LYNDON INTERPOLATION PROPERTY FOR EXTENSIONS OF **S4** AND INTERMEDIATE PROPOSITIONAL LOGICS

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Abstract. We study the Lyndon interpolation property (LIP) and the uniform LIP (ULIP) for extensions of **S4** and intermediate propositional logics. We prove that among the 18 consistent normal modal logics of finite height extending **S4** known to have CIP, 11 logics have LIP and 7 logics do not. We also prove that for intermediate propositional logics, the Craig interpolation property, LIP, and ULIP are equivalent.

§1. Introduction. The Craig Interpolation Theorem was first proved by Craig [2] for classical first-order predicate logic, and has been studied for many logics. Let $v(\varphi)$ denote the set of all propositional variables contained in a propositional formula φ. We say that a propositional logic L has the *Craig interpolation property* (CIP) iff for any formulas φ and ψ, if $L \vdash \varphi \to \psi$, then there exists a formula ρ such that $v(\rho) \subseteq v(\varphi) \cap v(\psi)$, $L \vdash \varphi \to \rho$, and $L \vdash \rho \to \psi$. CIP has been extensively investigated in modal and intermediate propositional logics. In particular, Maksimova [12] proved that there are exactly seven intermediate propositional logics having CIP. They are classical propositional logic **Cl**, **LS**, **LV**, $\mathbf{LP}_2$, the Gödel–Dummett logic **LC**, the Jankov logic **KC**, and intuitionistic propositional logic **Int**[1]. Moreover, Maksimova proved that there are at most 36 consistent normal extensions of **S4** having CIP, and that at least 30 consistent logics among them have CIP (see [5, 16]). In particular, there are exactly 18 consistent normal modal logics of finite height extending **S4** and having CIP (see Table 1[2]).

Lyndon [11] proved that classical first-order predicate logic enjoys a stronger interpolation property. Let $v^+(\varphi)$ (resp. $v^-(\varphi)$) be the set of all propositional variables occurring in a propositional formula φ positively (resp. negatively). We say that a propositional logic L has the *Lyndon interpolation property* (LIP) iff for any formulas φ and ψ, if $L \vdash \varphi \to \psi$, then there exists a formula ρ such that $v^\circ(\rho) \subseteq v^\circ(\varphi) \cap v^\circ(\psi)$ for $\circ \in \{+, -\}$, $L \vdash \varphi \to \rho$, and $L \vdash \rho \to \psi$. LIP for extensions of **S4** has been studied by Maksimova [15, 17], Fitting [3], and Kuznets [7, 8]. It is also verifiable that Shimura's proofs of CIP for some extensions of **S4** also

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[1]For the names of these logics, we adopt those presented in Gabbay and Maksimova's book [5].

[2]See Gabbay and Maksimova [5] for the detailed definitions of the logics included in the table.

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Logic	CIP	LIP
$\Gamma(\mathbf{Cl}, 1, 0)$, **Triv**	Maksimova [13]	Maksimova [15]
$\Gamma(\mathbf{Cl}, 2, 0)$	Maksimova [13]	$\times$ Maksimova [15]
$\Gamma(\mathbf{Cl}, \omega, 0)$, **S5**	Schumm [20]	Fitting [3]
$\Gamma(\mathbf{LS}, 1, 1)$, **GW.2**	Schumm [20]	Maksimova [15]
$\Gamma(\mathbf{LS}, 1, 2)$	Maksimova [13]	$\times$ Maksimova [15]
$\Gamma(\mathbf{LS}, 1, \omega)$	Maksimova [13]	$\times$ This article (Theorem 9.2)
$\Gamma(\mathbf{LS}, 2, 1)$	Maksimova [13]	This article (Theorem 6.2)
$\Gamma(\mathbf{LS}, \omega, 1)$, **S4.4**	Schumm [20]	Shimura [22]
$\Gamma(\mathbf{LV}, 1, 1)$, **GV**	Maksimova [13]	This article (Theorem 7.1)
$\Gamma(\mathbf{LV}, 1, 2)$	Maksimova [13]	$\times$ Maksimova [15]
$\Gamma(\mathbf{LV}, 1, \omega)$	Maksimova [13]	$\times$ This article (Theorem 9.2)
$\Gamma(\mathbf{LV}, 2, 1)$	Maksimova [13]	This article (Theorem 7.2)
$\Gamma(\mathbf{LV}, \omega, 1)$	Maksimova [13]	This article (Theorem 7.3)
$\Gamma(\mathbf{LP}_2, 1, 1)$, **GW**	Schumm [20]	Shimura [22]; Maksimova [17]
$\Gamma(\mathbf{LP}_2, 1, 2)$	Maksimova [13]	$\times$ Maksimova [15]
$\Gamma(\mathbf{LP}_2, 1, \omega)$	Maksimova [13]	$\times$ This article (Theorem 9.2)
$\Gamma(\mathbf{LP}_2, 2, 1)$	Maksimova [13]	This article (Theorem 8.2)
$\Gamma(\mathbf{LP}_2, \omega, 1)$	Schumm [20]	Shimura [22]

TABLE 1. CIP and LIP for consistent normal modal logics of finite height extending **S4**.

work with respect to LIP [22][3]. Among the 30 logics known to have CIP, it has been proved that 12 logics actually have LIP[4] and 4 logics do not. LIP for intermediate propositional logics has been studied by Maksimova [15, 17] and Kuznets and Lellmann [9, 10]. Among these seven logics known to have CIP, LIP for the six logics other than **LV** has already been proved[5] and the problem of LIP for **LV** was open (see Maksimova [17] and Table 2).

In the present article, we focus on the 18 extensions of **S4** of finite height listed in Table 1. The first goal of this article is to provide a complete description concerning LIP for these logics. We prove that five of them have LIP and three do not. Then, we conclude that among these 18 logics, 11 logics have LIP and 7 logics do not (see Table 1). Our strategy for proving LIP for these logics is to prove a stronger property, the uniform LIP (ULIP). We say that a propositional logic L has the ULIP iff for any formula φ and any finite sets P^+, P^- of propositional variables, there exists a formula θ satisfying the following three conditions:

[3]Shimura proved that CIP of **S4.4**, **GW**, and $\Gamma(\mathbf{LP}_2, \omega, 1)$ follows from that of **S5**, **Triv**, and **S5**, respectively. It is easily shown that his proofs also show that LIP of **S4.4**, **GW**, and $\Gamma(\mathbf{LP}_2, \omega, 1)$ follows from that of **S5**, **Triv**, and **S5**, respectively. The first explicit proof of LIP for **GW** was presented in Maksimova [17].

[4]These 12 logics are **Triv** [15], **S5** [3], **GW.2** [15], **S4.4** [22], **GW** [17, 22], $\Gamma(\mathbf{LP}_2, \omega, 1)$ [22], **S4.2** [7], **S4.1.2** [15], **Grz.2** [17], **S4** [3, 15], **S4.1** [15], and **Grz** [17].

[5]Maksimova [17] explicitly mentioned that LIP of $\mathbf{LP}_2$ immediately follows from that of **GW**. Even earlier, although implicitly, Shimura's results also yield the LIP of $\mathbf{LP}_2$.

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Logic	CIP	LIP	UIP	ULIP
Cl	Craig [2]	Lyndon [11]	✓	✓
LS	Maksimova [12]	Maksimova [15]	✓	✓
LV	Maksimova [12]	This article (Corollary 10.4)	✓	✓
$\mathbf{LP}_2$	Maksimova [12]	Shimura [22] Maksimova [17]	✓	✓
LC	Maksimova [12]	Kuznets and Lellmann [9]	✓	✓
KC	Gabbay [4]	Maksimova [15]	Maksimova [17]	This article (Corollary 10.12)
Int	Schütte [21]	Maksimova [15]	Pitts [19]	This article (Corollary 10.8)

TABLE 2. Interpolation properties for intermediate propositional logics.

1. $L \vdash \varphi \to \theta$,
2. $v^{\circ}(\theta) \subseteq v^{\circ}(\varphi) \setminus P^{\circ}$ for $\circ \in \{+,-\}$,
3. for any formula ψ with $L \vdash \varphi \to \psi$ and $v^{\circ}(\psi) \cap P^{\circ} = \emptyset$ for $\circ \in \{+,-\}$, we have $L \vdash \theta \to \psi$.

Here, θ is called a *uniform Lyndon interpolant* of (φ, P^+, P^-) in L. ULIP is a simultaneous strengthening of LIP and the uniform interpolation property (UIP), and the notion ULIP was introduced in [6]. In that paper, by extending the semantic technique for proving UIP developed by Visser [23], it was proved that the modal logics, such as **K**, **KT**, **KB**, **GL**, and **Grz**, have ULIP. We say that a logic L is *locally tabular* iff for any finite set P of propositional variables, there are finitely many formulas built from variables in P up to L-provable equivalence. It is known that LIP and ULIP are equivalent for every locally tabular logic (cf. [6]). Since the logics in Table 1 are all locally tabular, to prove LIP for these logics, we will adopt the method of proving stronger ULIP.

We also discuss LIP and ULIP for intermediate propositional logics. The problem of LIP for the modal logic **GV** was stated to be open in Maksimova [17], but in this article, by proving LIP for **GV**, we immediately obtain LIP for the intermediate logic **LV**. From this, we conclude that CIP and LIP are equivalent for intermediate propositional logics. Since the logics **Cl**, **LS**, **LV**, $\mathbf{LP}_2$, and **LC** are known to be locally tabular (cf. [1, p. 428]), they are shown to enjoy ULIP. ULIP for **Int** follows immediately from ULIP for **Grz**. In the present article, we prove ULIP for **Grz.2**, which implies that **KC** also has ULIP. From these investigations, we also conclude that CIP and ULIP are equivalent for intermediate propositional logics (see Table 2).

This article is organized as follows. In Section 2, we develop the basis of our method of proving ULIP for several logics. We present a simple proof of ULIP for classical propositional logic **Cl** using the basis of our method. Section 3 is devoted to developing a sufficient condition for modal logics to have ULIP. More precisely, we introduce the notion that a class of Kripke models enjoys n-IP and prove that a logic L has ULIP if L is sound and complete with respect to a class of models enjoying n-IP for some natural number n. In Section 4, we prepare some lemmas on matching between clusters of Kripke models, which are used in our proofs of ULIP

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for several logics. In Sections 5–8, we use our method to prove ULIP for 11 logics in Table 1, including six logics for which LIP has already been proved. In Section 9, we discuss the failure of LIP for some logics. At last, in Section 10, we discuss LIP and ULIP for intermediate propositional logics.

§2. ULIP for classical propositional logic. In this section, we develop the basis of our method for proving ULIP of several modal logics and we apply this basis to prove ULIP of classical propositional logic. For this section only, we assume that formulas mean formulas of classical propositional logic. The language of classical propositional logic consists of propositional variables, logical constant $\bot$, and logical connectives $\wedge, \vee, \neg$, and $\to$.

For each formula φ, we define the sets $v^+(\varphi)$ and $v^-(\varphi)$ recursively as follows:

- $v^+(p) = \{p\}$ and $v^-(p) = \emptyset$ for every propositional variable p,
- $v^\circ(\bot) = \emptyset$ for $\circ \in \{+, -\}$,
- $v^\circ(\varphi * \psi) = v^\circ(\varphi) \cup v^\circ(\psi)$ for $\circ \in \{+, -\}$ and $* \in \{\wedge, \vee\}$,
- $v^+(\neg\varphi) = v^-(\varphi)$ and $v^-(\neg\varphi) = v^+(\varphi)$,
- $v^+(\varphi \to \psi) = v^-(\varphi) \cup v^+(\psi)$ and $v^-(\varphi \to \psi) = v^+(\varphi) \cup v^-(\psi)$.

Also, let $v(\varphi) := v^+(\varphi) \cup v^-(\varphi)$.

DEFINITION 2.1. Let P^+ and P^- be any finite sets of propositional variables.

1. A formula φ is said to be a (P^+, P^-)-*formula* iff $v^\circ(\varphi) \subseteq P^\circ$ for $\circ \in \{+, -\}$.
2. Let $\Phi_0^{(P^+,P^-)}$ denote a fixed finite set of (P^+, P^-)-formulas such that for any (P^+, P^-)-formula φ, there exists a $\psi \in \Phi_0^{(P^+,P^-)}$ such that $\mathbf{Cl} \vdash \varphi \leftrightarrow \psi$. The existence of such a finite set $\Phi_0^{(P^+,P^-)}$ is easily proved. Here, the subscript 0 indicates box-free.
3. For every truth assignments V_0 and V_1 of formulas, we write
$$\mathsf{V}_0 \xrightarrow[0]{(P^+,P^-)} \mathsf{V}_1$$
iff for any (P^+, P^-)-formula φ, if $\mathsf{V}_0(\varphi) = 1$, then $\mathsf{V}_1(\varphi) = 1$.
4. For every truth assignment V, let $\chi_0^{(P^+,P^-)}[\mathsf{V}]$ denote the (P^+, P^-)-formula
$$\bigwedge\{\varphi \in \Phi_0^{(P^+,P^-)} \mid \mathsf{V}(\varphi) = 1\}.$$

The following lemma is easily verified, and therefore we use the lemma freely without referring to it.

LEMMA 2.2 (Cf. [6, Proposition 6]). *Let P^+ and P^- be any finite sets of propositional variables and V_0 and V_1 be any truth assignments. Then, the following are equivalent*:

1. $\mathsf{V}_0 \xrightarrow[0]{(P^+,P^-)} \mathsf{V}_1$.
2. $\mathsf{V}_1 \xrightarrow[0]{(P^-,P^+)} \mathsf{V}_0$.
3. $\mathsf{V}_1\big(\chi_0^{(P^+,P^-)}[\mathsf{V}_0]\big) = 1$.
4. $\mathsf{V}_0\big(\chi_0^{(P^-,P^+)}[\mathsf{V}_1]\big) = 1$.

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	P_0^+	P_1^+	P_2^+	P_0^-	P_1^-	P_2^-	
1	✓			✓			$\mathsf{V}_0(p) = 1$
2	✓				✓		$\mathsf{V}_0(p) = 1$
3	✓					✓	$\mathsf{V}_0(p) = 1$ or $\mathsf{V}_1(p) = 1$
4	✓						$\mathsf{V}_0(p) = 1$
5		✓		✓			$\mathsf{V}_0(p) = 1$
6		✓			✓		$\mathsf{V}_1(p) = 1$
7		✓				✓	$\mathsf{V}_1(p) = 1$
8		✓					$\mathsf{V}_0(p) = 1$
9			✓	✓			$\mathsf{V}_0(p) = 1$ and $\mathsf{V}_1(p) = 1$
10			✓		✓		$\mathsf{V}_1(p) = 1$
11			✓			✓	$\mathsf{V}_1(p) = 1$
12			✓				$\mathsf{V}_1(p) = 1$
13				✓			$\mathsf{V}_0(p) = 1$
14					✓		$\mathsf{V}_1(p) = 1$
15						✓	$\mathsf{V}_1(p) = 1$
16							$\mathsf{V}_0(p) = 1$

TABLE 3. The definition of V^*.

We prove the following theorem, which is an adaptation of a result proved in [6, Proof of Lemma 1] to the framework of this section.

THEOREM 2.3. *Let $P_0^+, P_1^+, P_2^+, P_0^-, P_1^-$, and P_2^- be finite sets of propositional variables such that P_0°, P_1°, and P_2° are pairwise disjoint for $\circ \in \{+,-\}$. Let V_0 and V_1 be truth assignments. If $\mathsf{V}_0 \xrightarrow[0]{(P_1^+,P_1^-)} \mathsf{V}_1$, then there exists a truth assignment V^* such that $\mathsf{V}_0 \xrightarrow[0]{(P_0^+\cup P_1^+,P_0^-\cup P_1^-)} \mathsf{V}^*$ and $\mathsf{V}^* \xrightarrow[0]{(P_1^+\cup P_2^+,P_1^-\cup P_2^-)} \mathsf{V}_1$.*

PROOF. Let V_0 and V_1 be any truth assignments such that $\mathsf{V}_0 \xrightarrow[0]{(P_1^+,P_1^-)} \mathsf{V}_1$. We define a truth assignment V^* by referring to Table 3. For each propositional variable p, let $\mathsf{V}^*(p) = 1$ iff p meets one of the conditions stated in the 16 rows of the table. For example, the fifth row in the table states the condition that $p \in P_1^+ \cap P_0^-$, $p \notin P_0^+ \cup P_2^+ \cup P_1^- \cup P_2^-$, and $\mathsf{V}_0(p) = 1$.

For the proof of $\mathsf{V}_0 \xrightarrow[0]{(P_0^+\cup P_1^+,P_0^-\cup P_1^-)} \mathsf{V}^*$, we simultaneously prove the following two conditions by induction on the construction of φ:

1. If φ is a $(P_0^+ \cup P_1^+, P_0^- \cup P_1^-)$-formula and $\mathsf{V}_0(\varphi) = 1$, then $\mathsf{V}^*(\varphi) = 1$.
2. If φ is a $(P_0^- \cup P_1^-, P_0^+ \cup P_1^+)$-formula and $\mathsf{V}_0(\varphi) = 0$, then $\mathsf{V}^*(\varphi) = 0$.

The case of $\bot$ trivially holds. We prove the case of a propositional variable p.

1. Suppose $p \in P_0^+ \cup P_1^+$ and $\mathsf{V}_0(p) = 1$. If $p \in P_0^+$, then p meets one of 1, 2, 3, and 4, and hence $\mathsf{V}^*(p) = 1$. If $p \in P_1^+$, then we have $\mathsf{V}_1(p) = 1$ because $\mathsf{V}_0 \xrightarrow[0]{(P_1^+,P_1^-)} \mathsf{V}_1$. Then, p meets one of 5, 6, 7, and 8, and thus we obtain $\mathsf{V}^*(p) = 1$.

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2. Suppose $p \in P_0^- \cup P_1^-$ and $\mathsf{V}^*(p) = 1$. In this case, p meets one of 1, 2, 5, 6, 9, 10, 13, and 14. If p meets one of 1, 2, 5, 9, and 13, then we have $\mathsf{V}_0(p) = 1$. If p meets one of 6, 10, and 14, then $p \in P_1^-$ and $\mathsf{V}_1(p) = 1$. Since $\mathsf{V}_0 \xrightarrow[0]{(P_1^+, P_1^-)} \mathsf{V}_1$, we obtain $\mathsf{V}_0(p) = 1$.

The cases of the Boolean combinations are easily proved by using the induction hypothesis.

For the proof of $\mathsf{V}^* \xrightarrow[0]{(P_1^+ \cup P_2^+, P_1^- \cup P_2^-)} \mathsf{V}_1$, we simultaneously prove the following two conditions by induction on the construction of φ:

1. If φ is a $(P_1^+ \cup P_2^+, P_1^- \cup P_2^-)$-formula and $\mathsf{V}^*(\varphi) = 1$, then $\mathsf{V}_1(\varphi) = 1$.
2. If φ is a $(P_1^- \cup P_2^-, P_1^+ \cup P_2^+)$-formula and $\mathsf{V}^*(\varphi) = 0$, then $\mathsf{V}_1(\varphi) = 0$.

We give only the proof of the case that φ is a propositional variable p.

1. Suppose $p \in P_1^+ \cup P_2^+$ and $\mathsf{V}^*(p) = 1$. Then, p meets one of 5, 6, 7, 8, 9, 10, 11, and 12. If p meets one of 6, 7, 9, 10, 11, and 12, then $\mathsf{V}_1(p) = 1$. If p meets one of 5 and 8, then $p \in P_1^+$ and $\mathsf{V}_0(p) = 1$. Since $\mathsf{V}_0 \xrightarrow[0]{(P_1^+, P_1^-)} \mathsf{V}_1$, we obtain $\mathsf{V}_1(p) = 1$.

2. Suppose $p \in P_1^- \cup P_2^-$ and $\mathsf{V}_1(p) = 1$. If $p \notin P_0^+ \cap P_1^-$, then p meets one of 3, 6, 7, 10, 11, 14, and 15, and hence $\mathsf{V}^*(p) = 1$. If $p \in P_0^+ \cap P_1^-$, then we have $\mathsf{V}_0(p) = 1$ because $\mathsf{V}_0 \xrightarrow[0]{(P_1^+, P_1^-)} \mathsf{V}_1$. Then p meets 2, and thus we get $\mathsf{V}^*(p) = 1$. ⊣

By using Theorem 2.3, we can easily prove ULIP for classical propositional logic **Cl**.

THEOREM 2.4. *The logic* **Cl** *has ULIP.*

PROOF. Let φ be any formula and P^+ and P^- be any finite sets of propositional variables. Let $P_0^\circ := P^\circ$ and $P_1^\circ := v^\circ(\varphi) \setminus P^\circ$ for $\circ \in \{+, -\}$. Let θ be the (P_1^+, P_1^-)-formula

$$\bigwedge \{\psi \in \Phi_0^{(P_1^+, P_1^-)} \mid \mathbf{Cl} \vdash \varphi \to \psi\}.$$

We prove that θ is a uniform Lyndon interpolant of (φ, P^+, P^-) in **Cl**. The conditions $\mathbf{Cl} \vdash \varphi \to \theta$ and $v^\circ(\theta) \subseteq v^\circ(\varphi) \setminus P^\circ$ for $\circ \in \{+, -\}$ are easily verified. Let ρ be any formula such that $\mathbf{Cl} \nvdash \theta \to \rho$ and $v^\circ(\rho) \cap P^\circ = \emptyset$ for $\circ \in \{+, -\}$. It suffices to prove $\mathbf{Cl} \nvdash \varphi \to \rho$. Let $P_2^\circ := v^\circ(\rho) \setminus v^\circ(\varphi)$ for $\circ \in \{+, -\}$. Since $v^\circ(\rho) \cap P^\circ = \emptyset$, we have that P_0°, P_1°, and P_2° are pairwise disjoint for $\circ \in \{+, -\}$.

Since $\mathbf{Cl} \nvdash \theta \to \rho$, there exists a truth assignment V_1 such that $\mathsf{V}_1(\theta) = 1$ and $\mathsf{V}_1(\rho) = 0$. Since $\mathsf{V}_1\big(\chi_0^{(P_1^-, P_1^+)}[\mathsf{V}_1]\big) = 1$, we have $\mathsf{V}_1\big(\theta \to \neg\chi_0^{(P_1^-, P_1^+)}[\mathsf{V}_1]\big) = 0$. Thus, $\mathbf{Cl} \nvdash \theta \to \neg\chi_0^{(P_1^-, P_1^+)}[\mathsf{V}_1]$. Since $\neg\chi_0^{(P_1^-, P_1^+)}[\mathsf{V}_1]$ is a (P_1^+, P_1^-)-formula, by the definition of θ, we obtain $\mathbf{Cl} \nvdash \varphi \to \neg\chi_0^{(P_1^-, P_1^+)}[\mathsf{V}_1]$. Hence, there exists a truth assignment V_0 such that $\mathsf{V}_0(\varphi) = 1$ and $\mathsf{V}_0\big(\chi_0^{(P_1^-, P_1^+)}[\mathsf{V}_1]\big) = 1$. By Lemma 2.2, we get $\mathsf{V}_0 \xrightarrow[0]{(P_1^+, P_1^-)} \mathsf{V}_1$.

It follows from Theorem 2.3 that there exists a truth assignment V^* such that $\mathsf{V}_0 \xrightarrow[0]{(P_0^+ \cup P_1^+, P_0^- \cup P_1^-)} \mathsf{V}^*$ and $\mathsf{V}^* \xrightarrow[0]{(P_1^+ \cup P_2^+, P_1^- \cup P_2^-)} \mathsf{V}_1$. Since φ is a $(P_0^+ \cup P_1^+, P_0^- \cup P_1^-)$-

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formula and ρ is a $(P_1^+ \cup P_2^+, P_1^- \cup P_2^-)$-formula, we obtain $\mathsf{V}^*(\varphi) = 1$ and $\mathsf{V}^*(\rho) = 0$. Therefore, $\mathsf{V}^*(\varphi \to \rho) = 0$. We conclude $\mathbf{Cl} \nvdash \varphi \to \rho$. ⊣

§3. A sufficient condition for ULIP in modal logic. In this section, we present a sufficient condition for modal logics to have ULIP. For this purpose, we extend Definition 2.1 to the framework of modal propositional logic. From now on, we assume that formulas mean formulas of modal propositional logic. The language of modal propositional logic is obtained from that of classical propositional logic by adding the unary modal operator $\Box$. Let $\Diamond\varphi$ be the abbreviation for $\neg\Box\neg\varphi$. The notions of positive variables and negative variables of formulas are extended to the language of modal propositional logic with the following clause:

- $v^\circ(\Box\varphi) = v^\circ(\varphi)$ for $\circ \in \{+,-\}$.

DEFINITION 3.1. For each formula φ, let $d(\varphi)$ denote the maximum number of nested occurrences of $\Box$ in φ. More precisely, $d(\varphi)$ is defined recursively as follows:

- $d(p) = d(\bot) = 0$, where p is a propositional variable p,
- $d(\neg\varphi) = d(\varphi)$,
- $d(\varphi \circ \psi) = \max\{d(\varphi), d(\psi)\}$ for $\circ \in \{\wedge, \vee, \to\}$,
- $d(\Box\varphi) = d(\varphi) + 1$.

DEFINITION 3.2.

- A pair (W, R) is called an (**S4**-)*Kripke frame* iff W is a non-empty set and R is a reflexive and transitive binary relation on W.
- A triple $(W, R, \Vdash)$ is called a *Kripke model* iff (W, R) is a Kripke frame and $\Vdash$ is a satisfaction relation between elements of W and formulas fulfilling the usual conditions for each propositional connective and the following condition:

$$(M, x) \Vdash \Box\varphi \iff \forall y(xRy \Rightarrow (M, y) \Vdash \varphi).$$

DEFINITION 3.3. Let P^+ and P^- be any finite sets of propositional variables and n be any natural number.

1. A formula φ is said to be a (P^+, P^-)-*formula* iff $v^\circ(\varphi) \subseteq P^\circ$ for $\circ \in \{+,-\}$.
2. Let $\Phi_n^{(P^+,P^-)}$ denote a fixed finite set of (P^+, P^-)-formulas φ with $d(\varphi) \leq n$ satisfying that for any (P^+, P^-)-formula ψ with $d(\psi) \leq n$, there exists a $\rho \in \Phi_n^{(P^+,P^-)}$ such that $\mathbf{K} \vdash \psi \leftrightarrow \rho$. The existence of such a finite set $\Phi_n^{(P^+,P^-)}$ is proved by induction on n.
3. For Kripke models $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ and elements $w_0 \in W_0$ and $w_1 \in W_1$, we write

$$(M_0, w_0) \xrightarrow[n]{(P^+,P^-)} (M_1, w_1)$$

iff for any (P^+, P^-)-formula φ with $d(\varphi) \leq n$, if $(M_0, w_0) \Vdash_0 \varphi$, then $(M_1, w_1) \Vdash_1 \varphi$.

4. For every Kripke model $M = (W, R, \Vdash)$ and element $w \in W$, let $\chi_n^{(P^+,P^-)}[M, w]$ denote the (P^+, P^-)-formula

$$\bigwedge\{\varphi \in \Phi_n^{(P^+,P^-)} \mid (M, w) \Vdash \varphi\}.$$

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LEMMA 3.4 (Cf. [6, Proposition 6]). *Let P^+ and P^- be any finite sets of propositional variables, $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ be Kripke models, $w_0 \in W_0$ and $w_1 \in W_1$, and n be any natural number. Then, the following are equivalent*:

1. $(M_0, w_0) \xrightarrow[n]{(P^+,P^-)} (M_1, w_1)$.
2. $(M_1, w_1) \xrightarrow[n]{(P^-,P^+)} (M_0, w_0)$.
3. $(M_1, w_1) \Vdash_1 \chi_n^{(P^+,P^-)}[M_0, w_0]$.
4. $(M_0, w_0) \Vdash_0 \chi_n^{(P^-,P^+)}[M_1, w_1]$.

DEFINITION 3.5 (p-morphisms). Let $F = (W, R)$ and $F' = (W', R')$ be Kripke frames. A mapping $f : W \to W'$ is called a *p-morphism* iff the following conditions hold:

- For any $x, y \in W$, if xRy, then $f(x)R'f(y)$.
- For any $x \in W$ and $w \in W'$, if $f(x)R'w$, then there exists $z \in W$ such that xRz and $f(z) = w$.

We introduce a key notion in this article.

DEFINITION 3.6 (n-IP). Let $\mathcal{C}$ be a class of Kripke models and n be a natural number. We say that $\mathcal{C}$ enjoys *n-IP* iff for any finite sets P^+ and P^- of propositional variables, any Kripke models $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ in $\mathcal{C}$, and any elements $w_0 \in W_0$ and $w_1 \in W_1$, if $(M_0, w_0) \xrightarrow[n]{(P^+,P^-)} (M_1, w_1)$, then there exist a Kripke frame $\mathcal{F}^* = (W^*, R^*)$, an element $w^* \in W^*$, and two p-morphisms $f_0 : W^* \to W_0$ and $f_1 : W^* \to W_1$ satisfying the following three conditions:

1. all Kripke models based on $\mathcal{F}^*$ are in $\mathcal{C}$,
2. $f_0(w^*) = w_0$ and $f_1(w^*) = w_1$,
3. for any $x^* \in W^*$, we have $(M_0, f_0(x^*)) \xrightarrow[0]{(P^+,P^-)} (M_1, f_1(x^*))$.

We are ready to prove our main instrument in the present article. The following theorem is an adaptation of [6, Theorem 2] for the purpose of this article, and it also extends the proof of Theorem 2.4 to modal logic. We note that our sufficient condition for ULIP resembles to Marx's condition on bisimulation products for CIP [18].

THEOREM 3.7. *If a modal logic L is sound and complete with respect to some class $\mathcal{C}$ of Kripke models enjoying n-IP for some natural number n, then L has ULIP.*

PROOF. Suppose that a class $\mathcal{C}$ of Kripke models enjoys n-IP and that L is sound and complete with respect to $\mathcal{C}$. Let φ be any formula and P^+ and P^- be any finite sets of propositional variables. Let $P_0^\circ := P^\circ$ and $P_1^\circ := v^\circ(\varphi) \setminus P^\circ$ for $\circ \in \{+, -\}$. Let θ be the (P_1^+, P_1^-)-formula

$$\bigwedge \{\psi \in \Phi_n^{(P_1^+,P_1^-)} \mid L \vdash \varphi \to \psi\}.$$

We prove that θ is a uniform Lyndon interpolant of (φ, P^+, P^-) in L. The conditions $L \vdash \varphi \to \theta$ and $v^\circ(\theta) \subseteq v^\circ(\varphi) \setminus P^\circ$ for $\circ \in \{+, -\}$ are easily verified. Let ρ be any formula such that $L \nvdash \theta \to \rho$ and $v^\circ(\rho) \cap P^\circ = \emptyset$ for $\circ \in \{+, -\}$. We prove $L \nvdash$

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$\varphi \to \rho$. Let $P_2^\circ := v^\circ(\rho) \setminus v^\circ(\varphi)$ for $\circ \in \{+,-\}$. Since $v^\circ(\rho) \cap P^\circ = \emptyset$, we have that P_0°, P_1°, and P_2° are pairwise disjoint for $\circ \in \{+,-\}$.

Since $L \nvdash \theta \to \rho$, by the completeness, there exists a Kripke model $M_1 = (W_1, R_1, \Vdash_1)$ in $\mathcal{C}$ and $w_1 \in W_1$ such that $(M_1, w_1) \Vdash_1 \theta$ and $(M_1, w_1) \nVdash_1 \rho$. Since $(M_1, w_1) \Vdash_1 \chi_n^{(P_1^-, P_1^+)}[M_1, w_1]$, we have

$$(M_1, w_1) \nVdash_1 \theta \to \neg\chi_n^{(P_1^-, P_1^+)}[M_1, w_1].$$

By the soundness, $L \nvdash \theta \to \neg\chi_n^{(P_1^-, P_1^+)}[M_1, w_1]$. Since $\neg\chi_n^{(P_1^-, P_1^+)}[M_1, w_1]$ is a (P_1^+, P_1^-)-formula, by the definition of θ, we obtain $L \nvdash \varphi \to \neg\chi_n^{(P_1^-, P_1^+)}[M_1, w_1]$. By the completeness, there exists a Kripke model $M_0 = (W_0, R_0, \Vdash_0)$ in $\mathcal{C}$ and $w_0 \in W_0$ such that $(M_0, w_0) \Vdash_0 \varphi$ and $(M_0, w_0) \Vdash_0 \chi_n^{(P_1^-, P_1^+)}[M_1, w_1]$. By Lemma 3.4, we get $(M_0, w_0) \xrightarrow[n]{(P_1^+, P_1^-)} (M_1, w_1)$.

Since $\mathcal{C}$ enjoys n-IP, there exists a Kripke frame $\mathcal{F}^* = (W^*, R^*)$, an element $w^* \in W^*$, and two p-morphisms $f_0 : W^* \to W_0$ and $f_1 : W^* \to W_1$ satisfying the following three conditions:

1. all Kripke models based on $\mathcal{F}^*$ are in $\mathcal{C}$,
2. $f_0(w^*) = w_0$ and $f_1(w^*) = w_1$,
3. for any $x^* \in W^*$, we have $(M_0, f_0(x^*)) \xrightarrow[0]{(P_1^+, P_1^-)} (M_1, f_1(x^*))$.

We define a Kripke model $M^* = (W^*, R^*, \Vdash^*)$ as follows: Let $x^* \in W^*$. Since $(M_0, f_0(x^*)) \xrightarrow[0]{(P_1^+, P_1^-)} (M_1, f_1(x^*))$, it follows from Theorem 2.3 that there exists a truth assignment V^* such that for any formula ψ with $d(\psi) = 0$, the following two conditions hold:

1. If ψ is a $(P_0^+ \cup P_1^+, P_0^- \cup P_1^-)$-formula and $(M_0, f_0(x^*)) \Vdash_0 \psi$, then $\mathsf{V}^*(\psi) = 1$.
2. If ψ is a $(P_1^+ \cup P_2^+, P_1^- \cup P_2^-)$-formula and $\mathsf{V}^*(\psi) = 1$, then $(M_1, f_1(x^*)) \Vdash_1 \psi$.

Let $(M^*, x^*) \Vdash^* p :\iff \mathsf{V}^*(p) = 1$.

By the first condition of n-IP, we have $M^* \in \mathcal{C}$.

We prove that for any formula ψ and any $x^* \in W^*$,

1. If ψ is a $(P_0^+ \cup P_1^+, P_0^- \cup P_1^-)$-formula and $(M_0, f_0(x^*)) \Vdash_0 \psi$, then $(M^*, x^*) \Vdash^* \psi$.
2. If ψ is a $(P_0^- \cup P_1^-, P_0^+ \cup P_1^+)$-formula and $(M_0, f_0(x^*)) \nVdash_0 \psi$, then $(M^*, x^*) \nVdash^* \psi$.

We simultaneously prove these two statements by induction on the construction of ψ. The case that $d(\psi) = 0$ directly follows from Theorem 2.3. The cases of propositional connectives easily follow from the induction hypothesis. So we give only the proof of the case that ψ is of the form $\Box\xi$.

1. Suppose that $\Box\xi$ is a $(P_0^+ \cup P_1^+, P_0^- \cup P_1^-)$-formula and $(M^*, x^*) \nVdash^* \Box\xi$. Then, there exists $y^* \in W^*$ such that $x^* R^* y^*$ and $(M^*, y^*) \nVdash^* \xi$. Since ξ is also a $(P_0^+ \cup P_1^+, P_0^- \cup P_1^-)$-formula, by the induction hypothesis, we have $(M_0, f_0(y^*)) \nVdash_0 \xi$. We have $f_0(x^*) R_0 f_0(y^*)$ because f_0 is a p-morphism, and so we obtain $(M_0, f_0(x^*)) \nVdash_0 \Box\xi$.

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2. Suppose that $\Box\xi$ is a $(P_0^- \cup P_1^-, P_0^+ \cup P_1^+)$-formula and $(M_0, f_0(x^*)) \nVdash_0 \Box\xi$. There exists $y_0 \in W_0$ such that $f_0(x^*)R_0y_0$ and $(M_0, y_0) \nVdash_0 \xi$. Since f_0 is a p-morphism, there exists $y^* \in W^*$ such that $x^*R^*y^*$ and $f_0(y^*) = y_0$. Since ξ is also a $(P_0^- \cup P_1^-, P_0^+ \cup P_1^+)$-formula, by the induction hypothesis, we have $(M^*, y^*) \nVdash^* \xi$. Hence, $(M^*, x^*) \nVdash^* \Box\xi$.

In the same way, we can prove that for any formula ψ and any $x^* \in W^*$,

- If ψ is a $(P_1^+ \cup P_2^+, P_1^- \cup P_2^-)$-formula and $(M^*, x^*) \Vdash {}^*\psi$, then $(M_1, f_1(x^*)) \Vdash_1 \psi$.

Since φ is a $(P_0^+ \cup P_1^+, P_0^- \cup P_1^-)$-formula and $(M_0, w_0) = (M_0, f_0(w^*)) \Vdash_0 \varphi$, we have $(M^*, w^*) \Vdash^* \varphi$. Also, since ρ is a $(P_1^+ \cup P_2^+, P_1^- \cup P_2^-)$-formula and $(M_1, w_1) = (M_1, f_1(w^*)) \nVdash_1 \rho$, we obtain $(M^*, w^*) \nVdash^* \rho$. Therefore, $(M^*, w^*) \nVdash^* \varphi \to \rho$. By the soundness, we conclude $L \nvdash \varphi \to \rho$. ⊣

REMARK 3.8. In Definition 3.6, if we label each $x^* \in W^*$ with the pair $(f_0(x^*), f_1(x^*))$ and we roughly write x^* as its label $(f_0(x^*), f_1(x^*))$, then the conditions required for $\mathcal{F}^*$ in the definition can be rewritten as follows without mentioning p-morphisms:

1. all Kripke models based on $\mathcal{F}^*$ are in $\mathcal{C}$,
2. $(w_0, w_1) \in W^*$,
3. if $(x_0, x_1)R^*(y_0, y_1)$, then $x_0R_0y_0$ and $x_1R_1y_1$,
4. if $(x_0, x_1) \in W^*$ and $x_0R_0y_0$, then $(x_0, x_1)R^*(y_0, y_1)$ for some $y_1 \in W_1$,
5. if $(x_0, x_1) \in W^*$ and $x_1R_1y_1$, then $(x_0, x_1)R^*(y_0, y_1)$ for some $y_0 \in W_0$,
6. if $(x_0, x_1) \in W^*$, then $(M_0, x_0) \xrightarrow[0]{(P^+, P^-)} (M_1, x_1)$.

Since there may be distinct elements of W^* with the same label, the above description is somewhat imprecise. For example, Clause 4 should be precisely as follows:

- if $x^* \in W^*$ has a label (x_0, x_1) and $x_0R_0y_0$, then there exist $y^* \in W^*$ and $y_1 \in W_1$ such that y^* has the label (y_0, y_1) and $x^*R^*y^*$.

Since this rough labeling notation makes it easy to check that a given model meets the required conditions, in the subsequent sections, we will adopt this notation to describe the elements of W^*.

§4. Some lemmas on matching between clusters. Before actually proving ULIP for several logics using the method developed in the last section, in this section, we present some lemmas on matching between clusters, which will be used in our proofs of ULIP.

For each Kripke frame (W, R), we say that a subset $I \subseteq W$ is a *cluster* iff I is an equivalence class of the equivalence relation R^s defined by $xR^sy :\iff xRy \,\&\, yRx$. All the modal logics for which we prove their ULIP in this article are extensions of $\Gamma(\mathbf{LP}_2, \omega, 1)$, and so we will basically deal only with finite Kripke models $(W, R, \Vdash)$ such that W may have or may not have the unique root element x and $W \setminus \{x\}$ is a disjoint union of finitely many clusters of final elements. For two such models $M_0 = (W_0, R_0, \Vdash_0)$, $M_1 = (W_1, R_1, \Vdash_1)$ and elements $w_0 \in W_0, w_1 \in W_1$, we discuss some conclusions that can be obtained from the assumption that $(M_0, w_0) \xrightarrow[n]{(P^+, P^-)} (M_1, w_1)$ holds. In this section, for $i \in \{0, 1\}$, let x_i be the root element of M_i and let

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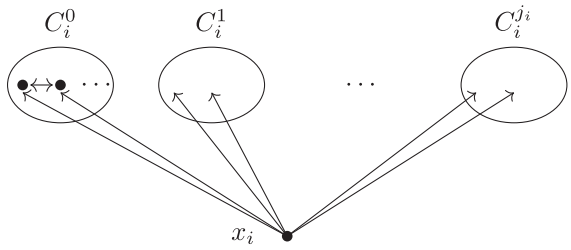


FIGURE 1. The model $M_i = (W_i, R_i, \Vdash_i)$.

$C_i^0, C_i^1, \ldots, C_i^{j_i}$ be all the clusters of final elements of M_i. The model M_i is visualized in Figure 1.

DEFINITION 4.1. We say that a cluster C_0^k of M_0 *matches* a cluster C_1^l of M_1 iff the following two conditions hold:

1. For any $u_0 \in C_0^k$, there exists $u_1 \in C_1^l$ such that $(M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$.
2. For any $u_1 \in C_1^l$, there exists $u_0 \in C_0^k$ such that $(M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$.

LEMMA 4.2. *For $w_0 \in C_0^k$ and $w_1 \in C_1^l$, if $(M_0, w_0) \xrightarrow[1]{(P^+,P^-)} (M_1, w_1)$, then C_0^k matches C_1^l.*

PROOF. Suppose $(M_0, w_0) \xrightarrow[1]{(P^+,P^-)} (M_1, w_1)$. We prove the two clauses of Definition 4.1.

1. For each $u_0 \in C_0^k$, since $(M_0, u_0) \Vdash_0 \chi_0^{(P^+,P^-)}[M_0, u_0]$, we have $(M_0, w_0) \Vdash_0 \Diamond\chi_0^{(P^+,P^-)}[M_0, u_0]$ because $w_0 R_0 u_0$. Since $(M_0, w_0) \xrightarrow[1]{(P^+,P^-)} (M_1, w_1)$, this formula is also true in (M_1, w_1). Thus, we find some $u_1 \in C_1^l$ such that $(M_1, u_1) \Vdash_1 \chi_0^{(P^+,P^-)}[M_0, u_0]$. By Lemma 3.4, we obtain $(M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$.

2. For each $u_1 \in C_1^l$, we have $(M_1, w_1) \Vdash_1 \Diamond\chi_0^{(P^-,P^+)}[M_1, u_1]$. Since $(M_0, w_0) \xrightarrow[1]{(P^+,P^-)} (M_1, w_1)$, this formula is also true in (M_0, w_0). Then as above, we find some $u_0 \in C_0^k$ such that $(M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$. ⊣

LEMMA 4.3. *For $w_0 \in C_0^k$, if $(M_0, w_0) \xrightarrow[2]{(P^+,P^-)} (M_1, x_1)$, then C_0^k matches all the clusters C_1^l of M_1.*

PROOF. Suppose $(M_0, w_0) \xrightarrow[2]{(P^+,P^-)} (M_1, x_1)$. Let C_1^l be any cluster of M_1 and u_1 be any element of C_1^l. Since $(M_1, u_1) \Vdash_1 \chi_1^{(P^-,P^+)}[M_1, u_1]$, we have $(M_1, x_1) \Vdash_1 \Diamond\chi_1^{(P^-,P^+)}[M_1, u_1]$. By the supposition, we have that this formula is also true in

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(M_0, w_0). So, there exists some $u_0 \in C_0^k$ such that $(M_0, u_0) \xrightarrow[1]{(P^+,P^-)} (M_1, u_1)$. By Lemma 4.2, we have that C_0^k matches C_1^l. ⊣

LEMMA 4.4. *If* $(M_0, x_0) \xrightarrow[3]{(P^+,P^-)} (M_1, x_1)$, *then the following two properties hold*:

1. *For any cluster* C_0^k *of* M_0, *there exists a cluster* C_1^l *of* M_1 *such that* C_0^k *matches* C_1^l.
2. *For any cluster* C_1^l *of* M_1, *there exists a cluster* C_0^k *of* M_0 *such that* C_0^k *matches* C_1^l.

PROOF. Suppose $(M_0, x_0) \xrightarrow[3]{(P^+,P^-)} (M_1, x_1)$. We give only a proof of the first clause and the second clause can be proved in the same way.

1. Let C_0^k be any cluster of M_0. For any $v_0 \in C_0^k$, we have

$$(M_0, v_0) \Vdash_0 \bigvee_{u_0 \in C_0^k} \chi_1^{(P^+,P^-)}[M_0, u_0]$$

and hence

$$(M_0, v_0) \Vdash_0 \Box \bigvee_{u_0 \in C_0^k} \chi_1^{(P^+,P^-)}[M_0, u_0].$$

Then,

$$(M_0, x_0) \Vdash_0 \Diamond\Box \bigvee_{u_0 \in C_0^k} \chi_1^{(P^+,P^-)}[M_0, u_0].$$

Since $(M_0, x_0) \xrightarrow[3]{(P^+,P^-)} (M_1, x_1)$, this formula is also true in (M_1, x_1). Then, for some $y_1 \in W_1$,

$$(M_1, y_1) \Vdash_1 \Box \bigvee_{u_0 \in C_0^k} \chi_1^{(P^+,P^-)}[M_0, u_0].$$

Hence, we find some cluster C_1^l of M_1 and an element $u_1 \in C_1^l$ such that

$$(M_1, u_1) \Vdash_1 \bigvee_{u_0 \in C_0^k} \chi_1^{(P^+,P^-)}[M_0, u_0].$$

Thus, there exists $u_0 \in C_0^k$ such that $(M_0, u_0) \xrightarrow[1]{(P^+,P^-)} (M_1, u_1)$. By Lemma 4.2, we conclude that C_0^k matches C_1^l. ⊣

LEMMA 4.5. *Let* $A = \{a_0, a_1\}$ *and* $B = \{b_0, b_1\}$ *be any sets consisting of two elements and let* $R \subseteq A \times B$ *be any binary relation. Suppose that* R *satisfies the following two conditions*:

1. *For any* $a \in A$, *there exists* $b \in B$ *such that* $(a, b) \in R$.
2. *For any* $b \in B$, *there exists* $a \in A$ *such that* $(a, b) \in R$.

Then, there exist $c, d \in B$ *such that* $c \neq d$, $(a_0, c) \in R$, *and* $(a_1, d) \in R$.

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PROOF. If both (a_0, b_0) and (a_1, b_1) are in R, then the statement holds for $c = b_0$ and $d = b_1$. If $(a_0, b_0) \notin R$ or $(a_1, b_1) \notin R$, then by considering the two conditions, we find that both (a_0, b_1) and (a_1, b_0) are in R, and so the statement holds for $c = b_1$ and $d = b_0$. ⊣

LEMMA 4.6. *If the clusters $C_0^k = \{y_0, z_0\}$ and $C_1^l = \{y_1, z_1\}$ consist of two elements and C_0^k matches C_1^l, then there exist $u_1, v_1 \in C_1^l$ such that $u_1 \neq v_1$ and*

$$(M_0, y_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1) \text{ and } (M_0, z_0) \xrightarrow[0]{(P^+,P^-)} (M_1, v_1).$$

PROOF. This directly follows from Definition 4.1 and Lemma 4.5. ⊣

LEMMA 4.7. *Suppose both M_0 and M_1 include exactly two clusters C_0^0, C_0^1 and C_1^0, C_1^1 of final elements, respectively. If $(M_0, x_0) \xrightarrow[3]{(P^+,P^-)} (M_1, x_1)$, then there exist distinct clusters D_1^0 and D_1^1 of M_1 such that C_0^k matches D_1^k for $k \in \{0, 1\}$.*

PROOF. This directly follows from Lemmas 4.4 and 4.5. ⊣

§5. Modal companions of Cl. In this section, we prove ULIP for the logics **Triv** and **S5** as test cases for our application of Theorem 3.7.

5.1. Triv. Let $\mathcal{C}_{\mathbf{Triv}}$ be the class of all Kripke models whose frame consists of a single reflexive world. The logic **Triv** is sound and complete with respect to $\mathcal{C}_{\mathbf{Triv}}$.

THEOREM 5.1 (Maksimova [15]). **Triv** *has ULIP.*

PROOF. We prove that $\mathcal{C}_{\mathbf{Triv}}$ enjoys 0-IP. Then, ULIP for **Triv** follows from Theorem 3.7. Let P^+ and P^- be any finite sets of propositional variables and let $M_0 = (\{w_0\}, R_0, \Vdash_0)$ and $M_1 = (\{w_1\}, R_1, \Vdash_1)$ be Kripke models in $\mathcal{C}_{\mathbf{Triv}}$, and suppose $(M_0, w_0) \xrightarrow[0]{(P^+,P^-)} (M_1, w_1)$. Let $W^* := \{(w_0, w_1)\}$ and $R^* := (W^*)^2$. Then, it is easy to check that $\mathcal{F}^* = (W^*, R^*)$ satisfies the conditions stated in Remark 3.8. ⊣

5.2. S5. Let $\mathcal{C}_{\mathbf{S5}}$ be the class of all Kripke models whose frame forms a finite cluster. Figure 2 visualizes such a frame. In the following, we assume that each diagram represents the Kripke frame of the transitive and reflexive closure of the arrows shown in the diagram. The logic **S5** is sound and complete with respect to $\mathcal{C}_{\mathbf{S5}}$.

THEOREM 5.2 (Fitting [3]). **S5** *has ULIP.*

PROOF. We prove that $\mathcal{C}_{\mathbf{S5}}$ enjoys 1-IP. Let P^+ and P^- be any finite sets of propositional variables and let $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ be

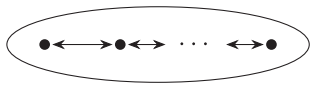


FIGURE 2. A frame of **S5**.

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Kripke models in $\mathcal{C}_{\mathbf{S5}}$, let $w_0 \in W_0$ and $w_1 \in W_1$, and suppose $(M_0, w_0) \xrightarrow[1]{(P^+,P^-)} (M_1, w_1)$. Then, both W_0 and W_1 are clusters of M_0 and M_1, respectively. By Lemma 4.2, we have that W_0 matches W_1.

Let $W^* := \{(x_0, x_1) \mid (M_0, x_0) \xrightarrow[0]{(P^+,P^-)} (M_1, x_1)\}$ and $R^* := (W^*)^2$. We check that $\mathcal{F}^* = (W^*, R^*)$ fulfills the conditions stated in Remark 3.8.

1. Since $\mathcal{F}^*$ forms a finite cluster, all Kripke models based on $\mathcal{F}^*$ are in $\mathcal{C}_{\mathbf{S5}}$.
2. $(w_0, w_1) \in W^*$ because $(M_0, w_0) \xrightarrow[0]{(P^+,P^-)} (M_1, w_1)$.
3. If $(x_0, x_1)R^*(y_0, y_1)$, then $x_0, y_0 \in W_0$ and $x_1, y_1 \in W_1$. Since both W_0 and W_1 form clusters, we obtain $x_0 R_0 y_0$ and $x_1 R_1 y_1$.
4. Suppose $(x_0, x_1) \in W^*$ and $x_0 R_0 y_0$. Since W_0 matches W_1, we find some $y_1 \in W_1$ such that $(M_0, y_0) \xrightarrow[0]{(P^+,P^-)} (M_1, y_1)$. Hence $(y_0, y_1) \in W^*$. Since $R^* = (W^*)^2$, we obtain $(x_0, x_1)R^*(y_0, y_1)$.
5. Suppose $(x_0, x_1) \in W^*$ and $x_1 R_1 y_1$. In the same way as above, we can find some $y_0 \in W_0$ such that $(y_0, y_1) \in W^*$. Then, $(x_0, x_1)R^*(y_0, y_1)$.
6. If $(x_0, x_1) \in W^*$, then $(M_0, x_0) \xrightarrow[0]{(P^+,P^-)} (M_1, x_1)$ by the definition of W^*.

It follows from Theorem 3.7 that **S5** has ULIP. ⊣

§6. Modal companions of LS. In this section, we prove that the three logics $\mathbf{GW.2} = \Gamma(\mathbf{LS}, 1, 1)$, $\Gamma(\mathbf{LS}, 2, 1)$, and $\mathbf{S4.4} = \Gamma(\mathbf{LS}, \omega, 1)$ have ULIP. In particular, LIP for $\Gamma(\mathbf{LS}, 2, 1)$ is new.

6.1. GW.2. Let $\mathcal{C}_{\mathbf{GW.2}}$ be the class of all Kripke models whose frames are of the form shown in Figure 3. The logic **GW.2** is sound and complete with respect to $\mathcal{C}_{\mathbf{GW.2}}$.

THEOREM 6.1 (Maksimova [15]). **GW.2** *has ULIP.*

PROOF. We prove that $\mathcal{C}_{\mathbf{GW.2}}$ enjoys 2-IP. Let P^+ and P^- be any finite sets of propositional variables and let $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ be Kripke models in $\mathcal{C}_{\mathbf{GW.2}}$, let $w_0 \in W_0$ and $w_1 \in W_1$, and suppose $(M_0, w_0) \xrightarrow[2]{(P^+,P^-)} (M_1, w_1)$. For $i \in \{0, 1\}$, let y_i be the final element of M_i.



FIGURE 3. The frame of **GW.2**.

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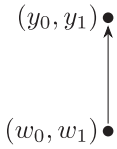


FIGURE 4. The frame (W^*, R^*).

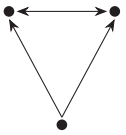


FIGURE 5. The frame of $\Gamma(\mathbf{LS}, 2, 1)$.

We show that $(M_0, y_0) \xrightarrow[0]{(P^+,P^-)} (M_1, y_1)$. Since $v_1 R_1 y_1$ for all $v_1 \in W_1$, we have $(M_1, w_1) \Vdash_1 \Box\Diamond\chi_0^{(P^-,P^+)}[M_1, y_1]$. Since $(M_0, w_0) \xrightarrow[2]{(P^+,P^-)} (M_1, w_1)$, we get $(M_0, w_0) \Vdash_0 \Box\Diamond\chi_0^{(P^-,P^+)}[M_1, y_1]$. Since $w_0 R_0 y_0$, we have $(M_0, y_0) \Vdash_0 \Diamond\chi_0^{(P^-,P^+)}[M_1, y_1]$, and hence $(M_0, y_0) \Vdash_0 \chi_0^{(P^-,P^+)}[M_1, y_1]$. By Lemma 3.4, $(M_0, y_0) \xrightarrow[0]{(P^+,P^-)} (M_1, y_1)$.

It is easy to check that the frame $\mathcal{F}^* = (W^*, R^*)$ drawn in Figure 4 satisfies all the required conditions stated in Remark 3.8. Note that in our rough labeling notation, the situation $(w_0, w_1) = (y_0, y_1)$ is allowed.

⊣

6.2. $\Gamma(\mathbf{LS}, 2, 1)$**.** Let $\mathcal{C}_{\Gamma(\mathbf{LS},2,1)}$ be the class of all Kripke models whose frames are of the form shown in Figure 5. The logic $\Gamma(\mathbf{LS}, 2, 1)$ is sound and complete with respect to $\mathcal{C}_{\Gamma(\mathbf{LS},2,1)}$.

THEOREM 6.2. $\Gamma(\mathbf{LS}, 2, 1)$ *has ULIP.*

PROOF. We prove that $\mathcal{C}_{\Gamma(\mathbf{LS},2,1)}$ enjoys 2-IP. Let P^+ and P^- be any finite sets of propositional variables and let $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ be Kripke models in $\mathcal{C}_{\Gamma(\mathbf{LS},2,1)}$, let $w_0 \in W_0$ and $w_1 \in W_1$, and suppose $(M_0, w_0) \xrightarrow[2]{(P^+,P^-)} (M_1, w_1)$. Let $C_i = \{y_i, z_i\}$ be the cluster of final elements of M_i for $i \in \{0, 1\}$.

We show that C_0 matches C_1. Let $u_1 \in C_1$. Since $v_1 R_1 u_1$ for all $v_1 \in W_1$ and $w_0 R_0 y_0$, in the same way as in the proof of Theorem 6.1, we have $(M_0, y_0) \Vdash_0 \Diamond\chi_0^{(P^-,P^+)}[M_1, u_1]$. Then, $\chi_0^{(P^-,P^+)}[M_1, u_1]$ is true in at least one of y_0 and z_0. Hence,

$$(M_0, y_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1) \text{ or } (M_0, z_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1). \tag{1}$$

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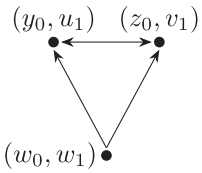


FIGURE 6. The frame (W^*, R^*).

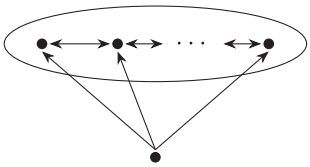


FIGURE 7. A frame of **S4.4**.

In a similar way, we can prove that for $u_0 \in \{y_0, z_0\}$,

$$(M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, y_1) \text{ or } (M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, z_1). \tag{2}$$

Since both C_0 and C_1 consist of two elements, by Lemma 4.6, there exist $u_1, v_1 \in C_1$ such that $(M_0, y_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$ and $(M_0, z_0) \xrightarrow[0]{(P^+,P^-)} (M_1, v_1)$

It is easy to check that the frame (W^*, R^*) in Figure 6 satisfies all the required conditions.

⊣

6.3. S4.4. Let $\mathcal{C}_{\mathbf{S4.4}}$ be the class of all finite Kripke models whose frames are of the form shown in Figure 7. The logic **S4.4** is sound and complete with respect to $\mathcal{C}_{\mathbf{S4.4}}$.

THEOREM 6.3 (Shimura [22]). **S4.4** *has ULIP.*

PROOF. We prove that $\mathcal{C}_{\mathbf{S4.4}}$ enjoys 2-IP. Let P^+ and P^- be any finite sets of propositional variables and let $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ be Kripke models in $\mathcal{C}_{\mathbf{S4.4}}$, let $w_0 \in W_0$ and $w_1 \in W_1$, and suppose $(M_0, w_0) \xrightarrow[2]{(P^+,P^-)} (M_1, w_1)$. Let C_i be the finite cluster of final elements of M_i for $i \in \{0, 1\}$.

We show that C_0 matches C_1. For each $v_0 \in C_0$, we have

$$(M_0, v_0) \Vdash_0 \bigvee_{u_0 \in C_0} \chi_0^{(P^+,P^-)}[M_0, u_0],$$

and hence

$$(M_0, v_0) \Vdash_0 \Box \bigvee_{u_0 \in C_0} \chi_0^{(P^+,P^-)}[M_0, u_0].$$

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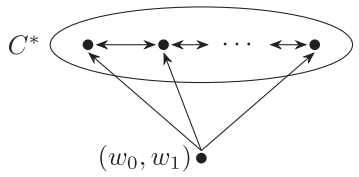


FIGURE 8. The frame (W^*, R^*).

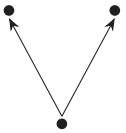


FIGURE 9. The frame of **GV**.

So,

$$(M_0, w_0) \Vdash_0 \Diamond\Box \bigvee_{u_0 \in C_0} \chi_0^{(P^+,P^-)}[M_0, u_0].$$

Since $(M_0, w_0) \xrightarrow[2]{(P^+,P^-)} (M_1, w_1)$, we obtain that this formula is also true in (M_1, w_1). It follows that for every $u_1 \in C_1$, we get

$$(M_1, u_1) \Vdash_1 \bigvee_{u_0 \in C_0} \chi_0^{(P^+,P^-)}[M_0, u_0],$$

and thus, there exists $u_0 \in C_0$ such that $(M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$.

In the same way, we can show that for any $u_0 \in C_0$, there exists $u_1 \in C_1$ such that $(M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$.

Let

$$C^* := \{(u_0, u_1) \mid u_0 \in C_0, u_1 \in C_1, \text{ and } (M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)\}.$$

Then, the frame (W^*, R^*) shown in Figure 8 satisfies all the required conditions.

⊣

§7. Modal companions of LV. In this section, we newly prove that the three logics $\mathbf{GV} = \Gamma(\mathbf{LV}, 1, 1)$, $\Gamma(\mathbf{LV}, 2, 1)$, and $\Gamma(\mathbf{LV}, \omega, 1)$ have ULIP. In particular, LIP for **GV** is mentioned in [17] as an open problem.

7.1. GV. Let $\mathcal{C}_{\mathbf{GV}}$ be the class of all Kripke models whose frames are of the form shown in Figure 9. The logic **GV** is sound and complete with respect to $\mathcal{C}_{\mathbf{GV}}$.

THEOREM 7.1. **GV** *has ULIP.*

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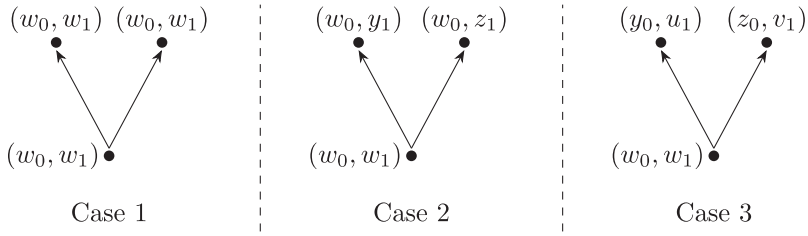


FIGURE 10. The frames (W^*, R^*).

PROOF. We prove that $\mathcal{C}_{\mathbf{GV}}$ enjoys 3-IP. Let P^+ and P^- be any finite sets of propositional variables and let $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ be Kripke models in $\mathcal{C}_{\mathbf{GV}}$, let $w_0 \in W_0$ and $w_1 \in W_1$, and suppose $(M_0, w_0) \xrightarrow[3]{(P^+,P^-)} (M_1, w_1)$. For $i \in \{0, 1\}$, let x_i be the root element of M_i and y_i, z_i be two distinct final elements of M_i. We distinguish the following three cases.

CASE 1. $w_0 \in \{y_0, z_0\}$ *and* $w_1 \in \{y_1, z_1\}$.

CASE 2. $w_0 \in \{y_0, z_0\}$ *and* $w_1 = x_1$.

For the case where $w_0 = x_0$ and $w_1 \in \{y_1, z_1\}$, we can draw the required frame (W^*, R^*) similar to that corresponding to Case 2 in Figure 10. In subsequent proofs, we will basically omit the mention of such similar cases. In Case 2, by Lemma 4.3, we have that $(M_0, w_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$ for $u_1 \in \{y_1, z_1\}$.

CASE 3. $w_0 = x_0$ *and* $w_1 = x_1$.

Since both M_0 and M_1 consist of two clusters, by Lemma 4.7, there exist two distinct final elements u_1 and v_1 of M_1 such that $(M_0, y_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$ and $(M_0, z_0) \xrightarrow[0]{(P^+,P^-)} (M_1, v_1)$.

Our frames (W^*, R^*) corresponding to each of the three cases are shown together in Figure 10. It is easy to check that these frames satisfy all the required conditions. ⊣

7.2. $\Gamma(\mathbf{LV}, 2, 1)$**.** Let $\mathcal{C}_{\Gamma(\mathbf{LV},2,1)}$ be the class of all Kripke models whose frames are of the form shown in Figure 11. The logic $\Gamma(\mathbf{LV}, 2, 1)$ is sound and complete with respect to $\mathcal{C}_{\Gamma(\mathbf{LV},2,1)}$.

THEOREM 7.2. $\Gamma(\mathbf{LV}, 2, 1)$ *has ULIP.*

PROOF. We prove that $\mathcal{C}_{\Gamma(\mathbf{LV},2,1)}$ enjoys 3-IP. Let P^+ and P^- be any finite sets of propositional variables and let $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ be Kripke models in $\mathcal{C}_{\Gamma(\mathbf{LV},2,1)}$, let $w_0 \in W_0$ and $w_1 \in W_1$, and suppose $(M_0, w_0) \xrightarrow[3]{(P^+,P^-)} (M_1, w_1)$. For $i \in \{0, 1\}$, let x_i be the root element of M_i and let $C_i^0 = \{y_i^0, z_i^0\}$ and

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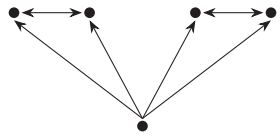


FIGURE 11. The frame of $\Gamma(\mathbf{LV}, 2, 1)$.

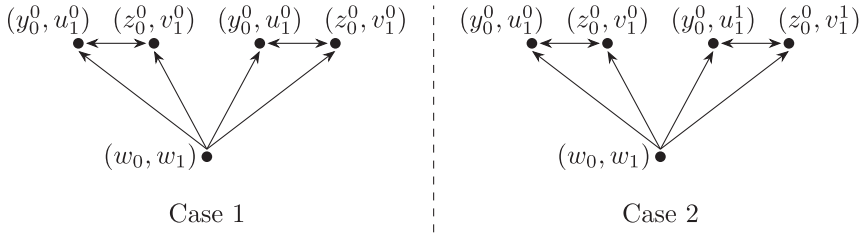


FIGURE 12. The frames (W^*, R^*) in Cases 1 and 2.

$C_i^1 = \{y_i^1, z_i^1\}$ be two distinct clusters of final elements of M_i. We distinguish the following three cases.

CASE 1. $w_0 \in C_0^0$ *and* $w_1 \in C_1^0$.

By Lemma 4.2, we have that C_0^0 matches C_1^0. Since each cluster consists of two elements, by Lemma 4.6, there exist $u_1^0, v_1^0 \in C_1^0$ such that $(M_0, y_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^0)$ and $(M_0, z_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, v_1^0)$.

CASE 2. $w_0 \in C_0^0$ *and* $w_1 = x_1$.

By Lemma 4.3, C_0^0 matches both C_1^0 and C_1^1. Then, by Lemma 4.6, for $k \in \{0, 1\}$, there exist $u_1^k, v_1^k \in C_1^k$ such that $(M_0, y_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^k)$ and $(M_0, z_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, v_1^k)$.

CASE 3. $w_0 = x_0$ *and* $w_1 = x_1$.

Since both M_0 and M_1 consist of two clusters, by Lemma 4.7, there exist two distinct clusters D_1^0 and D_1^1 of final elements of M_1 such that for $k \in \{0, 1\}$, C_0^k matches D_1^k. Also, since each cluster consists of two elements, by Lemma 4.6, for $k \in \{0, 1\}$, there exist $u_1^k, v_1^k \in D_1^k$ such that $(M_0, y_0^k) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^k)$ and $(M_0, z_0^k) \xrightarrow[0]{(P^+,P^-)} (M_1, v_1^k)$.

Our frames (W^*, R^*) corresponding to Cases 1 and 2 are shown together in Figure 12. Figure 13 shows our frame (W^*, R^*) corresponding to Case 3. These frames satisfy all the required conditions.

⊣

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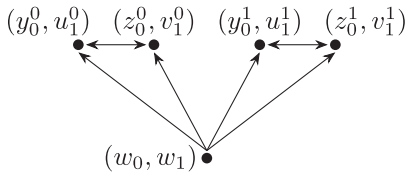


FIGURE 13. The frame (W^*, R^*) in Case 3.

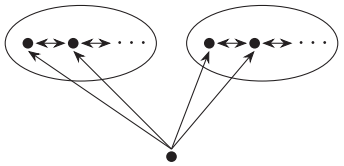


FIGURE 14. A frame of $\Gamma(\mathbf{LV}, \omega, 1)$.

7.3. $\Gamma(\mathbf{LV}, \omega, 1)$**.** Let $\mathcal{C}_{\Gamma(\mathbf{LV},\omega,1)}$ be the class of all finite Kripke models whose frames are of the form shown in Figure 14. The logic $\Gamma(\mathbf{LV}, \omega, 1)$ is sound and complete with respect to $\mathcal{C}_{\Gamma(\mathbf{LV},\omega,1)}$.

THEOREM 7.3. $\Gamma(\mathbf{LV}, \omega, 1)$ *has ULIP.*

PROOF. We prove that $\mathcal{C}_{\Gamma(\mathbf{LV},\omega,1)}$ enjoys 3-IP. Let P^+ and P^- be any finite sets of propositional variables and let $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ be Kripke models in $\mathcal{C}_{\Gamma(\mathbf{LV},\omega,1)}$, let $w_0 \in W_0$ and $w_1 \in W_1$, and suppose $(M_0, w_0) \xrightarrow[3]{(P^+,P^-)} (M_1, w_1)$. For $i \in \{0, 1\}$, let x_i be the root element of M_i and let C_i^0 and C_i^1 be two distinct clusters of final elements of M_i. We distinguish the following three cases.

CASE 1. $w_0 \in C_0^0$ *and* $w_1 \in C_1^0$.

By Lemma 4.2, C_0^0 matches C_1^0. Let

$$C^* := \{(u_0^0, u_1^0) \mid u_0^0 \in C_0^0, u_1^0 \in C_1^0, \text{ and } (M_0, u_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^0)\}.$$

CASE 2. $w_0 \in C_0^0$ *and* $w_1 = x_1$.

By Lemma 4.3, C_0^0 matches both C_1^0 and C_1^1. Let

$$C^{*,0} := \{(u_0^0, u_1^0) \mid u_0^0 \in C_0^0, u_1^0 \in C_1^0, \text{ and } (M_0, u_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^0)\},$$

$$C^{*,1} := \{(u_0^0, u_1^1) \mid u_0^0 \in C_0^0, u_1^1 \in C_1^1, \text{ and } (M_0, u_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^1)\}.$$

CASE 3. $w_0 = x_0$ *and* $w_1 = x_1$.

Since both M_0 and M_1 consist of two clusters, by Lemma 4.7, there exist two distinct clusters D_1^0 and D_1^1 of final elements of M_1 such that for $k \in \{0, 1\}$, C_0^k

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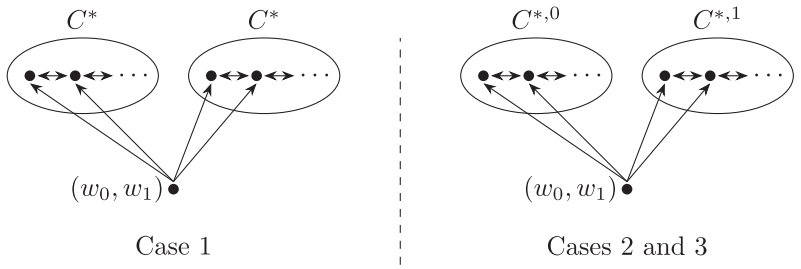


FIGURE 15. The frame (W^*, R^*).

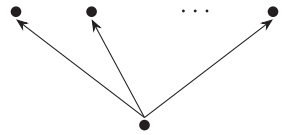


FIGURE 16. A frame of **GW**.

matches D_1^k. Thus, let

$$C^{*,0} := \{(u_0^0, u_1^0) \mid u_0^0 \in C_0^0, u_1^0 \in D_1^0, \text{ and } (M_0, u_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^0)\},$$

$$C^{*,1} := \{(v_0^0, v_1^1) \mid v_0^0 \in C_0^1, v_1^1 \in D_1^1, \text{ and } (M_0, v_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, v_1^1)\}.$$

Our frames (W^*, R^*) corresponding to these three cases are shown together in Figure 15. We can easily check that these frames satisfy all the required conditions. ⊣

§8. Modal companions of $\mathbf{LP}_2$. In this section, we prove that the three logics $\mathbf{GW} = \Gamma(\mathbf{LP}_2, 1, 1)$, $\Gamma(\mathbf{LP}_2, 2, 1)$, and $\Gamma(\mathbf{LP}_2, \omega, 1)$ have ULIP. In particular, LIP for $\Gamma(\mathbf{LP}_2, 2, 1)$ is new.

8.1. GW. Let $\mathcal{C}_{\mathbf{GW}}$ be the class of all finite Kripke models whose frames are of the form shown in Figure 16. The logic **GW** is sound and complete with respect to $\mathcal{C}_{\mathbf{GW}}$.

THEOREM 8.1 (Shimura [22]; Maksimova [17]). **GW** *has ULIP.*

PROOF. We prove that $\mathcal{C}_{\mathbf{GW}}$ enjoys 3-IP. Let P^+ and P^- be any finite sets of propositional variables and let $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ be Kripke models in $\mathcal{C}_{\mathbf{GW}}$, let $w_0 \in W_0$ and $w_1 \in W_1$, and suppose $(M_0, w_0) \xrightarrow[3]{(P^+,P^-)} (M_1, w_1)$. For $i \in \{0, 1\}$, let x_i be the root element of M_i and let F_i be the set of all final elements of M_i. We distinguish the following three cases.

CASE 1. $w_0 \in F_0$ *and* $w_1 \in F_1$.

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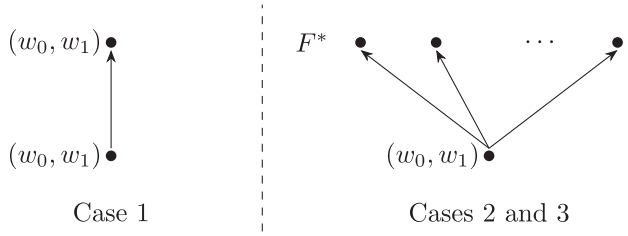


FIGURE 17. The frames (W^*, R^*).

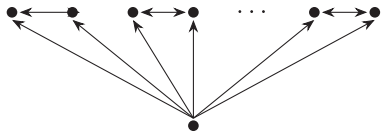


FIGURE 18. A frame of $\Gamma(\mathbf{LP}_2, 2, 1)$.

CASE 2. $w_0 \in F_0$ *and* $w_1 = x_1$.

By Lemma 4.3, for all $u_1 \in F_1$, we have that $(M_0, w_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$. Let $F^* := \{(w_0, u_1) \mid u_1 \in F_1\}$.

CASE 3. $w_0 = x_0$ *and* $w_1 = x_1$.

By Lemma 4.4, we obtain the following properties:

1. For any $u_0 \in F_0$, there exists $u_1 \in F_1$ such that $(M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$.
2. For any $u_1 \in F_1$, there exists $u_0 \in F_0$ such that $(M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)$.

Let

$$F^* := \{(u_0, u_1) \mid u_0 \in F_0, u_1 \in F_1, \text{ and } (M_0, u_0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1)\}.$$

Our frames (W^*, R^*) corresponding to these three cases are shown together in Figure 17. It is easy to check that these frames satisfy all the required conditions. ⊣

8.2. $\Gamma(\mathbf{LP}_2, 2, 1)$**.** Let $\mathcal{C}_{\Gamma(\mathbf{LP}_2,2,1)}$ be the class of all finite Kripke models whose frames are of the form shown in Figure 18. The logic $\Gamma(\mathbf{LP}_2, 2, 1)$ is sound and complete with respect to $\mathcal{C}_{\Gamma(\mathbf{LP}_2,2,1)}$.

THEOREM 8.2. $\Gamma(\mathbf{LP}_2, 2, 1)$ *has ULIP.*

PROOF. We prove that $\mathcal{C}_{\Gamma(\mathbf{LP}_2,2,1)}$ enjoys 3-IP. Let P^+ and P^- be any finite sets of propositional variables and let $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ be Kripke models in $\mathcal{C}_{\Gamma(\mathbf{LP}_2,2,1)}$, let $w_0 \in W_0$ and $w_1 \in W_1$, and suppose

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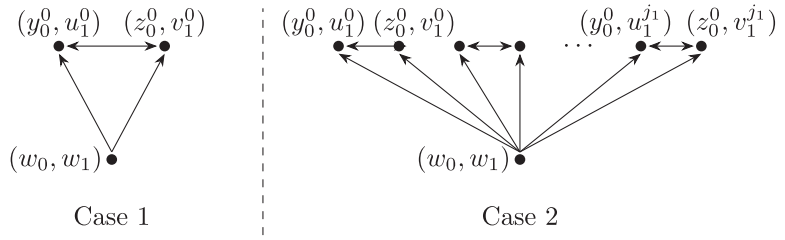


FIGURE 19. The frames (W^*, R^*) in Cases 1 and 2.

$(M_0, w_0) \xrightarrow[3]{(P^+,P^-)} (M_1, w_1)$. For $i \in \{0, 1\}$, let x_i be the root element of M_i and let $C_i^0, C_i^1, \ldots, C_i^{j_i}$ be all the clusters of final elements of M_i. Also let $C_i^k = \{y_i^k, z_i^k\}$. We distinguish the following three cases.

CASE 1. $w_0 \in C_0^0$ *and* $w_1 \in C_1^0$.

By Lemma 4.2, C_0^0 matches C_1^0. Since these clusters consist of two elements, by Lemma 4.6, there exist $u_1^0, v_1^0 \in C_1^0$ such that $(M_0, y_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^0)$ and $(M_0, z_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, v_1^0)$.

CASE 2. $w_0 \in C_0^0$ *and* $w_1 = x_1$.

By Lemma 4.3, C_0^0 matches all the clusters C_1^k of final elements of M_1. Since every cluster consists of two elements, by Lemma 4.6, for each cluster C_1^k of M_1, there exist $u_1^k, v_1^k \in C_1^k$ such that $(M_0, y_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^k)$ and $(M_0, z_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, v_1^k)$.

CASE 3. $w_0 = x_0$ *and* $w_1 = x_1$.

By Lemmas 4.4 and 4.6, we obtain the following two properties:

1. For any cluster C_0^k of M_0, there exist a cluster $C_1^{k'}$ of M_1 and $u_1^{k'}, v_1^{k'} \in C_1^{k'}$ such that $u_1^{k'} \neq v_1^{k'}$ and
$$(M_0, y_0^k) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^{k'}) \text{ and } (M_0, z_0^k) \xrightarrow[0]{(P^+,P^-)} (M_1, v_1^{k'}).$$
2. For any cluster C_1^k of M_1, there exist a cluster $C_0^{k'}$ of M_0 and $u_0^{k'}, v_0^{k'} \in C_0^{k'}$ such that $u_0^{k'} \neq v_0^{k'}$ and
$$(M_0, u_0^{k'}) \xrightarrow[0]{(P^+,P^-)} (M_1, y_1^k) \text{ and } (M_0, v_0^{k'}) \xrightarrow[0]{(P^+,P^-)} (M_1, z_1^k).$$

Our frames (W^*, R^*) corresponding to Cases 1 and 2 are drawn in Figure 19. Our frame (W^*, R^*) corresponding to Case 3 is shown in Figure 20. These frames satisfy all the required conditions.

⊣

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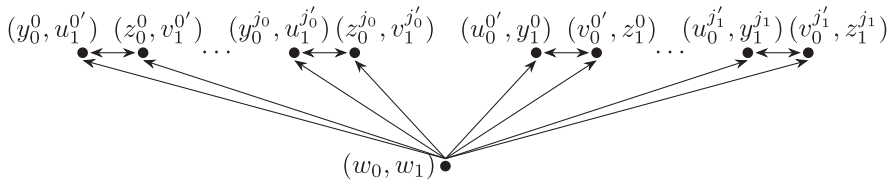


FIGURE 20. The frame (W^*, R^*) in Case 3.

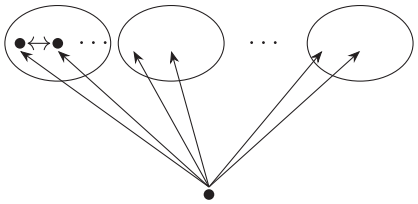


FIGURE 21. A frame of $\Gamma(\mathbf{LP}_2, \omega, 1)$.

8.3. $\Gamma(\mathbf{LP}_2, \omega, 1)$**.** Let $\mathcal{C}_{\Gamma(\mathbf{LP}_2,\omega,1)}$ be the class of all finite Kripke models whose frames are of the form shown in Figure 21. The logic $\Gamma(\mathbf{LP}_2, \omega, 1)$ is sound and complete with respect to $\mathcal{C}_{\Gamma(\mathbf{LP}_2,\omega,1)}$.

THEOREM 8.3 (Shimura [22]). $\Gamma(\mathbf{LP}_2, \omega, 1)$ *has ULIP.*

PROOF. We prove that $\mathcal{C}_{\Gamma(\mathbf{LP}_2,\omega,1)}$ enjoys 3-IP. Let P^+ and P^- be any finite sets of propositional variables and let $M_0 = (W_0, R_0, \Vdash_0)$ and $M_1 = (W_1, R_1, \Vdash_1)$ be Kripke models in $\mathcal{C}_{\Gamma(\mathbf{LP}_2,\omega,1)}$, let $w_0 \in W_0$ and $w_1 \in W_1$, and suppose $(M_0, w_0) \xrightarrow[3]{(P^+,P^-)} (M_1, w_1)$. For $i \in \{0, 1\}$, let x_i be the root element of M_i and let $C_i^0, C_i^1, \ldots, C_i^{j_i}$ be all the clusters of final elements of M_i. We distinguish the following three cases.

CASE 1. $w_0 \in C_0^0$ *and* $w_1 \in C_1^0$.

By Lemma 4.2, C_0^0 matches C_1^0. Let

$$C^* := \{(u_0^0, u_1^0) \mid u_0^0 \in C_0^0, u_1^0 \in C_1^0, \text{ and } (M_0, u_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^0)\}.$$

CASE 2. $w_0 \in C_0^0$ *and* $w_1 = x_1$.

By Lemma 4.3, C_0^0 matches all the clusters C_1^k of final elements of M_1. Then, for each cluster C_1^k of M_1, let

$$C^{*,k} = \{(u_0^0, u_1^k) \mid u_0^0 \in C_0^0, u_1^k \in C_1^k, \text{ and } (M_0, u_0^0) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^k)\}.$$

CASE 3. $w_0 = x_0$ *and* $w_1 = x_1$.

By Lemma 4.4, we obtain the following two properties:

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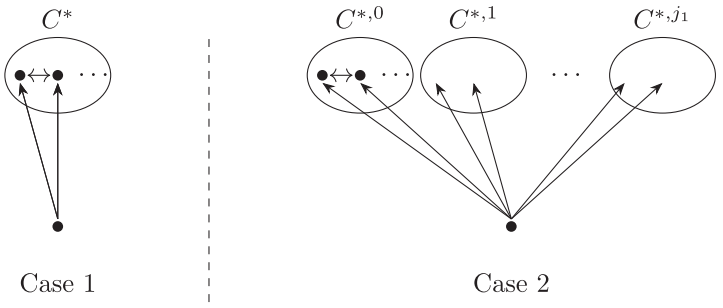


FIGURE 22. The frames (W^*, R^*) in Cases 1 and 2.

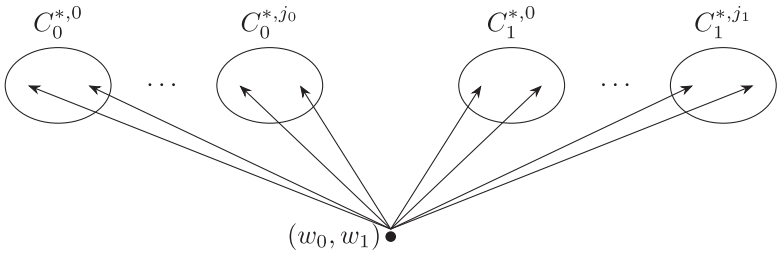


FIGURE 23. The frame (W^*, R^*) in Case 3.

1. For any cluster C_0^k of M_0, there exists a cluster $C_1^{k'}$ of M_1 such that C_0^k matches $C_1^{k'}$.
2. For any cluster C_1^k of M_1, there exists a cluster $C_0^{k'}$ of M_0 such that $C_0^{k'}$ matches C_1^k.

For each cluster C_0^k of M_0, let

$$C_0^{*,k} := \{(u_0^k, u_1^{k'}) \mid u_0^k \in C_0^k, u_1^{k'} \in C_1^{k'}, \text{ and } (M_0, u_0^k) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^{k'})\}.$$

Also, for each cluster C_1^k of M_1, let

$$C_1^{*,k} := \{(u_0^{k'}, u_1^k) \mid u_0^{k'} \in C_0^{k'}, u_1^k \in C_1^k, \text{ and } (M_0, u_0^{k'}) \xrightarrow[0]{(P^+,P^-)} (M_1, u_1^k)\}.$$

Our frames (W^*, R^*) corresponding to Cases 1 and 2 are drawn in Figure 22. Our frame (W^*, R^*) corresponding to Case 3 is shown in Figure 23. We can check that these frames satisfy all the required conditions.

⊣

§9. Failure of LIP. In this section, we investigate the failure of LIP. Maksimova [15] proved that each of the logics $\Gamma(\mathbf{LP}_2, 1, 2)$, $\Gamma(\mathbf{LV}, 1, 2)$, $\Gamma(\mathbf{LS}, 1, 2)$, and $\Gamma(\mathbf{Cl}, 2, 0)$ has CIP but does not have LIP (see also [5, 16]). We prove that the

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three further logics $\Gamma(\mathbf{LP}_2, 1, \omega)$, $\Gamma(\mathbf{LV}, 1, \omega)$, and $\Gamma(\mathbf{LS}, 1, \omega)$ also do not have LIP. Moreover, we will prove this in a form that includes a part of Maksimova's results, namely, we prove that every logic L satisfying $\Gamma(\mathbf{LP}_2, 1, \omega) \subseteq L \subseteq \Gamma(\mathbf{LS}, 1, 2)$ does not have LIP.

$$\begin{array}{ccccc} \Gamma(\mathbf{LP}_2, 1, 2) & \subseteq & \Gamma(\mathbf{LV}, 1, 2) & \subseteq & \Gamma(\mathbf{LS}, 1, 2) \\ \curlyvee & & \curlyvee & & \curlyvee \\ \Gamma(\mathbf{LP}_2, 1, \omega) & \subseteq & \Gamma(\mathbf{LV}, 1, \omega) & \subseteq & \Gamma(\mathbf{LS}, 1, \omega). \end{array}$$

Before proving our theorem, we prepare the following lemma.

LEMMA 9.1. *Over the logic* $\Gamma(\mathbf{LS}, 1, 2)$, *every formula* φ *satisfying* $v^+(\varphi) \subseteq \{p\}$ *and* $v^-(\varphi) = \emptyset$ *is provably equivalent to one of the following eight formulas*:

$$\bot, \quad \Box p, \quad p \wedge \Box\Diamond p, \quad \Box\Diamond p, \quad p, \quad p \vee \Box\Diamond p, \quad \Diamond p, \quad \top.$$

PROOF. The logic $\Gamma(\mathbf{LS}, 1, 2)$ is characterized by the Kripke frame shown in Figure 24. It is easy to see that $\Gamma(\mathbf{LS}, 1, 2)$ contains $\Box\Diamond p \leftrightarrow \Diamond\Box p$.

The diagram in Figure 25 shows the implications between these eight formulas over **S4**.

We prove the lemma by induction on the construction of φ. If φ is one of p, $\bot$, and $\top$, then the lemma is trivial.

Suppose that the lemma holds for φ_0 and φ_1. It is easy to show that $\varphi_0 \wedge \varphi_1$ and $\varphi_0 \vee \varphi_1$ are equivalent to one of the eight formulas. The following list of equivalences shows that $\Box\varphi_0$ and $\Diamond\varphi_0$ are also equivalent to some of the eight formulas.

$\Box\bot \leftrightarrow \bot$	$\Diamond\bot \leftrightarrow \bot$
$\Box\Box p \leftrightarrow \Box p$	$\Diamond\Box p \leftrightarrow \Box\Diamond p$
$\Box(p \wedge \Box\Diamond p) \leftrightarrow \Box p$	$\Diamond(p \wedge \Box\Diamond p) \leftrightarrow \Box\Diamond p$
$\Box p \leftrightarrow \Box p$	$\Diamond p \leftrightarrow \Diamond p$
$\Box\Box\Diamond p \leftrightarrow \Box\Diamond p$	$\Diamond\Box\Diamond p \leftrightarrow \Box\Diamond p$
$\Box(p \vee \Box\Diamond p) \leftrightarrow \Box\Diamond p$	$\Diamond(p \vee \Box\Diamond p) \leftrightarrow \Diamond p$
$\Box\Diamond p \leftrightarrow \Box\Diamond p$	$\Diamond\Diamond p \leftrightarrow \Diamond p$
$\Box\top \leftrightarrow \top$	$\Diamond\top \leftrightarrow \top$

⊣

Here, we are ready to prove the theorem.

THEOREM 9.2. *Let* L *be any logic satisfying* $\Gamma(\mathbf{LP}_2, 1, \omega) \subseteq L \subseteq \Gamma(\mathbf{LS}, 1, 2)$. *Then*, L *does not have LIP. In particular, each of the logics* $\Gamma(\mathbf{LP}_2, 1, \omega)$, $\Gamma(\mathbf{LP}_2, 1, 2)$, $\Gamma(\mathbf{LV}, 1, \omega)$, $\Gamma(\mathbf{LV}, 1, 2)$, $\Gamma(\mathbf{LS}, 1, \omega)$, *and* $\Gamma(\mathbf{LS}, 1, 2)$ *has CIP but does not have LIP.*

PROOF. At first, we prove

$$\Gamma(\mathbf{LP}_2, 1, \omega) \vdash p \wedge \Box(\Box\neg p \vee p) \rightarrow \Box(p \vee q \vee \Box\neg q).$$

Let $(W, R, \Vdash)$ be any finite Kripke model whose frame is of the form shown in Figure 26. The logic $\Gamma(\mathbf{LP}_2, 1, \omega)$ is characterized by the class of all such Kripke frames.

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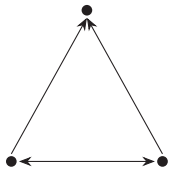


FIGURE 24. The frame of $\Gamma(\mathbf{LS}, 1, 2)$.

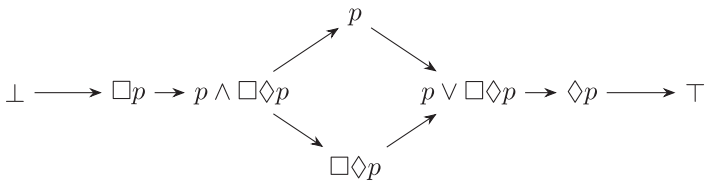


FIGURE 25. The implications between the eight formulas over **S4**.

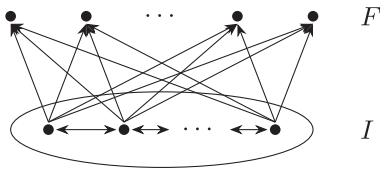


FIGURE 26. A frame of $\Gamma(\mathbf{LP}_2, 1, \omega)$.

Let F be the set of all final elements of the model, and I be the cluster $W \setminus F$ of inner elements. It suffices to show that $p \wedge \Box(\Box\neg p \vee p) \to \Box(p \vee q \vee \Box\neg q)$ is valid in $(W, R, \Vdash)$. For $x \in F$, we have $x \Vdash p \wedge \Box(\Box\neg p \vee p) \to \Box(p \vee q \vee \Box\neg q)$ because $x \Vdash p \to \Box p$. For $x \in I$, assume $x \Vdash p \wedge \Box(\Box\neg p \vee p)$. Let $y \in W$ be any element such that xRy. If $y \in I$, then $y \Vdash \Diamond p \wedge (\Box\neg p \vee p)$, and so $y \Vdash p$. If $y \in F$, then $y \Vdash q \vee \Box\neg q$ holds. Hence, we obtain $x \Vdash \Box(p \vee q \vee \Box\neg q)$. We have shown that $p \wedge \Box(\Box\neg p \vee p) \to \Box(p \vee q \vee \Box\neg q)$ is valid in $(W, R, \Vdash)$.

Suppose, towards a contradiction, that the logic L has LIP. Then, there would exist a Lyndon interpolant θ of $p \wedge \Box(\Box\neg p \vee p) \to \Box(p \vee q \vee \Box\neg q)$. Since $L \subseteq \Gamma(\mathbf{LS}, 1, 2)$, we would have:

1. $\Gamma(\mathbf{LS}, 1, 2) \vdash p \wedge \Box(\Box\neg p \vee p) \to \theta$.
2. $\Gamma(\mathbf{LS}, 1, 2) \vdash \theta \to \Box(p \vee q \vee \Box\neg q)$.
3. $v^+(\theta) \subseteq \{p\}$ and $v^-(\theta) = \emptyset$.

Here, we consider the model $M_0 = (W_0, R_0, \Vdash_0)$ of $\Gamma(\mathbf{LS}, 1, 2)$ as drawn in Figure 27:

It is easy to see that $(M_0, x_0) \Vdash_0 p \wedge \Box(\Box\neg p \vee p)$. Thus, we have $(M_0, x_0) \Vdash_0 \theta$ because $\Gamma(\mathbf{LS}, 1, 2) \vdash p \wedge \Box(\Box\neg p \vee p) \to \theta$. On the other hand, since $(M_0, x_0) \nVdash_0$

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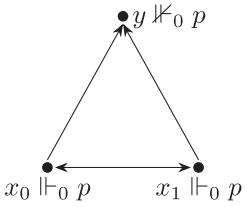


FIGURE 27. The model M_0.

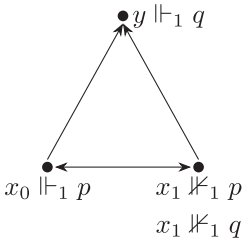


FIGURE 28. The model M_1.

$\Box\Diamond p$, we have that θ is equivalent to one of p, $p \vee \Box\Diamond p$, $\Diamond p$, and $\top$ over $\Gamma(\mathbf{LS}, 1, 2)$ by Lemma 9.1. So, $\Gamma(\mathbf{LS}, 1, 2) \vdash p \to \theta$.

Next, we consider the model $M_1 = (W_0, R_0, \Vdash_1)$ of $\Gamma(\mathbf{LS}, 1, 2)$ as drawn in Figure 28.

Since $(M_1, x_0) \Vdash_1 p$, we have $(M_1, x_0) \Vdash_1 \theta$. On the other hand, $(M_1, x_1) \nVdash_1 p$, $(M_1, x_1) \nVdash_1 q$, and $(M_1, x_1) \nVdash_1 \Box\neg q$ imply that $(M_1, x_0) \nVdash_1 \Box(p \vee q \vee \Box\neg q)$. This contradicts $\Gamma(\mathbf{LS}, 1, 2) \vdash \theta \to \Box(p \vee q \vee \Box\neg q)$.

Therefore, L does not have LIP. ⊣

§10. LIP and ULIP for intermediate propositional logics. In this section, we discuss LIP and ULIP for intermediate propositional logics. In particular, we will prove the following theorem.

THEOREM 10.1. *For any intermediate propositional logic L, the following are equivalent*:

1. *L has CIP.*
2. *L has UIP.*
3. *L has LIP.*
4. *L has ULIP.*

The equivalence $(1 \Leftrightarrow 2)$ is already known [17]. So, we will prove the equivalences $(1 \Leftrightarrow 3)$ and $(1 \Leftrightarrow 4)$.

First, we consider LIP. LIP for intermediate logics directly follows from LIP for modal logics through Gödel's translation.

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DEFINITION 10.2 (Gödel's translation). We define the translation T from formulas of intermediate logic to formulas of modal logic as follows:

- $\mathsf{T}(p) = \Box p$ for propositional variables p,
- $\mathsf{T}(\bot) = \bot$,
- $\mathsf{T}(\varphi \wedge \psi) = \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi)$,
- $\mathsf{T}(\varphi \vee \psi) = \mathsf{T}(\varphi) \vee \mathsf{T}(\psi)$,
- $\mathsf{T}(\neg\varphi) = \Box\neg\mathsf{T}(\varphi)$,
- $\mathsf{T}(\varphi \to \psi) = \Box(\mathsf{T}(\varphi) \to \mathsf{T}(\psi))$.

For any intermediate logic L and normal modal logic M extending **S4**, we say that M is a *modal companion* of L iff for any formula φ of intermediate logic, $L \vdash \varphi \iff M \vdash \mathsf{T}(\varphi)$.

FACT 10.3 (Maksimova [15]). *For any intermediate propositional logic L, if some modal companion of L has LIP, then L also has LIP.*

For example, LIP for **Cl**, **LS**, $\mathbf{LP}_2$, **KC**, and **Int** follows from LIP for **S5**, **GW.2**, **GW**, **S4.2**, and **S4**, respectively. Since all modal companions of **LC** do not have CIP [14], Kuznets and Lellmann [9, 10] directly proved LIP for **LC** without using Fact 10.3. Among the intermediate logics having CIP, only the logic **LV** was not known to have LIP or not (cf. [17]). Since we proved that a modal companion **GV** of **LV** has LIP (Theorem 7.1), we obtain the following corollary, which yields the equivalence $(1 \Leftrightarrow 3)$ of Theorem 10.1.

COROLLARY 10.4. **LV** *has LIP.*

Next, we consider ULIP. We say that a logic L is *locally tabular* iff for any finite set P of propositional variables, there are finitely many formulas built from variables in P up to L-provable equivalence. For example, it is known that **Cl**, **LS**, **LV**, $\mathbf{LP}_2$, and **LC** are locally tabular. For locally tabular logics, LIP and ULIP coincide.

FACT 10.5 (Kurahashi [6, Proposition 3]). *If a locally tabular logic L has LIP, then L also has ULIP.*

So, we obtain the following corollary.

COROLLARY 10.6. *The logics* **Cl**, **LS**, **LV**, $\mathbf{LP}_2$, *and* **LC** *have ULIP.*

Therefore, we only need to consider ULIP for **Int** and **KC**. Here, we prove an analogue of Fact 10.3.

PROPOSITION 10.7. *For any intermediate propositional logic L, if some modal companion of L has ULIP, then L also has ULIP.*

PROOF. Suppose that a modal companion M of L has ULIP. Let φ be any formula and P^+, P^- be any finite sets of propositional variables. Let θ be a uniform Lyndon interpolant of $(\mathsf{T}(\varphi), P^+, P^-)$ in M. Let s be the uniform substitution such that $\mathsf{s}(p) = \Box p$. It is easy to show that there exists a formula ξ of intermediate logic satisfying the following conditions (cf. [1, Theorem 14.9]):

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1. $\mathbf{S4} \vdash \Box \mathsf{s}(\theta) \leftrightarrow \mathsf{T}(\xi)$,
2. $v^{\circ}(\xi) = v^{\circ}(\theta)$ for $\circ \in \{+, -\}$.

We show that ξ is a uniform Lyndon interpolant of (φ, P^+, P^-) in L.

1. Since $M \vdash \mathsf{T}(\varphi) \to \theta$ and $\mathbf{S4} \vdash \mathsf{T}(\varphi) \leftrightarrow \Box \mathsf{T}(\varphi)$, we have $M \vdash \mathsf{T}(\varphi) \to \Box\theta$. Since M is closed under applying uniform substitutions and $\mathbf{S4} \vdash \mathsf{T}(\varphi) \leftrightarrow \mathsf{s}\left(\mathsf{T}(\varphi)\right)$, we obtain $M \vdash \mathsf{T}(\varphi) \to \Box \mathsf{s}(\theta)$. Hence, $M \vdash \mathsf{T}(\varphi) \to \mathsf{T}(\xi)$, and so $M \vdash \mathsf{T}(\varphi \to \xi)$. Since M is a modal companion of L, we get $L \vdash \varphi \to \xi$.

2. For $\circ \in \{+, -\}$, we have $v^{\circ}(\xi) = v^{\circ}(\theta) \subseteq v^{\circ}\left(\mathsf{T}(\varphi)\right) \setminus P^{\circ} = v^{\circ}(\varphi) \setminus P^{\circ}$.

3. Let ψ be any formula such that $L \vdash \varphi \to \psi$ and $v^{\circ}(\psi) \cap P^{\circ} = \emptyset$. Then, $M \vdash \mathsf{T}(\varphi) \to \mathsf{T}(\psi)$. Since $v^{\circ}\left(\mathsf{T}(\psi)\right) \cap P^{\circ} = v^{\circ}(\psi) \cap P^{\circ} = \emptyset$, we obtain $M \vdash \theta \to \mathsf{T}(\psi)$. Then, in a similar way as above, we obtain $M \vdash \mathsf{T}(\xi \to \psi)$. Therefore, $L \vdash \xi \to \psi$. ⊣

It was proved in [6] that a modal companion **Grz** of **Int** has ULIP. So, we immediately obtain the following corollary.

COROLLARY 10.8. **Int** *has ULIP.*

Finally, we focus on ULIP for **KC**. For this, it suffices to prove ULIP for a modal companion **Grz.2** of **KC**. Let $\alpha(p)$ be the formula $\Box(\Box\Diamond p \leftrightarrow \Diamond\Box p)$, then **Grz.2** is the least normal extension of **Grz** containing $\alpha(p)$. We use the following facts.

FACT 10.9 (Cf. [5, Lemma 5.32 and Proposition 5.35]). *For any formula φ with $v(\varphi) = \{q_0, \dots, q_{k-1}\}$, if* $\mathbf{Grz.2} \vdash \varphi$, *then*

$$\mathbf{Grz} \vdash \alpha(\bot) \land \alpha(q_0) \land \cdots \land \alpha(q_{k-1}) \to \varphi.$$

FACT 10.10 (Cf. [5, Lemma 5.33]). *Let L be a normal modal logic and $\varphi(p), \psi$ and ρ be any formulas such that $L \vdash \psi \to \rho$.*

1. *If $p \notin v^{-}(\varphi)$, then $L \vdash \varphi(\psi) \to \varphi(\rho)$.*
2. *If $p \notin v^{+}(\varphi)$, then $L \vdash \varphi(\rho) \to \varphi(\psi)$.*

THEOREM 10.11. **Grz.2** *has ULIP.*

PROOF. Let φ be any formula and P^+, P^- be any finite sets of propositional variables. If $\circ$ is $+$ (resp. $-$), then let $\bullet$ denote $-$ (resp. $+$). Let X be the union of the following three sets:

$$v^{+}(\varphi) \cap v^{-}(\varphi), \quad (v^{+}(\varphi) \setminus v^{-}(\varphi)) \cap P^{-}, \quad (v^{-}(\varphi) \setminus v^{+}(\varphi)) \cap P^{+}.$$

Also let Y be the union of the following two sets:

$$v^{+}(\varphi) \setminus (v^{-}(\varphi) \cup P^{-}), \quad v^{-}(\varphi) \setminus (v^{+}(\varphi) \cup P^{+}).$$

Then, $v(\varphi)$ is the disjoint union of X and Y.

By ULIP for **Grz**, we find a uniform Lyndon interpolant θ of

$$\left(\alpha(\bot) \land \bigwedge_{p \in X} \alpha(p) \land \varphi, P^+, P^-\right)$$

in **Grz**. We prove that θ is also a uniform Lyndon interpolant of (φ, P^+, P^-) in **Grz.2**.

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1. Since $\mathbf{Grz} \vdash \alpha(\bot) \land \bigwedge_{p\in X} \alpha(p) \land \varphi \to \theta$, we get $\mathbf{Grz.2} \vdash \varphi \to \theta$.
2. We have:

$$\begin{aligned} v^\circ(\theta) &\subseteq v^\circ\left(\alpha(\bot) \land \bigwedge_{p\in X} \alpha(p) \land \varphi\right) \setminus P^\circ \\ &= \left[v^\circ(\varphi) \cup \left\{\left((v^\bullet(\varphi) \setminus v^\circ(\varphi)\right) \cap P^\circ\right\}\right] \setminus P^\circ \\ &= v^\circ(\varphi) \setminus P^\circ. \end{aligned}$$

3. Let ψ be any formula such that $\mathbf{Grz.2} \vdash \varphi \to \psi$ and $v^\circ(\psi) \cap P^\circ = \emptyset$. By Fact 10.9, we have

$$\mathbf{Grz} \vdash \alpha(\bot) \land \bigwedge_{p\in v(\varphi)\cup v(\psi)} \alpha(p) \land \varphi \to \psi.$$

Since $v(\varphi) \cup v(\psi)$ is the disjoint union of X, Y, and $v(\psi) \setminus v(\varphi)$, we obtain

$$\mathbf{Grz} \vdash \alpha(\bot) \land \bigwedge_{p\in X} \alpha(p) \land \varphi \to \Big(\bigwedge_{q\in Y\cup(v(\psi)\setminus v(\varphi))} \alpha(q) \to \psi\Big). \tag{3}$$

Let s be the uniform substitution defined as follows:

$$\mathsf{s}(p) := \begin{cases} \top & \text{if } p \notin v^-(\varphi) \cup v^+(\psi), \\ \bot & \text{else if } p \notin v^+(\varphi) \cup v^-(\psi), \\ p & \text{otherwise.} \end{cases}$$

Since $\mathbf{K} \vdash p \to \top$ and $\mathbf{K} \vdash \bot \to p$, by Fact 10.10, we have $\mathbf{K} \vdash \varphi \to \mathsf{s}(\varphi)$ and $\mathbf{K} \vdash \mathsf{s}(\psi) \to \psi$. Since $\mathbf{Grz} \vdash \alpha(\top)$, we have

$$\mathbf{Grz} \vdash \alpha(\bot) \land \bigwedge_{p\in X} \alpha(p) \land \varphi \to \mathsf{s}\left(\alpha(\bot) \land \bigwedge_{p\in X} \alpha(p) \land \varphi\right).$$

Let Z be the intersection of the following three sets:

$$Y \cup (v(\psi) \setminus v(\varphi)), \quad v^-(\varphi) \cup v^+(\psi), \quad v^+(\varphi) \cup v^-(\psi).$$

Then, it is shown that Z is included in the union of the following three sets:

$$\begin{aligned} &(v^+(\varphi) \cap v^+(\psi)) \setminus (v^-(\varphi) \cup P^-), \quad (v^-(\varphi) \cap v^-(\psi)) \setminus (v^+(\varphi) \cup P^+), \\ &\qquad\qquad\qquad\qquad\qquad\qquad (v^+(\psi) \cap v^-(\psi)) \setminus v(\varphi). \end{aligned}$$

Moreover, since $v^\circ(\psi) \cap P^\circ = \emptyset$, we have $Z \subseteq (v^+(\psi) \setminus P^-) \cup (v^-(\psi) \setminus P^+)$. Then, by applying the uniform substitution s to (3), it follows from the above observations that

$$\mathbf{Grz} \vdash \alpha(\bot) \land \bigwedge_{p\in X} \alpha(p) \land \varphi \to \left(\bigwedge_{\substack{q\in v^+(\psi)\setminus P^- \\ \text{or } q\in v^-(\psi)\setminus P^+}} \alpha(q) \to \psi \right).$$

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We have:

$$\begin{aligned}
&v^\circ\left(\bigwedge_{\substack{q\in v^+(\psi)\setminus P^-\\ \text{or } q\in v^-(\psi)\setminus P^+}} \alpha(q)\to\psi\right)\cap P^\circ\\
&=\left[(v^+(\psi)\setminus P^-)\cup(v^-(\psi)\setminus P^+)\cup v^\circ(\psi)\right]\cap P^\circ\\
&=\left[(v^\bullet(\psi)\setminus P^\circ)\cup v^\circ(\psi)\right]\cap P^\circ\\
&=v^\circ(\psi)\cap P^\circ\\
&=\emptyset.
\end{aligned}$$

Therefore, we obtain

$$\mathbf{Grz}\vdash\theta\to\left(\bigwedge_{\substack{q\in v^+(\psi)\setminus P^-\\ \text{or } q\in v^-(\psi)\setminus P^+}} \alpha(q)\to\psi\right).$$

We conclude $\mathbf{Grz.2}\vdash\theta\to\psi$. ⊣

COROLLARY 10.12. **KC** *has ULIP.*

We have proved the equivalence $(1\Leftrightarrow 4)$ of Theorem 10.1.

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