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TRANSIENT BEHAVIOR OF PROCESSES IN THE TCP PARADIGM

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This article derives the transient behavior of the stochastic process $X(t)$ for which in earlier articles it was proven that for $\alpha < 1$, the process $(X(t))^{1/(1-\alpha)}$ (approximately, under low drop probability or ECN marking probability) describes the behavior of the congestion windows in certain transport protocols in the so-called TCP paradigm. The transient distribution is found explicitly and is particularly transparent for moments like $E[(X(t))^k|X(0)]$. The purpose of this article is to be part of the mathematical foundation when comparing the many protocols in the TCP paradigm.

1. INTRODUCTION

In this article we study the process $(X(t))_{-\infty<t<+\infty}$ obtained as follows: There is a Poisson process with intensity λ of which the points are denoted by $(\tau_k)_{k=-\infty}^{+\infty}$, and there is a number $0 < c < 1$. In between the points τ_k, the process $X(\cdot)$ behaves as

$$\frac{d}{dt}X(t) = 1, \tag{1.1}$$

and in the points τ_k, we have

$$X(\tau_k^+) = cX(\tau_k^-). \tag{1.2}$$

If $\lambda = 1$, we denote this process by $X^*(t)$.

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Our interest in this process is due to its connection with existing and proposed internet transport protocols. In [34,35] and [4] we proposed a class of potential such transport protocols and made a first attempt to identify the more promising protocols in that class. The class contains classical TCP as well as Kelly's scalable TCP; see [23,24].

If the parameter β in that class of protocols is chosen equal to 1 and the drop probability p or ECN (explicit congestion notification) marking probability p is assumed to be constant, the protocols just referred to lead to the discrete-time stochastic process $(W_n)_{n=0}^{\infty}$ as follows.

Let $(U_n)_{n=0}^{\infty}$ be independent and identically distributed (i.i.d.) random variables, each distributed uniformly [0, 1]. Let p be a probability, $0 < p < 1$. Define the i.i.d. random variables $\chi_{p,n}$ by

$$\chi_{p,n} = \begin{cases} \text{success} & \text{if } U_n \geq p \\ \text{failure} & \text{if } U_n < p. \end{cases} \tag{1.3}$$

The process $W_{p,n}$ then is defined by

$$W_{p,n+1} = \begin{cases} W_{p,n} + c_1 (W_{p,n})^{\alpha} & \text{if } \chi_{p,n} = \text{success} \\ \max(bW_{p,n}, \ell) & \text{if } \chi_{p,n} = \text{failure}, \end{cases} \tag{1.4}$$

where $\alpha < 1$, $c_1 > 0$, $0 < b < 1$, and $\ell \geq 0$. In classical TCP, $\alpha = -1$, $c_1 = 1$, and $b = \frac{1}{2}$. In scalable TCP, $\alpha = 0$. In various versions of TCP, ℓ usually is 1 MSS (maximum segment size; i.e., maximum data size of a TCP packet) or a few MSSs. p, of course, stands for the probability that a packet will be lost, or possibly marked in the ECN mechanism. See [19,20,43] for a description of ECN.

In the limit for $p \downarrow 0$, the value of ℓ is not relevant, as long as it is constant. $W_{p,n}$ stands for the congestion window after the nth successful (or unmarked) acknowledgment.

The purpose of this article is to be part of the mathematical foundation for a comparison of the many protocols in the TCP paradigm. This includes finding the better combinations of $\alpha < 1$ and $0 < c < 1$ but also includes study of desirable values of the "slow start threshold" parameter. Such a study must also include the effects of a delay of 1 RTT (round-trip time) in the feedback. A start on the latter is made in [37].

References 39 and 40 prove stationarity of the process $W_{p,n}$. Curiously, if ($\ell = 0$ and $\alpha < 0$), uniqueness of the stationary distribution has not been proven (yet?).

Reference 39 proves that if ($\ell = 0$ and $\alpha < 0$), then for this stationary distribution $E[(W_p)^r] < \infty$ if

$$-\frac{|\log(p)|}{|\log(b)|} < r < \frac{|\log(p)|}{|\alpha \log(b)|},$$

else it is infinite; if ($\ell = 0$ and $0 \leq \alpha < 1$), $E[(W_p)^r] < \infty$ as long as

$$-\frac{|\log(p)|}{|\log(b)|} < r < \infty,$$

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else it is infinite. The same methods can be used to prove that if $\ell > 0$, then $E[(W_p)^r] < \infty$ for all values of r, regardless of the value of $\alpha < 1$.

References 34 and 35 conjecture that, for $p \downarrow 0$, the process $[p/c_1(1-\alpha)]$ $(W_{p,[t/p]})^{1-\alpha}$ converges weakly to the process $X^*(t)$ with $c = b^{1-\alpha}$ and that the stationary distributions correspondingly converge to the stationary distributions of the limiting process $X^*(\cdot)$. In fact, [38], although not formulating the conjectures, uses them without proof.

These conjectures were proven in [40]. In addition, [36] obtains rate of convergence results for the convergence of the stationary distributions to the stationary distribution of the limiting process $X^*(\cdot)$; see also below.

The stationary distribution of the process $X(\cdot)$ was established in [38]. For example, if Z has that stationary distribution, it has the form

$$Z = \sum_{k=0}^{\infty} c^k E_k, \tag{1.5}$$

where $(E_k)_{k=0}^{\infty}$ are i.i.d. random variables, all exponentially distributed with parameter λ, and for all (even complex) r,

$$M(r) = E[Z^r] = \lambda^{-r}\Gamma(r+1)\prod_{k=1}^{\infty}\frac{(1-c^{r+k})}{(1-c^k)}. \tag{1.6}$$

If r is a real integer, this expression simplifies; see [38]. If $\lambda = 1$, we denote Z by Z^* and $M(r)$ by $M^*(r)$.

In [36] it was investigated under what circumstances

$$\lim_{p\downarrow 0} E\left[\left(\frac{p}{c_1(1-\alpha)}(W_p)^{1-\alpha}\right)^r\right] = M^*(r). \tag{1.7}$$

for the stationary distributions.

It was proven that if ($\ell = 0$ and $\alpha \le 0$), then (1.7) holds for all $-\infty < r < +\infty$. If ($\ell = 0$ and $0 < \alpha < 1$), (1.7) was proven to hold for all $r \ge 0$. There is little doubt that it also holds for $r < 0$. The same methods can be used to prove that if $\ell > 0$, then (1.7) holds for all $-\infty < r < +\infty$, regardless of the value of α. Reference 36 also contains rate of convergence results for (1.7).

In this article we will obtain the transient behavior of the process $(X(\cdot))$, and thus for p small, the approximate transient behavior of the congestion window in transport protocols in the proposed class.

Mathematical analysis of TCP and TCP-like protocols has been an active area of research since 1996. Reference 28 by now is of mostly historical interest. In addition to the ones already mentioned, notable papers are [2–11,16–18,25–27,29,41,42]. Of these, [17] contains an excellent review of the literature up to about 2004. Reference 26 studies the transient behavior of a generalization of the process $X(t)$,

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where $Q_k = X(\tau_k^+)/X(\tau_k^-)$ is random instead of deterministic; see also the remark following (3.10).

Although mostly a mathematical work on the weak convergence issues for $p \downarrow 0$, [40] contains some more information on the recent literature. In addition to the articles just mentioned, pride of ownership compels us to mention [12–15,30–33].

It must be noted that the "time" used in this article is packet-time; that is, time is counted in data packets sent (or acknowledgements received). Several other articles studying protocols in the TCP paradigm use "clock-time," where time typically is counted in RTTs. If packet-time is denoted as t_P and clock-time is denoted as t_C, we have

$$dt_P = W\, dt_C, \tag{1.8}$$

where W is the congestion window (in MSSs) at time t_P (time t_C). The "packet-time stationary distribution" and the "clock-time stationary distribution" are, of course, not the same (in packet-time, we sample faster when the congestion window is larger). Reference 38 gives the relationship between the two stationary distributions: If W_P and W_C have respectively the "packet-time" stationary distribution and the "clock-time" stationary distribution, then

$$dP\{W_P \le w\} = \frac{w\, dP\{W_C \le w\}}{E[W_C]}, \qquad dP\{W_C \le w\} = \frac{dP\{W_P \le w\}}{wE[1/W_P]}. \tag{1.9}$$

Use of clock-time with a constant (independent of the current window size) "probability per RTT" (e.g. "hazard rate") of a drop event leads to an interesting modeling issue: Implicitly it assumes a probability of drop per packet, which is inversely proportional to the current window size W. This means that two or more TCP flows between the same source–destination pairs most of the time are assumed to have different drop probabilities per packet.

The model described thus far assumes that the congestion window is the only constraint on the amount of data "in flight" (data sent but no acknowledgment received yet by the sender), that the congestion window is expressed in MSSs, that (practically) all data packets contain 1 MSS worth of data, and that every individual data packet causes an acknowledgment to be sent (no delayed acknowledgments).

If desired, delayed acknowledgments can easily be incorporated into the model. Equation (1.8) indicates that we assume that the actual time (clock-time) between acknowledgment n and acknowledgment $n+1$ can (away from error recovery, etc.) be assumed to be equal to $1/W_{p,n}$ of an RTT. To send N packets, numbered $k+1, k+2, \ldots, k+N$, takes (assuming only a negligible amount of time is spent in time-outs and recovery) about

$$\frac{1}{W_{p,k}} + \frac{1}{W_{p,k+1}} + \cdots + \frac{1}{W_{p,k+N-1}} \text{ RTTs.} \tag{1.10}$$

Expression (1.10) is, of course, a decent approximation only if N is large. N should be large compared with $\max_{k \le j \le k+N} W_{p,j}$.

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We can condition on $W_{p,k} = w$ and take expected values. Thus, we get that the expected amount of time to move MSSs $k+1, \ldots, k+N$ given $W_{p,k} = w$ is approximately

$$E\left[\frac{1}{W_{p,k}} + \cdots + \frac{1}{W_{p,k+N-1}} \middle| W_{p,k} = w\right] \text{ RTTs.} \tag{1.11}$$

Using the weak convergence results stated earlier, we see that with

$$\rho = \frac{1}{1-\alpha}, \qquad x = \frac{pw^{1-\alpha}}{c_1(1-\alpha)}, \tag{1.12}$$

the expected duration in (1.11) for p small is approximately equal to

$$\frac{p^{\alpha/(1-\alpha)}}{(c_1(1-\alpha))^{\rho}} E\left[\int_{pk}^{p(k+N)} \frac{dt}{(X^*(t))^{\rho}} \middle| X^*(pk) = x\right]. \tag{1.13}$$

For interesting results, we should have that pN is at least $\Omega(1)$. For a good understanding of (1.13), it is important to remember that for fixed N (residual number of packets), pN goes to zero when p goes to zero.

Expression (1.13) also proves, again, a result known since 1996: The expected long-range throughput for a large file, with p small, should be approximately

$$\left(\frac{c_1(1-\alpha)}{p}\right)^{\rho} \left(E[(Z^*)^{-\rho}]\right)^{-1} \text{ MSSs/RTT,} \tag{1.14}$$

where Z^* has the stationary distribution of the process $X^*(\cdot)$. Equation (1.6) thus can be used to compute the expected long-range throughput. This results holds, of course, only if the assumptions of the model hold. For example, a home-user with a DSL modem will never get a throughput higher than the DSL link allows. Also, when there is a limit on the "flight size" (e.g. receive window or send window), this leads to limits on the achievable throughput. On the other hand, (1.14) does not require that RTTs be constant.

This article will not directly derive explicit expressions for entities like (1.13), but it will also derive explicit expressions for many transient entities such as $E[(X(t))^k | X(0) = x]$ and for entities like

$$\int_0^{\infty} (E[(X(t))^r | X(0) = x_1] - E[(X(t))^r | X(0) = x_2])\, dt, \tag{1.15}$$

$$\int_0^{\infty} (E[(X(t))^r | X(0) = x] - E[(X(t))^r | X(0) = Z])\, dt, \tag{1.16}$$

where, of course, Z has the stationary distribution as in (1.5) and is independent of the Poisson process (τ_k), so that for all $t \geq 0$, $(X(t)|X(0) = Z)$ has that stationary

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distribution and for

$$\frac{d}{dx}\int_0^\infty (E[(X(t))^r|X(0)=x]-E[(X(t))^r|X(0)=x_0])\,dt. \qquad \textbf{(1.17)}$$

In Sections 3–7 the integrals (1.15)–(1.17) will be redefined as expected values of certain stochastic integrals. In the notation of those sections, (1.15) is denoted as $D(x_1,x_2,r,0)$, (1.16) is denoted as $D(x_1,Z,r,0)$, and (1.17) is denoted as $D(x,x_0,r,0)d/dx$. There will be a fourth parameter (β), which is important in the derivations and which is zero in (1.15)–(1.17).

With $\rho=1/(1-\alpha)$,

$$\frac{p^{\alpha/(1-\alpha)}}{(c_1(1-\alpha))^\rho}D(x_1,x_2,-\rho,0)$$

is the expected difference in duration until completion, in RTTs, for a (very) large file, between starting with an initial window $w_1=(c_1(1-\alpha)x_1/p)^\rho$ and starting with an initial window $w_2=(c_1(1-\alpha)x_2/p)^\rho$. Similarly,

$$\frac{p^{\alpha/(1-\alpha)}}{(c_1(1-\alpha))^\rho}D(x,Z,-\rho,0)$$

describes the difference in expected duration for a large file, between starting with a window $w=(c_1(1-\alpha)x/p)^\rho$ and starting with a window in the stationary distribution. Since $-\rho<0$, $D(x,Z,-\rho,0)\to-\infty$ if $x\to\infty$.

Expression (1.17) similarly describes the marginal expected advantage, for ftp-ing a very large file, of starting with a slightly larger congestion window.

For a large file consisting of N packets (MSSs), if we start with the window in the stationary distribution, the expected time until completion is $N(p/c_1(1-\alpha))^\rho E[(Z^*)^{-\rho}]$ RTTs. If we start with a window $w=(c_1(1-\alpha)x/p)^\rho$, the expected duration therefore will approximately be

$$N\left(\frac{p}{c_1(1-\alpha)}\right)^\rho E[(Z^*)^{-\rho}]+\frac{p^{\alpha/(1-\alpha)}}{(c_1(1-\alpha))^\rho}D(x,Z,-\rho,0)\text{ RTTs.} \qquad \textbf{(1.18)}$$

For p (small, fixed) and x fixed, (1.18) becomes accurate if $N\to\infty$. If x is of the order of $E[Z^*]$, it probably is enough that N is large compared with $1/p(1-c)$. If x is larger, it probably is enough that N is large compared with

$$\frac{1}{p}\left(\frac{1}{1-c}+\frac{\log(x)}{|\log(c)|}\right).$$

These, and many others, are issues to be studied in a more engineering-oriented work.

The above might be useful, for example, in protocols using a "slow start-like" mechanisms, in choosing the analog of a "slow start threshold" parameter value.

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Results will be obtained by two different methods: "elementary" methods and methods based on Laplace transform techniques.

Section 2 contains some preliminaries. The next five sections (3–7) use only elementary techniques to obtain results for various moments and related entities. Laplace transform methods will be introduced in Section 8. Real-domain results for transient densities of $(X(t)|X(0) = x)$ will be given in Section 9. Their proofs depend on Laplace transforms from Section 8. Section 10 contains results for moments that could not be obtained by elementary methods but require the results from Section 9.

Appendix A contains some results on special functions that will be used throughout this work. Appendix B derives a result needed in Section 10.

2. PRELIMINARIES

Let $N(t)$ be the number of points of the Poisson process in the time interval $(0, t]$. It is well known that

$$P\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}. \tag{2.1}$$

One of the main objectives of this article is to derive the joint distribution of $(N(t), X(t))$ given $X(0) = x$.

A special role is played by the process $X(t)$ with $X(0) = 0$. We denote this process by $Y(t)$ and use the process $Y(t)$ to study the process $X(t)$, with, of course, the understanding that both processes are generated by the identical sequence (τ_k). With this understanding, we have the identity

$$X(t) = (Y(t) + c^{N(t)} X(0)), \tag{2.2}$$

which we often write as

$$X(t) = \sum_{k=0}^{\infty} (Y(t) + c^k X(0)) \chi(N(t) = k). \tag{2.3}$$

It must be kept in mind that $Y(t)$ and $N(t)$ are (very!) dependent on each other but, of course, independent of $X(0)$.

It is useful to observe that

$$tc^{N(t)} \le Y(t) \le t, \qquad (t + x)c^{N(t)} \le Y(t) + xc^{N(t)} \le t + xc^{N(t)} \le t + x \tag{2.4}$$

for all $t \ge 0, x \ge 0$. This implies that for all $-\infty < \beta < +\infty$,

$$t^r e^{-\lambda t(1-c^{\beta+r})} \le E[(Y(t))^r c^{\beta N(t)}] \le t^r e^{-\lambda t(1-c^{\beta})} \tag{2.5}$$

as long as $r \ge 0$ and

$$t^r e^{-\lambda t(1-c^{\beta})} \le E[(Y(t))^r c^{\beta N(t)}] \le t^r e^{-\lambda t(1-c^{\beta}+r)} \tag{2.6}$$

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as long as $r \le 0$. It also shows that with probability 1 for every $0 < \epsilon < \infty$ and $-\infty < \beta < +\infty$,

$$\int_0^{\epsilon} (Y(t))^r c^{\beta N(t)}\, dt \begin{cases} < \infty & \text{if } r > -1 \\ = \infty & \text{if } r \le -1, \end{cases} \tag{2.7}$$

whereas as long as $x > 0$, similar integrals for $(Y(t) + xc^{N(t)})^r c^{\beta N(t)}$ exist and are finite for all r, even complex r. Similar results hold for the integrals of $E[(Y(t))^r c^{\beta N(t)}]$ and $E[(Y(t) + xc^{N(t)})^r c^{\beta N(t)}]$.

In Section 8 we will study transforms like

$$\psi(x, t, s, z) = E[\exp\{-s(Y(t) + xc^{N(t)})\} z^{N(T)}]. \tag{2.8}$$

Often, we write z as c^{β}. In Section 9 we will use those transforms to obtain real-domain information about transient densities. Before we proceed to transform techniques, we restrict ourselves in the next five sections to elementary (real-domain) methods.

3. ELEMENTARY METHODS FOR EXPECTED VALUES

A useful observation is that for any r and β,

$$\frac{d}{dt} E[(X(t))^r c^{\beta N(t)}] = rE[(X(t))^{r-1} c^{\beta N(t)}] - \lambda(1 - c^{r+\beta}) E[(X(t))^r c^{\beta N(t)}]. \tag{3.1}$$

For $r = 1$, this reduces to

$$\frac{d}{dt} E[X(t) c^{\beta N(t)}] = e^{-\lambda t(1-c^{\beta})} - \lambda(1 - c^{1+\beta}) E[X(t) c^{\beta N(t)}], \tag{3.2}$$

which yields

$$E[X(t) c^{\beta N(t)}] = \left(X(0) - \frac{1}{\lambda c^{\beta}(1-c)} \right) e^{-\lambda t(1-c^{1+\beta})} + \frac{e^{-\lambda t(1-c^{\beta})}}{\lambda c^{\beta}(1-c)}. \tag{3.3}$$

Equation (3.1) with $r = 2$ and (3.3) show that

$$\begin{aligned} E[(X(t))^2 c^{\beta N(t)}] = {} & \frac{2}{\lambda^2 c^{2\beta}(1-c)(1-c^2)} \exp\{-\lambda t(1-c^{\beta})\} \\ & + \frac{2}{\lambda c^{1+\beta}(1-c)} \left(E[X(0)] - \frac{1}{\lambda c^{\beta}(1-c)} \right) \exp\{-\lambda t(1-c^{1+\beta})\} \\ & + \left(E[(X(0))^2 - \frac{2E[X(0)]}{\lambda c^{1+\beta}(1-c)} + \frac{2}{\lambda^2 c^{1+2\beta}(1-c)(1-c^2)} \right) \\ & \times \exp\{-\lambda t(1-c^{2+\beta})\}. \end{aligned} \tag{3.4}$$

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We easily see that, in general, for k a positive integer,

$$E[(X(t))^k c^{\beta N(t)}] = \sum_{j=0}^{k} C_{j,k} \exp\{-\lambda t(1 - c^{j+\beta})\}, \tag{3.5}$$

where

$$C_{0,0} = 1,$$

$$C_{j,k} = \frac{k!}{j!\lambda^{k-j} c^{(k-j)(j+\beta)}(1-c)(1-c^2)\cdots(1-c^{k-j})} C_{j,j} \quad \text{if } 0 \le j \le k \tag{3.6}$$

(which if $j = k$ is a trivial identity), and

$$\sum_{j=0}^{k} C_{j,k} = \sum_{j=0}^{k} \frac{k!}{j!\lambda^{k-j} c^{(k-j)(j+\beta)}(1-c)(1-c^2)\cdots(1-c^{k-j})} C_{j,j} = E[(X(0))^k]. \tag{3.7}$$

We can rewrite (3.7) as

$$\sum_{j=0}^{k} \frac{1}{c^{(k-j)j}(1-c)\cdots(1-c^{k-j})} \frac{(\lambda c^\beta)^j C_{j,j}}{j!} = \frac{(\lambda c^\beta)^k E[(X(0))^k]}{k!} \tag{3.8}$$

to make it more obvious which infinite triangular matrix to invert. Using (3.8) and (3.6), we obtain

$$\begin{aligned} \frac{(\lambda c^\beta)^k C_{k,k}}{k!} &= \sum_{j=0}^{k} \frac{(-1)^{k-j}}{c^{j(k-j)+(k-j)(k-j-1)/2}(1-c)(1-c^2)\cdots(1-c^{k-j})} \frac{(\lambda c^\beta)^j E[(X(0))^j]}{j!} \\ &= \sum_{j=0}^{k} \frac{(-1)^j}{c^{j(k-j)+j(j-1)/2}(1-c)(1-c^2)\cdots(1-c^j)} \frac{(\lambda c^\beta)^{k-j} E[(X(0))^{k-j}]}{(k-j)!}. \end{aligned} \tag{3.9}$$

The proof of (3.9) amounts to inverting an infinite triangular matrix and uses (A.6) (with $\mu = k - 1$) in Appendix A.

Substituting (3.9) into (3.6) and the results into (3.5) gives

$$\begin{aligned} \frac{(\lambda c^\beta)^k E[(X(t))^k c^{\beta N(t)}]}{k!} &= \sum_{j=0}^{k} \frac{(\lambda c^\beta)^j E[(X(0))^j]}{j!} c^{-j(k-j)} \\ &\quad \times \left(\sum_{\nu=j}^{k} \frac{(-1)^{\nu-j} c^{\frac{1}{2}(\nu-j)(\nu-j-1)-(\nu-j)(k-j-1)} \exp\{-\lambda t(1-c^{\nu+\beta})\}}{(1-c)\cdots(1-c^{k-\nu}).(1-c)\cdots(1-c^{\nu-j})} \right) \end{aligned} \tag{3.10}$$

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In the special case that $\beta = 0$, (3.10) is identical to Theorem 8 in [26] in the special case (in the language of that work) that $P\{Q = c\} = 1$.

Equation (3.10) makes it easy to verify (again using (A.6)) that for $t \downarrow 0$, it reduces to $E[(X(0))^k] = E[(X(0))^k]$.

Using Lemma A.2 in Appendix A, (3.10) can be rewritten as

$$\frac{\lambda^k}{k!}E[(X(t))^k c^{\beta N(t)}] = e^{-\lambda t}\sum_{\nu=0}^{k}\frac{\lambda^\nu}{\nu!}E[(X(0))^\nu] \times \sum_{\mu=0}^{\infty}\frac{(\lambda t)^{\mu+k-\nu}}{(\mu+k-\nu)!}c^{\mu(\nu+\beta)}\frac{(1-c^{\mu+1})\cdots(1-c^{\mu+k-\nu})}{(1-c)\cdots(1-c^{k-\nu})}. \tag{3.11}$$

Using (A.12), we can rewrite (3.11) as

$$\frac{\lambda^k}{k!}E[(X(t))^k c^{\beta N(t)}] = e^{-\lambda t}\sum_{\nu=0}^{k}\frac{\lambda^\nu}{\nu!}E[(X(0))^\nu] \times \sum_{\mu=0}^{\infty}\frac{(\lambda t)^{\mu+k-\nu}}{(\mu+k-\nu)!}c^{\mu(\nu+\beta)}\frac{(1-c^{\mu-\nu+1})\cdots(1-c^{\mu-\nu+\mu})}{(1-c)\cdots(1-c^{\mu})}. \tag{3.12}$$

It sometimes is useful to write (3.12) as

$$\frac{\lambda^k}{k!}E[(X(t))^k c^{\beta N(t)}] = e^{-\lambda t}\sum_{\nu=0}^{k}\frac{\lambda^\nu}{\nu!}E[(X(0))^\nu] \times \sum_{\mu=0}^{\infty}\frac{(\lambda t)^{\mu+k-\nu}}{(\mu+k-\nu)!}c^{\mu\beta}\frac{(c^\nu-c^{k+1})\cdots(c^\nu-c^{k+\mu})}{(1-c)\cdots(1-c^{\mu})}. \tag{3.13}$$

In a formal sense, we can write (3.13) as

$$\frac{\lambda^k}{k!}E[(X(t))^k c^{\beta N(t)}] = e^{-\lambda t}\sum_{\nu=0}^{\infty}\frac{\lambda^\nu}{\nu!}E[(X(0))^\nu] \times \sum_{\mu=0}^{\infty}\frac{(\lambda t)^{\mu+k-\nu}}{\Gamma(\mu+k-\nu+1)}c^{\mu\beta}\frac{(c^\nu-c^{k+1})\cdots(c^\nu-c^{k+\mu})}{(1-c)\cdots(1-c^{\mu})}; \tag{3.14}$$

namely if $\nu \geq k+1$, then $1/\Gamma(\mu+k-\nu+1) = 0$ if $\mu \leq \nu-k-1$ and the product in the inner sum equals zero if $\mu \geq \nu-k$. In Section 10 we will see that as long as

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$X(0) \le t$, we have, for all r, even complex,

$$\frac{\lambda^r}{\Gamma(r+1)} E[(X(t))^r c^{\beta N(t)}] = e^{-\lambda t} \sum_{\nu=0}^{\infty} \frac{\lambda^\nu}{\nu!} E[(X(0))^\nu] \sum_{\mu=0}^{\infty} \frac{(\lambda t)^{\mu+r-\nu}}{\Gamma(\mu+r-\nu+1)} \times c^{\mu\beta} \frac{(c^\nu - c^{r+1}) \cdots (c^\nu - c^{r+\mu})}{(1-c)\cdots(1-c^\mu)}. \tag{3.15}$$

Equation (3.15) converges absolutely as long as $X(0) \le t$. It diverges if $P\{X(0) > t\} > 0$. In the special case that $X(0) = 0$, (3.10) gives

$$\frac{(\lambda c^\beta)^k E[(Y(t))^k c^{\beta N(t)}]}{k!} = \sum_{\nu=0}^{k} \frac{(-1)^\nu c^{\frac{1}{2}\nu(\nu-1)-\nu(k-1)} \exp\{-\lambda t(1-c^{\nu+\beta})\}}{(1-c)\cdots(1-c^\nu)\cdot(1-c)\cdots(1-c^{k-\nu})}, \tag{3.16}$$

and, similarly, (3.12), or (3.16) with Lemma A2, gives

$$\frac{\lambda^k}{k!} E[(Y(t))^k c^{\beta N(t)}] = e^{-\lambda t} \sum_{\mu=0}^{\infty} \frac{(\lambda t)^{\mu+k}}{(\mu+k)!} c^{\mu\beta} \frac{(1-c^{k+1})\cdots(1-c^{k+\mu})}{(1-c)\cdots(1-c^\mu)}. \tag{3.17}$$

In Section 10 we will see that (3.17) can be generalized to the situation where k is not a nonnegative integer. For all r, even complex,

$$\frac{\lambda^r}{\Gamma(r+1)} E[(Y(t))^r c^{\beta N(t)}] = e^{-\lambda t} \sum_{\mu=0}^{\infty} \frac{(\lambda t)^{\mu+r}}{\Gamma(\mu+r+1)} c^{\mu\beta} \frac{(1-c^{r+1})\cdots(1-c^{r+\mu})}{(1-c)\cdots(1-c^\mu)}. \tag{3.18}$$

We recognize (3.18) as (3.15) with $X(0) = 0$. In fact, (3.18) will be proven before (3.15) and is used in the proof of (3.15).

By choosing $\beta = 0$, we obtain the behavior of $E[(X(t))^k]$ and $E[(Y(t))^k]$, and (3.5) shows that the "relaxation time" of the process $X(t)$ is $1/\lambda(1-c)$: If $X(0)$ is not too large compared with $E[Z]$, the time it takes for $X(t)$ to "reach the stationary distribution" is a moderate multiple of $1/\lambda(1-c)$.

Equations (3.12) and (3.17) show that

$$\frac{\lambda^{k+1}}{k!} \int_0^\infty E[((Y(t) + xc^{N(t)})^k - (Y(t))^k) c^{\beta N(t)}] dt = \sum_{\nu=1}^{k} \frac{(\lambda x)^\nu}{\nu!} \sum_{\mu=0}^{\infty} c^{\mu(\nu+\beta)} \frac{(1-c^{k-\nu+1})\cdots(1-c^{k-\nu+\mu})}{(1-c)\cdots(1-c^\mu)}. \tag{3.19}$$

The integral, and the sum, in (3.19) converge as long as $Re(\beta) > -1$. More on convergence of this kind of integrals will be discussed in Sections 5–7. Using (A.2) (which

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in this case requires $Re(\beta) > -1$), we rewrite (3.19) as

$$\frac{\lambda^{k+1}}{k!}\int_0^\infty E[((Y(t)+xc^{N(t)})^k-(Y(t))^k)c^{\beta N(t)}]dt=\sum_{\nu=1}^{k}\frac{(\lambda x)^\nu}{\nu!}\prod_{\mu=0}^{\infty}\frac{1-c^{k+\beta+1+\mu}}{1-c^{\nu+\beta+\mu}}, \tag{3.20}$$

which, since $\nu \le k$ and both are nonnegative integers, can be written as

$$\frac{\lambda^{k+1}}{k!}E\left[\int_0^\infty((Y(t)+xc^{N(t)})^k-(Y(t))^k)c^{\beta N(t)}dt\right] = \sum_{\nu=1}^{k}\frac{(\lambda x)^\nu}{\nu!}\frac{1}{(1-c^{\nu+\beta})(1-c^{\nu+\beta+1})\cdots(1-c^{k+\beta})}. \tag{3.21}$$

Equations (3.20) and (3.21) are identities as long as $Re(\beta) > -1$. If β is real and $\beta \le -1$, the integrals are infinite but the sums on the right-hand sides (RHSs) are the analytic continuations of the expressions for $Re(\beta) > -1$. As long as β is real, the interchange of expected value and integration is warranted by Fubini's theorem (all functions involved are nonnegative).

By using (3.21) twice, with different values of x, we obtain expressions like (1.15); however, with the constraint that r must be a nonnegative integer k. One of our goals is to generalize (3.21) to general r. It must be noted that in none of the three equations (3.19)–(3.21) can we change the upper limit of the sum from k to ∞. Therefore, to find the entities described in (1.15), we must obtain new methods. This will be done in Sections 5–7.

A second useful insight (beyond (3.1)) is that for any r,

$$\begin{aligned}\frac{d}{dt}E[(X(t))^r\chi(N(t)=k)] &= \lambda c^r E[(X(t))^r\chi(N(t)=k-1)]\\ &\quad + rE[(X(t))^{r-1}\chi(N(t)=k)]\\ &\quad - \lambda E[(X(t))^r\chi(N(t)=k)].\end{aligned} \tag{3.22}$$

For $k=0$, obviously

$$E[(X(t))^r\chi(N(t)=0)|X(0)=x] = (x+t)^r\exp\{-\lambda t\}. \tag{3.23}$$

For $r=1$, (3.22) reduces to

$$\begin{aligned}\frac{d}{dt}E[X(t)\chi(N(t)=k)] &= \lambda cE[X(t)\chi(N(t)=k-1)]\\ &\quad + P\{N(t)=k\} - \lambda E[X(t)\chi(N(t)=k)].\end{aligned} \tag{3.24}$$

This yields

$$E[X(t)\chi(N(t)=k)|X(0)=x] = \left(x\frac{(c\lambda t)^k}{k!}+\frac{(1-c^{k+1})}{\lambda(1-c)}\frac{(\lambda t)^{k+1}}{(k+1)!}\right)e^{-\lambda t}. \tag{3.25}$$

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The proofs of the last few statements are left for the reader.

A third potentially useful insight is that for $m \in \{0, 1, 2, \ldots\}$,

$$\left(\frac{d}{dx}\right)^m E[(X(t))^r c^{\beta N(t)} | X(0) = x] = \left(\frac{d}{dx}\right)^m E[(Y(t) + xc^{N(t)})^r c^{\beta N(t)}]$$

$$= r(r-1)\cdots(r-m+1)$$

$$\times E[(Y(t) + xc^{N(t)})^{r-m} c^{(\beta+m)N(t)}]. \quad \textbf{(3.26)}$$

4. STOCHASTIC DOMINANCE AND BOUNDS

Let $X_1(t), X_2(t), \ldots$ be processes as in the previous sections, with identical $(Y(t))_{t\geq 0}$ but different starting values $X_1(0), X_2(0), \ldots$, with, of course, $(X_1(0), X_2(0), \ldots)$ independent of $(Y(t))_{t\geq 0}$. By (2.2), if $X_1(0)$ is (stochastically) less than $X_2(0)$, then, for all $t \geq 0$, $X_1(t)$ is (stochastically) less than $X_2(t)$.

Similarly, if $X_3(0)$ has the distribution of Z in (1.5), then, for all $t \geq 0$, $X_3(t) = Y(t) + X_3(0)c^{N(t)}$ also has the distribution of Z.

Hence, if also $X_1(0)$ is stochastically less than $X_3(0) = Z$ and Z is stochastically less than $X_2(0)$, then, for all $t \geq 0$, $X_1(t) = Y(t) + X_1(0)c^{N(t)}$ is stochastically less than Z and Z is stochastically less than $X_2(t) = Y(t) + X_2(0)c^{N(t)}$.

We conjecture that, in this case, actually $X_1(t) = Y(t) + X_1(0)c^{N(t)}$ is stochastically increasing in t and $X_2(t) = Y(t) + X_2(0)c^{N(t)}$ is stochastically decreasing in t, in both cases, of course, with the distribution of Z as limit distribution (for $t \uparrow \infty$). Thus far we have been unable to prove this conjecture.

However, the following is enough for the needs of this article.

LEMMA 1: *$(Y(t))_{t\geq 0}$ is stochastically increasing in t, with, for limit distribution ($t \uparrow \infty$), the distribution of Z.*

PROOF: Let $0 < t_1 < t_2$. Then

$$Y(t_2) = Z(t_2 - t_1) + Y(t_1)c^{N(t_2)-N(t_1)} > Z(t_2 - t_1), \quad \textbf{(4.1)}$$

where $Z(t_2 - t_1)$ has the same distribution as $Y(t_2 - t_1)$. Hence, $Y(t_2)$ is stochastically larger than $Y(t_2 - t_1)$ for all $0 < t_1 < t_2$. This proves the lemma. ■

Clearly, if $r > 0$, then $E[(Y(t))^r]$ increases from 0 at $t = 0$ to $M(r)$ at $t = \infty$, and if $r < 0$, then $E[(Y(t))^r]$ decreases from $+\infty$ at $t = 0$ to $M(r)$ at $t = \infty$. More

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quantitatively, we have the following. If $r \geq 0$, then

$$t^r e^{-\lambda t(1-c^r)} \leq E[(Y(t)^r] \leq \min(M(r), t^r), \tag{4.2}$$

whereas if $r \leq 0$, then

$$\max(M(r), t^r) \leq E[(Y(t))^r] \leq t^r e^{\lambda t(c^r-1)}, \tag{4.3}$$

and in both cases,

$$\lim_{t\to\infty} E[(Y(t))^r] = M(r). \tag{4.4}$$

For the next result we need an intermediate step.

observation: If U and V are nonnegative random variables and $\eta > 0$, $r \geq 0$, then

$$E[(U+V)^r] = E[(U+V)^r \chi(V < \eta U)] + E\left[(U+V)^r \chi\left(U \leq \frac{1}{\eta} V\right)\right]$$

$$\leq (1+\eta)^r E[U^r] + \left(\frac{1+\eta}{\eta}\right)^r E[V^r]. \tag{4.5}$$

The result still holds if $r \leq 0$. (The proof requires a trivial modification.) However, in that case it is not of much use in the following.

LEMMA 2: *Let $X(0)$ be random (nonnegative), independent of $(Y(t))_{t\geq 0}$. If now $r \geq 0$, then for all $\eta(t) > 0$,*

$$\max(E[(Y(t))^r], E[(X(0))^r] e^{-\lambda t(1-c^r)})$$

$$\leq E[(Y(t) + X(0)c^{N(t)})^r] \leq (1+\eta(t))^r$$

$$\times \left(E[(Y(t))^r] + \left(\frac{1+\eta(t)}{\eta(t)}\right)^r E[(X(0))^r] e^{-\lambda t(1-c^r)}\right), \tag{4.6}$$

whereas if $r \leq 0$, then

$$E[(Y(t) + X(0)c^{N(t)})^r] \leq \min(E[(Y(t))^r], E[(X(0))^r] e^{\lambda t(c^r-1)}). \tag{4.7}$$

In (4.6) we can choose (for example) $\eta(t) = e^{-(\lambda t/2r)}(1-c^r)$ to get a bound on how fast $E[(Y(t) + X(0)c^{N(t)})^r]$ approaches $E[(Y(t))^r]$ if $t \to \infty$. By substituting (4.2) and (4.3) into (4.6) and (4.7) and using (4.4), we easily prove the following.

THEOREM 1A: *For all $0 < \epsilon < \infty$, $0 \leq x < \infty$, and $-\infty < r < +\infty$, the integral*

$$\int_\epsilon^\infty (Y(t) + xc^{N(t)})^r c^{\beta N(t)}\, dt \tag{4.8}$$

is finite with probability 1 and has finite expected value, as long as $\beta > 0$. If $\beta \leq 0$, the integral, with probability 1, is infinite.

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Clearly, in Theorem 1A we can allow β, r to be complex-valued. If $\mathrm{Re}(\beta) > 0$, then for all r, the integral converges with probability 1 and has a well-defined expected value. If $\mathrm{Re}(\beta) \leq 0$, then for all r, the integral does not converge.

THEOREM 1B: *Unless* ($x = 0$ *and* $\mathrm{Re}(r) \leq -1$)*, the integral*

$$\int_0^{\epsilon} (Y(t) + xc^{N(t)})^r c^{\beta N(t)}\, dt \tag{4.9}$$

($0 < \epsilon < \infty, 0 \leq x < \infty, -\infty < r < +\infty, -\infty < \beta < +\infty$) *is finite with probability 1 and has finite expected value. If, simultaneously,* $x = 0$ *and* $\mathrm{Re}(r) \leq -1$*, the integral diverges with probability* 1. (*See also* (2.7).)

PROOF: Considering the results earlier in this section, the only statement that still needs proof is that if β is real and $\beta \leq 0$, the integral in (4.8) is infinite with probability 1. Because of the construction of the process $(X(t))$, convergence of the integral (4.8) requires that $\lim_{t\to\infty}(X(t))^r c^{\beta N(t)} = 0$. This clearly has probability 0 if $\mathrm{Re}(\beta) \leq 0$. ■

Paraphrasing Theorem 1: *The critical point* $t = \infty$ *leads to divergence if and only if* $\mathrm{Re}(\beta) \leq 0$. *The critical point* $t = 0$ *leads to divergence if and only if* ($x = 0$ *and* $\mathrm{Re}(r) \leq -1$).

We see the following.

LEMMA 3: *If* $\mathrm{Re}(\beta) > 0$ *and* $x > 0$*, the expected value*

$$E\left[\int_0^{\infty} (Y(t) + xc^{N(t)})^r c^{\beta N(t)}\, dt\right] \tag{4.10}$$

is analytic in r on the whole complex plane, whereas (β) > 0 *and* $x = 0$*, it is analytic in r on the half-plane* $\mathrm{Re}(r) > -1$.

LEMMA 4: *Similarly, if* $\mathrm{Re}(\beta) > -1$ *and* $x > 0$*,*

$$\frac{d}{dx} E\left[\int_0^{\infty} (Y(t) + xc^{N(t)})^r c^{\beta N(t)}\, dt\right] = rE\left[\int_0^{\infty} (Y(t) + xc^{N(t)})^{r-1} c^{(\beta+1)N(t)}\, dt\right] \tag{4.11}$$

is an entire function of r and, hence,

$$\begin{aligned}
&\int_0^{\infty} ((Y(t) + x_1 c^{N(t)})^r - (Y(t) + x_2 c^{N(t)})^r) c^{\beta N(t)}\, dt \\
&\quad = r \int_0^{\infty} \int_{x_2}^{x_1} (Y(t) + uc^{N(t)})^{r-1} c^{(\beta+1)N(t)}\, du\, dt \\
&\quad = r \int_{x_2}^{x_1} \int_0^{\infty} (Y(t) + uc^{N(t)})^{r-1} c^{(\beta+1)N(t)}\, dt\, du
\end{aligned} \tag{4.12}$$

is an entire function in r as long as $\mathrm{Re}(\beta) > -1$.

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5. QUANTITATIVE RESULTS FOR EXPECTED VALUE OF INTEGRALS

Let $\tau_k (k \geq 1)$ be the kth event of the Poisson process after time 0. Even though (with probability 1) time 0 is not an event of the Poisson process, we let $\tau_0 = 0$ and $E_k = \tau_{k+1} - \tau_k$, so that $(E_k)_{k=0}^{\infty}$ are i.i.d., exponentially distributed with parameter λ. Now, as long as $r \neq -1$,

$$\begin{aligned}\int_0^{\infty} (Y(t) + xc^{N(t)})^r c^{\beta N(t)}\, dt &= \sum_{k=0}^{\infty} \int_{\tau_k}^{\tau_{k+1}} (Y(t) + xc^k)^r c^{\beta k}\, dt \\ &= \sum_{k=0}^{\infty} \frac{c^{\beta k}}{r+1} ((Y(\tau_k^+) + (\tau_{k+1} - \tau_k) + xc^k)^{r+1} \\ &\quad - (Y(\tau_k^+) + xc^k)^{r+1}) \\ &= \sum_{k=0}^{\infty} \frac{c^{\beta k}}{r+1} (Y(\tau_{k+1}^-) + xc^k)^{r+1} - \frac{x^{r+1}}{r+1} \\ &\quad - \sum_{k=1}^{\infty} \frac{c^{\beta k}}{r+1} (cY(\tau_k^-) + xc^k)^{r+1} \\ &= -\frac{x^{r+1}}{r+1} + \frac{1 - c^{r+\beta+1}}{r+1} \sum_{k=0}^{\infty} c^{\beta k} (Y(\tau_{k+1}^-) + xc^k)^{r+1}. \end{aligned} \tag{5.1}$$

It is easily seen that as long as $\mathrm{Re}(\beta) > 0$, the rearranging of the above terms is warranted. Next, we take expected values in (5.1). Clearly,

$$Y(\tau_{k+1}^-) = \sum_{j=0}^{k} c^j E_{k-j}, \tag{5.2}$$

so that $Y(\tau_{k+1}^-)$ has Laplace transform

$$\prod_{j=0}^{k} \frac{1}{1 + c^j s/\lambda} = \sum_{j=0}^{k} \frac{A_{j,k}}{1 + c^j s/\lambda} \tag{5.3}$$

and density

$$\sum_{j=0}^{k} A_{j,k} \frac{\lambda}{c^j} e^{-(\lambda/c^j)y}, \tag{5.4}$$

where

$$A_{j,k} = \prod_{\substack{\nu \neq j \\ 0 \leq \nu \leq k}} \frac{1}{1 - c^{\nu - j}} = \frac{(-1)^j c^{\frac{1}{2} j(j+1)}}{(1-c)\cdots(1-c^j)(1-c)\cdots(1-c^{k-j})}. \tag{5.5}$$

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It is interesting to compare $A_{j,k}$ with R_j [38]. $A_{j,k} = R_j \prod_{\nu=1}^{\infty}(1 - c^{k-j+\nu})$ and $R_j = \lim_{k\to\infty} A_{j,k}$. The random variable Z in (1.5) has density

$$\sum_{j=0}^{\infty} R_j \frac{\lambda}{c^j} e^{-(\lambda/c^j)z}. \tag{5.6}$$

We observe that by (A.4),

$$\sum_{j=0}^{\infty} |A_{j,j+\nu}| c^{j(r+1+\beta)} = \frac{1}{(1-c)\cdots(1-c^{\nu})} \prod_{\mu=0}^{\infty}(1 + c^{r+2+\beta+\mu}), \tag{5.7}$$

which, on occasion, can be used to prove absolute convergence, and we observe that

$$\sum_{j=0}^{k} A_{j,k} = 1, \qquad \sum_{j=0}^{\infty} R_j = 1, \tag{5.8}$$

$$\sum_{j=0}^{\infty} A_{j,j+k} z^j = \frac{\prod_{\nu=1}^{\infty}(1 - c^{\nu} z)}{(1-c)\cdots(1-c^k)}, \tag{5.9}$$

$$\begin{aligned} &- (1 - c^{\beta}) \sum_{k=0}^{\infty} c^{\beta k} \sum_{j=0}^{k} j A_{j,k} \\ &\quad = \left(\prod_{\nu=0}^{\infty} \frac{1}{1 - c^{\beta+1+\nu}} \right) \left(\sum_{j=1}^{\infty} \frac{(-1)^{j-1} j c^{\frac{1}{2}j(j+1)} c^{\beta j}}{(1-c)\cdots(1-c^j)} \right) \\ &\quad = \left(\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} c^{\frac{1}{2}\nu(\nu+1)} c^{\beta\nu}}{(1-c)\cdots(1-c^{\nu})} \right)^{-1} \left(\sum_{j=1}^{\infty} \frac{(-1)^{j-1} j c^{\frac{1}{2}j(j+1)} c^{\beta j}}{(1-c)\cdots(1-c^j)} \right) \\ &\quad = \sum_{j=1}^{\infty} \frac{c^{\theta+j}}{1 - c^{\beta+j}}. \end{aligned} \tag{5.10}$$

Equations (5.10) uses (A.14). In (5.10), the last but one expression is the one to be used for numerical purposes, but in formulas we will use the more compact last expression.

Using the above results and the fact (see [1]) that

$$\int_0^{\infty} e^{-y} \log(y)\, dy = \Gamma'(1) = -\gamma = -0.57721566490\ldots, \tag{5.11}$$

where γ is Euler's constant, we straightforwardly obtain that

$$E[\log(Z)] = -\gamma - \log(\lambda) - (\log(c)) \sum_{j=1}^{\infty} \frac{c^j}{1 - c^j}. \tag{5.12}$$

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By (5.1)–(5.3),

$$E\left[\int_0^\infty (Y(t)+xc^{N(t)})^r c^{\beta N(t)}\,dt\right] = -\frac{x^{r+1}}{r+1} + \frac{1-c^{r+\beta+1}}{r+1}\sum_{k=0}^{\infty} c^{\beta k}\sum_{j=0}^{k} A_{j,k}$$
$$\times \int_0^\infty (y+xc^k)^{r+1}\frac{\lambda}{c^j}e^{(\lambda/c^j)y}\,dy$$
$$= -\frac{x^{r+1}}{r+1} + \frac{1-c^{r+\beta+1}}{(r+1)\lambda^{r+1}}\sum_{k=0}^{\infty} c^{\beta k}\sum_{j=0}^{k} A_{j,k}c^{j(r+1)}$$
$$\times \int_0^\infty (v+\lambda xc^{k-j})^{r+1}e^{-v}\,dv$$
$$= -\frac{x^{r+1}}{r+1} + \frac{1-c^{r+\beta+1}}{(r+1)\lambda^{r+1}}\sum_{k=0}^{\infty} c^{\beta k}\sum_{j=0}^{k} A_{j,k}c^{j(r+1)}$$
$$\times e^{+\lambda xc^{k-j}}\int_{\lambda xc^{k-j}}^\infty u^{r+1}e^{-u}\,du. \tag{5.13}$$

We can rewrite (5.13) with either the lower incomplete gamma functions $\gamma(a,x)$ or with the upper incomplete Gamma functions $\Gamma(a,x)$; see, for example, [1]. Since possibly $\mathrm{Re}(r) < -2$, we use the upper incomplete Gamma functions and get

$$E\left[\int_0^\infty (Y(t)+xc^{N(t)})^r c^{\beta N(t)}\,dt\right] = -\frac{x^{r+1}}{r+1} + \frac{1-c^{r+\beta+1}}{(r+1)\lambda^{r+1}}\sum_{\nu=0}^{\infty} c^{\beta\nu}$$
$$\times \int_0^\infty ((v+\lambda xc^\nu)^{r+1}e^{-v}dv)\sum_{j=0}^{\infty} A_{j,j+\nu}c^{j(r+1+\beta)}$$
$$= -\frac{x^{r+1}}{r+1} + \frac{1}{(r+1)\lambda^{r+1}}\left(\prod_{\mu=0}^{\infty}(1-c^{r+1+\beta+\mu})\right)$$
$$\times \sum_{\nu=0}^{\infty}\frac{c^{\beta\nu}\int_0^\infty (v+\lambda xc^\nu)^{r+1}e^{-v}\,dv}{(1-c)\cdots(1-c^\nu)}$$
$$= -\frac{x^{r+1}}{r+1} + \frac{1}{(r+1)\lambda^{r+1}}\left(\prod_{\mu=0}^{\infty}(1-c^{r+1+\beta+\mu})\right)$$
$$\times \sum_{\nu=0}^{\infty}\frac{c^{\beta\nu}e^{\lambda xc^\nu}(r+2,\lambda xc^\nu)}{(1-c)\cdots(1-c^\nu)}. \tag{5.14}$$

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The last but one step uses (A.13). To check whether the change in order of summation in (5.14) is warranted, we use (5.7) and observe that (unless ($x = 0$ and $\mathrm{Re}(r) \le -1$)) the infinite sum on the RHS of (5.14) is finite when $\mathrm{Re}(r+\beta) > -2$ and infinite when r and β are real and $r + \beta \le -2$. The latter follows from the fact (easily verified by partial integration) that for $\lambda x > 0$, $0 < c < 1$, and a real,

$$\lim_{\nu\to\infty} c^{-a\nu}\Gamma(a, \lambda x c^{\nu}) = \begin{cases} +\infty & \text{if } a \ge 0 \\ \dfrac{1}{|a|(\lambda x)^{|a|}} & \text{if } a < 0. \end{cases} \tag{5.15}$$

Combining this with (5.7), we see that reordering of terms in the sums in (5.14) is warranted as long as $\mathrm{Re}(r+\beta) > -2$. Therefore, (5.14) holds as long as ($\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(r+\beta) > -2$). The last expression in (5.14) will be useful in the next section.

If not only $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(r+\beta) > -2$ but also $\mathrm{Re}(r) > -2$, we can use the fact that $\Gamma(r+2, \lambda x c^{\nu}) = \Gamma(r+2) - \gamma(r+2, \lambda x c^{\nu})$ to obtain (also using Lemma A.4 and other results from Appendix A)

$$E\left[\int_0^{\infty} (Y(t) + xc^{N(t)})^r C^{\beta N(t)}\, dt\right] = \frac{\Gamma(r+1)}{\lambda^{r+1}} \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!} \left(\prod_{\mu=0}^{\infty} \frac{1 - c^{r+1+\beta+\mu}}{1 - c^{n+\beta+\mu}}\right)$$

$$= -\frac{1}{\lambda^{r+1}} \sum_{n=1}^{\infty} \frac{(\lambda x)^{r+n}}{(r+1)\cdots(r+n)} \left(\prod_{\mu=1}^{n-1} (1 - c^{r+\beta+\mu})\right). \tag{5.16}$$

It must be noted that on the RHS of (5.16), both components have singularities in $r = -1, -2, \ldots$. These singularities annihilate each other: The left-hand side (LHS) in (5.16) is an entire function of r (as long as $\mathrm{Re}(\beta) > 0$ and $x > 0$). Thus, although the proof of (5.16) depends on $\mathrm{Re}(r) > -2$, the result, by analytic continuation, is valid for all r and can be used to compute the expected value of the integral as long as $\mathrm{Re}(\beta) > 0$ and r is not a negative integer. As function of β, the first component on the RHS has singularities (simple poles) in $\beta = 0, -1, -2, \ldots$.

If r is a nonnegative integer, $r = k$, (5.16) becomes

$$E\left[\int_0^{\infty} (Y(t) + xc^{N(t)})^k c^{\beta N(t)}\, dt\right] = \frac{k!}{\lambda^{k+1}} \sum_{\nu=0}^{k} \frac{(\lambda x)^{\nu}}{\nu!} \frac{1}{(1 - c^{\nu+\beta}) \cdots (1 - c^{k+\beta})}, \tag{5.17}$$

from which we obtain an alternative derivation of (3.21).

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From (5.16) we easily see that for every $k \in \{0, 1, 2, \ldots\}$, $r \notin \{-k, -k+1, \ldots, -1\}$,

$$E\left[\int_0^\infty (Y(t) + xc^{N(t)})^r c^{\beta N(t)}\, dt\right]$$

$$= -\frac{1}{\lambda^{r+1}} \sum_{j=1}^{k} \frac{(\lambda x^{r+j})}{r+j} \left(\prod_{\mu=1}^{j-1} \frac{1 - c^{r+\beta+\mu}}{r+\mu}\right)$$

$$+ \lambda^k \left(\prod_{\mu=1}^{k} \frac{1 - c^{r+\beta+\mu}}{r+\mu}\right) E\left[\int_0^\infty (Y(t) + xc^{N(t)})^{r+k} c^{\beta N(t)}\, dt\right]. \tag{5.18}$$

An alternate way to prove (5.18) is directly from (5.14), using the observation that $e^z \Gamma(r, x) = -x^r/r + (x^r/r)\Gamma(r+1, x)$.

By continuity, (5.18) shows that for $k \in \{2, 3, \ldots\}$,

$$E\left[\int_0^\infty (Y(t) + xc^{N(t)})^{-k} c^{\beta N(t)}\, dt\right]$$

$$= -\sum_{j=1}^{k-1} \lambda^{j-1} \frac{x^{-k+j}}{-k+j} \left(\prod_{\mu=1}^{j-1} \frac{1 - c^{-k+\beta+\mu}}{-k+\mu}\right)$$

$$+ \lambda^{k-1} \left(\prod_{\mu=1}^{k-1} \frac{1 - c^{-k+\beta+\mu}}{-k+\mu}\right) E\left[\int_0^\infty (Y(t) + xc^{N(t)})^{-1} c^{\beta N(t)}\, dt\right] \tag{5.19}$$

We will now study the case $r = -1$. If $r = -1$, we obtain, following the same recipe as in (5.1) and so forth,

$$\int_0^\infty \frac{c^{\beta N(t)}}{Y(t) + xc^{N(t)}}\, dt = -\log(x) - \frac{c^\beta \log(c)}{1 - c^\beta} + (1 - c^\beta) \sum_{k=0}^{\infty} c^{\beta k} \log(Y(\tau_k^-) + xc^k); \tag{5.20}$$

hence,

$$E\left[\int_0^\infty \frac{c^{\beta N(t)}}{Y(t) + xc^{N(t)}}\, dt\right]$$

$$= -\log(x) - \frac{c^\beta \log(c)}{1 - c^\beta} + (1 - c^\beta) \sum_{k=0}^{\infty} c^{\beta k} [\log(Y(\tau_k^-) + xc^k)]. \tag{5.21}$$

Since ordinary convergence implies convergence in the sense of Abel, we have

$$\lim_{\beta \downarrow 0} (1 - c^\beta) \sum_{k=0}^{\infty} c^{\beta k} E[\log(Y(\tau_k^-) + xc^k)] = E[\log(Z)], \tag{5.22}$$

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independent of $x \geq 0$, where, of course, Z has the distribution as in (1.5). For many purposes there is no need to work out (5.21) in more detail. In case more detail is needed, we easily obtain (using ideas similar to Lemma A.4 and the proof of (5.16))

$$\begin{aligned} E\left[\int_0^\infty \frac{c^{\beta N(t)}}{Y(t)+xc^{N(t)}}\,dt\right] &= -\log(\lambda x) - \log(c)\left(\sum_{j=0}^\infty \frac{c^{\beta+j}}{1-c^{\beta+j}}\right) \\ &\quad + \left(\prod_{j=0}^\infty (1-c^{\beta+j})\right)\sum_{\nu=0}^\infty \frac{c^{\beta\nu}}{(1-c)\cdots(1-c^\nu)} \\ &\quad \times \int_0^\infty (\log(v+\lambda x c^\nu))e^{-v}\,dv \\ &= -\log(\lambda x) - \gamma\sum_{\nu=0}^\infty \frac{(\lambda x)^\nu}{\nu!}(1-c^\beta)\cdots(1-c^{\beta+\nu-1}) \\ &\quad - \log(c)\left(\sum_{j=0}^\infty \frac{c^{\beta+j}}{1-c^{\beta+j}}\right) \\ &\quad - \left(\prod_{j=0}^\infty (1-c^{\beta+j})\right)\sum_{\nu=0}^\infty \frac{c^{\beta\nu}e^{\lambda x c^\nu}\int_0^{\lambda x c^\nu}\log(u)e^{-u}\,du}{(1-c)\cdots(1-c^\nu)}. \end{aligned} \tag{5.23}$$

We can rewrite further using Lemma A.5. Such a rewrite shows that

$$\sum_{\nu=0}^\infty \frac{c^{\beta\nu}e^{\lambda x c^\nu}\int_0^{\lambda x c^\nu}\log(u)e^{-u}\,du}{(1-c)\cdots(1-c^\nu)} \tag{5.24}$$

as a function of β is analytic on $\mathrm{Re}(\beta) > -1$ and has a pole of order 2 in $\beta = -1$, so that

$$\left(\prod_{j=0}^\infty (1-c^{\beta+j})\right)\sum_{\nu=0}^\infty \frac{c^{\beta\nu}e^{\lambda x c^\nu}\int_0^{\lambda x c^\nu}\log(u)e^{-u}\,du}{(1-c)\cdots(1-c^\nu)}. \tag{5.25}$$

as a function of β is analytic on $\mathrm{Re}(\beta) > -1$, has a pole of order 1 in $\beta = -1$, and, for $\beta = 0$, is zero for all values of x.

We can substitute (5.23) into (5.19) to take care of the situation $r \in \{-2, -3, \ldots\}$.

If the initial value $X(0)$ is random, we can take expected values in (5.14), (5.16), (5.18), and (5.19).

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The most interesting such situation is where $X(0)$ has the distribution of Z in (1.5), so that all $X(t)$ have that same distribution. However, in that situation, simply replacing x^ν by $E[Z^\nu]$ and using (1.6) does not work because in (5.16) it leads to $\infty - \infty$. Instead, we use the fact that (for $\mathrm{Re}(r) > -2$, and with the substitution $x = v + \lambda z c^\nu$ and $y = v - \lambda z c^\nu$)

$$E\left[\int_0^\infty (v+\lambda Z c^\nu)^{r+1} e^{-v}\,dv\right] = \sum_{j=0}^\infty R_j \frac{\lambda}{c^j} \int_0^\infty \int_0^\infty (v+\lambda z c^\nu)^{r+1} e^{-v} e^{-\frac{\lambda}{c^j} z}\,dv\,dz$$

$$= \sum_{j=0}^\infty \frac{R_j}{2c^{\nu+j}} \int_0^\infty x^{r+1} e^{-(x/2)(1/(c^\nu+j)+1)}$$

$$\times \int_{-x}^{+x} e^{(y/2)(1/(c^\nu+j)-1)}\,dy\,dx$$

$$= \Gamma(r+2) \sum_{j=0}^\infty R_j \frac{1-c^{(r+2)(\nu+j)}}{1-c^{\nu+j}}, \tag{5.26}$$

where, if $\nu = j = 0$, $(1-c^{(r+2)(\nu+j)})/(1-c^{\nu+j})$ is interpreted as $(r+2)$.

For the case $r = -1$, we similarly obtain

$$E\left[\int_0^\infty (\log(v+\lambda Z c^\nu)) e^{-v}\,dv\right] = -\gamma - (\log(c)) \sum_{j=0}^\infty R_j \frac{(\nu+j)c^{\nu+j}}{1-c^{\nu+j}}, \tag{5.27}$$

where, if $\nu = j = 0$, $(\nu+j)/(1-c^{\nu+j})$ is interpreted as $-1/\log(c)$.

Equations (5.26) and (5.27) can be substituted into (5.14), (5.18), (5.19), and (5.23). For $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(r+\beta) > -2$, $r \neq -1$, this yields

$$E\left[\int_0^\infty (Y(t) + Z c^{N(t)})^r c^{\beta N(t)}\,dt\right] = -\frac{E[Z^{r+1}]}{r+1} + \frac{1}{(r+1)\lambda^{r+1}} \left(\prod_{\mu=0}^\infty (1-c^{r+1+\beta+\mu})\right)$$

$$\times \sum_{\nu=0}^\infty \frac{c^{\beta\nu} E\left[\int_0^\infty (v+\lambda Z c^\nu)^{r+1} e^{-v} dv\right]}{(1-c)\cdots(1-c^\nu)}$$

$$= \frac{\Gamma(r+1)}{\lambda^{r+1}} \left(\prod_{k=1}^\infty \frac{1-c^{r+1+k}}{1-c^k}\right)$$

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$$+\frac{\Gamma(r+1)}{\lambda^{r+1}}\left(\prod_{k=1}^{\infty}\frac{1-c^{r+\beta+k}}{1-c^k}\right)$$

$$\times\sum_{\mu=0}^{\infty}c^{\beta\mu}\frac{1-c^{(r+2)\mu}}{1-c^{\mu}}\frac{(1-c^{1-\beta})\cdots(1-c^{\mu-\beta})}{(1-c)\cdots(1-c^{\mu})}, \tag{5.28}$$

where, if $\mu=0$, $(1-c^{(r+2)\mu})/(1-c^{\mu})$ is interpreted as $(r+2)$.

The first component on the RHS of (5.28) is a function of r only and has only one singularity (a simple pole in $r=-1$). The other singularities of $\Gamma(r+1)$ are annihilated by the infinite product. The infinite sum in the second component of the RHS of (5.28) converges for $\mathrm{Re}(\beta)>0$ and $\mathrm{Re}(r+\beta)>-2$, We have not succeeded in sufficiently simplifying that infinite sum in the second component of the RHS of (5.28). Because of the first remark following (5.28) the second component has, for $\mathrm{Re}(\beta)>0$, as a function of r, only one singularity, namely a simple pole in $r=-1$.

Similarly, (5.27) yields

$$E\left[\int_0^{\infty}\frac{c^{\beta N(t)}}{(Y(t)+Zc^{N(t)})}\,dt\right]=-E[\log(\lambda Z)]-(\log(c))$$

$$\times\sum_{j=0}^{\infty}\frac{c^{\beta+j}}{1-c^{\beta+j}}+\left(\prod_{j=0}^{\infty}(1-c^{\beta+j})\right)$$

$$\times\sum_{\nu=0}^{\infty}\frac{c^{\beta\nu}E\left[\int_0^{\infty}(\log(\nu+\lambda Zc^{\nu}))e^{-\nu}\,d\nu\right]}{(1-c)\cdots(1-c^{\nu})}$$

$$=(\log(c))\left(\left(\sum_{j=1}^{\infty}\frac{c^j}{1-c^j}\right)-\left(\sum_{j=0}^{\infty}\frac{c^{\beta+j}}{1-c^{\beta+j}}\right)\right)$$

$$-(\log(c))\left(\prod_{j=0}^{\infty}\frac{1-c^{\beta+j}}{1-c^{j+1}}\right)$$

$$\times\sum_{\mu=0}^{\infty}\frac{\mu c^{(1+\beta)\mu}}{1-c^{\mu}}\frac{(1-c^{1-\beta})\cdots(1-c^{\mu-\beta})}{(1-c)\cdots(1-c^{\mu})}, \tag{5.29}$$

where, if $\mu=0$, $\mu/(1-c^{\mu})$ is interpreted as $-1/\log(c)$. We have not succeeded in sufficiently simplifying the infinite sum in the second component of the RHS of (5.29). It converges for $\mathrm{Re}(\beta)>-1$. The first component converges for $\mathrm{Re}(\beta)>0$.

It might be of interest to observe that with $\beta=1$, the last expression reduces to

$$E\left[\int_0^{\infty}\frac{c^{N(t)}}{Y(t)+Zc^{N(t)}}\,dt\right]=1. \tag{5.30}$$

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We can use (5.18) to obtain that for any $n \in \{0, 1, \cdots\}$,

$$E\left[\int_0^\infty (Y(t) + Zc^{N(t)})^r c^{\beta N(t)}\, dt\right]$$
$$= -\sum_{j=1}^{n} \lambda^{j-1} \frac{E[Z^{r+j}]}{r+j} \left(\prod_{\mu=1}^{j-1} \frac{1 - c^{r+\beta+\mu}}{r+\mu}\right)$$
$$+ \lambda^n \left(\prod_{\mu=1}^{n} \frac{1 - c^{r+\beta+\mu}}{r+\mu}\right) E\left[\int_0^\infty (Y(t) + Zc^{N(t)})^{r+n} c^{\beta N(t)}\, dt\right]. \tag{5.31}$$

For $r = -k \in \{-2, -3, \ldots\}$, we choose $n = k - 1$ and use (5.29). Otherwise, we choose n such that $\mathrm{Re}(r + n + \beta) > -2$. In that case, it is an interesting problem to choose the "best" value of n.

It must be noted that although in (5.28)–(5.31), $X(t) = (Y(t) + Zc^{N(t)})$ always has the stationary distribution (i.e., the distribution of Z), no such simple result holds for $X(t)^r c^{\beta N(t)} = (Y(t) + Zc^{N(t)})^r c^{\beta N(t)}$.

Because "ordinary" convergence implies convergence in the sense of Abel, (5.15) also shows (for convenience of the reader, we changed the sign of r) that if $\mathrm{Re}(r) > 2$, then

$$E\left[\int_0^\infty \frac{1}{(Y(t) + xc^{N(t)})^r} c^{(r-2)N(t)}\, dt\right] = \lim_{\beta \downarrow r-2} \left[\int_0^\infty \frac{1}{(Y(t) + xc^{N(t)})^r} c^{\beta N(t)}\, dt\right]$$
$$= \frac{1}{(r-1)x^{r-1}} + \frac{(1-c)\lambda}{c(r-1)(r-2)x^{r-2}}. \tag{5.32}$$

Another case where (5.14) gives a simple answer is the case $r + \beta = -1$. For ease of reading, we again change the sign of r and obtain the following: If $\mathrm{Re}(r) > 1$, then

$$E\left[\int_0^\infty \frac{1}{(Y(t) + xc^{N(t)})^r} c^{(r-2)N(t)}\, dt\right] = \frac{1}{(r-1)x^{r-1}}. \tag{5.33}$$

6. EXPECTED VALUES OF INTEGRALS OF DIFFERENCES

In this section we obtain expressions for

$$D(x_1, x_2, r, \beta) = E\left[\int_0^\infty ((Y(t) + x_1 c^{N(t)})^r - (Y(t) + x_2 c^{N(t)})^r) c^{\beta N(t)}\, dt\right] \tag{6.1}$$

and

$$D(x, Z, r, \beta) = E\left[\int_0^\infty ((Y(t) + xc^{N(t)})^r - (Y(t) + Zc^{N(t)})^r) c^{\beta N(t)}\, dt\right]; \tag{6.2}$$

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that is, we evaluate integrals of differences as in (5.1) and their expected values and also similar differences between starting with $X(0) = x$ and starting with $X(0) = Z$.

From (5.16) we immediately obtain

$$D(x_1, x_2, r, \beta) = \frac{\Gamma(r+1)}{\lambda^{r+1}} \left(\prod_{\mu=1}^{\infty} \frac{1 - c^{r+\beta+\mu}}{1 - c^{\beta+\mu}} \right) \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} (x_1^n - x_2^n) \left(\prod_{\mu=1}^{n-1} (1 - c^{\beta+\mu}) \right)$$

$$- \frac{1}{\lambda^{r+1}} \sum_{n=1}^{\infty} \frac{\lambda^{r+n} (x_1^{r+n} - x_2^{r+n})}{(r+1) \cdots (r+n)} \left(\prod_{\mu=1}^{n-1} (1 - c^{r+\beta+\mu}) \right). \tag{6.3}$$

In (6.3), if $r + n = 0$, then $(x_1^{r+n} - x_2^{r+n})/(r+n)$ is interpreted as $\log(x_1/x_2)$. Both components on the RHS still have singularities in $r = -1, -2, \ldots$. These singularities still annihilate each other. As a function of β, the first component on the RHS of (6.3) has singularities (simple poles) in $\beta = -1, -2, \ldots$.

For $r = -1$, we obtain from (5.23)

$$D(x_1, x_2, -1, \beta) = -\log\left(\frac{x_1}{x_2}\right) - \gamma \sum_{\nu=1}^{\infty} \frac{\lambda^\nu (x_1^\nu - x_2^\nu)}{\nu!} (1 - c^\beta) \cdots (1 - c^{\beta+\nu-1})$$

$$- \left(\prod_{j=0}^{\infty} (1 - c^{\beta+j}) \right)$$

$$\times \sum_{\nu=0}^{\infty} \frac{c^{\beta\nu} \left(e^{\lambda x_1 c^\nu} \int_0^{\lambda x_1 c^\nu} (\log(u)) e^{-u} du - e^{\lambda_2 c^\nu} \int_0^{\lambda x_2 c^\nu} (\log(u)) e^{-u}\, du \right)}{(1-c) \cdots (1 - c^\nu)} \tag{6.4}$$

For $r \in \{-2, -3, \ldots\}$, we use (5.19) and (6.4). We leave this result as an exercise for the reader.

In order to prepare for (1.16), we obtain, for $\mathrm{Re}(r) > -2$, $r \neq -1$, and $\mathrm{Re}(\beta) > -1$,

$$E\left[\int_0^{\infty} ((Y(t) - xc^{N(t)})^r - (Y(t) - Zc^{N(t)})^r) c^{\beta N(t)}\, dt \right]$$

$$= -\frac{x^{r+1} - E[Z^{r+1}]}{r+1} + \frac{1}{(r+1)\lambda^{r+1}} \left(\prod_{\mu=0}^{\infty} (1 - c^{r+1+\beta+\mu}) \right)$$

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$$\times \sum_{\nu=0}^{\infty} \frac{c^{\beta\nu}}{(1-c)\cdots(1-c^{\nu})} \left(\int_0^{\infty} (\nu + \lambda x c^{\nu})^{r+1} e^{-\nu}\, d\nu \right.$$

$$\left. - \Gamma(r+2) \sum_{j=0}^{\infty} R_j \frac{1-c^{(r+2)(\nu+j)}}{1-c^{\nu+j}} \right). \tag{6.5}$$

Now,

$$\int_0^{\infty} (\nu + \lambda x c^{\nu})^{r+1} e^{-\nu}\, d\nu - \Gamma(r+2) \sum_{j=0}^{\infty} R_j \frac{1-c^{(r+2)(\nu+j)}}{1-c^{\nu+j}}$$

$$= e^{\lambda x c^{\nu}} \Gamma(r+2) - e^{\lambda x c^{\nu}} \gamma(r+2, \lambda x c^{\nu}) - \Gamma(r+2) \sum_{j=0}^{\infty} R_j \frac{1-c^{(r+2)(\nu+j)}}{1-c^{\nu+j}}$$

$$= (e^{\lambda x c^{\nu}} - 1)\Gamma(r+2) - e^{\lambda x c^{\nu}} \gamma(r+2, \lambda x c^{\nu}) - \Gamma(r+2) \left(1 - \sum_{j=0}^{\infty} R_j \frac{1-c^{(r+2)(\nu+j)}}{1-c^{\nu+j}} \right)$$

$$= \Gamma(r+2)(e^{\lambda x c^{\nu}} - 1) - e^{\lambda x c^{\nu}} \gamma(r+2, \lambda x c^{\nu}) - \Gamma(r+2) \sum_{j=0}^{\infty} R_j c^{\nu+j} \frac{1-c^{(r+1)(\nu+j)}}{1-c^{\nu+j}}, \tag{6.6}$$

which after some (by now standard) arithmetic yields

$$E\left[\int_0^{\infty} ((Y(t) - x c^{N(t)})^r - (Y(t) - Z c^{N(t)})^r) c^{\beta N(t)}\, dt \right]$$

$$= \frac{\Gamma(r+1)}{\lambda^{r+1}} \sum_{n=1}^{\infty} \frac{(\lambda x)^n}{n!} \left(\prod_{\mu=0}^{\infty} \frac{1-c^{r+1+\beta+\mu}}{1-c^{n+\beta+\mu}} \right)$$

$$- \frac{1}{\lambda^{r+1}} \sum_{n=1}^{\infty} \frac{(\lambda x)^{r+n}}{(r+1)\cdots(r+n)} (1-c^{r+1+\beta}) \cdots (1-c^{r+n-1+\beta})$$

$$+ \frac{\Gamma(r+1)}{\lambda^{r+1}} \left(\prod_{\mu=1}^{\infty} \frac{1-c^{r+1+\mu}}{1-c^{\mu}} \right)$$

$$- \frac{\Gamma(r+1)}{\lambda^{r+1}} \left(\prod_{\mu=1}^{\infty} \frac{1-c^{r+\beta+\mu}}{1-c^{\mu}} \right)$$

$$\times \sum_{n=0}^{\infty} c^{(1+\beta)n} \frac{1-c^{(r+1)n}}{1-c^{n}} \frac{(1-c^{1-\beta}) \cdots (1-c^{n-\beta})}{(1-c)\cdots(1-c^{n})}, \tag{6.7}$$

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where, if $n = 0$, $(1 - c^{(r+1)n})/(1 - c^n)$ is interpreted as $r + 1$. Whereas the derivation uses the fact that $\mathrm{Re}(r) > -2$, we see by analytic continuation that the result holds for $\mathrm{Re}(\beta) > -1$ and $\mathrm{Re}(r + \beta) > -2$, $r \notin \{-1, -2, \ldots\}$.

For $r = -1$, we combine (5.23) and (5.29) and obtain

$$
\begin{aligned}
&E\left[\int_0^\infty \left(\frac{1}{Y(t) + xc^{N(t)}} - \frac{1}{Y(t) + Zc^{N(t)}}\right) c^{\beta N(t)}\, dt\right] \\
&\quad = -\log(\lambda x) - \gamma \sum_{\nu=0}^{\infty} \frac{(\lambda x)^\nu}{\nu!}(1 - c^\theta)\cdots(1 - c^{\beta+\nu-1}) - (\log(c)) \sum_{j=1}^{\infty} \frac{c_j}{1 - c_j} \\
&\qquad - \left(\prod_{j=0}^{\infty}(1 - c^{\beta+j})\right) \sum_{\nu=0}^{\infty} \frac{c^{\beta\nu} e^{\lambda x c^\nu} \int_0^{\lambda x c^\nu} \log(u) e^{-u}\, du}{(1 - c)\cdots(1 - c^\nu)} \\
&\qquad - \log(c) \left(\prod_{j=0}^{\infty} \frac{1 - c^{\beta+j}}{1 - c^{j+1}}\right) \sum_{\mu=0}^{\infty} \frac{\mu c^{(1+\beta)\mu}}{1 - c^\mu} \frac{(1 - c^{1-\beta})\cdots(1 - c^{\mu-\beta})}{1 - c^\mu}, \qquad \textbf{(6.8)}
\end{aligned}
$$

where, if $\mu = 0$, $\mu/(1 - c^\mu)$ still is interpreted as $-1/\log(c)$. Equation (6.8) converges for $\mathrm{Re}(\beta) > -1$.

The case

$$
E\left[\int_0^\infty ((Y(t) + xc^{N(t)})^r - (Y(t) + Zc^{N(t)})^r) c^{\beta N(t)}\, dt\right] \qquad \textbf{(6.9)}
$$

with $\mathrm{Re}(\beta) > -1$ and $\mathrm{Re}(r + \beta) \le -2$ is left as an exercise for the reader.

Note that (6.3), (6.4), (6.7), and (6.8), as functions of β, are analytic on $\mathrm{Re}(\beta) > -1$, so that in the next section setting $\beta = 0$ will not be a problem.

7. THE CASE $\beta = 0$

Equation (6.3) with $\beta = 0$ yields

$$
\begin{aligned}
D(x_1, x_2, r, 0) &= \frac{\Gamma(r+1)}{\lambda^{r+1}} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!}(x_1^n - x_2^n) \left(\prod_{\mu=1}^{\infty} \frac{1 - c^{r+\mu}}{1 - c^{n-1+\mu}}\right) \\
&\quad - \sum_{n=1}^{\infty} \lambda^{n-1} \frac{x_1^{r+n} - x_2^{r+n}}{r+n} \left(\prod_{\mu=1}^{n-1} \frac{1 - c^{r+\mu}}{r+\mu}\right), \qquad \textbf{(7.1)}
\end{aligned}
$$

where, if $r + n = 0$, $(x_1^{r+n} - x_2^{r+n})/(r + n)$ is interpreted as $\log(x_1/x_2)$, and if $r + \mu = 0$, $(1 - c^{r+\mu})/(r + \mu)$ is interpreted as $-\log(c)$.

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If $r = -k \in \{-1, -2, \ldots\}$, then $(\Gamma(r+1))(1-c^{r+k})$ is interpreted as $(-1)^k \log(c)/(k-1)!$. If $r = k \in \{0, 1, 2, \ldots\}$ (7.1) reduces to

$$D(x_1, x_2, k, 0) = \frac{k!}{\lambda^{k+1}} \sum_{\nu=1}^{k} \lambda^\nu \frac{x_1^\nu - x_2^\nu}{\nu!} \frac{1}{(1-c^\nu)\cdots(1-c^k)}. \tag{7.2}$$

If $r = -1$, (7.1) reduces to

$$D(x_1, x_2, -1, 0) = -\log\left(\frac{x_1}{x_2}\right). \tag{7.3}$$

The case $D(x_1, x_2, -k, 0)$ with $k \in \{2, 3, \ldots\}$ is left as an exercise for the reader. From (6.7) we see that for $\mathrm{Re}(r) > -2$, $r \neq -1$,

$$\begin{aligned} D(x, Z, r, 0) = {} & \frac{\Gamma(r+1)}{\lambda^{r+1}} \sum_{n=1}^{\infty} \frac{(\lambda x)^n}{n!} \left(\prod_{\mu=0}^{\infty} \frac{1-c^{r+1-\mu}}{1-c^{n+\beta+\mu}} \right) \\ & - \frac{1}{\lambda^{r+1}} \sum_{n=1}^{\infty} \frac{(\lambda x)^{r+n}}{(r+1)\cdots(r+n)} (1-c^{r+1})\cdots(1-c^{r+n-1}) \\ & + \frac{\Gamma(r+1)}{\lambda^{r+1}} \left(\prod_{\mu=1}^{\infty} \frac{1-c^{r+1+\mu}}{1-c^\mu} \right) \\ & - \frac{\Gamma(r+1)}{\lambda^{r+1}} \left(\prod_{\mu=1}^{\infty} \frac{1-c^{r+\mu}}{1-c^\mu} \right) \sum_{n=0}^{\infty} c^n \frac{1-c^{(r+1)n}}{1-c^n}, \end{aligned} \tag{7.4}$$

where, if $n = 0$, $(1-c^{(r+1)n})/(1-c^n)$ is interpreted as $r+1$.

For $r = -1$, (6.8) gives

$$D(x, Z, -1, 0) = -\log(\lambda x) - \gamma - (\log(c)) \sum_{j=1}^{\infty} \frac{c^j}{1-c^j}. \tag{7.5}$$

The case of $D(x, Z, r, 0)$ with $\mathrm{Re}(r) \leq -2$ is left as an exercise for the reader.

8. TRANSFORM TECHNIQUES

We define

$$\psi(x, t, s, z) = E[\exp\{-s(Y(t) + xc^{N(t)})\} z^{N(t)}]. \tag{8.1}$$

Let y be the *last* point of the Poisson process $(\tau_k)_k$ in the interval $(0, t]$. With probability $e^{-\lambda t}$, there is no such point. This gives

$$\psi(x, t, s, z) = e^{-\lambda t} e^{-s(x+t)} + \lambda z \int_0^t e^{-(s+\lambda)(t-y)} \psi(x, y, cs, z)\, dy. \tag{8.2}$$

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We define

$$\eta(x,\sigma,s,z) = \int_0^\infty e^{-\sigma t}\psi(x,t,s,z)\,dt. \tag{8.3}$$

With (8.2), this gives

$$\eta(x,\sigma,s,z) = \frac{1}{\sigma+\lambda+s}(e^{-sx} + \lambda z\,\eta(x,\sigma,cs,z)). \tag{8.4}$$

Repeating this procedure, we see that for all $k \in \{1,2,\ldots\}$,

$$\eta(x,\sigma,s,z) = \sum_{j=0}^{k-1} \frac{(\lambda z)^j \exp\{-sxc^j\}}{\prod_{v=0}^{j}(\sigma+\lambda+c^v s)} + \frac{(\lambda z)^k}{\prod_{v=0}^{k-1}(\sigma+\lambda+c^v s)}\eta(x,\sigma,c^k s,z). \tag{8.5}$$

For $|z| \le 1$, $\mathrm{Re}(\sigma) > 0$, and $\mathrm{Re}(s) > 0$, this gives

$$\eta(x,\sigma,s,z) = \sum_{j=0}^{\infty} \frac{(\lambda z)^j \exp\{-sxc^j\}}{\prod_{v=0}^{j}(\sigma+\lambda+c^v s)}. \tag{8.6}$$

For use in the the proof of Theorem 2 in the next section we rewrite (8.6) as

$$\eta(x,\sigma,s,z) = \sum_{k=0}^{\infty}\left(\frac{\lambda z}{\sigma+\lambda}\right)^k \frac{e^{-sxc^k}}{(\sigma+\lambda)\prod_{j=0}^{k}(1+[s/(\sigma+\lambda)]c^j)}. \tag{8.7}$$

By partial fraction expansion.we then prove that

$$\begin{aligned}\eta(x,\sigma,s,z) = {} & \frac{e^{-sx}}{\sigma+\lambda+s} + \sum_{k=1}^{\infty} z^k e^{-sxc^k} \times \sum_{j=0}^{k}(-1)^j c^{\frac{1}{2}j(j+1)} \\ & \times \left(\prod_{v=1}^{j}\frac{1}{1-c^v}\prod_{v=1}^{k-j}\frac{1}{c-v}\right)\left(\frac{\lambda}{\sigma+\lambda}\right)^k \frac{1}{\sigma+\lambda+sc^j}.\end{aligned} \tag{8.8}$$

Hence,

$$\begin{aligned}\psi(x,t,s,z) = {} & e^{-sx}e^{-t(\lambda+s)} + e^{-\lambda t}\sum_{k=1}^{\infty}(\lambda z)^k \sum_{j=0}^{k}(-1)^j c^{\frac{1}{2}j(j+1)} \\ & \times \left(\prod_{v=1}^{j}\frac{1}{1-c^v}\prod_{v=1}^{k-j}\frac{1}{1-c^v}\right) e^{-sxc^k}\int_0^t \frac{y^{k-1}}{(k-1)!}e^{-(t-y)sc^j}\,dy.\end{aligned} \tag{8.9}$$

In Section 9 we will formulate real-domain results for transient densities, which are a direct consequence of (8.9).

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Before going to those transient densities, we use (8.5) to give an alternative proof of (1.5). Using (3.3) and (3.25), we obtain that for $|z| \le 1, s \downarrow 0$,

$$\begin{aligned}\psi(x,t,0,z) - \psi(x,t,s,z) &= \exp\{-\lambda t(1-z)\} - \psi(x,t,s,z)\\ &= s\left(\frac{\exp\{\lambda t(1-z)\}}{\lambda z(1-c)} + \left(x - \frac{1}{\lambda z(1-c)}\right)\right.\\ &\quad \left.\times \exp\{-\lambda t(1-cz)\}\right) + o(s)(s \downarrow 0), \qquad \textbf{(8.10)}\end{aligned}$$

where, for all $0 < x_0 < \infty$, the $o(s)$ is uniform on $|z| \le 1$, $0 \le x \le x_0$, $0 \le t < \infty$. Hence, for $k \in \{1, 2, \ldots\}$, we have

$$\begin{aligned}\eta(x,\sigma,s,z) &= \sum_{j=0}^{k-1} \frac{(\lambda z)^j \exp\{-sxc^j\}}{\prod_{\nu=0}^{j}(\sigma + \lambda + c^\nu s)}\\ &\quad + \frac{(\lambda z)^k}{\prod_{\nu=0}^{k-1}(\sigma + \lambda + c^\nu s)}\left(\frac{1}{\sigma + \lambda(1-z)} - \frac{sc^k}{\sigma + \lambda(1-cz)}\right.\\ &\quad \left.\times\left(x + \frac{1}{\sigma + \lambda(1-c)}\right) + \frac{o(sc^k)(k \uparrow \infty)}{\sigma}\right), \qquad \textbf{(8.11)}\end{aligned}$$

where, for every $0 < s_0 < \infty$, $0 < x_0 < \infty$, the $o(sc^k)$ is uniform on $|z| \le 1$, $0 \le x \le x_0$, $0 < s \le s_0$, $0 \le \sigma < \infty$. Hence,

$$\psi(x,\infty,s,1) = \lim_{\sigma \downarrow 0} \sigma\eta(x,\sigma,s,1) = \left(\prod_{\nu=0}^{k-1} \frac{1}{1 + s\frac{c^\nu}{\lambda}}\right)(1 + o(sc^k))(k \uparrow \infty). \qquad \textbf{(8.12)}$$

We (again) obtain (1.5) by letting $k \uparrow \infty$.

9. TRANSIENT DISTRIBUTIONS

The following lemma makes it easier to relate the various formulations of the main result in this section.

LEMMA 5: *For all $k \in \{1, 2, \ldots\}$ and all t and u,*

$$\sum_{j=0}^{k} (-1)^j c^{\frac{1}{2}j(j+1)} c^{-kj} \left(\prod_{\nu=1}^{j} \frac{1}{1-c^\nu}\right)\left(\prod_{\nu=1}^{k-j} \frac{1}{1-c^\nu}\right)(tc^j + u)^{k-1} = 0. \qquad \textbf{(9.1)}$$

This lemma will be proven in Appendix A.

In the following, it always is understood that $x \ge 0$ and $t > 0$. The main result in this section is the following:

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THEOREM 2: *For $k \geq 1$, choose any $y_{\min}$ and $y_{\max}$ with*

$$-\infty \leq y_{\min} \leq (x+t)c^k < xc^k + t \leq y_{\min} \leq +\infty \tag{9.2}$$

(*where $y_{\min}$ and $y_{\max}$ are allowed to depend on k*). *Then (for $k \geq 1$)*

$$\begin{aligned}
&\frac{d}{dy}P\{N(t)=k, X(t)\leq y | X(0)=x\} \\
&\quad = \frac{e^{-\lambda t}\lambda^k}{(k-1)!}\sum_{j=0}^{k}(-1)^j c^{\frac{1}{2}j(j+1)}c^{-kj}\left(\prod_{\nu=1}^{j}\frac{1}{1-c^\nu}\right)\left(\prod_{\nu=1}^{k-j}\frac{1}{1-c^\nu}\right) \\
&\qquad \times (xc^k + tc^j - y)^{k-1}\chi(\, y_{\min} < y < xc^k + tc^j) \\
&\quad = \frac{e^{-\lambda t}\lambda^k}{(k-1)!}\sum_{k}^{j=0}(-1)^{k-j} c^{\frac{1}{2}j(j+1)}c^{-kj}\left(\prod_{\nu=1}^{j}\frac{1}{1-c^\nu}\right)\left(\prod_{\nu=1}^{k-j}\frac{1}{1-c^\nu}\right) \\
&\qquad \times (\, y - (xc^k + tc^j))^{k-1}\chi(xc^k + tc^j < y < y_{\max}).
\end{aligned} \tag{9.3}$$

Because of (9.1), the actual values of $y_{\min}$ and $y_{\max}$ do not matter as long as they satisfy (9.2).

For $k = 0$,

$$P\{N(t)=0, X(t)\leq y | X(0)=x\}\begin{cases} 0 & \text{if } y < x+t \\ e^{-\lambda t} & \text{if } y \geq x+t. \end{cases} \tag{9.4}$$

For $y_{\min} < y < y_{\max}$ and $k \geq 1$, choose $n = n(\, y, k)$ such that

$$n = \begin{cases} k & \text{if } y \leq (x+t)c^k \\ \text{such that } xc^k + tc^{n+1} < y \leq xc^k - tc^n & \text{if } (x+t)c^k < y \leq xc^k + t \\ -1 & \text{if } xc^k + t < y. \end{cases} \tag{9.5}$$

The two expressions in (9.3) are reconciled by observing that

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^j c^{\frac{1}{2}j(j+1)}c^{-kj}\left(\prod_{\nu=1}^{j}\frac{1}{1-c^\nu}\right)\left(\prod_{\nu=1}^{k-j}\frac{1}{1-c^\nu}\right) \\
&\quad \times (xc^k + tc^j - y)^{k-1}\chi(\, y_{\min} < y < xc^k + tc^j) \\
&\qquad = \sum_{j=0}^{n}(-1)^j c^{\frac{1}{2}j(j+1)}c^{-kj}\left(\prod_{\nu=1}^{j}\frac{1}{1-c^\nu}\right)\left(\prod_{\nu=1}^{k-j}\frac{1}{1-c^\nu}\right)(xc^k + tc^j - y)^{k-1}
\end{aligned}$$

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$$= -\sum_{j=n+1}^{k} (-1)^j c^{\frac{1}{2}j(j+1)} c^{-kj} \left(\prod_{\nu=1}^{j} \frac{1}{1-c^\nu}\right)\left(\prod_{\nu=1}^{k-j} \frac{1}{1-c^\nu}\right)(xc^k + tc^j - y)^{k-1}$$

$$= -\sum_{j=n+1}^{k} (-1)^{k-j} c^{\frac{1}{2}j(j+1)} c^{-kj} \left(\prod_{\nu=1}^{j} \frac{1}{1-c^\nu}\right)\left(\prod_{\nu=1}^{k-j} \frac{1}{1-c^\nu}\right)(\, y - (xc^k + tc^j))^{k-1}$$

$$= \sum_{j=0}^{k} (-1)^{k-j} c^{\frac{1}{2}j(j+1)} c^{-kj} \left(\prod_{\nu=1}^{j} \frac{1}{1-c^\nu}\right)\left(\prod_{\nu=1}^{k-j} \frac{1}{1-c^\nu}\right)$$

$$\times (\, y - (xc^k + tc^j))^{k-1} \chi(xc^k + tc^j < y < y_{\max}). \tag{9.6}$$

For numerical purposes, we, of course, usually choose $y_{\min} = (x+t)c^k$ and $y_{\max} = xc^k + t$, and if $n+1 < k-n$, we use the sum from zero to n, whereas if $n+1 > k-n$, we use the sum from $n+1$ to k. If $n+1 = k-n$, there does not seem to be a reason to prefer one sum over the other.

The first formulation of Theorem 2, with $y_{\min} = xc^k$, is a direct consequence of (8.9). The second formulation, and formulations with more general $y_{\min}$ and $y_{\max}$, then follow from Lemma 5.

10. MOMENTS FROM THEOREM 2

We define

$$\mu(x,t,k,r) = E[(X(t))^r | X(0) = x, N(t) = k] \tag{10.1}$$

and

$$\begin{aligned} \nu(x,t,k,r) &= E[(X(t))^r \chi(N(t) = k) | X(0) = x] \\ &= \frac{(\lambda t)^k}{k!} e^{-\lambda t} \nu(x,t,k,r). \end{aligned} \tag{10.2}$$

In Theorem 2, if we choose $x = 0$ and $y_{\min} = 0$, we get, for $k \geq 1$,

$$E[(Y(t))^r \chi(N(t) = k)] = \frac{e^{-\lambda t} \lambda^k}{(k-1)!} \sum_{j=0}^{k} (-1)^j c^{\frac{1}{2}j(j+1)} c^{-kj}$$

$$\times \left(\prod_{\nu=1}^{j} \frac{1}{1-c^\nu}\right)\left(\prod_{\nu=1}^{k-j} \frac{1}{1-c^\nu}\right)$$

$$\times \int_{xc^k}^{xc^k + tc^j} (\, y - xc^k)^r (xc^k + tc^j - y)^{k-1}\, dy$$

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$$= e^{-\lambda t} t^r \frac{(\lambda t)^k}{(r+1)(r+2)\cdots(r+k)} \times \sum_{j=0}^{k} (-1)^j c^{\frac{1}{2}(j-1)j} c^{(r+1)j} \left(\prod_{\nu=1}^{j} \frac{1}{1-c^\nu} \right) \left(\prod_{\nu=1}^{k-j} \frac{1}{1-c^\nu} \right)$$

$$= e^{-\lambda t} t^r \frac{(\lambda t)^k}{(r+1)(r+2)\cdots(r+k)} \times \frac{(1-c^{r+1})(1-c^{r+2})\cdots(1-c^{r+k})}{(1-c)(1-c^2)\cdots(1-c^k)}. \tag{10.3}$$

The last step uses Lemma A.1 in Appendix A.

We find that the same formula holds for $k = 0$. Thus, we have proven the following.

THEOREM 3:

$$\nu(0,t,k,r) = E[(Y(t))^r \chi(N(t) = k)] = e^{-\lambda t} \frac{\lambda^k t^{k+r} \Gamma(r+1)}{\Gamma(r+k+1)} \frac{(1-c^{r+1})(1-c^{r+2})\cdots(1-c^{r+k})}{(1-c)(1-c^2)\cdots(1-c^k)}, \tag{10.4}$$

$$\mu(0,t,k,r) = E[(Y(t))^r | N(t) = k] = \Gamma(r+1) t^r \frac{k!}{\Gamma(k+r+1)} \frac{(1-c^{r+1})(1-c^{r+2})\cdots(1-c^{r+k})}{(1-c)(1-c^2)\cdots(1-c^k)}. \tag{10.5}$$

It is easily seen that if $t \uparrow \infty$, and $k \uparrow \infty$ while $k/t \to \lambda$, indeed $\mu(0,t,k,r)$ converges to $E[(X(\infty))^r]$ as in (1.6).

Equation (10.4) also gives

$$E[(Y(t))^r c^{\beta N(t)}] = \sum_{j=0}^{\infty} \nu(0,t,j,r) c^{\beta j} = \Gamma(r+1) t^r e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t c^\beta)^j}{\Gamma(j+r+1)} \times \frac{(1-c^{r+1})(1-c^{r+2})\cdots(1-c^{r+j})}{(1-c)(1-c^2)\cdots(1-c^j)}. \tag{10.6}$$

For $r = k$, integer (and nonnegative), we already had (3.17). We observe that (10.6) reduces to (3.17) if r is a nonnegative integer.

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If r is a nonnegative integer, then (for later use, we write the finite binomial sum, j from zero to r, as an infinite sum) we have

$$\begin{aligned}\mu(x,t,k,r) &= E[(Y(t)+xc^{N(t)})^r|N(t)=k]\\ &= \sum_{j=0}^{\infty}\binom{r}{j}(xc^k)^{r-j}\mu(0,t,k,j)\\ &= \Gamma(r+1)(xc^k)^r\sum_{j=0}^{\infty}\frac{(xc^k)^{-j}t^j}{\Gamma(r-j+1)}\frac{k!}{(k+j)!}\\ &\quad\times\frac{(1-c^{j+1})(1-c^{j+2})\cdots(1-c^{j+k})}{(1-c)(1-c^2)\cdots(1-c^k)}\\ &= (xc^k)^r\sum_{j=0}^{\infty}\frac{r(r-1)\cdots(r-j+1)}{(k+1)(k+2)\cdots(k+j)}\left(\frac{t}{xc^k}\right)^j\\ &\quad\times\frac{(1-c^{j+1})(1-c^{j+2})\cdots(1-c^{j+k})}{(1-c)(1-c^2)\cdots(1-c^k)}. \qquad \textbf{(10.7)}\end{aligned}$$

If r is not a nonnegative integer, (10.7) becomes an infinite sum and we have to check when it converges and when it is an identity.

For (10.7) to be valid it is sufficient that $P\{0\le Y(t)<xc^k\}=1$. Since $0\le Y(t)\le t$, it therefore is sufficient that $0\le t<xc^k$. In fact, (10.7) converges absolutely as long as $0\le t\le xc^k$ (we leave this as an exercise for the reader). From the material in Appendix B, (10.7) then actually holds as long as either r is a nonnegative integer (finite sum) or $0\le t\le xc^k$ (absolutely converging sum).

A similar argument can be made if $0\le xc^k\le t$. Since $0\le xc^k<t$ does not guarantee that $0\le xc^k<Y(t)$, this argument depends more heavily on the material in Appendix B. Nevertheless, we have that as long as either r is a nonnegative integer (finite sum) or $0\le xc^k\le t$ (absolutely converging sum), then

$$\begin{aligned}\mu(x,t,k,r) &= E[(Y(t)+xc^{N(t)})^r|N(t)=k]\\ &= \sum_{j=0}^{\infty}\binom{r}{j}(xc^k)^j\mu(0,t,k,r-j)\\ &= \Gamma(r+1)t^r\\ &\quad\times\sum_{j=0}^{\infty}\frac{(xc^k)^j t^{-j}}{j!}\frac{k!}{\Gamma(k+r-j+1)}\\ &\quad\times\frac{(1-c^{r-j+1})(1-c^{r-j+2})\cdots(1-c^{r-j+k})}{(1-c)(1-c^2)\cdots(1-c^k)}\end{aligned}$$

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$$= \frac{k!\Gamma(r+1)}{\Gamma(r+k+1)} t^r \times \sum_{j=0}^{\infty} \frac{r(r+k)(r+k-1)\cdots(r+k-j+1)}{j!} \left(\frac{x}{t}\right)^j \times \frac{(c^j - c^{r-j+1})(c^j - c^{r-j+2})\cdots(c^j - c^{r-j+k})}{(1-c)(1-c^2)\cdots(1-c^k)}. \quad \textbf{(10.8)}$$

If r is a nonnegative integer, then (10.7) and (10.8) both are finite sums and they are identical.

If $0 \le x \le t$, then $0 \le xc^k \le t$ for all $k \ge 0$. Thus, for $0 \le x \le l$, we can multiply (10.8) by $P\{N(t) = k\}$ and sum over k, thus obtaining that as long as either r is a nonnegative integer or $0 \le x \le t$, we have

$$E[(X(t))^r c^{\beta N(t)} | X(0) = x] = \Gamma(r+1) t^r e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(x/t)^j}{j!} \times \sum_{\nu=0}^{\infty} \frac{(c^j \lambda t c^\beta)^\nu}{\Gamma(\nu + r - j + 1)} \frac{(1-c^{r-j+1})(1-c^{r-j+2})\cdots(1-c^{r-j+\nu})}{(1-c)(1-c^2)\cdots(1-c^\nu)}. \quad \textbf{(10.9)}$$

If r is a nonnegative integer, the outer sum in (10.9) is a finite sum (j from zero to r) and is identical to (3.12), If r is not a nonnegative integer but $0 \le x \le t$, then (10.9) is an absolutely converging infinite sum and still is an identity. The latter statement is left as an exercise for the reader.

If $0 < t \le x$, we still have that $0 \le xc^k < t$ for k sufficiently large. Therefore, we can not use (10.7) the same way we used (10.8). It might be possible to obtain useful results by splitting the sum into the ranges

$$k < \frac{\log(x) - \log(t)}{|\log(c)|} \quad \text{and} \quad k \ge \frac{\log(x) - \log(t)}{|\log(c)|}.$$

We have not tried this.

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APPENDIX A
Some Auxiliary Results

LEMMA A.1: *For all* $k \in \{0, 1, \ldots\}$ *and all* a,

$$\frac{(1-a)(1-ca)(1-c^2a)\cdots(1-c^{k-1}a)}{(1-c)(1-c^2)(1-c^3)\cdots(1-c^k)} = \sum_{j=0}^{k}(-1)^j c^{\frac{1}{2}(j-1)j} a^j \prod_{v=1}^{j}\frac{1}{1-c^v}\prod_{v=1}^{k-j}\frac{1}{1-c^v}. \tag{A.1}$$

PROOF: A classical result in hypergeometric series (see, e.g., [40, pp. 7–8] for a proof and historical notes) is that for $|z| < 1$, $|c| < 1$, and a arbitrary,

$$\prod_{k=0}^{\infty}\frac{1-c^k az}{1-c^k z} = \sum_{k=0}^{\infty}\left(\prod_{j=0}^{k-1}\frac{1-c^j a}{1-c^{j+1}}\right)z^k. \tag{A.2}$$

We also know (see [38] for a simple proof or [40] for more background information) that for $|c| < 1$,

$$\prod_{k=0}^{\infty}\frac{1}{1+c^k z} = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{(1-c)(1-c^2)\ldots(1-c^k)} \tag{A.3}$$

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and

$$\prod_{k=0}^{\infty}(1+c^k z)=\sum_{k=0}^{\infty}\frac{c^{\frac{1}{2}(k-1)k}zk}{(1-c)(1-c^2)\dots(1-c^k)}. \tag{A.4}$$

The last two results also give a power series for (A.2):

$$\prod_{k=0}^{\infty}\frac{1}{1-c^k z}\prod_{k=0}^{\infty}(1-c^k az)=\sum_{k=0}^{\infty}z^k\sum_{j=0}^{k}(-1)^j c^{\frac{1}{2}(j-1)j}a^j \times\prod_{\nu=1}^{j}\frac{1}{1-c^\nu}\prod_{\nu=1}^{k-j}\frac{1}{1-c^\nu}. \tag{A.5}$$

Equating coefficients in the power series proves the result. ■

Equation (A.1) shows that

$$\sum_{j=0}^{k}(-1)^j c^{\frac{1}{2}(j-1)j}a^j\prod_{\nu=1}^{j}\frac{1}{1-c^\nu}\prod_{\nu=1}^{k-j}\frac{1}{1-c^\nu} =\begin{cases}\delta_{0,k} & \text{if } a=1=c^0\\ 0 & \text{if } k\geq 1 \text{ and } a=c^{-\mu},\mu\in\{0,1,2,\dots,k-1\}\\ \prod_{j=1}^{k-1}\frac{1-c^{\mu+j}}{1-c^j} & \text{if } a=c^\mu,\mu\in\{1,2,\dots\}\\ >0 & \text{if } -1<c<+1 \text{ and } -1<a<+1.\end{cases} \tag{A.6}$$

If $0<c<1$, we also know that the expression in (A.1) is positive for all $-\infty<a<1$.

With the binomial formula, the second case in (A.6) shows that if k, μ, and κ are integers with $1\leq\mu\leq k$ and $0\leq\kappa\leq\mu-1$, then

$$\sum_{j=0}^{k}(-1)^j c^{\frac{1}{2}j(j+1)}e^{-\mu j}\left(\prod_{\nu=1}^{j}\frac{1}{1-c^\nu}\prod_{\nu=1}^{k-j}\frac{1}{1-c^\nu}\right)(rc^j+\beta)^\kappa=0 \quad \text{for all } r,\beta. \tag{A.7}$$

This proves the lemma in Section 9.

Lemma A.2: *For all $|c|<1$, all z, and all κ, $k\in\{0,1,2,\dots\}$ with $\kappa\leq k$,*

$$\begin{aligned}&\sum_{\mu=0}^{k}(-1)^\mu c^{\frac{1}{2}\mu(\mu-1)-\mu(\kappa-1)}e^{zc^\mu}\left(\prod_{\nu=1}^{\mu}\frac{1}{1-c^\nu}\right)\left(\prod_{\nu=1}^{k-\mu}\frac{1}{1-c^\nu}\right)\\ &\quad=\sum_{j=\kappa}^{\infty}\frac{z^j}{j!}\prod_{\nu=1}^{k}\frac{1-c^{j-\kappa+\nu}}{1-c^\nu}\\ &\quad=\sum_{j=0}^{\infty}\frac{z^{j+\kappa}}{(j+\kappa)!}\prod_{\nu=1}^{k}\frac{1-c^{j+\nu}}{1-c^\nu}.\end{aligned} \tag{A.8}$$

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PROOF: The result trivially holds if $k = \kappa = 0$. If $k \geq 1$, write, on the LHS, e^{zc^μ} as a power series in zc^μ, change the order of summation, and use (A.6) and (A.1). ■

Similar results can, of course, be obtained if e^{zc^μ} is replaced by $f(zc^\mu)$, where $f(\cdot)$ can be written as a power series.

Somewhat similar results are the following lemmas

LEMMA A.3:

$$\sum_{\nu=0}^{\infty} \frac{c^{\beta\nu} e^{\lambda x c^\nu}}{(1-c)\cdots(1-c^n u)} = \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} \sum_{\nu=0}^{\infty} \frac{c^{(\beta+k)\nu}}{(1-c)\cdots(1-c^\nu)}$$
$$= \left(\prod_{\nu=0}^{\infty} \frac{1}{1-c^{\beta+\nu}} \right) \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} (1-c^\beta)\cdots(1-c^{k-1+\beta}), \quad \textbf{(A.9)}$$

LEMMA A.4:

$$e^{\lambda x c^\nu} \gamma(r+2, \lambda x c^\nu) = (\lambda x c^\nu)^{r+2} \int_0^1 s^{r+2-1} e^{(1-s)\lambda x c^\nu} ds$$
$$= (\lambda x c^\nu)^{r+2} \int_0^1 s^{r+2-1} \sum_{n=0}^{\infty} \frac{(1-s)^n (\lambda x c^\nu)^n}{n!} ds$$
$$= \sum_{n=0}^{\infty} \frac{(\lambda x c^\nu)^{r+2+n}}{n!} B(r+2, n+1)$$
$$= \sum_{n=0}^{\infty} \frac{(\lambda x c^\nu)^{r+2+n}}{(r+2)\cdots(r+2+n)}. \quad \textbf{(A.10)}$$

LEMMA A.5:

$$e^{\lambda x c^\nu} \int_0^{\lambda x c^\nu} (\log(u)) e^{-u} du = (e^{\lambda x c^\nu} - 1)\log(\lambda x c^\nu) - \sum_{k=1}^{\infty} \frac{(\lambda x c^\nu)^k}{k!} \sum_{n=1}^{k} \frac{1}{n}. \quad \textbf{(A.11)}$$

The following is often useful to rewrite expressions such as those in this article.

Observation A.1: If m and n are nonnegative integers, then

$$\frac{(1-c^{m+1})\cdots(1-c^{m+n})}{(1-c)\cdots(1-c^n)} = \frac{(1-c^{n+1})\cdots(1-c^{n+m})}{(1-c)\cdots(1-c^m)}. \quad \textbf{(A.12)}$$

This observation is easily verified by considering the two cases ($m \leq n$) and ($m \geq n$)4.

By changing the sign of z in (A.4), we obtain

$$\prod_{j=0}^{\infty} (1 - c^j z) = \sum_{j=0}^{\infty} (-1)^j \frac{c^{\frac{1}{2}(j-1)j} z^j}{(1-c)(1-c^2)\ldots(1-c^j)}, \quad \textbf{(A.13)}$$

which we recognize as the limit of (A.1) for $k \to \infty$. By differentiating (A.13) with respect to z, we obtain the following.

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Observation A.2:

$$\left(\sum_{j=0}^{\infty}\frac{c^j}{1-c^jz}\right)\left(\prod_{j=0}^{\infty}(1-c^jz)\right)=\sum_{j=1}^{\infty}\frac{j(-1)^{j-1}c^{\frac{1}{2}(j-1)j}z^{j-1}}{(1-c)\cdots(1-c^j)}. \tag{A.14}$$

APPENDIX B
Moments and Power Series

Suppose we have a function $f(\cdot)$ that has the power series

$$f(x)=\sum_{j=0}^{\infty}\frac{f^{(j)}(a)}{k!}(x-a)^j \tag{B.1}$$

with radius of convergence $R>0$. Suppose X is a random variable. When can we conclude that

$$E[f(X)]=\sum_{j=0}^{\infty}\frac{f^{(j)}(a)}{j!}E[(X-a)^j]? \tag{B.2}$$

Clearly, a sufficient condition is that there exists a $0<r<R$ with $P\{|X-a|<r\}=1$. However, (B.2) often holds even if $P\{|X-a|>R\}>0$. In that case, to prevent trivial exceptions, we always assume that $E[|X-a|^j]<\infty$, for all $j\geq 0$.

It is well known that as long as $f^{(n+1)}(\cdot)$ exists and is continuous over the necessary range,

$$f(x)=\sum_{j=0}^{n}\frac{f^{(j)}(a)}{j!}(x-a)^j+\int_a^x\frac{(x-u)^n}{n!}f^{(n+1)}(u)\,du; \tag{B.3}$$

therefore,

$$E[f(X)]=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}E[(X-a)^k]+E\left[\int_a^X\frac{(X-u)^n}{n!}f^{(n+1)}(u)\,du\right]. \tag{B.4}$$

Under the above conditions, a necessary and sufficient condition for (B.2) to hold is that

$$\lim_{n\to\infty}E\left[\int_a^X\frac{(X-u)^n}{n!}f^{(n+1)}(u)\,du\right]=0. \tag{B.5}$$

We consider the special case $f(x)=(C+x)^r=C^r(1+x/C)^r=x^r(1+C/x)^r$ with $C>0$. We see that as long as $E[X^j]<\infty$, for all $j\geq 0$,

$$E[(C+X)^r]=\sum_{j=0}^{\infty}\binom{r}{j}C^{r-j}E[X^j] \tag{B.6}$$

if and only if

$$\lim_{n\to\infty}\binom{r}{n}(r-n)E\left[\int_0^{X/C}\left(\frac{X}{C}-u\right)^n(1+u)^{r-(n+1)}\,du\right]=0, \tag{B.7}$$

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whereas as long as $E[X^{r-j}] < \infty$ for all $j \geq 0$,

$$E[(C+X)^r] = \sum_{j=0}^{\infty} \binom{r}{j} E[X^{r-j}]C^j \tag{B.8}$$

if and only if

$$\lim_{n\to\infty} \binom{r}{n} (r-n)E\left[X^r \int_0^{C/X} \left(\frac{C}{X} - u\right)^n (1+u)^{r-(n+1)}\, du\right] = 0. \tag{B.9}$$

If $P\{X \geq 0\} = 1$, (B.6) holds as long as

$$\lim_{n\to\infty} \binom{r}{n+i} E\left[\left(\frac{X}{C}\right)^{n+1}\right] = 0 \tag{B.10}$$

(take $n+1 > r$). If $P\{X > 0\} = 1$, (B.8) holds as long as

$$\lim_{n\to\infty} \binom{r}{n+1} E\left[\left(\frac{X}{C}\right)^{r-(n+1)}\right] = 0 \tag{B.11}$$

(take $n+1 > r$).

The upshot is that as long as $C > 0$ and X is a nonnegative random variable, each of (B.6) and (B.8) holds as long as the series on the RHS converges.

Moreover, we know that if $C > 0$, X is a nonnegative random variable, and $n+1 \geq r$, then

$$\left|E[(C+X)^r] - \sum_{j=0}^{n} \binom{r}{j} C^{r-j}E[X^j]\right| \leq \left|\binom{r}{n+1} C^{r-(n+1)}E[X^{n+1}]\right|, \tag{B.12}$$

whereas if $C > 0$, X is a positive random variable, and $n+1 \geq r$, then

$$\left|E[(C+X)^r] - \sum_{j=0}^{n} \binom{r}{j} C^{j}E[X^{r-j}]\right| \leq \left|\binom{r}{n+1} C^{n+1}E[X^{r-(n+1)}]\right|. \tag{B.13}$$

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