D(x_1, x_2, x_3)$ and $D(x_1', x_2', x_3')$,
2. $\gamma_{x_1} = \gamma$ and $\gamma_{x_1'} = \gamma'$.

We now have that (**) implies that

(1) if $q \in \mathsf{Ea}_{\delta_{\mathcal{R}}}^{\mathcal{R}[g]}$ then $\mathcal{R}[g, (x_1, x_2, x_3)] \vDash q \Vdash$ '$U_\nu = U_{x_2}$',
(2) if $q \in \mathsf{Ea}_{\delta_{\mathcal{R}}}^{\mathcal{R}[g]}$ then $\mathcal{R}[g, (x_1', x_2', x_3')] \vDash q \Vdash$ '$U_\nu = U_{x_2'}$',

Therefore, we get that

(3) $\mathcal{R}[g, (x_1, x_2, x_3)][x] \vDash$ '$U_\nu = U_{x_2}$' and $\mathcal{R}[g, (x_1', x_2', x_3')][x] \vDash$ '$U_\nu = U_{x_2'}$'.

We now use (3) and clause 12 of Section 3 to make the following implications showing that $x \in A_{\gamma'}$ (recall that we chose $x \in A_\gamma - A_{\gamma'}$):

$$\begin{aligned} U_{x_2} = A_\gamma &\rightarrow x \in (U_{x_2})^{\mathcal{R}[g, x, (x_1, x_2, x_3)]} \\ &\rightarrow x \in (U_\nu)^{\mathcal{R}[g, x, (x_1, x_2, x_3)]} \\ &\rightarrow x \in U_\nu \end{aligned}$$

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$$\rightarrow x \in (U_v)^{\mathcal{R}[g,x,(x_1',x_2',x_3')]}$$
$$\rightarrow x \in (U_{x_2'})^{\mathcal{R}[g,x,(x_1',x_2',x_3')]}$$
$$\rightarrow x \in U_{x_2'}$$
$$\rightarrow x \in A_{\gamma'}$$

Since we now have that the function $f(\gamma) = (p_{\gamma,\infty}, e_{\gamma,\infty})$ is an injection, $|\pi_{\mathcal{M},\infty}(\mathcal{Q})| = \delta^1_{2n+2}$.

However, because $\mathsf{Ord} \cap \mathcal{Q}$ is a cutpoint of $\mathcal{M}$, $\pi_{\mathcal{M},\infty}(\mathcal{Q})$ is the direct limit of all countable iterates of $\mathcal{Q}$, which we denoted by $\mathcal{M}_\infty(\mathcal{Q})$ (see clause 3 of Review 2.2). It then follows from clause 2 of Lemma 4.5 that $|\mathcal{M}_\infty(\mathcal{Q})| < \delta^1_{2n+2}$. Indeed, consider the set $E = \{u \in \mathbb{R} : u \text{ codes a pair } (\mathcal{R}_u, \iota_u) \text{ such that } \mathcal{R}_u \text{ is a complete iterate of } \mathcal{Q} \text{ and } \iota_u \in \mathsf{Ord} \cap \mathcal{R}_u\}$ and for $u, u' \in E$, set $u \leq^*_0 u'$ if and only if letting $\mathcal{R}$ be the complete common iterate of $\mathcal{R}_u$ and $\mathcal{R}_{u'}$, $\pi_{\mathcal{R}_u,\mathcal{R}}(\iota_u) \leq \pi_{\mathcal{R}_{u'},\mathcal{R}}(\iota_{u'})$. It now follows from clause 2 of Lemma 4.5 that $\leq^*_0$ is $\Sigma^1_{2n+2}(u_0)$ where u_0 codes $\mathcal{Q}$. Hence, the ordinal length of $\leq^*_0$ is $< \delta^1_{2n+2}$[69], and therefore $|\mathcal{M}_\infty(\mathcal{Q})| < \delta^1_{2n+2}$.

5. Conclusion

Acknowledgements. I wish to thank John Steel for introducing me to [8] so many years ago. The work carried out in [29], which answers most of the questions raised in the addendum of [8], was done while I was Steel's PhD student. The addendum of [8] appeared in the unpublished version of Hjorth's paper by the same title available on his web site. I am grateful to Derek Levinson for a list of typos and corrections, and indebted to the referee for a long list of corrections.

This paper is about 10 years late. Sometime in 2010, while I was a postdoc at UCLA, Hjorth suggested that two of us work on improving the results of [29], and in particular compute $b_{2n+1,0}$ (which [29] conjectures to be δ^1_{2n+2}) and also prove Kechris' 2nd Conjecture. In [8], Hjorth showed that $b_{1,0} = \delta^1_2 = \omega_2$. The issues we encountered were the familiar ones: proving Kechris-Martin for Π^1_5 and beyond, and avoiding fine boundedness arguments involving Σ^1_1 relations and admissible ordinals. Unfortunately, on January 13 of 2011, Greg Hjorth unexpectedly passed away, and the project has remained unfinished. In the Spring of 2021 it became apparent that the use of Kechris-Martin in [8] is unnecessary.

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[69] Here we are using Kunen-Martin theorem, see [13, Theorem 2.6].

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