A NOTE ON A THEOREM OF BURNSIDE

By K. W. GRUENBERG

Received 1 August 1951

Burnside (see (1), Theorem 170) proved the following theorem: If a finite group G is expressible as the product of two prime-power subgroups H and K, i.e. $G = HK$, then G is necessarily soluble. It has been conjectured that the same result holds even when the subgroups H and K are only supposed to be nilpotent. In this note we prove an easy special case of this conjecture; we show that G is soluble, provided H is a prime-power subgroup and K a nilpotent subgroup. The particular case of this, in which H is a prime-power subgroup, K is Abelian and the orders of H and K are coprime, has been dealt with by Szép (2).

We may put our result as follows: *If a finite group G has a nilpotent subgroup K of prime-power index p^r, then G is soluble.* For, if $G = HK$, where H is a prime-power subgroup and K a nilpotent subgroup, then the index $(G:K)$ divides the order of H, since it is equal to $(H:H \cap K)$. Conversely, if K is of index p^r in G, then for any Sylow p-subgroup H of G, we have $G = HK$.

We prove our result by induction on the order of G. Since K is nilpotent, its centre is greater than 1. Let x be an element, not equal to 1, of the centre of K. Then either x lies in the centre of G, or the number of conjugates to x in G is a prime-power. In either case, G is not simple (see (1), Theorem 169). So let N be a proper normal subgroup of G. Then KN/N is a nilpotent subgroup of G/N, and KN is of prime-power index in G, since $K \leqslant KN \leqslant G$. Consequently G/N is soluble by the induction hypothesis. But $K \cap N$ is a nilpotent subgroup of N, and $(N : K \cap N) = (KN : K)$ is a prime power. Therefore N is also soluble by the induction hypothesis. Hence G is soluble.

I want to thank Mr P. Hall for his helpful comments.

REFERENCES

(1) Speiser, A. *Theorie der Gruppen*, 3rd ed. (Berlin, 1937).
(2) Szép, J. *Publicationes mathematicae*, 2 (1951), 43–5.

Magdalene College
Cambridge

https://doi.org/10.1017/S030500410002750X Published online by Cambridge University Press