Introduction

How do people make decisions? Restricting this philosophical question to its utility side, mathematical economics came up with utility theory. The key point is that the rational decision maker always aims to maximise some utility function that depends on the states of nature and the possible decisions; the choice of such a function may depend on personal views, laws, and society attitudes towards possible decisions and the risk they entail.

Historically, one of the first mathematical decision problems relates to the St. Petersburg paradox. This concerns repetitions of a fair-coin game conducted until heads comes up first. If this happens in the first round, the player wins 2 dollars, if it happens in the second round he wins 4 dollars, and with each subsequent round, the bet is increased by a factor of two. The gain X has infinite expectation, so, if the expectation is chosen as the utility, then it would be sensible to pay any amount for entering this game. This is naturally wrong, and Daniel Bernoulli (1700–1782) wrote in relation to this (see Bernoulli, 1954, p. 24) that

The determination of the value of an item must not be based on the price, but rather on the utility it yields.

Further, he mentioned that the winning of a certain amount is much more valuable for a poor man than for a rich one. Daniel Bernoulli suggested using the logarithmic utility function to settle the above paradox, since the expectation of $\log X$ should better reflect the utility of the above game.

In economics, the numerical value of utility is often not essential; most important is the order of possible decisions. A higher utility corresponds to lower loss (or risk), and so it is possible to reach a decision by comparing risks. In finance, on the other hand, it is usually important to go beyond comparison and quantify risk (or utility) in monetary terms.

Decisions taken by financial institutions may lead to substantial losses and

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 12 Nov 2026 at 10:49:55, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009710916

eventually to the bankruptcy of an institution which does not possess the necessary capital reserves in order to offset eventual losses. Due to a subsequent chain reaction affecting other financial institutions linked to the bankrupt one, this may affect entire financial sectors. It is common to quantify risk in terms of money needed to be set aside in order to avoid financial defaults within a given time horizon. Governmental organisations have been created to enforce such requirements. On the mathematical side, this called for the study of monetary utility functions, or equivalently, of monetary risk measures.

Basic courses in financial mathematics and portfolio choice bring us to the conclusion that properly hedged portfolios grow according to the risk-neutral interest rate. However, in practice, the portfolio managers or traders believe that their particular choice of assets will lead to a better performance than the risk-free rate. Such a claim is always associated with accepting some risks that might lead either to a gain or to a loss, affecting the overall position of the financial institution.

Therefore, taking risks is an integral and absolutely indispensable part of the operations of most financial institutions. Thus, in order to ensure the secure operation of financial institutions, they have to set aside some reserves that offset possible losses resulting from poor performance of their risky investments or other risky transactions. The calculation of risk measures aims to determine such capital reserves.

The mathematical theory of risk measures was boosted by the seminal paper of Artzner et al. (1999), who suggested axiomatic properties of risk measures, that opened connections to other parts of mathematics like functional analysis, convexity, game theory, and non-additive measures. Risk measures are defined as functionals of random variables which satisfy some structural properties, most importantly convexity.

This book starts with basic properties of risk measures, then develops dual representations of convex risk measures, which are then specialised to the cases of law-determined risk measures and comonotonic additive ones. In the course of this, important special risk measures are mentioned.

In this book we consider risk measures defined on a rather general space, with a particular attention devoted to risk measures on $\boldsymbol{L}^p$-spaces and their dual representations.

Following our main line of duality theory, we also revisit some basic concepts of the theory of risk measures and sometimes adjust the names of some of them. Most importantly, we systematically speak of cash additivity instead of cash invariance, and advocate for the use of the term law-determined instead of law-invariant. In both cases, the properties are not invariance properties in a

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 12 Nov 2026 at 10:49:55, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009710916

mathematical sense, and we believe that the terminology we chose is both, more precise and more intuitive.

Guide to the Book

This book aims to serve a lecture course on risk measures. It consists of 12 chapters, each of them consisting of a main part and exercises accompanied by additional material with in-depth discussion of the related mathematical concepts and in some cases an extra part touching upon the financial motivation. A solutions manual for exercises is available as an online tool from the publisher's WEB page for this book at `www.cambridge.org/9781009710961`.

The main part of each chapter roughly corresponds to a 2-hour lecture, in some cases to two lectures, while the extra part aims for self-study or can be used to extend the scope of the lectures. In this way, this book is suitable for courses of varying numbers of hours. The additional material is never used in the main part and exercises, so that the main part can be consistently followed neglecting the additional material.

It is assumed that the readers (or students) know major facts from measure-theoretic probability theory, and elementary concepts from functional analysis.

Risk measures (or the utility functions closely related to them) are the main subject of the books by Pflug and Römisch (2007) and Delbaen (2012). The book by Föllmer and Schied (2004) devoted to discrete time finance contains substantial parts concerning risk measures. Further topics on risk measures, mostly in relation to actuarial science, are covered by Denuit et al. (2005) and McNeil et al. (2015).

Understanding risk is facilitated by studying general ideas from financial mathematics, with Hull (2000) being an ideal reference. For mathematical background related to functional analysis, we usually rely on and cite Dunford and Schwartz (1988), Werner (2002), and Aliprantis and Border (2006).

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 12 Nov 2026 at 10:49:55, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009710916