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A New Two-Grid Method for Expanded Mixed Finite Element Solution of Nonlinear Reaction Diffusion Equations

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Abstract. In the paper, we present an efficient two-grid method for the approximation of two-dimensional nonlinear reaction-diffusion equations using a expanded mixed finite-element method. We transfer the nonlinear reaction diffusion equation into first order nonlinear equations. The solution of the nonlinear system on the fine space is reduced to the solutions of two small (one linear and one non-linear) systems on the coarse space and a linear system on the fine space. Moreover, we obtain the error estimation for the two-grid algorithm. It is showed that coarse space can be extremely coarse and achieve asymptotically optimal approximation as long as the mesh sizes satisfy $h^{k+1}=\mathcal{O}(H^{3k+1})$. An numerical example is also given to illustrate the effectiveness of the algorithm.

AMS subject classifications: 65N30, 65N15, 65M12

Key words: Error estimation, mixed finite elements, reaction-diffusion equations, two-grid methods.

1 Introduction

In this paper, we consider the following nonlinear reaction-diffusion equations

$$sp_t-\nabla\cdot(K(p)\nabla p)=f(x,t),\quad (x,t)\in\Omega\times J, \tag{1.1}$$

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- with initial condition:

$$p(x,0)=p^0(x), \quad x\in\Omega, \tag{1.2}$$

- and boundary condition:

$$(K(p)\nabla p)\cdot\boldsymbol{\nu}=0, \quad (x,t)\in\partial\Omega\times J, \tag{1.3}$$

where $\Omega\subset R^2$ is a polygonal domain with boundary $\partial\Omega$, $\boldsymbol{\nu}$ is the unit exterior normal to $\partial\Omega$, $p_t=\partial p/\partial t$, $J=(0,T]$ and K is a square integrable, symmetric, uniformly positive definite tensor defined on Ω. They often arise from fist-order mass balance and constitutive laws governing a potential p and a related flux $\boldsymbol{u}$:

$$s\frac{\partial p}{\partial t}+\nabla\cdot\boldsymbol{u}=f(x,t), \tag{1.4a}$$

$$K^{-1}(p)\boldsymbol{u}+\nabla p=0. \tag{1.4b}$$

Reaction diffusion equations have received a great deal of attention in numerous applications noteworthy among them being mathematical models of hydrologic and biogeochemical phenomena [15, 19]. In the case, p is the unknown pressure head; s is the compressibility coefficient; $\boldsymbol{u}$ is the Darcy velocity of the water; and $f(p)$ models external flow rate. Mixed finite element methods have been found be very important for solving the ground-water problem. In fact, there is substantial literature on the application in ground-water flow. For example, there are many applications of mixed methods to miscible displacement problems that describe two-phase flow in a petroleum reservoir [14]. In mixed finite element formulation, both the pressure and the flux (Darcy velocity), or displacements and stresses, are approximated simultaneously etc., see [1, 2, 14, 20]. In groundwater hydrology, we have known that the coefficient in the pressure formulation may tend to zero because of low permeability, hence its reciprocal is not readily usable as in standard mixed finite element methods.

To linearize the resulting disctete equations, we use the two-grid method, which was first introduced by Xu [22, 23], as a discretization method for non-symmetric indefinite and non-linear problems. Chen and Huang [7] presented a multilevel iterative method for solving finite element equations of non-linear singular two-point boundary value problems in 1994, Huang and Xue [17] applied the multilevel linearization to the convergence analysis of finite element approximations for Ginzburg-Landau model of D-wave superconductors. Huang [16] constructed a multilevel successive iteration method for non-linear elliptic problems. This approach was applied to non-linear parabolic equations with the expanded mixed finite element method by Dawson and Wheeler [12] and with a finite difference scheme based on the expanded mixed finite element method by Dawson et al [13]. Wu and Allen [21] discussed the case where f is the nonlinear term by the two-grid expanded mixed finite element method. Chen [8–11] made many works of two-grid method for reaction diffusion equations. He and Li [4] considered three iterative methods for solving the stationary Navier-Stokes equations. Hou and his coauthors [5, 6]

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constructed postprocessing fourier Galerkin method and applied two-level finite element method for the Navier-Stokes equations based on a new projection.

In this paper, we present two-grid methods of nonlinear reaction-diffusion equations using the (expanded) mixed finite element method, we design a three-stage algorithm by using Newton iteration and correction techniques. To linearize the resulting discrete equations, we use Newton iteration idea on the fine grid in our approaches. Firstly, we resolve an original non-linear problem on the coarse mesh, Then linear problem is solved by Newton iteration on the fine mesh, finally we make a correction on the coarse mesh. The two-grid idea is derived from Xu's work [22,23] and Chen's work [8], we extend the ideas to the mixed finite element method. Moreover, we obtain the error estimates for the two-grid algorithm. It is showed that coarse space can be extremely coarse and it can be achieved asymptotically optimal approximation as long as the mesh sizes satisfy $h^{k+1}=\mathcal{O}(H^{3k+1})$. An example is also given to illustrate the theoretical results.

This paper is organized into six sections. We establish notation and some approximation results in Section 2. The one-grid scheme is presented in Section 3 along with error estimates. In Section 4, an algorithm of two-grid methods and their errors estimates is obtained. A numerical experiment is designed in Section 5 to verify the our theoretic result of the main algorithm. Brief summary and remarks on the further work are concluded in Section 6.

2 Some notation and approximation assumptions

In the following, we introduce some notations and projection approximation properties.

Let $W^{m,p}(\Omega)$ be the standard soblev spaces and $L^p(\Omega)$ be the standard Banach space. Specially, let $H^m(\Omega)=W^{m,2}(\Omega)$ and $H_0^m=W_0^{m,2}(\Omega)$. For simplicity, let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ or $(L^2(\Omega))^2$ inner product, scalar and vector. In the same time, we denote $\|\cdot\|_m=\|\cdot\|_{m,2}$ and $\|\cdot\|=\|\cdot\|_{0,2}$.

Let

$$\boldsymbol{W}=L^2(\Omega),\quad \boldsymbol{V}=H(\Omega,\mathrm{div})=\left\{\boldsymbol{v}\in(L^2(\Omega))^2:\nabla\cdot\boldsymbol{v}\in L^2(\Omega)\right\},\quad \widetilde{\boldsymbol{V}}=\boldsymbol{V}\cap\{\boldsymbol{v}\cdot\nu=0\}.$$

We will consider quasi-uniform partition of Ω into rectangles or triangles with the partition step h in two dimensions, we use mixed finite element spaces such as the RT [14] spaces of order k, RT_k to form $\boldsymbol{V}_h$ and $\boldsymbol{W}_h$, discrete subspaces of $\boldsymbol{V}$ and $\boldsymbol{W}$. We define $\widetilde{\boldsymbol{V}}_h=\widetilde{\boldsymbol{V}}\cap\boldsymbol{V}_h$.

The following inclusion holds for RT spaces or BDM spaces used here [1,20]:

$$\nabla\cdot\boldsymbol{v}_h\in\boldsymbol{W}_h,\quad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h.$$

To analyze the discretization on a time interval $(0,T)$, let $N>0$, $\Delta t=T/N$ and $t^n=n\Delta t$. Set $\phi^n=\phi(\cdot,t^n)$, $\partial_t\phi^n=\frac{\phi^n-\phi^{n-1}}{\Delta t}$, and discrete L^2-norm

$$\|\phi\|_{l^2(0,T)}:=\Big(\sum_{n=1}^{N}\Delta t\|\phi^n\|^2\Big)^{1/2},\quad \|\phi\|_{l^2(0,t^m)}:=\Big(\sum_{n=1}^{m}\Delta t\|\phi^n\|^2\Big)^{1/2},$$

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where $1 \le m \le N$.

In the convergence analysis, we shall employ standard L^2 projections Q_h onto the spaces $\boldsymbol{W}_h$ or $\boldsymbol{V}_h$. That is to say, for $g \in L^2(\Omega)$ and for $\boldsymbol{q} \in (L^2(\Omega))^2$ we have

$$(Q_h g, w_h) = (g, w_h), \qquad \forall w_h \in \boldsymbol{W}_h, \tag{2.1a}$$

$$(Q_h \boldsymbol{q}, \boldsymbol{v}_h) = (\boldsymbol{q}, \boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.1b}$$

We also use the well known the standard mixed finite element projection operator $\Pi_h : H^1(\Omega) \times H^1(\Omega) \longrightarrow \boldsymbol{V}_h$ such that for $\boldsymbol{q} \in H(\mathrm{div}, \Omega)$

$$(\nabla \cdot \Pi_h \boldsymbol{q}, w_h) = (\nabla \cdot \boldsymbol{q}, w_h), \quad \forall w_h \in \boldsymbol{W}_h. \tag{2.2}$$

We assume that

$$\mathrm{div}\Pi_h = Q_h, \quad \mathrm{div} : \boldsymbol{V} \xrightarrow{\text{onto}} \boldsymbol{W}_{\mathrm{h}}.$$

Throughout the following, assume that $1 < q \le \infty$. We have the following stable and approximation properties [1, 20]. For $\psi \in W^{k+1,q}(\Omega)$ (or $\psi \in (W^{k+1,q}(\Omega))^2$), there exists a positive constant C independent of h such that

$$\|Q_h \psi\|_{0,q} \le C \|\psi\|_{0,q}, \qquad 2 \le q \le \infty, \tag{2.3a}$$

$$\|\psi - Q_h \psi\|_{0,q} \le C \|\psi\|_{r,q} h^r, \qquad 0 \le r \le k+1, \tag{2.3b}$$

$$\|\psi - Q_h \psi\|_{\infty} \le C \|\psi\|_{\infty} h^r, \qquad 0 \le r \le k+1. \tag{2.3c}$$

We also have the approximation properties of Π_h: for $\boldsymbol{q} \in (W^{k+1,q}(\Omega))^2$,

$$\|\boldsymbol{q} - \Pi_h \boldsymbol{q}\|_{0,q} \le C \|\boldsymbol{q}\|_{r,q} h^r, \quad 0 \le r \le k+1. \tag{2.4}$$

We give the following inverse property on $\boldsymbol{V}_h$, that is to say, for $\boldsymbol{v}_h \in \boldsymbol{V}_h$

$$\|\boldsymbol{v}_h\|_{\infty} \le C h^{-1} \, \|\boldsymbol{v}_h\|. \tag{2.5}$$

These properties are known to hold for the Raviart-Thomas-Nedelec and Brezzi-Douglas-Marini spaces mentioned above from [1, 20]. In this paper C shall denote a generic constant. For convenience we will also assume that the solution is sufficiently smooth, uniformly in time, and that the maximum index of convegence $k+1$ is attained.

In the paper, in order to derive main estimates we make the following assumptions:

1. We assume that the reaction term f is smooth function defined on Ω.
2. We assume that $f : \Omega \times R \to R$ is triple continuously differentiable function with bounded derivatives through the third order and $K(p)$ is a square-integrable symmetric, uniformly positive definite tensor.
3. There exist positive constants K_* and K^* such that for $z \in R^2$
$$K_* \|z\|^2 \le z^T K(x,s) z \le K^* \|z\|^2, \quad x \in \Omega, \quad s \in R.$$
4. For some integer $k \ge 1$, we assume that the solution function $(p, \boldsymbol{u})$ of (1.4a)-(1.4b) have the following regularity:
$$p \in W_{\infty}^2(0, T; W^{k+1,4}(\Omega)), \quad \boldsymbol{u} \in (W_{\infty}^2(0, T; W^{k+1,4}(\Omega)))^2.$$

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3 Error estimate for mixed finite-element method

We set the following three variables, the pressure p, the gradient $\mathbf{q}=-\nabla p$, and the flux $\boldsymbol{u}=K(p)\mathbf{q}$. For the sake of simplicity, we assume that $s=1$, we define the weak form of (1.1)-(1.3) as $(p,\boldsymbol{q},\boldsymbol{u})\in \boldsymbol{W}\times\boldsymbol{V}\times\widetilde{\boldsymbol{V}}$ satisfying

$$(p_t,w)+(\nabla\cdot\boldsymbol{u},w)=(f,w), \qquad \forall w\in\boldsymbol{W}, \tag{3.1a}$$
$$(\boldsymbol{q},\boldsymbol{v})=(p,\nabla\cdot\boldsymbol{v}), \qquad \forall \boldsymbol{v}\in\widetilde{\boldsymbol{V}}, \tag{3.1b}$$
$$(\boldsymbol{u},\boldsymbol{v})=(K(p)\boldsymbol{q},\boldsymbol{v}), \qquad \forall \boldsymbol{v}\in\boldsymbol{V}. \tag{3.1c}$$

Now, by the above some notations we can establish the following discrete time expanded mixed finite element approximation of problem (3.1a)-(3.1c) in RT_k element. Given $(p_h^0,\boldsymbol{q}_h^0,\boldsymbol{u}_h^0)\in\boldsymbol{W}_h\times\boldsymbol{V}_h\times\widetilde{\boldsymbol{V}}_h$, for $n=1,\cdots,m$, we seek $(p_h^n,\boldsymbol{q}_h^n,\boldsymbol{u}_h^n)\in\boldsymbol{W}_h\times\boldsymbol{V}_h\times\widetilde{\boldsymbol{V}}_h$ satisfy

$$\left(\frac{p_h^n-p_h^{n-1}}{\Delta t},w_h\right)+(\nabla\cdot\boldsymbol{u}_h^n,w_h)=(f^n,w_h), \qquad \forall w_h\in\boldsymbol{W}_h, \tag{3.2a}$$
$$(\boldsymbol{q}_h^n,\boldsymbol{v}_h)=(p_h^n,\nabla\cdot\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h\in\widetilde{\boldsymbol{V}}_h, \tag{3.2b}$$
$$(\boldsymbol{u}_h^n,\boldsymbol{v}_h)=(K(p_h^n)\boldsymbol{q}_h^n,\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h. \tag{3.2c}$$

The proofs of existence and uniqueness of the solution to the nonlinear algebraic system (3.2b)-(3.2c) are given in the reference [22].

At the time of $t=t^n$, we rewrite (3.1a)-(3.1c) in the following form

$$(p_t^n,w_h)+(\nabla\cdot\boldsymbol{u}^n,w_h)=(f^n,w_h), \qquad \forall w_h\in\boldsymbol{W}_h, \tag{3.3a}$$
$$(\boldsymbol{q}^n,\boldsymbol{v}_h)=(p^n,\nabla\cdot\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h\in\widetilde{\boldsymbol{V}}_h, \tag{3.3b}$$
$$(\boldsymbol{u}^n,\boldsymbol{v}_h)=(K(p^n)\boldsymbol{q}^n,\boldsymbol{v}_n), \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h. \tag{3.3c}$$

Using the definition of Π_h, Q_h and $\nabla\cdot\boldsymbol{v}_h\in\boldsymbol{W}_h$, we can obtain

$$(\partial_t Q_h p^n,w_h)+(\nabla\cdot\Pi_h\boldsymbol{u}^n,w_h)=(f^n,w_h)+(\partial_t p^n-p_t^n,w_h), \qquad \forall w_h\in\boldsymbol{W}_h, \tag{3.4a}$$
$$(Q_h\boldsymbol{q}^n,\boldsymbol{v}_h)=(Q_h p^n,\nabla\cdot\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h\in\widetilde{\boldsymbol{V}}_h, \tag{3.4b}$$
$$(\Pi_h\boldsymbol{u}^n,\boldsymbol{v}_h)=(K(p^n)\boldsymbol{q}^n,\boldsymbol{v}_n)+(\Pi_h\boldsymbol{u}^n-u^n,\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h. \tag{3.4c}$$

Set

$$\gamma^n=Q_h p^n-p_h^n,\quad \eta^n=Q_h\boldsymbol{q}^n-\boldsymbol{q}_h^n,\quad \xi^n=\Pi_h\boldsymbol{u}^n-\boldsymbol{u}_h^n.$$

By using (3.2a)-(3.2c) and (3.4a)-(3.4c), and letting $w_h=\gamma^n$, $\boldsymbol{v}_h=\xi^n$, $\boldsymbol{v}_h=\eta^n$ in the first, second and third equations, respectively, we have

$$(\partial_t\gamma^n,\gamma^n)+(\nabla\cdot\xi^n,\gamma^n)=(\partial_t p^n-p_t^n,\gamma^n), \tag{3.5a}$$
$$(\eta^n,\xi^n)=(\gamma^n,\nabla\cdot\xi^n), \tag{3.5b}$$
$$(\xi^n,\eta^n)=(k(p^n)\boldsymbol{q}^n-k(p_n^n)\boldsymbol{q}_h^n,\eta^n)+(\Pi_h\boldsymbol{u}^n-u^n,\eta^n). \tag{3.5c}$$

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We see the right term of (3.5c)

$$\begin{aligned}&(K(p^n)\boldsymbol{q}^n-K(p_h^n)\boldsymbol{q}_h^n,\eta^n)\\&=(K(p_h^n)\eta^n,\eta^n)+(K(p_h^n)(q^n-Q_hq^n),\eta^n)-((K(p_h^n)-K(p^n))\boldsymbol{q}^n,\eta^n),\end{aligned}\tag{3.6}$$

adding the (3.5a) to (3.5b), and subtracting the (3.5c), we can get

$$\begin{aligned}(\partial_t\gamma^n,\gamma^n)+(K(p_h^n)\eta^n,\eta^n)=&(\partial_tp^n-p_t^n,\gamma^n)-(K(p_h^n)(q^n-Q_hq^n),\eta^n)\\&+((K(p_h^n)-K(p^n))\boldsymbol{q}^n,\eta^n)+(u^n-\Pi_h\boldsymbol{u}^n,\eta^n).\end{aligned}\tag{3.7}$$

Firstly, we note an inequality

$$-\frac{\|\gamma^{n-1}\|^2-\|\gamma^n\|^2}{2}\le-(\gamma^n,\gamma^n),$$

and

$$(K(p_h^n)\eta^n,\eta^n)=\|K(p_h^n)^{1/2}\eta^n\|^2.$$

Secondly, we bound the right-hand side of relation (3.7), we know

$$\begin{aligned}|(\partial_tp^n-p_t^n,\gamma^n)|=&(\Delta t)^{-1}\Big|\Big(\int_{t^{n-1}}^{t^n}(t-t^{n-1})p_{tt}(\cdot,s)ds,\gamma^n\Big)\Big|\\&\le\int_{t^{n-1}}^{t^n}\|p_{tt}(\cdot,s)\|ds\|\gamma^n\|\le C(\Delta t)^2+\|\gamma^n\|^2,\end{aligned}$$

and

$$\begin{aligned}((K(p_h^n)-K(p^n))\boldsymbol{q}^n,\eta^n)\le&C\|p^n-p_h^n\|\|\eta^n\|\\\le&C(\|p^n-Q_hp^n\|+\|Q_hp^n-p_h^n\|)\|\eta^n\|\\\le&C(\|\gamma^n\|^2+\|p^n-Q_hp^n\|^2)+\delta\|\eta^n\|^2.\end{aligned}$$

Combining the above analysis, we may obtain

$$\begin{aligned}&\frac{\|\gamma^n\|^2-\|\gamma^{n-1}\|^2}{2\Delta t}+\|K(p_h^n)^{1/2}\eta^n\|^2\\&\le\Big(\frac{\gamma^n-\gamma^{n-1}}{\Delta t},\gamma^n\Big)+(K(p_h^n)\eta^n,\eta^n)\\&\le C((\Delta t)^2+\|\gamma^n\|^2)+3\delta\|\eta^n\|^2+C(\|u^n-\Pi_h\boldsymbol{u}^n\|^2+\|q^n-Q_hq^n\|^2+\|p^n-Q_hp^n\|^2),\end{aligned}$$

where δ is a small constant and $3\delta\le K_*/2$. Using the properties (2.3b) and (2.4), we have

$$\frac{\|\gamma^n\|^2-\|\gamma^{n-1}\|^2}{2\Delta t}+\|K(p_h^n)^{1/2}\eta^n\|^2\le C((\Delta t)^2+h^{2k+2})+C\|\gamma^n\|^2+3\delta\|\eta^n\|^2.\tag{3.8}$$

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We next multiply both sides of (3.8) by $2\Delta t$, and sum on n, $n=1,\cdots,m$, we have the following error estimate by applying Gronwall's Lemma and choosing the initial function $p_h^0=Q_hp^0$,

$$\|\gamma^m\|^2+\sum_{n=1}^{m}\Delta t\|K(p_h^n)^{1/2}\eta^n\|^2\leq C((\Delta t)^2+h^{2k+2}),$$

that is to say,

$$\|p_h^m-Q_hp^m\|+\Big(\sum_{n=1}^{m}\Delta t\|K_*^{1/2}(\boldsymbol{q}_h^n-Q_h\boldsymbol{q}^n)\|^2\Big)^{1/2}\leq C(\Delta t+h^{k+1}).$$

From the above analysis, using the triangle inequality, we can obtain the error estimate for the mixed finite-element method, seeing the following Theorem:

Theorem 3.1. *Let $(p_h^n,\boldsymbol{q}_h^n,\boldsymbol{u}_h^n)\in\boldsymbol{W}_h\times\boldsymbol{V}_h\times\widetilde{\boldsymbol{V}}_h$ be the solution of the mixed finite-element equation for $n\geq 1$. If $p_h^0=R_hp^0$, then there exists a positive C independent of h such that*

$$\|p^m-p_h^m\|+\Big(\sum_{n=1}^{m}\Delta t\|K_*^{1/2}(\boldsymbol{q}^n-\boldsymbol{q}_h^n)\|^2\Big)^{1/2}\leq C(\Delta t+h^{k+1}). \tag{3.9}$$

4 Two-grid method and error estimate for mixed finite-element method

In this section, we shall present the main algorithm. To linearize the nonlinear equations, we consider two grid scheme based on Newton iteration idea and correction technique, Such nonlinear equations is only solved on the coarse grid and a fine grid approximation is obtained by solving a linearized equations on the fine mesh. Then we correct the fine grid approximation on the coarse grid. the method has three steps as follows:

Step 1: On the coarse grid T_H, solve $(p_H^n,\boldsymbol{q}_H^n,\boldsymbol{u}_H^n)\in\boldsymbol{W}_H\times\boldsymbol{V}_H\times\widetilde{\boldsymbol{V}}_H$ to satisfy the following nonlinear system:

$$\Big(\frac{p_H^n-p_H^{n-1}}{\Delta t},w_H\Big)+(\nabla\cdot\boldsymbol{u}_H^n,w_H)=(f^n,w_H),\qquad\forall w_H\in\boldsymbol{W}_H, \tag{4.1a}$$

$$(\boldsymbol{q}_H^n,\boldsymbol{v}_H)=(p_H^n,\nabla\cdot\boldsymbol{v}_H),\qquad\forall\boldsymbol{v}_H\in\widetilde{\boldsymbol{V}}_H, \tag{4.1b}$$

$$(K(p_H^n)\boldsymbol{q}_H^n,\boldsymbol{v}_H)=(\boldsymbol{u}_H^n,\boldsymbol{v}_H),\qquad\forall\boldsymbol{v}_H\in\boldsymbol{V}_H. \tag{4.1c}$$

Step 2: On the fine grid T_h, solve $(p_h^n,\boldsymbol{q}_h^n,\boldsymbol{u}_h^n)\in\boldsymbol{W}_h\times\boldsymbol{V}_h\times\widetilde{\boldsymbol{V}}_h$ to satisfy the following linear system:

$$\Big(\frac{p_h^n-p_h^{n-1}}{\Delta t},w_h\Big)+(\nabla\cdot\boldsymbol{u}_h^n,w_h)=(f^n,w_h),\qquad\forall w_H\in\boldsymbol{W}_h, \tag{4.2a}$$

$$(\boldsymbol{q}_h^n,\boldsymbol{v}_h)=(p_h^n,\nabla\cdot\boldsymbol{v}_h),\qquad\forall\boldsymbol{v}_H\in\widetilde{\boldsymbol{V}}_h, \tag{4.2b}$$

$$(K(p_H^n)\boldsymbol{q}_h^n+(K'(p_H^n)(p_h^n-p_H^n)\boldsymbol{q}_H^n,\boldsymbol{v}_h)=(\boldsymbol{u}_h^n,\boldsymbol{v}_h),\qquad\forall\boldsymbol{v}_H\in\boldsymbol{V}_h. \tag{4.2c}$$

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Step 3: On the coarse grid T_H, solve $(e_H^n, \boldsymbol{g}_H^n, \boldsymbol{z}_H^n) \in \boldsymbol{W}_H \times \boldsymbol{V}_H \times \widetilde{\boldsymbol{V}}_H$ to satisfy the following linear system:

$$\left(\frac{e_H^n - e_H^{n-1}}{\Delta t}, w_H\right) + (\nabla \cdot \boldsymbol{z}_H^n, w_H) = 0, \qquad \forall w_H \in \boldsymbol{W}_H, \tag{4.3a}$$

$$(\boldsymbol{g}_H^n, \boldsymbol{v}_H) = (e_H^n, \nabla \cdot \boldsymbol{v}_H), \qquad \forall \boldsymbol{v}_H \in \widetilde{\boldsymbol{V}}_H, \tag{4.3b}$$

$$\begin{aligned}&(K(p_H^n)\boldsymbol{g}_H^n + K'(p_H^n)(p_h^n - p_H^n)(\boldsymbol{q}_h^n - \boldsymbol{q}_H^n)\\ &\quad + K'(p_H^n)e_H^n \boldsymbol{q}_H^n + \frac{1}{2}K''(p_H^n)(p_h^n - p_H^n)^2 \boldsymbol{q}_H^n, \boldsymbol{v}_H) = (\boldsymbol{z}_H^n, \boldsymbol{v}_H), \qquad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H.\end{aligned} \tag{4.3c}$$

In our algorithm, the two grid solutions $(\tilde{p}_h^n, \tilde{\boldsymbol{q}}_h^n, \tilde{\boldsymbol{u}}_h^n) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times \widetilde{\boldsymbol{V}}_h$ are defined as follows:

$$\tilde{p}_h^n = p_h^n + e_H^n, \quad \tilde{\boldsymbol{q}}_h^n = \boldsymbol{q}_h^n + \boldsymbol{g}_H^n, \quad \tilde{\boldsymbol{u}}_h^n = \boldsymbol{u}_h^n + \boldsymbol{z}_H^n.$$

In the following, we recall some results in work [12]. Firstly, we see the following Lemma:

Lemma 4.1. *Let $(p_h^n, \boldsymbol{q}_h^n, \boldsymbol{u}_h^n) \in \boldsymbol{W}_h \times \boldsymbol{V}_h \times \widetilde{\boldsymbol{V}}_h$, we choose the initial function $p_h^0 = Q_h p^0$, then for $1 \le n \le m$ and $k \ge 1$, there exists a positive constant C independent of h such that*

$$\|Q_h p^m - p_h^m\| + \Big(\sum_{n=1}^m \Delta t \|K_*^{1/2}(Q_h \boldsymbol{q}^n - \boldsymbol{q}_h^n)\|^2\Big)^{1/2} \le C(h^{k+1} + H^{2k+1} + \Delta t),$$

$$\|p^m - p_h^m\| + \Big(\sum_{n=1}^m \Delta t \|K_*^{1/2}(\boldsymbol{q}^n - \boldsymbol{q}_h^n)\|^2\Big)^{1/2} \le C(h^{k+1} + H^{2k+1} + \Delta t).$$

Using the inverse estimates, we have

$$\|\boldsymbol{q}_H^n\|_\infty \le \|\boldsymbol{q}^n - \boldsymbol{q}_H^n\|_\infty + \|\boldsymbol{q}^n\|_\infty \le H^{-1}\|\boldsymbol{q}^n - \boldsymbol{q}_H^n\| + C.$$

Adding (4.2a)-(4.2c) to (4.3a)-(4.3c), we have

$$\left(\frac{\tilde{p}_h^n - \tilde{p}_h^{n-1}}{\Delta t}, w_h\right) + (\nabla \cdot \tilde{\boldsymbol{u}}_h^n, w_h) = (f^n, w_h), \tag{4.4a}$$

$$(\tilde{\boldsymbol{q}}_h^n, \boldsymbol{v}_h) = (\tilde{p}_h^n, \nabla \cdot \boldsymbol{v}_h), \tag{4.4b}$$

$$(\tilde{\boldsymbol{u}}_h^n, \boldsymbol{v}_h) = (\boldsymbol{H}, \boldsymbol{v}_h), \tag{4.4c}$$

where

$$\begin{aligned}\boldsymbol{H} =& K(p_H^n)\tilde{\boldsymbol{q}}_h^n + K'(p_H^n)(\tilde{p}_h^n - p_H^n)\boldsymbol{q}_H^n\\ &+ K'(p_H^n)(p_h^n - p_H^n)(\boldsymbol{q}_h^n - \boldsymbol{q}_H^n) + \frac{1}{2}K''(p_H^n)(p_h^n - p_H^n)^2 \boldsymbol{q}_H^n.\end{aligned}$$

Setting

$$\xi^n = Q_h p - \tilde{p}_h^n, \quad \gamma^n = Q_h \boldsymbol{q}^n - \tilde{\boldsymbol{q}}_h^n, \quad \sigma^n = \Pi_h \boldsymbol{u}^n - \tilde{\boldsymbol{u}}_h^n.$$

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Subtracting weak formulation (4.4a) from (3.4a), (4.4b) from (3.4b), and (4.4c) from (3.4c), setting $w_h=\xi^n$ and $v_h=\sigma^n$, $v_h=\gamma^n$ in the first, second and third equations respectively, we have

$$\begin{aligned}&(\partial_t\xi^n,\xi^n)+(\nabla\cdot\sigma^n,\xi^n)=(\partial_t p^n-p_t^n,\xi^n),\\&(\gamma^n,\sigma^n)=(\xi^n,\nabla\cdot\sigma^n),\\&(\sigma^n,\gamma^n)=(\Pi_h\boldsymbol{u}^n-\boldsymbol{u}^n,\gamma^n)+(K(p^n)\boldsymbol{q}^n,\gamma^n)-(H,\gamma^n).\end{aligned}$$

Combining the above equations, adding the first and second equations, and reducing the finally equation, using Taylor expansion, we can

$$\begin{aligned}&\left(\frac{\xi^n-\xi^{n-1}}{\Delta t},\xi^n\right)\\&=(\partial_t p^n-p_t^n,\xi^n)-(\Pi_h\boldsymbol{u}^n-\boldsymbol{u}^n,\gamma^n)\\&\quad+(K(p^n)(Q_h\boldsymbol{q}^n-\boldsymbol{q}^n),\gamma^n)-(K(p^n)Q_h\boldsymbol{q}^n,\gamma^n)+(\boldsymbol{H},\gamma^n),\end{aligned}\tag{4.5}$$

using Taylor expansion, we can

$$K(p^n)=K(p_H^n)+K'(p_H^n)(p^n-p_H^n)+\frac{1}{2}K''(p_H^n)(p^n-p_H^n)^2+\frac{1}{6}K'''(\bar{p})(p^n-p_H^n)^3,$$

for Eq. (4.5), we have

$$\begin{aligned}&\left(\frac{\xi^n-\xi^{n-1}}{\Delta t},\xi^n\right)\\&=(\partial_t p^n-p_t^n,\xi^n)-(\Pi_h\boldsymbol{u}^n-\boldsymbol{u}^n,\gamma^n)-(K(p_H^n)\gamma^n,\gamma^n)\\&\quad+(K(p^n)(Q_h\boldsymbol{q}^n-\boldsymbol{q}^n),\gamma^n)+J_1+J_2+J_3,\end{aligned}\tag{4.6}$$

where J_1, J_2 and J_3 are defined as follows

$$\begin{aligned}J_1&=(K'(p_H^n)(\tilde{p}_h^n-p_H^n)\boldsymbol{q}_H^n,\gamma^n)-(K'(p_H^n)(p^n-p_H^n)Q_h\boldsymbol{q}^n,\gamma^n)\\&\quad+(K'(p_H^n)(p_h^n-p_H^n)(\boldsymbol{q}_h^n-\boldsymbol{q}_H^n),\gamma^n),\\J_2&=\left(\frac{1}{2}K''(p_H^n)(p_h^n-p_H^n)^2\boldsymbol{q}_H^n,\gamma^n\right)-\left(\frac{1}{2}K''(p_H^n)(p^n-p_H^n)^2Q_h\boldsymbol{q},\gamma^n\right),\\J_3&=-\left(\frac{1}{6}K'''(\bar{p})(p^n-p_H^n)^3Q_h\boldsymbol{q}^n,\gamma^n\right),\end{aligned}$$

where $\bar{p}$ is between p^n and p_H^n.

We see the left term of Eq. (4.6),

$$\begin{aligned}J_1=&(K'(p_H^n)(\tilde{p}_h^n-p^n)\boldsymbol{q}_H^n+K'(p_H^n)(p^n-p_H^n)\boldsymbol{q}_H^n-K'(p_H^n)(p^n-p_H^n)Q_h\boldsymbol{q}^n\\&+K'(p_H^n)(p_h^n-p_H^n)(\boldsymbol{q}_h^n-Q_h\boldsymbol{q}^n),\gamma^n)+K'(p_H^n)(p_h^n-p_H^n)(Q_h\boldsymbol{q}^n-\boldsymbol{q}_H^n),\gamma^n)\\=&(K'(p_H^n)(Q_hp^n-p^n-\xi^n)(\boldsymbol{q}^n-\boldsymbol{q}^n+\boldsymbol{q}_H^n),\gamma^n)\\&+(K'(p_H^n)(p_h^n-Q_hp^n+Q_hp^n-p^n)(Q_h\boldsymbol{q}^n-\boldsymbol{q}^n),\gamma^n)\\&+(K'(p_H^n)(p_h^n-Q_hp^n+Q_hp^n-p^n)(\boldsymbol{q}^n-\boldsymbol{q}_H^n),\gamma^n)\\&+K'(p_H^n)(p_h^n-Q_hp^n+Q_hp^n-p^n+p^n-p_H^n)(\boldsymbol{q}_h^n-Q_h\boldsymbol{q}^n),\gamma^n),\end{aligned}$$

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and

$$\begin{aligned}J_2=&\Big(\frac{1}{2}K''(p_H^n)(p_h^n-p_H^n)^2\boldsymbol{q}_H^n-\frac{1}{2}K''(p_H^n)(p^n-p_H^n)^2Q_h\boldsymbol{q}^n,\gamma^n\Big)\\=&\Big(\frac{1}{2}K''(p_H^n)(p^n-p_H^n)^2(\boldsymbol{q}_H^n-Q_h\boldsymbol{q}^n),\gamma^n\Big)\\&+\Big(\frac{1}{2}K''(p_H^n)(p_h^n-p^n)(2(p^n-p_H^n)+(p_h^n-p^n))\boldsymbol{q}_H^n,\gamma^n\Big),\end{aligned}$$

we have

$$\begin{aligned}J_2\le&\Big(\frac{1}{2}K''(p_H^n)(p^n-p_H^n)^2(\boldsymbol{q}_H^n-\boldsymbol{q}^n+\boldsymbol{q}^n-Q_h\boldsymbol{q}^n),\gamma^n\Big)\\&+(K''(p_H^n)(p_h^n-Q_hp^n)^2\boldsymbol{q}^n,\gamma^n)+(K''(p_H^n)(p_h^n-Q_hp^n)^2(\boldsymbol{q}_H^n-\boldsymbol{q}^n),\gamma^n)\\&+(K''(p_H^n)(Q_hp^n-p^n)^2\boldsymbol{q}^n,\gamma^n)+(K''(p_H^n)(Q_hp^n-p^n)^2(\boldsymbol{q}_H^n-\boldsymbol{q}^n),\gamma^n)\\&+(K''(p_H^n)(\tilde{p}_h^n-Q_hp^n)(p^n-p_H^n)(\boldsymbol{q}_H^n-\boldsymbol{q}^n+\boldsymbol{q}^n,\gamma^n)\\&+(K''(p_H^n)(Q_hp^n-p^n)(p^n-p_H^n)(\boldsymbol{q}_H^n-\boldsymbol{q}^n+\boldsymbol{q}^n),\gamma^n).\end{aligned}$$

Therefore, using Holder's inequality, we take N^* is the time step at which $\|\xi^m\|$ achieves its maximum values. From the above equation, we deduce that

$$\begin{aligned}&\frac{\|\xi^n\|^2-\|\xi^{n-1}\|^2}{2\Delta t}+\|K(p_H^n)^{1/2}\gamma^n\|^2\\\le&\Big(\frac{\xi^n-\xi^{n-1}}{\Delta t},\xi^n\Big)+(K(p_H^n)\gamma^n,\gamma^n)\\\le&\|(\partial_tp^n-p_t^n,\xi^n)\|+\delta\|\gamma^n\|^2+C\|\Pi_h\boldsymbol{u}^n-\boldsymbol{u}^n\|^2+\|Q_h\boldsymbol{q}^n-\boldsymbol{q}^n\|^2\\&+C\|\xi^n\|^2+C\|\xi^n\|\|\gamma^n\|\|\boldsymbol{q}^n-Q_h\boldsymbol{q}^n\|_\infty+T_1+T_2+T_3+T_4,\end{aligned}\tag{4.7}$$

where T_1, T_2, T_3 and T_4 are defined as follows.

We will consider the last four terms of (4.7). The first of these can be bounded as follows:

$$\begin{aligned}T_1=&C\|p_h^n-Q_hp^n\|^2\|\boldsymbol{q}^n\|_\infty^2+C\|Q_hp^n-p^n\|_\infty^2\|\boldsymbol{q}^n-\boldsymbol{q}_H^n\|^2\\&+C\|p_h^n-Q_hp^n\|^2\|Q_h\boldsymbol{q}^n-\boldsymbol{q}^n\|_\infty^2+\|Q_hp_h-p^n\|_\infty^2\|Q_h\boldsymbol{q}^n-\boldsymbol{q}^n\|^2\\&+C\|p_h^n-Q_hp^n\|^2\|\boldsymbol{q}^n-\boldsymbol{q}_H^n\|_\infty^2+\|Q_hp_h-p^n\|_\infty^2\|Q_hp^n-p^n\|^2\\&+C\|p_h^n-Q_hp^n\|_\infty^2\|\tilde{\boldsymbol{q}}_h^n-Q_h\boldsymbol{q}^n\|^2+\|Q_hp^n-p^n\|_\infty^2\|\boldsymbol{q}_h^n-Q_h\boldsymbol{q}^n\|^2\\&+C\|p^n-p_H^n\|_\infty^2\|\boldsymbol{q}_h^n-Q_h\boldsymbol{q}^n\|^2,\end{aligned}$$

by Lemma 4.1 and the projection properties, we have

$$\begin{aligned}T_1\le&C[h^{2(k+1)}+h^{2(k+1)}\|\boldsymbol{q}^n-\boldsymbol{q}_H^n\|^2+h^{2(k+1)}(h^{k+1}+H^{2k+1}+(\Delta t)^2)^2\\&+(h^{k+1}+H^{2k+1}+(\Delta t)^2)^2(H^{-1}(H^{k+1}+\Delta t))^2)]\\&+C[h^{2k+2}+h^{-2}(h^{k+1}+H^{2k+1}+(\Delta t)^2)^2+(H^{-1}(H^{k+1}+\Delta t))^2)]\|\boldsymbol{q}_h^n-Q_h\boldsymbol{q}^n\|^2\end{aligned}$$

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$$\begin{aligned}&\leq C(h^{2k+2}+H^{6k+2}+(\Delta t)^2)+Ch^{2(k+1)}\|\boldsymbol{q}^n-\boldsymbol{q}_H^n\|^2\\&\quad+C[h^{2k+2}+h^{-2}(h^{k+1}+H^{2k+1}+(\Delta t)^2)^2+(H^{-1}(H^{k+1}+\Delta t))^2)]\|\boldsymbol{q}_h^n-Q_h\boldsymbol{q}^n\|^2\\&\leq C(h^{2k+2}+H^{6k+2}+(\Delta t)^2).\end{aligned}\tag{4.8}$$

For T_2,

$$\begin{aligned}T_2=&C\|(p^n-p_H^n)^2(\boldsymbol{q}_H^n-\boldsymbol{q}^n)\|^2+C\|(p^n-p_H^n)^2(\boldsymbol{q}^n-Q_h\boldsymbol{q}^n)\|^2\\&+C\|(p_h^n-Q_hp^n)^2\boldsymbol{q}^n\|^2+C\|(p_h^n-Q_hp^n)^2(\boldsymbol{q}_H^n-\boldsymbol{q}^n)\|^2\\&+C\|(c)^2\boldsymbol{q}^n\|^2+C\|(Q_hp^n-p^n)^2(\boldsymbol{q}_H^n-\boldsymbol{q}^n)\|^2\\&+C\|(p_h^n-Q_hp^n)(p^n-p_H^n)(\boldsymbol{q}_H^n-\boldsymbol{q}^n+\boldsymbol{q}^n)\|^2\\&+C\|(Q_hp^n-p^n)(p^n-p_H^n)(\boldsymbol{q}_H^n-\boldsymbol{q}^n+\boldsymbol{q}^n)\|^2,\end{aligned}$$

we have

$$\begin{aligned}T_2\leq&C\|p^n-p_H^n\|_\infty^4\|\boldsymbol{q}_H^n-\boldsymbol{q}^n\|^2+C\|p^n-p_H^n\|_\infty^2\|p^n-p_H^n\|^2\|\boldsymbol{q}^n-Q_h\boldsymbol{q}^n\|^2\\&+C\|p_h^n-Q_hp^n\|_\infty^2\|\tilde{p}_h^n-Q_hp^n\|^2+C\|p_h^n-Q_hp^n\|_\infty^4\|\boldsymbol{q}_H^n-\boldsymbol{q}^n\|^2\\&+C\|Q_hp^n-p^n\|_\infty^2\|Q_hp^n-p^n\|^2+C\|Q_hp^n-p^n\|_\infty^4\|\boldsymbol{q}_H^n-\boldsymbol{q}^n\|^2\\&+C\|p^n-p_H^n\|_\infty^2(\|\tilde{p}_h^n-Q_hp^n\|^2+C\|\tilde{p}_h^n-Q_hp^n\|_\infty^2\|\boldsymbol{q}_H^n-\boldsymbol{q}^n\|^2)\\&+C\|Q_hp^n-p^n\|_\infty^2\|p^n-p_H^n\|^2+C\|Q_hp^n-p^n\|_\infty^2\|p^n-p_H^n\|_\infty^2\|\boldsymbol{q}_H^n-\boldsymbol{q}^n\|^2,\end{aligned}$$

we apply the above Lemma and the projection properties, thus

$$\begin{aligned}T_2\leq&CH^{-4}(H^{k+1}+\Delta t))^4\|\boldsymbol{q}^n-\boldsymbol{q}_H^n\|^2+h^{2k+2}H^{-2}(H^{k+1}+\Delta t))^4\\&+h^{-2}(h^{k+1}+H^{2k+1}+(\Delta t)^2)^4+h^{-2d}(h^{k+1}+H^{2k+1}+(\Delta t)^2)^4\|\boldsymbol{q}^n-\boldsymbol{q}_H^n\|^2\\&+h^{4k+4}(C+\|\boldsymbol{q}^n-\boldsymbol{q}_H^n\|^2)+(h^{k+1}+H^{2k+1}+(\Delta t)^2)^2H^{-2}(H^{k+1}+\Delta t))^2\\&+\left\{h^{-2}(h^{k+1}+H^{2k+1}+(\Delta t)^2)^2+h^{2k+2}\right\}H^{-2}(H^{k+1}+\Delta t))^2\|\boldsymbol{q}^n-\boldsymbol{q}_H^n\|^2\\&+Ch^{2k+2}(H^{k+1}+\Delta t))^2\leq C(h^{2k+2}+H^{6k+2}+(\Delta t)^2),\end{aligned}\tag{4.9}$$

we can get estimate as to T_3,

$$\begin{aligned}T_3=&C\|(p^n-p_H^n)^3Q_h\boldsymbol{q}^n\|^2\leq C\|(p^n-p_H^n)\|_\infty^4\|(p^n-p_H^n)\|^2\\&\leq CH^{-4}(H^{k+1}+\Delta t))^6.\end{aligned}\tag{4.10}$$

The last term in (4.7) can be bounded by the inverse assumption,

$$T_4=C\|\xi^n\|\|\gamma^n\|\|\boldsymbol{q}^n-\boldsymbol{q}_H^n\|.\tag{4.11}$$

Combing Eq. (4.7) and the above analysis, where $\delta\leq K_*/2$, we now multiply the equation by $2\delta t$ and sum from $n=1$ to M, $M\leq N$. When choosing $M=N^*$, where N^* is the time

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step at which $\|\xi^n\|$ achieves its maximum values. Using approximation properties and Theorem 3.1, we have

$$\begin{aligned}&\|\xi^{N^*}\|^2-\|\xi^0\|^2+\sum_{n=1}^{N^*}\Delta t\|K(p_H^n)^{1/2}\gamma^n\|^2\\&\leq C\Delta t^2+\delta\sum_{n=1}^{N^*}\Delta t\|\gamma^n\|^2+Ch^{2k+2}(\|\boldsymbol{u}\|_{L^*}^2+\|\boldsymbol{q}\|_{L^*}^2)+C\sum_{n=1}^{N^*}\Delta t\|\xi^n\|^2+G_1+G_2,\end{aligned}\tag{4.12}$$

where $L^*=l^2((0,T),W_4^{k+1})$, G_1 and G_2 are defined by Eqs. (4.8)-(4.11),

$$G_1=C\sum_{n=1}^{N^*}\Delta t(h^{2k+2}+H^{6k+2}+(\Delta t)^2),$$

by Theorem 3.1 and the inverse assumption (2.5), we obtain

$$\begin{aligned}G_2&=\sum_{n=1}^{N^*}\Delta t\|\xi^n\|\|\gamma^n\|\|\boldsymbol{q}^n-\boldsymbol{q}_H^n\|_\infty\\&\leq K^*\|\xi^{N^*}\|\Big(\sum_{n=1}^{N^*}\Delta t\|\gamma^n\|^2\Big)^{1/2}\Big(\sum_{n=1}^{N^*}\Delta t\|\boldsymbol{q}^n-\boldsymbol{q}_H^n\|_\infty^2\Big)^{1/2}\\&\leq K^*\|\xi^{N^*}\|\Big(\sum_{n=1}^{N^*}\Delta t\|\gamma^n\|^2\Big)^{1/2}(H^{-1}(H^{k+1}+\Delta t)).\end{aligned}$$

For $k\geq 1$, H and Δt can be chosen sufficiently small so that

$$G_2\leq 1/2\|\xi^{N^*}\|^2+K^*/8\sum_{n=1}^{N^*}\Delta t\|\gamma^n\|^2,$$

choosing $\tilde{p}_h^0=Q_hp^0$, from Gronwall Lemma, we see that for Δt and H sufficiently small, we have

$$\|\xi^{N^*}\|\leq C(h^{k+1}+H^{3k+1}+\Delta t).$$

Combing the above analysis, taking $M=N^*$, we deduce that

$$\|\xi^M\|+\Big(\sum_{n=1}^{M}\Delta t\|K_*^{1/2}\sigma^n\|^2\Big)^{1/2}\leq C(h^{k+1}+H^{3k+1}+\Delta t).$$

Applying the triangle inequality and approximation properties, we obtain the following Theorem,

Theorem 4.1. *Let $(\tilde{p}_h^n,\tilde{\boldsymbol{q}}_h^n,\tilde{\boldsymbol{u}}_h^n)\in\boldsymbol{W}_h\times\boldsymbol{V}_h\times\widetilde{\boldsymbol{V}}_h$ be the solution of the two-grid method for* (4.1a)-(4.3c)*, if we choose the initial function $\tilde{p}_h^0=Q_hp^0$, then there exists a positive C independent of h such that*

$$\|p^m-\tilde{p}_h^m\|+\Big(\sum_{n=1}^{m}\Delta t\|K_*^{1/2}(\boldsymbol{q}^n-\tilde{\boldsymbol{q}}_h^n)\|^2\Big)^{1/2}\leq C(h^{k+1}+H^{3k+1}+\Delta t).\tag{4.13}$$

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5 Numerical example

In this section, we shall demonstrate the efficiency of our algorithm for nonlinear reaction diffusion problems. we consider the following nonlinear parabolic problem:

$$\begin{cases} p_t-\nabla\cdot(K(p)\nabla p)=f(x,t), & (x,t)\in\Omega\times J,\\ p(x,0)=0, & (x,t)\in\Omega\times\{t=0\},\\ (K(p)\nabla p)\cdot\boldsymbol{\nu}=0, & (x,t)\in\partial\Omega\times J, \end{cases} \tag{5.1}$$

where $\Omega=[0,1]^2$, $t\in[0,T]$,

$$K(p)=\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix},$$

and $f(x,t)$ is suitably chosen such that the exact solution is $p(x,t)=t^2(x_1^2(x_1-1)^2+x_2^2(x_2-1)^2)$.

The domain Ω is uniformly divided by the triangulations of mesh size H and h, for the sake of simplicity, time step $\tau=1.0e-4$, $T=1.0e-1$, In Table 1, we can find that the two grid methods and Newton iterative have the same accuracy, but Two grid methods have much less cost time than Newton iterative, since we just have to solve a small scale nonlinear problem. This phenomenon shows that two grid algorithm is a very effective algorithm when it comes to deal with the nonlinear problems. In Table 2, we present numerical results for error estimate of $\|p^n-\tilde{p}_h^n\|$, $\|\boldsymbol{q}^n-\tilde{\boldsymbol{q}}_h^n\|_V$ and $\|\boldsymbol{u}^n-\tilde{\boldsymbol{u}}_h^n\|_V$, respectively. The numerical results confirm our theoretical expectation. As shown in Figs. 1 and 2, we can see the exact solution p and the two grid solution $\tilde{p}_h$ where $h=1/64$, in Figs. 3-5, we can observe that error of two grid solution $\tilde{p}_h,\tilde{\boldsymbol{q}}_h,\tilde{\boldsymbol{u}}_h$ varied with time where $h=1/64$. From the Figs. 6-8, we can find the error are satisfied in our two grid method.

Table 1: Error and CPU time of the two-grid method and Newton iterative.

H	h	Newton iterative		Two grid method	
1/2	1/8	5.7106e-5	7.1483s	5.7106e-5	4.3311s
1/4	1/64	7.2401e-6	534.0673s	7.2401e-6	337.2160s
1/8	1/256	1.9980e-6	42390.6423s	1.9980e-6	17778.5854s

Table 2: Error of the two-grid method.

H	h	$\|p^n-\tilde{p}_h^n\|$	$\|\boldsymbol{q}^n-\tilde{\boldsymbol{q}}_h^n\|_V$	$\|\boldsymbol{u}^n-\tilde{\boldsymbol{u}}_h^n\|_V$
1/2	1/8	5.7106e-5	4.4046e-4	3.0359e-7
1/4	1/64	7.2401e-6	5.7064e-5	3.9089e-8
1/8	1/256	1.9980e-6	1.4427e-5	1.0155e-8

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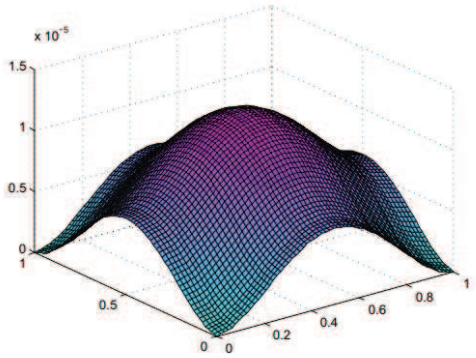


Figure 1: The exact solution $pT=1.0e-1$, $h=1/64$, $\tau=1.0e-4$.

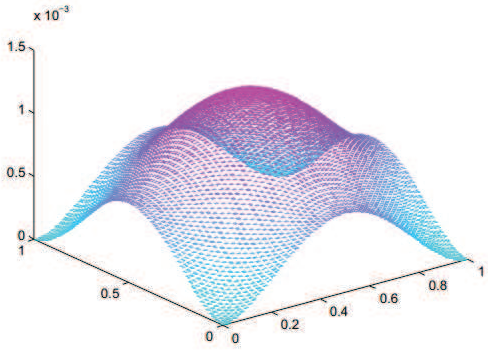


Figure 2: Two grid solution $\tilde{p}_hT=1.0e-1$, $h=1/64$, $\tau=1.0e-4$.

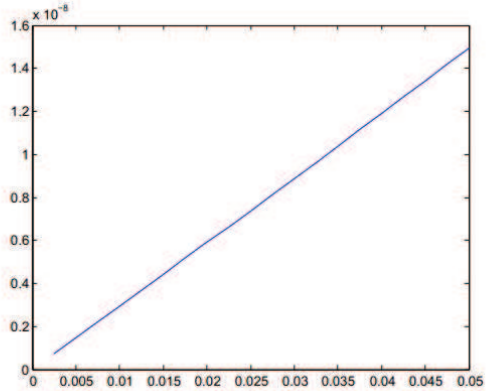


Figure 3: Error of Two grid solution $\tilde{p}_hT=1.0e-1$, $h=1/64$, $\tau=1.0e-4$.

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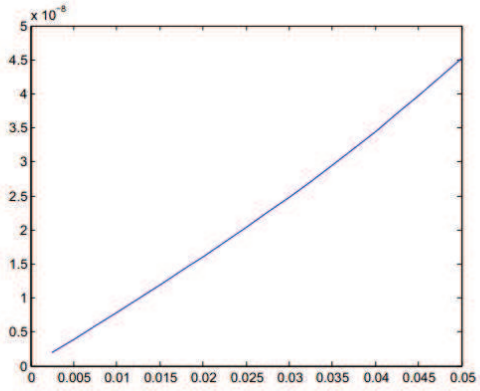


Figure 4: Error of Two grid solution $\tilde{q}_h T=1.0e-1$, $h=1/64$, $\tau=1.0e-4$.

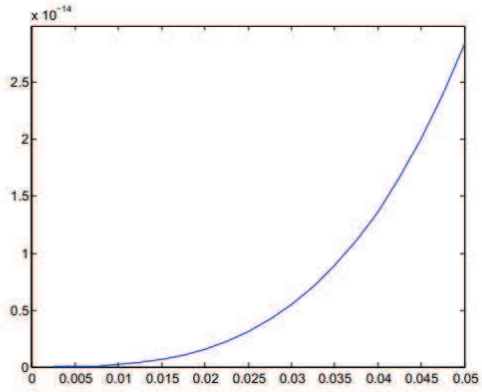


Figure 5: Error of Two grid solution $\tilde{u}_h T=1.0e-1$, $h=1/64$, $\tau=1.0e-4$.

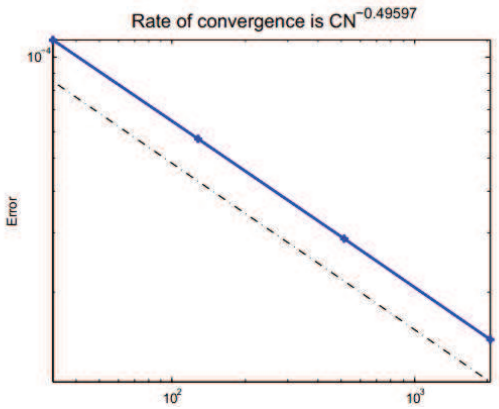


Figure 6: order of Two grid solution $\tilde{p}_h T=1.0e-1$, $\tau=1.0e-4$.

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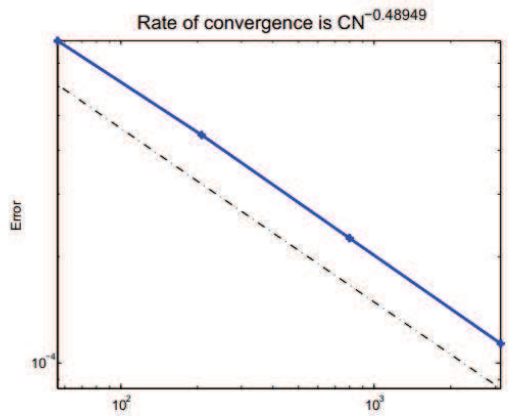


Figure 7: order of Two grid solution $\tilde{q}_h T=1.0e-1$, $\tau=1.0e-4$.

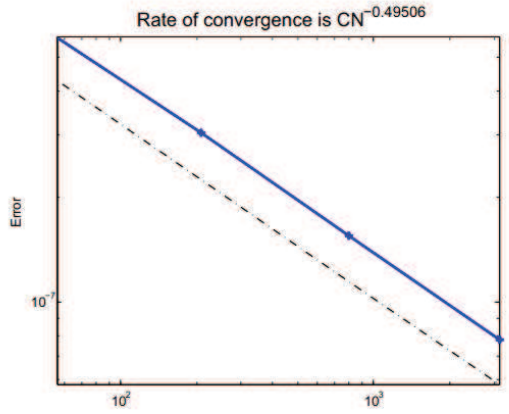


Figure 8: order of Two grid solution $\tilde{u}_h T=1.0e-1$, $\tau=1.0e-4$.

6 Conclusions and extensions

In this paper, we present an algorithm of the two-grid method for nonlinear reaction-diffusion equations discretized by expanded mixed finite element methods. The key feature of the two-grid method is that we use the Newton iteration idea on the fine grid and correction technique on the coarse grid. Our algorithm improve the error estimate for the application of the two-grid method. It is shown theoretically that coarse space can be extremely coarse and preserve asymptotically optimal-order estimates for the expanded mixed finite element method discretization error as long as the error satisfy $\mathcal{O}(h^{k+1}+H^{3k+1}+\Delta t)$. Furthermore, we can also obtain the error of $\mathcal{O}(h^{k+1}+H^{3k+3-d/2}+\Delta t)$ by expanding the algorithm to d-dimension ($d\geq 2$) space, As a result, solving such a large class of non-linear equation will not be much more difficult than solving single linearized equation. Computational results for the approaches

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described here are also provided. Moreover, we shall discuss the case of that both the coefficient matrix function K and f are non-linear.

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