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# Rigidity of diagonally embedded triangle groups

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*Abstract.* We show local rigidity of hyperbolic triangle groups generated by reflections in pairs of $n$-dimensional subspaces of $\mathbb{R}^{2n}$ obtained by composition of the geometric representation in $\mathsf{PGL}(2,\mathbb{R})$ with the diagonal embeddings into $\mathsf{PGL}(2n,\mathbb{R})$ and $\mathsf{PSp}^{\pm}(2n,\mathbb{R})$.

## 1 Introduction

We investigate subgroups of $\mathsf{PGL}(2n,\mathbb{R})$ generated by reflections in pairs $U, V$ of half-dimensional subspaces and their deformation properties. Such a reflection is an element of order two in $\mathsf{PGL}(2n,\mathbb{R})$, which has eigenvalues $\pm 1$, each with multiplicity $n$.

Denote by

$$\Delta(k_1,k_2,k_3) = \left\langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^{k_1} = (bc)^{k_2} = (ac)^{k_3} = 1 \right\rangle$$

the triangle group with parameters $k_1, k_2, k_3$. A triangle group is called *hyperbolic* if it can be realized as a discrete subgroup of isometries of the hyperbolic plane generated by reflections in the sides of a triangle. This is equivalent to the inequality $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} < 1$. In the upper half plane model of the hyperbolic plane, the isometry group identifies with the projective general linear group $\mathsf{PGL}(2,\mathbb{R})$ and the boundary at infinity with the projective line $\mathbb{RP}^1 = \mathsf{Gr}(1,2)$. The geometric representation $\rho_{\mathrm{geom}} : \Delta(k_1,k_2,k_3) \to \mathsf{PGL}(2,\mathbb{R})$ obtained by mapping the generators $a, b, c$ to the reflections in the sides of a hyperbolic triangle with interior angles $\frac{\pi}{k_1}, \frac{\pi}{k_2}, \frac{\pi}{k_3}$ is an example of a group generated by reflections in pairs of half-dimensional subspaces of $\mathbb{R}^2$. In this case, the subspaces are the endpoints at infinity of the hyperbolic geodesics extending the sides of the triangle.

One way to obtain a triangle group that is generated by reflections in half-dimensional subspaces of $\mathbb{R}^{2n}$ is by postcomposition with the irreducible representation $\rho_{\mathrm{irr}} : \mathsf{PGL}(2,\mathbb{R}) \to \mathsf{PGL}(2n,\mathbb{R})$. In [8], Long and Thistlethwaite compute the dimension of the $\mathsf{PGL}(m,\mathbb{R})$-*Hitchin component*, the connected component of representations of a triangle group containing $\rho_{\mathrm{irr}} \circ \rho_{\mathrm{geom}}$. The analog for the groups $\mathsf{Sp}^{\pm}(2n,\mathbb{R})$ was computed by Weir [10]. Except in a few cases of small dimension, these components have positive dimension.

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In their recent paper [1], Alessandrini, Lee, and Schaffhauser initiate the study of *higher Teichmüller spaces* for all orbifold surfaces. They prove that the Hitchin component of any orbifold is homeomorphic to a ball, and extend the dimension counts of Long and Thistlethwaite to all orbifold surface groups.

In this paper, we will be interested in representations of triangle groups that factor through the diagonal embedding of $\mathsf{PGL}(2,\mathbb{R})$ into $\mathsf{PGL}(2n,\mathbb{R})$. One motivation for the study of these examples is that they are examples of *Anosov representations*, which generalize Hitchin representations and were introduced by Labourie in [7] and extended to all hyperbolic groups by Guichard–Wienhard in [5]. Anosov representations share some of the properties of Hitchin representations; for example, they form an open subset of the character variety and are all discrete and faithful. By contrast with the Hitchin case, for representations factoring through the diagonal embedding, we obtain the following rigidity result.

***Theorem 1*** *Let $\Gamma = \Delta(k_1,k_2,k_3)$ be a hyperbolic triangle group and let $n \geq 1$. Let $i_{2n}$ denote the diagonal representation $\mathsf{PGL}(2,\mathbb{R}) \to \mathsf{PGL}(2n,\mathbb{R})$. Then any continuous deformation of the composition $i_{2n} \circ \rho_{\text{geom}}$ in $\mathsf{Hom}(\Gamma, \mathsf{PGL}(2n,\mathbb{R}))$ is conjugate to $i_{2n} \circ \rho_{\text{geom}}$.*

Alessandrini, Lee, and Schaffhauser introduce a notion of *expected dimension* of a component of the character variety for an orbifold surface group. They show that this count corresponds to the actual dimension in the case of the Hitchin components. For the component containing the diagonal representation as in Theorem 1, this expected dimension count is negative.

They also obtain results about Hitchin representations into $\mathsf{PSp}^{\pm}(2n,\mathbb{R})$. They prove that except for $k_1, k_2 \leq 3$ in $\mathsf{PSp}^{\pm}(4,\mathbb{R})$, Hitchin representations of triangle groups into $\mathsf{PSp}^{\pm}(2n,\mathbb{R})$ admit non-trivial deformations.

Using the same techniques as for $\mathsf{PGL}(2n,\mathbb{R})$, we find many examples that are rigid in arbitrary symplectic groups. This is because there are many non-conjugate diagonal representations of $\mathsf{PGL}(2,\mathbb{R})$ into $\mathsf{PSp}^{\pm}(2n,\mathbb{R})$. These diagonal representations are parameterized by the possible signatures $(p,q)$ of a nondegenerate symmetric bilinear form on $\mathbb{R}^n$ up to exchanging $p$ and $q$.

***Theorem 2*** *Let $\Gamma = \Delta(k_1,k_2,k_3)$ be a hyperbolic triangle group. Then, for any $(p,q)$-diagonal representation $i_{(p,q)}$, any deformation of the composition $i_{(p,q)} \circ \rho_{\text{geom}}$ in $\mathsf{Hom}(\Gamma, \mathsf{PSp}^{\pm}(2n,\mathbb{R}))$ is conjugate to $i_{(p,q)} \circ \rho_{\text{geom}}$. In particular, there are at least $\lceil \frac{n+1}{2} \rceil$ isolated points in the character variety*

$$\chi\big(\Gamma, \mathsf{PSp}^{\pm}(2n,\mathbb{R})\big) = \mathsf{Hom}\big(\Gamma, \mathsf{PSp}^{\pm}(2n,\mathbb{R})\big)/\mathsf{Inn}\big(\mathsf{PSp}^{\pm}(2n,\mathbb{R})\big).$$

We obtain a stronger result for the positive-definite case.

***Theorem 3*** *Let $\Gamma = \Delta(k_1,k_2,k_3)$ be a hyperbolic triangle group. Let $\rho : \Gamma \to \mathsf{PSp}^{\pm}(2n,\mathbb{R})$ be a representation mapping the generators $a,b,c$ of $\Gamma$ to reflections in pairs of Lagrangian subspaces. If there are three eigenspaces $L_a, L_b, L_c$ of the images*

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*$\rho(a), \rho(b), \rho(c)$ such that the Maslov form of the triple is positive-definite, then $\rho$ is locally rigid.*

For the fundamental group of a closed surface, the composition of a discrete and faithful representation into $\mathsf{PGL}(2,\mathbb{R})$ with the positive definite diagonal embedding $i_{(n,0)}$ into $\mathsf{PSp}^{\pm}(2n,\mathbb{R})$ gives rise to a maximal representation, that is, a representation with maximal Toledo invariant. Spaces of maximal representations and Hitchin components are examples of *higher Teichmüller theories*, connected components of the space of representations that consist entirely of discrete and faithful representations. For a recent survey on higher Teichmüller theories; see [11].

As a corollary of Theorem 2, for a hyperbolic triangle group, there are no non-Fuchsian representations in the *diagonal component* of maximal representations into $\mathsf{PSp}^{\pm}(2n,\mathbb{R})$. This is in contrast to the surface group case where the analogous component always contains Zariski-dense representations [2, 4].

The strategy of the proof will be the use of an endomorphism-valued invariant of quadruples of subspaces generalizing the cross-ratio of four points in $\mathbb{RP}^1$. For diagonally embedded representations, these invariants are scalar multiples of the identity. This fact, together with the rigidity of finite order elements, is what allows us to conclude that the representations are rigid.

The $\mathsf{PSp}^{\pm}(4,\mathbb{R})$ case was proved by R Hoban in his thesis [6] using a similar invariant, with a small gap in the proof for the signature $(1,1)$ case.

## 2 Preliminaries

Let $V$ be a real vector space of even dimension $2n$. For any vector space $U$, we denote by $\mathsf{GL}(U)$ the linear automorphism group of $U$. Denote by $\mathsf{Gr}(n,2n)$ the Grassmannian of $n$-dimensional subspaces in $V$. Let $U, W \in \mathsf{Gr}(n,2n)$ be transverse subspaces. The projection to $U$ according to the splitting $V = U \oplus W$ will be denoted by $\pi_U^W$.

***Definition 2.1*** The *reflection* in the pair $U, W$ is the order 2 element of $\mathsf{PGL}(V)$ defined by

$$\mathsf{R}_U^W = \pi_U^W - \pi_W^U.$$

The motivating example for this definition is the case $V = \mathbb{R}^2$. The Grassmannian $\mathsf{Gr}(1,2)$ is the projective line, and a reflection $\mathsf{R}_{\ell_1}^{\ell_2} \in \mathsf{PGL}(2,\mathbb{R})$ is also a hyperbolic reflection in the upper half plane model of the hyperbolic plane that pointwise fixes the geodesic with endpoints $\ell_1, \ell_2 \in \mathbb{RP}^1 \cong \partial_\infty \mathbb{H}^2$.

Given a splitting $V = U \oplus W$ into half-dimensional subspaces and a linear map $f : U \to W$, we will denote by $\mathsf{Graph}(f)$ the subspace $\{u + f(u) \mid u \in U\}$. Any $n$-dimensional subspace that is transverse to $W$ is the graph of a unique linear map in this way. The subspace $\mathsf{Graph}(f)$ is transverse to $U$ if and only if $f$ is invertible, and in this case, $\mathsf{Graph}(f) = \mathsf{Graph}(f^{-1})$.

The main invariant that we use to prove rigidity is a generalized cross-ratio.

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***Definition 2.2*** Let $U_1, U_2, U_3, U_4 \in \mathsf{Gr}(n, 2n)$ such that $U_1, U_2$ and $U_3, U_4$ are transverse pairs. The *cross-ratio* $[U_1, U_2; U_3, U_4]$ is the $\mathsf{GL}(U_1)$-conjugacy class of the endomorphism of $U_1$ defined by $\pi_{U_1}^{U_2} \circ \pi_{U_3}^{U_4}|_{U_1}$.

When $V$ is 2-dimensional, $\mathsf{Gr}(1,2) \cong \mathbb{RP}^1$ and the generalized cross-ratio of four lines is an endomorphism of a 1-dimensional real vector space, which when interpreted as a real number coincides with the classical cross-ratio.

***Notation 2.3*** *When comparing endomorphisms $f_1, f_2$ of different vector spaces $U_1, U_2$, we will use the notation $f_1 \sim f_2$ to mean that there exists an isomorphism $g : U_1 \to U_2$ such that $f_2 = g \circ f_1 \circ g^{-1}$.*

***Proposition 2.4*** *Let $U_0, U_\infty, U_1, U_2, U_3, U_4 \in \mathsf{Gr}(n, 2n)$ be pairwise transverse and write $U_i = \mathsf{Graph}(f_i)$ for some linear maps $f_i : U_0 \to U_\infty$ and $i = 1, 2, 3, 4$. Then the cross-ratios can be computed as follows:*

$$[U_0, U_2; U_\infty, U_4] = f_2^{-1} f_4,$$
$$[U_\infty, U_2; U_3, U_4] = (f_3 - f_2)(f_3 - f_4)^{-1},$$

*and*

$$[U_1, U_2; U_3, U_4] \sim (f_1 - f_2)^{-1}(f_3 - f_2)(f_3 - f_4)^{-1}(f_1 - f_4),$$

*where the equal sign denotes equality of conjugacy classes.*

**Proof** For the first identity, let $v \in U_0$. Decomposing $v$ according to the splitting $V = U_\infty \oplus U_4$, there exists $u \in U_\infty$ and $u' \in U_0$ such that $v = u + (u' + f_4(u')) = u' + (u + f_4(u'))$.

Therefore, $u = -f_4(u')$ and $u' = v$, so $\pi_{U_\infty}^{U_4}(v) = u = -f_4(v)$.

Similarly, for any $u \in U_\infty$, we have $\pi_{U_0}^{U_2}(u) = -f_2^{-1}(u)$. Hence,

$$[U_0, U_2; U_\infty, U_4] = \pi_{U_0}^{U_2} \pi_{U_\infty}^{U_4}|_{U_0} = f_2^{-1} \circ f_4.$$

For the second identity, let $v \in U_\infty$. If $v = (u_3 + f_3(u_3)) + (u_4 + f_4(u_4))$, then $u_4 = -u_3$, and so $u_3 = (f_3 - f_4)^{-1}(v)$, so $\pi_{U_3}^{U_4}|_{U_\infty} = (I + f_3)(f_3 - f_4)^{-1}$. Then if $u_3 + f_3(u_3) = u_\infty + (u_2 + f_2(u_2))$ for some $u_\infty \in U_\infty$ and $u_2 \in U_2$, we get $u_3 = u_2$, and hence $u_\infty = (f_3 - f_2)u_3$, from which we conclude $\pi_{U_\infty}^{U_2}(u_3 + f_3(u_3)) = (f_3 - f_2)u_3$, proving the claim.

Finally, for the third identity, note that the map $v \to v + f_i(v)$ is an isomorphism from $U_0$ to $U_i$. Decomposing an element $v + f_i(v)$ with respect to the splitting $V = U_j \oplus U_k$, we find that if

$$v + f_i(v) = \big(v_j + f_j(v_j)\big) + \big(v_k + f_k(v_k)\big),$$

where $v_j, v_k \in U_0$, then $v_j = (f_j - f_k)^{-1}(f_i - f_k)(v)$. In other words, the projection satisfies

$$\pi_{U_j}^{U_k}(v + f_i(v)) = (I + f_j)\big((f_j - f_k)^{-1}(f_i - f_k)(v)\big).$$

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Therefore,

$$\pi_{U_1}^{U_2}\pi_{U_3}^{U_4}(v + f_1(v)) = (I + f_1)\big((f_1 - f_2)^{-1}(f_3 - f_2)(f_3 - f_4)^{-1}(f_1 - f_4)(v)\big).$$

In other words, when seen as an endomorphism of $U_0$ via the isomorphism $I + f_1 : U_0 \to U_1$, the cross-ratio $[U_1, U_2; U_3, U_4]$ is given by the composition $(f_1 - f_2)^{-1}(f_3 - f_2)(f_3 - f_4)^{-1}(f_1 - f_4)$. □

The definition, together with the previous proposition, imply that this generalized cross-ratio has the following symmetries whenever $U_1, U_2, U_3, U_4$ are pairwise transverse and for any $g \in \mathsf{GL}(V)$:

$$\begin{aligned}
[gU_1, gU_2; gU_3, gU_4] &= g[U_1, U_2; U_3, U_4]g^{-1},\\
[U_1, U_2; U_4, U_3] &= I - [U_1, U_2; U_3, U_4],\\
[U_1, U_4; U_3, U_2] &= [U_1, U_2; U_3, U_4]^{-1},\\
[U_3, U_2; U_1, U_4] &\sim [U_1, U_2; U_3, U_4]^{-1}.
\end{aligned}$$

## 3 Elements of Finite Order

Since each reflection we consider is uniquely determined by a pair of transverse subspaces of dimension $n$, the cross-ratio provides invariants of the linear transformations obtained by composing two reflections. Let us describe the cross-ratios of elements of finite order that can arise in this way.

***Proposition 3.1*** *The characteristic polynomial $p_R(\lambda)$ of the composition $T = \mathsf{R}_{U_1}^{U_2} \circ \mathsf{R}_{U_3}^{U_4}$ is related to the characteristic polynomial $p_C(\lambda)$ of the cross-ratio*

$$C = [U_1, U_2; U_3, U_4]$$

*by the following equation:*

$$p_T(\lambda) = (-4\lambda)^n p_C\Big(\frac{(\lambda+1)^2}{4\lambda}\Big).$$

**Proof** Consider a basis $e_1, \dots, e_{2n}$ of $V$ such that

$$\begin{aligned}
U_1 &= \langle e_1 \dots e_n\rangle, & U_2 &= \mathsf{Graph}(f),\\
U_3 &= \langle e_{n+1} \dots e_{2n}\rangle, & U_4 &= \mathsf{Graph}(g),
\end{aligned}$$

where $f, g : U_1 \to U_3$ are linear maps. We will denote by $A, B$ the respective matrix expressions of $f, g$ with respect to the bases $e_1 \dots, e_n$ and $e_{n+1} \dots e_{2n}$.

With this normalization, the cross-ratio $C$ has matrix expression $A^{-1}B$, and the reflections satisfy

$$\mathsf{R}_{U_1}^{U_2} = \begin{pmatrix} I & -2A^{-1} \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \mathsf{R}_{U_3}^{U_4} = \begin{pmatrix} -I & 0 \\ -2B & I \end{pmatrix}.$$

Their composition is therefore

$$T = \mathsf{R}_{U_1}^{U_2} \circ \mathsf{R}_{U_3}^{U_4} = \begin{pmatrix} 4A^{-1}B - I & -2A^{-1} \\ 2B & -I \end{pmatrix}.$$

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Using the block determinant formula

$$\det\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \det(XW - YZ),$$

which is valid whenever $Z$ and $W$ commute, we get

$$\begin{aligned}\det(T - \lambda I) &= \det\left((\lambda+1)^2 I - 4\lambda A^{-1}B\right) \\ &= (-4\lambda)^n \det\left(\frac{-(\lambda+1)^2}{4\lambda} I + A^{-1}B\right).\end{aligned}$$

■

***Proposition 3.2*** *Let $U_1, U_2, U_3, U_4$ be pairwise transverse $n$-dimensional subspaces. If the composition of reflections $T = \mathsf{R}_{U_1}^{U_2} \circ \mathsf{R}_{U_3}^{U_4}$ has order $N$ with $N \geq 2$ in $\mathsf{PGL}(2n, \mathbb{R})$, then each eigenvalue $\mu_i$ of the cross-ratio $C = [U_1, U_2; U_3, U_4]$ is of the form $\mu_i = \sin^2(k_i \pi/(2N))$ for $k_i \in \mathbb{Z}$.*

**Proof** If $T^N = \pm I$, all eigenvalues of $T$ must be $N$-th roots of 1 or $-1$. Then, by Proposition 3.1, any eigenvalue $\mu$ of $C$ must satisfy

$$\lambda^2 + 2(2\mu - 1)\lambda + 1 = 0$$

for some $\lambda$ an $N$-th root of $\pm 1$.

Solving for $\mu$, we find

$$\mu = \frac{1}{2}\left(1 - \frac{\lambda + \lambda^{-1}}{2}\right),$$

so if $\lambda = e^{i\frac{k\pi}{N}}$, we obtain

$$\mu = \frac{1}{2}\left(1 - \cos\left(\frac{k\pi}{N}\right)\right) = \sin^2\left(\frac{k\pi}{2N}\right).$$

■

## 4 Configurations of 6-tuples

Let $(U_1^+, U_1^-)$, $(U_2^+, U_2^-)$, $(U_3^+, U_3^-)$ be three pairs of half-dimensional subspaces in $V$ and denote by $R_i = \mathsf{R}_{U_i^+}^{U_i^-}$ the associated reflections. Assume that the subspaces $U_i^\pm$ are pairwise transverse. We can associate three cross-ratios to this configuration:

$$\begin{aligned} C_1 &= [U_2^+, U_2^-; U_3^+, U_3^-], \\ C_2 &= [U_1^+, U_1^-; U_3^+, U_3^-], \\ C_3 &= [U_1^+, U_1^-; U_2^+, U_2^-]. \end{aligned}$$

***Definition 4.1*** The *configuration space* of pairwise transverse 6-tuples in $\mathsf{Gr}(n, 2n)$ is

$$\mathsf{Conf}^{(6)}(\mathsf{Gr}(n,2n)) := \left\{(U_1^+, U_1^-, U_2^+, U_2^-, U_3^+, U_3^-) \in \mathsf{Gr}(n,2n)^6\right\}/\mathsf{PGL}(2n,\mathbb{R}),$$

where $\mathsf{PGL}(2n, \mathbb{R})$ acts diagonally.

The three cross-ratios $C_1, C_2, C_3$ above define a map

$$\mathscr{C} : \mathsf{Conf}^{(6)}(\mathsf{Gr}(n,2n)) \longrightarrow \left(\mathsf{GL}(n,\mathbb{R})/\mathsf{Inn}(\mathsf{GL}(n,\mathbb{R}))\right)^3$$

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with values in triples of conjugacy classes of $\mathsf{GL}(n,\mathbb{R})$. We will denote by

$$\mathsf{Conf}^{(6)}_{C_1,C_2,C_3}(\mathsf{Gr}(n,2n)) := \mathscr{C}^{-1}(C_1,C_2,C_3)$$

the fiber of this map over the triple $(C_1,C_2,C_3)$.

Let $\rho : \Delta(k_1,k_2,k_3) \to \mathsf{PGL}(2n,\mathbb{R})$ be a representation of the $(k_1,k_2,k_3)$ hyperbolic triangle group such that $R_1 = \rho(a)$, $R_2 = \rho(b)$, and $R_3 = \rho(c)$ are reflections in half-dimensional subspaces $U_i^{\pm}$ as above. Then $R_1, R_2, R_3$ satisfy the relations of the triangle group; that is, the compositions $R_1R_2$, $R_2R_3$, and $R_1R_3$ have respective orders $k_1, k_2$, and $k_3$.

Note the following important observation.

***Fact 4.2*** *A path of elements of finite order $k$ in $\mathsf{PGL}(2n,\mathbb{R})$ must be a path of conjugations.*

In other words, every point of the character variety $\chi(C_k,\mathsf{PGL}(2n,\mathbb{R}))$ of a finite cyclic group $C_k$ is isolated (in fact the character variety is finite). This is due to the fact that finite order elements are diagonalizable over $\mathbb{C}$, and their eigenvalues are $k$-th roots of unity, of which there are only finitely many. Hence, for any order $k$ element $M$, there is a basis of $V$ in which $M$ is block diagonal with blocks of size 1 or 2, and there are only finitely many choices for these blocks and their multiplicities.

This observation implies that for any continuous path $\rho_t : \Delta(k_1,k_2,k_3) \to \mathsf{PGL}(2n,\mathbb{R})$ where $0 \le t \le 1$ and $\rho_0 = \rho$, the images of the generators $\rho_t(a), \rho_t(b), \rho_t(c)$ are reflections in pairs of half planes for all values of $t$.

The products of two generators $\rho_t(ab)$, $\rho_t(bc)$, and $\rho_t(ac)$ also have finite order and therefore stay within the same conjugacy classes. Hence, by Proposition 3.2, the cross-ratios $C_1, C_2, C_3$ stay constant for all $t$.

Hence, we can associate with a path of representations a path in the space

$$\mathsf{Conf}^{(6)}_{C_1,C_2,C_3}(\mathsf{Gr}(n,2n)),$$

namely the path of eigenspaces of the images of the generators. This path is invariant when conjugating the path of representations by an element of $\mathsf{PGL}(2n,\mathbb{R})$, and so only depends on the image of the path $\rho_t$ in the character variety. This path also uniquely determines the path of representations up to conjugation, since the generating reflections are uniquely determined by their eigenspaces. The strategy of proof for the rigidity theorems will therefore be to show that in certain cases, these paths in $\mathsf{Conf}^{(6)}_{C_1,C_2,C_3}(\mathsf{Gr}(n,2n))$ are constant.

The rest of this section will consist of the proof of the following proposition.

***Proposition 4.3*** *Let $\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}$. If the polynomial*

$$\lambda_3(1-\lambda_2)x^2 + (\lambda_1+\lambda_2-\lambda_3-1)x + (1-\lambda_1)$$

*has distinct real roots, then the configuration space*

$$\mathsf{Conf}^{(6)}_{\lambda_1 I,\lambda_2 I,\lambda_3 I}\big(\mathsf{Gr}(n,2n)\big)$$

*is finite.*

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Without loss of generality, we can assume the following:

$$\begin{aligned} U_1^+ &= \langle e_1, \dots, e_n \rangle, \\ U_2^+ &= \langle e_{n+1}, \dots, e_{2n} \rangle, \\ U_3^+ &= \mathsf{Graph}(N), \end{aligned}$$

where $N : U_1^+ \to U_2^+$ is an invertible linear map. We can write the remaining three subspaces similarly as graphs $U_1^- = \mathsf{Graph}(M_1)$, $U_2^- = \mathsf{Graph}(M_2)$, and $U_3^- = \mathsf{Graph}(M_3)$ of linear maps $M_i : U_1^+ \to U_2^+$.

Then the cross-ratios $C_i$ can be computed as functions of $M_1, M_2, M_3$ using Proposition 2.4:

$$\begin{aligned} C_1 &= (N - M_2)(N - M_3)^{-1} \\ C_2 &= M_1^{-1}(N - M_1)(N - M_3)^{-1} M_3 \\ C_3 &= M_1^{-1} M_2 \end{aligned} \tag{4.1}$$

***Lemma 4.4*** *If $C_1$, $C_2$, $C_3$ are scalar multiplies of the identity $C_i = \lambda_i I$, then $A_1 = N^{-1}M_1$ must satisfy the quadratic equation*

$$\lambda_3(1 - \lambda_2)A_1^2 + (\lambda_1 + \lambda_2 - \lambda_3 - 1)A_1 + (1 - \lambda_1)I = 0.$$

**Proof** Assume $C_i = \lambda_i I$. Then $M_2 = \lambda_3 M_1$. Substituting in the equation for $C_1$, we get

$$\lambda_1(N - M_3) = N - \lambda_3 M_1,$$

which implies

$$\lambda_1 M_3 = (\lambda_1 - 1)N + \lambda_3 M_1.$$

Finally, substituting in the equation for $C_2$, we find

$$\lambda_2 I = M_1^{-1}(N - M_1)\Big(N - \frac{\lambda_1 - 1}{\lambda_1}N - \frac{\lambda_3}{\lambda_1}M_1\Big)^{-1}\Big(\frac{\lambda_1 - 1}{\lambda_1}N + \frac{\lambda_3}{\lambda_1}M_1\Big).$$

This last equation simplifies to

$$\lambda_3(1 - \lambda_2)A_1^2 + (\lambda_1 + \lambda_2 - \lambda_3 - 1)A_1 + (1 - \lambda_1)I = 0. \qquad \blacksquare$$

Since the discriminant of the polynomial

$$\lambda_3(1 - \lambda_2)x^2 + (\lambda_1 + \lambda_2 - \lambda_3 - 1)x + (1 - \lambda_1)$$

appearing in the previous proposition will be important in what follows, we will denote it by

$$\begin{aligned} \phi(\lambda_1, \lambda_2, \lambda_3) = 1 &- 2(\lambda_1 + \lambda_2 + \lambda_3) + 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3) \\ &+ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 4\lambda_1\lambda_2\lambda_3. \end{aligned}$$

We now prove Proposition 4.3. By Lemma 4.4, with the normalizations of subspaces $U_i^\pm$ as above, the minimal polynomial of $A_1 = N^{-1}M_1$ is of degree at most two. By hypothesis, it has distinct real roots, and therefore $A_1$ is diagonalizable over $\mathbb{R}$. By

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changing basis in both $U_1^+$ and $U_2^+$, we can assume that $N$ is given by the identity matrix, and therefore that $M_1 = A_1$ is diagonalizable. After this change of basis and while keeping $N$ given by the identity matrix, there is still freedom to apply a change of basis to $U_1^+$ as long as we apply the corresponding change of basis to $U_2^+$ through the isomorphism given by $N$. We can use this change of basis to diagonalize $M_1$. The stabilizer of the pair $(U_1^+, U_2^+)$ in $\mathsf{PGL}(2n, \mathbb{R})$ acts on graphs precisely by change of basis in $U_1^+$ and $U_2^+$, and hence we can act by this stabilizer to assume that $N$ is given by the identity matrix and $M_1$ is diagonal. The matrix equation

$$\lambda_3(1-\lambda_2)M_1^2 + (\lambda_1+\lambda_2-\lambda_3-1)M_1 + (1-\lambda_1)I = 0$$

then translates into $n$ real quadratic equations for the diagonal entries of $M_1$, each of which has 2 solutions. The cross-ratio equations (4.1) then uniquely determine $M_2$ and $M_3$, finishing the proof of Proposition 4.3.

## 5 Triangle Groups

In this section, we first recall some facts about the geometric representation of a hyperbolic triangle group, and then prove the main local rigidity theorem for diagonally embedded representations into $\mathsf{PGL}(2n, \mathbb{R})$.

### 5.1 Geometric Representations in $\mathsf{PGL}(2, \mathbb{R})$

The Lie group $\mathsf{PGL}(2, \mathbb{R})$ identifies naturally with the isometry group of the hyperbolic plane. The *geometric representation* of a hyperbolic triangle group $\Delta(k_1, k_2, k_3)$ maps the generators to the three reflections in the sides of a triangle with interior angles $\frac{\pi}{k_i}$. It is uniquely defined up to conjugation, since hyperbolic triangles are determined up to isometry by their angles.

A different model for this geometric representation will be useful. Consider for some $x, y, z \in \mathbb{R}$ the *Gram matrix*

$$M = \begin{pmatrix} 1 & -x & -y \\ -x & 1 & -z \\ -y & -z & 1 \end{pmatrix}.$$

Let $\Gamma$ be the reflection group generated by orthogonal reflections

$$v \longmapsto v - 2(v^t M e_i)e_i$$

for the bilinear form given by $M$ in the canonical basis vectors $e_1, e_2, e_3$ (for the general theory of groups generated by linear reflections, see, for instance, [9]). The group $\Gamma$ preserves the bilinear form given by $M$. Assuming $x < 1$, the determinant

$$\det(M) = 1 - x^2 - y^2 - z^2 - 2xyz$$

is negative if and only if the signature of $M$ is $(2,1)$, in which case the group $\Gamma$ acts by hyperbolic isometries on the hyperboloid model of the hyperbolic plane.

The canonical or geometric representation of a triangle group is obtained when letting $x = \cos(\frac{\pi}{k_1})$, $y = \cos(\frac{\pi}{k_2})$, and $z = \cos(\frac{\pi}{k_3})$. That is, with this choice of $x, y, z$, the bilinear form given by $M$ has signature $(2,1)$, and the reflections in the basis

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vectors as above are the hyperbolic reflections in the sides of a triangle with interior angles $\frac{\pi}{k_i}$ in the hyperboloid model. Hence, the polynomial $q(x,y,z) = 1 - x^2 - y^2 - z^2 - 2xyz$ is negative whenever $x = \cos(\frac{\pi}{k_1})$, $y = \cos(\frac{\pi}{k_2})$, and $z = \cos(\frac{\pi}{k_3})$ with $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} < 1$.

The discriminant $\phi(\lambda_1, \lambda_2, \lambda_3)$ of the polynomial in Proposition 4.3 is related to the determinant of this Gram matrix by

$$\phi(\lambda_1, \lambda_2, \lambda_3) = -\frac{1}{4} q(1 - 2\lambda_1, 1 - 2\lambda_2, 1 - 2\lambda_3).$$

***Lemma 5.1*** *The discriminant $\phi(\lambda_1, \lambda_2, \lambda_3)$ is strictly positive for $\lambda_i = \sin^2(\frac{\pi}{2k_i})$ for $k_i$ satisfying $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} < 1$.*

**Proof** By the observations above, we have

$$q(1 - 2\lambda_1, 1 - 2\lambda_2, 1 - 2\lambda_3) = q\Big(\cos\Big(\frac{\pi}{k_1}\Big), \cos\Big(\frac{\pi}{k_2}\Big), \cos\Big(\frac{\pi}{k_3}\Big)\Big) < 0,$$

and so

$$\phi(\lambda_1, \lambda_2, \lambda_3) = -\frac{1}{4} q(1 - 2\lambda_1, 1 - 2\lambda_2, 1 - 2\lambda_3) > 0.$$

■

For the geometric representation $\rho_{\text{geom}} : \Delta(k_1, k_2, k_3) \to \mathsf{PGL}(2, \mathbb{R})$, we can use the normalization of the previous section with $n = 1$. The matrices $M_i$ are scalars, which we will denote by $m_i$, and the cross-ratios are real numbers, which we denote by $c_i$.

***Proposition 5.2*** *For the geometric representation of $\Delta(k_1, k_2, k_3)$, there is an ordering of the eigenspaces of the generators such that the cross-ratios have values $c_i = \sin^2(\frac{\pi}{2k_i})$.*

**Proof** Fix an orientation on the boundary at infinity of the hyperbolic plane and assume that the points $l_1^+, l_2^-, l_1^-, l_2^+ \in \partial\mathbb{H}^2$ are placed in that order. Then the angle between the two intersecting geodesics $\ell_1$, $\ell_2$ with respective endpoints $(l_1^+, l_1^-)$, $(l_2^+, l_2^-)$ in the hyperbolic plane is related to the cross-ratio of their endpoints by the formula

$$\cos(\theta) = 1 - 2[l_1^+, l_1^-; l_2^+, l_2^-],$$

where the cross-ratio can be computed as a cross-ratio of four points in $\mathbb{RP}^1 = \partial_\infty\mathbb{H}^2$ in the upper half plane model.

Note that the angle $\theta$ is always the angle given by the oriented arc between the boundary points $l_1^+$ and $l_2^-$ (or equivalently $l_1^-$ and $l_2^+$). Let $\Delta$ be a triangle in $\mathbb{H}^2$ with interior angles $\theta_1 = \frac{\pi}{k_1}$, $\theta_2 = \frac{\pi}{k_2}$, $\theta_3 = \frac{\pi}{k_3}$. Order the endpoints $l_i^\pm$ of the geodesics extending the edges of $\Delta$ as in Figure 1, so that

$$[l_1^+, l_1^-; l_2^+, l_2^-] = \frac{1 - \cos(\theta_3)}{2} = \sin^2\Big(\frac{\theta_3}{2}\Big),$$
$$[l_2^+, l_2^-; l_3^+, l_3^-] = \frac{1 - \cos(\theta_1)}{2} = \sin^2\Big(\frac{\theta_1}{2}\Big),$$

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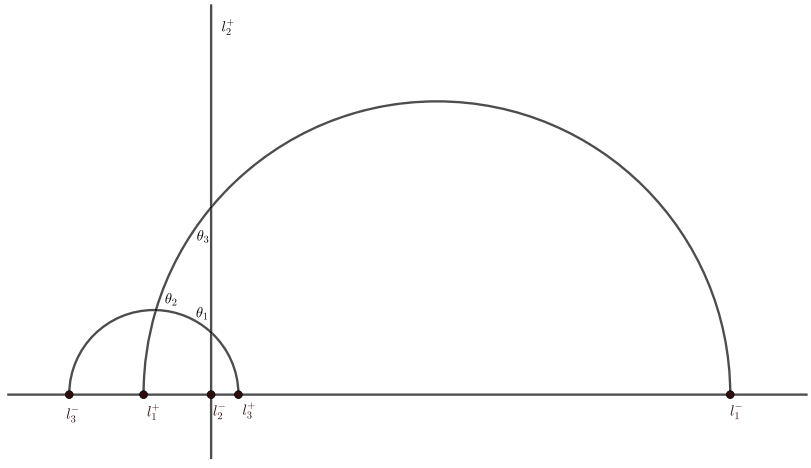



*Figure 1*: A hyperbolic triangle with labeled endpoints. The point $l_2^+$ is the point at infinity.

$$[l_1^+, l_1^-; l_3^+, l_3^-] = \frac{1-\cos(\theta_2)}{2} = \sin^2\left(\frac{\theta_2}{2}\right). \qquad \blacksquare$$

## 5.2 Diagonal Representations Into $\mathsf{PGL}(2n,\mathbb{R})$

Consider the diagonal embedding $i_{2n} : \mathsf{PGL}(2,\mathbb{R}) \to \mathsf{PGL}(2n,\mathbb{R})$, which maps an element $A \in \mathsf{PGL}(2,\mathbb{R})$ to the diagonal action of $A$ on $(\mathbb{R}^2)^n$. Let $\xi_{2n} : \mathbb{RP}^1 \to \mathsf{Gr}(n,2n)$ denote the map

$$\xi_{2n}(p) = \mathsf{span}\big(\iota_1(p), \dots, \iota_n(p)\big),$$

where $\iota_j$ is the inclusion into the $j$-th factor of $\mathbb{R}^{2n} = \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2$. The map $\xi_{2n}$ is $i_{2n}$-equivariant.

***Lemma 5.3*** *Let $\mathsf{R}_{l_1}^{l_2} \in \mathsf{PGL}(2,\mathbb{R})$ be the reflection in the pair of lines $l_1, l_2$. Then $i_{2n}(\mathsf{R}_{l_1}^{l_2}) = \mathsf{R}_{\xi_{2n}(l_1)}^{\xi_{2n}(l_2)}$ is the reflection in their images by $\xi_{2n}$.*

**Proof** Since the reflection $\mathsf{R}_{l_1}^{l_2}$ acts by the identity on $l_1$ and $-1$ on $l_2$, its image by $i_{2n}$ acts by the identity on $\xi_{2n}(l_1)$ and $-I$ on $\xi_{2n}(l_2)$. $\blacksquare$

***Lemma 5.4*** *Let $l_1, l_2, l_3, l_4 \in \mathbb{RP}^1$ be pairwise transverse with a given cross-ratio $[l_1, l_2; l_3, l_4] = c$. Then the generalized cross-ratio satisfies*

$$\big[\xi_{2n}(l_1), \xi_{2n}(l_2); \xi_{2n}(l_3), \xi_{2n}(l_4)\big] = cI.$$

**Proof** Fix a basis $e_1, e_2$ of $\mathbb{R}^2$ and the corresponding basis $e_1^j = \iota_j(e_1)$, $e_2^j = \iota_j(e_2)$ for $j = 1, \dots, n$. By transitivity of $\mathsf{PGL}(2,\mathbb{R})$ on pairs and equivariance of $\xi_{2n}$, we can assume that $l_1 = \mathsf{span}(e_1)$ and $l_3 = \mathsf{span}(e_2)$. Then we write $l_2 = \mathsf{Graph}(f)$ and $l_4 = \mathsf{Graph}(g)$ so that $c = \frac{g}{f}$.

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This implies that $\xi_{2n}(l_1) = \mathsf{span}(e_1^1, \ldots, e_1^n)$ and $\xi_{2n}(l_3) = \mathsf{span}(e_2^1, \ldots, e_2^n)$. Moreover, $\xi_{2n}(l_2) = \mathsf{Graph}(F)$ and $\xi_{2n}(l_4) = \mathsf{Graph}(G)$ where

$$F(v_1 \oplus \cdots \oplus v_n) = f(v_1) \oplus \cdots \oplus f(v_n)$$

and

$$G(v_1 \oplus \cdots \oplus v_n) = g(v_1) \oplus \cdots \oplus g(v_n).$$

We find

$$F^{-1} \circ G(v_1 \oplus \cdots \oplus v_n) = \frac{g}{f}(v_1 \oplus \cdots \oplus v_n) = cI,$$

and hence

$$\left[\xi_{2n}(l_1), \xi_{2n}(l_2); \xi_{2n}(l_3), \xi_{2n}(l_4)\right] = F^{-1}G = cI. \qquad \blacksquare$$

***Theorem 5.5*** *Let $\Gamma = \Delta(k_1, k_2, k_3)$ be a hyperbolic triangle group. Then the representations obtained by composing the geometric representation with the diagonal representation*

$$i_{2n} : \mathsf{PGL}(2, \mathbb{R}) \longrightarrow \mathsf{PGL}(2n, \mathbb{R})$$

*are locally rigid.*

**Proof** Consider the configuration of six subspaces $U_1^{\pm}, U_2^{\pm}, U_3^{\pm}$ of $\mathbb{R}^{2n}$ corresponding to the eigenspaces of the generators. By Lemma 5.4 and Proposition 5.2, the three cross-ratios

$$C_1 = [U_2^+, U_2^-; U_3^+, U_3^-],$$
$$C_2 = [U_1^+, U_1^-; U_3^+, U_3^-],$$
$$C_3 = [U_1^+, U_1^-; U_2^+, U_2^-],$$

satisfy $C_i = \sin^2(\frac{\pi}{2k_i})I$.

Let $\rho_t : \Gamma \to \mathsf{PGL}(2n, \mathbb{R})$ be a continuous family of representations such that $\rho_0 = i_{2n} \circ \rho_{\mathrm{geom}}$. Denote by $U_i^{\pm}(t)$ the corresponding paths of eigenspaces, and by $C_i(t)$ the cross-ratios as above.

The finite order of each composition of generators $\rho_t(ab), \rho_t(bc), \rho_t(ac)$ is fixed, and since there are only finitely many conjugacy classes of elements of a fixed order in $\mathsf{PGL}(2n, \mathbb{R})$, these conjugacy classes must remain fixed throughout the deformation $\rho_t$. Hence, by Proposition 3.2, the cross-ratios $C_1(t), C_2(t), C_3(t)$ must stay constant. By Lemma 5.1, the eigenvalues of the polynomial

$$\lambda_3(1-\lambda_2)x^2 + (\lambda_1 + \lambda_2 - \lambda_3 - 1)x + (1 - \lambda_1)$$

are real and distinct, and so Proposition 4.3 applies, and the configuration $U_1^{\pm}(t)$, $U_2^{\pm}(t)$, $U_3^{\pm}(t)$ remains fixed up to the diagonal action of $\mathsf{PGL}(2n, \mathbb{R})$. We conclude that $\rho_t$ is conjugate to $i_{2n} \circ \rho_{\mathrm{geom}}$. $\blacksquare$

The *expected dimension*, as defined in [1], is a heuristic dimension count for connected components of an orbifold surface character variety. Let $\Gamma = \Delta(k_1, k_2, k_3)$

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and let $r_1, r_2, r_3$ be conjugacy classes of elements in $\mathsf{PGL}(n,\mathbb{R})$, which have order $k_1, k_2, k_3$, respectively. Then the expected dimension of the component of the character variety that maps $(ab), (bc), (ac)$ to the respective conjugacy classes $r_1, r_2, r_3$ is defined to be

$$\dim_e(\chi_{r_1,r_2,r_3}(\Gamma, \mathsf{PGL}(n,\mathbb{R}))) := \frac{1}{2}\left(\sum_{i=1}^{3}\dim(r_i) - 2\dim \mathsf{PGL}(n,\mathbb{R})\right).$$

For a diagonal representation into $\mathsf{PGL}(2n,\mathbb{R})$, the conjugacy classes of the images of $(ab), (bc), (ac)$ have a representative in $\mathsf{SL}(2n,\mathbb{R})$ that has two eigenvalues $\zeta, \zeta^{-1}$, each with multiplicity $n$. Therefore, the centralizer in $\mathsf{PGL}(2n,\mathbb{R})$ of any element $M \in r_i$ is $(2n^2-1)$-dimensional, and we find

$$\begin{aligned}\dim(r_i) &= \dim\left(\mathsf{PGL}(2n,\mathbb{R})\right) - \dim\left(Z_{\mathsf{PGL}(2n,\mathbb{R})}(M)\right)\\ &= (4n^2-1) - (2n^2-1)\\ &= 2n^2.\end{aligned}$$

The expected dimension for the component containing diagonal representations is thus

$$\frac{1}{2}\left(6n^2 - 2\dim \mathsf{PGL}(n,\mathbb{R})\right) = \frac{1}{2}\left(6n^2 - 2(4n^2-1)\right) = -n^2+1,$$

which is negative unless $n = 1$. Therefore, even though in this case the dimension count is not exact, it still accurately predicts local rigidity.

## 6 Diagonal Representations Into $\mathsf{PSp}^{\pm}(2n,\mathbb{R})$

After recalling the relevant definitions for Lagrangian reflections in symplectic vector spaces, we will prove Theorem 2 using a method completely analogous to the $\mathsf{PGL}(2n,\mathbb{R})$ case.

Let $(V,\omega)$ be a real symplectic vector space of dimension $2n$ and let $\mathsf{Sp}(V,\omega)$ be its group of symplectic automorphisms. We will denote by $\mathsf{Sp}^{\pm}(V,\omega)$ the group of linear automorphisms $A$ of $V$ that satisfy $A^*\omega = \pm\omega$. We denote by $\mathsf{PSp}(V,\omega)$ and $\mathsf{PSp}^{\pm}(V,\omega)$ the respective quotients by the centers of these groups. When $V = \mathbb{R}^n$ and $\omega$ is the standard symplectic form, we will use the notation $\mathsf{Sp}(2n,\mathbb{R}) = \mathsf{Sp}(\mathbb{R}^{2n},\omega)$ and its variations.

We first recall the theory of $\mathsf{PSp}^{\pm}(V,\omega)$ orbits of transverse tuples of Lagrangians using the Maslov index. We roughly follow the exposition in [3, Chapter 5].

***Definition 6.1*** A *Lagrangian subspace* $L \subset V$ is an $n$-dimensional subspace such that $\omega|_L = 0$. The space of all Lagrangians will be denoted by $\mathsf{Lag}(V)$ and called the *Lagrangian Grassmannian*. It is a homogeneous space of $\mathsf{PSp}^{\pm}(V,\omega)$.

Let $L, L'$ be transverse Lagrangian subspaces of $V$. Then we will call the reflection $\mathsf{R}_L^{L'}$ a *Lagrangian reflection*.

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***Proposition 6.2*** *A Lagrangian reflection is anti-symplectic; that is,*

$$\left(\mathsf{R}_L^{L'}\right)^* \omega = -\omega.$$

*In particular,* $\mathsf{R}_L^{L'} \in \mathsf{PSp}^{\pm}(V, \omega)$.

**Proof** We have that

$$\begin{aligned}\omega(\mathsf{R}_L^{L'} u, \mathsf{R}_L^{L'} v) &= \omega(\pi_L^{L'} u - \pi_{L'}^{L} u,\ \pi_L^{L'} v - \pi_{L'}^{L} v)\\ &= -\omega(\pi_L^{L'} u, \pi_{L'}^{L} v) - \omega(\pi_{L'}^{L} u, \pi_L^{L'} v)\\ &= -\omega(\pi_L^{L'} u + \pi_{L'}^{L} u, \pi_L^{L'} v + \pi_{L'}^{L} v)\\ &= -\omega(u, v).\end{aligned}$$

■

Given any vector space $U$ of dimension $n$, the standard construction of the symplectic structure on the cotangent bundle of a manifold gives a symplectic vector space structure on $U \oplus U^*$. Explicitly, the symplectic form is given by

$$\omega_U(u \oplus \alpha, u' \oplus \alpha') := \alpha'(u) - \alpha(u').$$

Linear maps $f \in \mathsf{Hom}(U, U^*)$ parameterize $n$-dimensional subspaces of $U \oplus U^*$ that are transverse to $U^*$ via the graph construction $\mathsf{Graph}(f) = \{u \oplus f(u) \mid u \in U\}$. The subspace $\mathsf{Graph}(f)$ is Lagrangian for $\omega_U$ if and only if the map $f$ is *symmetric* in the following sense: $f(u)v = f(v)u$ for all $u, v \in U$. This is the standard notion of symmetric bilinear form on $U$ when identifying a bilinear form on $U$ with an element of $\mathsf{Hom}(U, U^*)$. The symmetric bilinear form is nondegenerate if and only if the corresponding graph is also transverse to $U$.

A pair of transverse Lagrangians $L_1, L_2$ in $V$ defines an isomorphism between the symplectic vector space $(V, \omega)$ and $(L_1 \oplus L_1^*, \omega_{L_1})$. The isomorphism is given by $V \cong L_1 \oplus L_2 \cong L_1 \oplus L_1^*$ where the last isomorphism is the identity on $L_1$ and $v \to \omega(v, -)$ on $L_2$. To see that this is an isomorphism, it suffices to check

$$\omega_{L_1}\big(u \oplus \omega(v, \cdot), u' \oplus \omega(v', \cdot)\big) = \omega(v', u) - \omega(v, u') = \omega(u \oplus v, u' \oplus v').$$

***Definition 6.3*** Let $L_1, L_2, L_3$ be pairwise transverse Lagrangians in $V$. Identify $V \cong L_1 \oplus L_3$ with $L_1 \oplus L_1^*$ as above. The *Maslov form* $B_{L_1,L_2,L_3}$ is the nondegenerate symmetric bilinear form on $L_1$ corresponding to $L_2$. The *Maslov index* $\mathsf{M}(L_1, L_2, L_3)$ is the signature of $B_{L_1,L_2,L_3}$.

The stabilizer in $\mathsf{Sp}(V, \omega)$ of the pair $L_1, L_3$ is isomorphic to $\mathsf{GL}(L_1)$, and its action on Lagrangians transverse to $L_3$ corresponds to the action on bilinear forms $B$ by the usual change of basis action $gB(u, v) = B(g^{-1}u, g^{-1}v)$.

The Maslov index is a complete $\mathsf{Sp}(V, \omega)$-invariant for triples of pairwise transverse Lagrangians. This means that any transverse triple of Lagrangians $L_1, L_2, L_3$ can be written as $L_1, L_2, \mathsf{Graph}(I_{p,q})$, where $I_{p,q}$ is the standard diagonal matrix representing a bilinear form of signature $(p, q)$. The action of an anti-symplectic

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element in $g \in \mathsf{Sp}^{\pm}(V)$ reverses the signature; that is, if $\mathsf{M}(L_1, L_2, L_3) = (p, q)$, then $\mathsf{M}(gL_1, gL_2, gL_3) = (q, p)$.

Because of the additional structure given by the Maslov index, there is more than one conjugacy class of diagonal representation of $\mathsf{PGL}(2, \mathbb{R})$ into $\mathsf{PSp}^{\pm}(2n, \mathbb{R})$. In order to distinguish these representations, we introduce the following notion.

***Definition 6.4*** Let $(p, q)$ be a pair of natural numbers with $p + q = n$. Let $L_1, L_2$ be a fixed pair of transverse Lagrangians in $(V, \omega)$ and let $B$ be a bilinear form of signature $(p, q)$ on $L_1$. The $(p, q)$*-circle* defined by $L_1, L_2, B$ is the collection of Lagrangians that correspond to multiples of $B$ :

$$\{\mathsf{Graph}(\lambda B) \subset V \mid \lambda \in \mathbb{R}\} \cup \{L_2\},$$

where the graph of a bilinear form means the graph of the corresponding linear map through the isomorphisms $V \cong L_1 \oplus L_2 \cong L_1 \oplus L_1^*$ used when defining the Maslov form.

When identifying Lagrangians transverse to a fixed pair $L_1, L_2$ with nondegenerate bilinear forms, the problem of classifying quadruples of pairwise transverse Lagrangians translates to the problem of simultaneous diagonalization of bilinear forms. The following theorem tells us precisely when this is possible.

***Theorem 6.5*** ([12]) *Let $q_1, q_2$ be a pair of nondegenerate quadratic forms on a real vector space $V$. Denote by $\phi_1, \phi_2$ the isomorphisms $V \to V^*$ induced respectively by $q_1, q_2$. Then $q_1, q_2$ are simultaneously diagonalizable over $\mathbb{R}$ if and only if the endomorphism $\phi_1^{-1}\phi_2$ is diagonalizable over $\mathbb{R}$.*

We will now show how to construct "diagonal" homomorphisms $\mathsf{PGL}(2, \mathbb{R}) \to \mathsf{PSp}(V)$ that preserve a $(p, q)$-circle for any signature $(p, q)$. These will be the analogs of the diagonal embedding $i_{2n}$ in the $\mathsf{PGL}(2n, \mathbb{R})$ case. Note that $\mathsf{PSL}(2, \mathbb{R}) \cong \mathsf{PSp}(2, \mathbb{R})$ and $\mathsf{PGL}(2, \mathbb{R}) \cong \mathsf{PSp}^{\pm}(2, \mathbb{R})$.

Let $(U, \omega)$ be a 2-dimensional real symplectic vector space. Let $(W, b)$ be an $n$-dimensional real vector space equipped with a nondegenerate bilinear form $b$ of signature $(p, q)$. The vector space $U \otimes W$ is a symplectic vector space when equipped with the form $\omega \otimes b$.

***Proposition 6.6*** *Let $u \in U$ be a nonzero vector. Define*

$$\xi(u) := \{u \otimes w \mid w \in W\}.$$

*Then $\xi(u)$ is a Lagrangian subspace that only depends on the projective equivalence class of $u$ and the collection $S = \{\xi(u) \mid u \in U - \{0\}\}$ is a $(p, q)$-circle in $\mathsf{Lag}(U \otimes W)$.*

**Proof** The fact that $\xi(ku) = \xi(u)$ for all $k \in \mathbb{R} - \{0\}$ follows from bilinearity of the tensor product. We compute, for all $u \in U$ and $w, w' \in W$:

$$(\omega \otimes b)(u \otimes w, u \otimes w') = \omega(u, u)b(w, w') = 0,$$

so $\xi(u)$ is a Lagrangian subspace.

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We now prove that $S$ is a $(p,q)$-circle. Let $(e_1, e_2)$ be a basis of $U$. Then $\xi(e_1)$ and $\xi(e_2)$ are transverse Lagrangians. A projective equivalence class in $U$ is either represented by $e_1 + te_2$ for some $t \in \mathbb{R}$ or by $e_2$. The map $f : \xi(e_1) \to \xi(e_2)$ given by $f(e_1 \otimes w) \mapsto e_2 \otimes w$ is symmetric, and $\xi(e_1 + te_2) = \mathsf{Graph}(tf)$. We conclude that $S$ is the $(p,q)$-circle defined by $\xi(e_1), \xi(e_2), f$. ∎

Now, since $\mathrm{Aut}(\omega) \cong \mathsf{Sp}(2,\mathbb{R})$ and $\mathrm{Aut}(b) \cong O(p,q)$, the above construction defines a homomorphism

$$\mathsf{Sp}(2,\mathbb{R}) \times \mathrm{O}(p,q) \longrightarrow \mathsf{Sp}(U \otimes W),$$

which preserves the $(p,q)$-circle $S$. More precisely, $\mathsf{Sp}(2,\mathbb{R}) = \mathsf{SL}(2,\mathbb{R})$ acts on this circle equivariantly with respect to the map $\xi : \mathbb{P}(U) \to \mathsf{Lag}(U \otimes W)$ and $\mathrm{O}(p,q)$ acts trivially. We will denote by $i_{(p,q)}$ the representation of $\mathsf{SL}(2,\mathbb{R})$ obtained by restricting to $\mathsf{SL}(2,\mathbb{R}) \times \{I\}$, and call it the *diagonal* $(p,q)$ *representation* of $\mathsf{SL}(2,\mathbb{R})$. More generally, if $f : U \otimes W \to V$ is an isomorphism of symplectic vector spaces, we say that $f(S)$ is a $(p,q)$-circle in $V$ and $f \circ i_{(p,q)} \circ f^{-1}$ is a diagonal $(p,q)$ representation.

***Lemma 6.7*** *Let $l_1, l_2, l_3, l_4$ be pairwise transverse in $\mathbb{P}(U)$. The generalized cross-ratio satisfies*

$$\big[\xi(l_1), \xi(l_2); \xi(l_3), \xi(l_4)\big] = [l_1, l_2; l_3, l_4] \otimes I.$$

**Proof** Write $l_2, l_4$ as the graphs of $f, g : l_1 \to l_3$, respectively. Then $[l_1, l_2, l_3, l_4] = f^{-1}g$. Observe that $\xi(l_2)$ and $\xi(l_4)$ are the graphs of $f \otimes I$, $g \otimes I$, respectively, as maps from $\xi(l_1)$ to $\xi(l_3)$. The lemma then follows from the cross-ratio formula for graphs in Proposition 2.4. ∎

***Proposition 6.8*** *The homomorphism $i_{(p,q)}$ extends to a homomorphism of the double covers $\mathsf{Sp}^{\pm}(2,\mathbb{R}) \to \mathsf{Sp}^{\pm}(U \otimes W)$. The Lagrangians reflections in $\mathsf{Sp}^{\pm}(2,\mathbb{R})$ map to Lagrangian reflections.*

**Proof** The extension of the homomorphism is clear. For the second part, let $R \in \mathsf{Sp}^{\pm}(2,\mathbb{R})$ be a reflection. There exists a decomposition $U = \ell \oplus \ell'$ in which $R = 1 \oplus (-1)$. This implies that $i_{(p,q)}(R)|_{\xi(\ell)} = 1$ and $i_{(p,q)}(R)|_{\xi(\ell')} = -1$. Since $\xi(\ell)$ and $\xi(\ell')$ are transverse subspaces, this proves the claim. ∎

Finally, any $(p,q)$ diagonal representation $i_{(p,q)}$ into $\mathsf{Sp}(V,\omega)$ passes down to the quotients by the respective centers and induces a homomorphism, which we also denote $i_{(p,q)} : \mathsf{PSp}^{\pm}(2,\mathbb{R}) \to \mathsf{PSp}^{\pm}(V,\omega)$.

Since the group $\mathsf{PSp}^{\pm}(V,\omega)$ acts transitively on triples with the same Maslov index, it also acts transitively on the set of $(p,q)$-circles in $\mathsf{Lag}(V)$. Using this transitivity we can construct a diagonal $(p,q)$ representation $j : \mathsf{PSp}^{\pm}(2,\mathbb{R}) \to \mathsf{PSp}^{\pm}(V,\omega)$ and an equivariant map $\xi : \mathbb{P}(\mathbb{R}^2) \to \mathsf{Lag}(V)$ with image $S$ for any $(p,q)$-circle $S$. In particular, the stabilizer in $\mathsf{PSp}^{\pm}(V,\omega)$ of a $(p,q)$-circle contains a subgroup isomorphic to $\mathsf{PSp}^{\pm}(2,\mathbb{R})$ that acts transitively on distinct triples (since $\mathsf{PSp}^{\pm}(2,\mathbb{R})$ acts transitively on distinct triples of $\mathbb{P}(\mathbb{R}^2)$). This shows that any triple

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of distinct Lagrangians on a $(p,q)$-circle either has Maslov index $(p,q)$ or $(q,p)$, depending on its cyclic ordering.

***Proposition 6.9*** *Three Lagrangians $L_1, L_2, L_3 \subset V$ are contained in a unique $(p,q)$-circle, where $(p,q) = \mathsf{M}(L_1, L_2, L_3)$.*

**Proof** By definition of the Maslov index and of a $(p,q)$-circle, the triple $L_1, L_2, L_3$ is contained in the $(p,q)$-circle defined by the splitting $V = L_1 \oplus L_3$ and the bilinear form of signature $(p,q)$ corresponding to $L_2$. Denote that $(p,q)$-circle by $S$.

Now, assume $L_1, L_2, L_3$ are contained in a different $(p,q)$-circle $S'$, which is defined by three Lagrangians $L_1', L_2', L_3'$. Since the stabilizer of a $(p,q)$-circle acts transitively on distinct triples, there is an element in the stabilizer of $S$ that maps $L_1, L_2, L_3$ to $L_1', L_2', L_3'$, and hence maps $S$ to $S'$, proving that $S = S'$. ∎

***Proposition 6.10*** *Let $L_1, L_2, L_3, L_4$ be Lagrangians lying on a common $(p,q)$-circle $S$. Then $\mathsf{R}_{L_1}^{L_2}\mathsf{R}_{L_3}^{L_4}$ preserves $S$ and lies in the image of a $(p,q)$ diagonal representation of $\mathsf{PSp}^{\pm}(2,\mathbb{R})$ that preserves $S$.*

**Proof** Let $\xi$ be the $i_{(p,q)}$-equivariant map with image $S$. Since $L_1, \dots, L_4$ lie on $S$, we can write them as $L_i = \xi(\ell_i)$ for $\ell_i \in \mathsf{Lag}(U)$. By the proof of Proposition 6.8, $\mathsf{R}_{L_i}^{L_j} = \mathsf{R}_{\xi(\ell_i)}^{\xi(\ell_j)}$, which is in the image of $i_{(p,q)}$. Hence, their composition is also in this image. ∎

As in Section 4, denote by $\mathsf{Conf}^{(6)}_{C_1,C_2,C_3}(\mathsf{Lag}(V))$ the space of $\mathsf{PSp}^{\pm}(V,\omega)$-orbits of pairwise transverse 6-tuples of Lagrangians $L_1^{\pm}, L_2^{\pm}, L_3^{\pm}$ in $V$ such that

$$\begin{aligned} C_1 &= [L_2^+, L_2^-; L_3^+, L_3^-], \\ C_2 &= [L_1^+, L_1^-; L_3^+, L_3^-], \\ C_3 &= [L_1^+, L_1^-; L_2^+, L_2^-]. \end{aligned}$$

The analog of Proposition 4.3 for the symplectic case still holds.

***Proposition 6.11*** *Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. If the polynomial*

$$\lambda_3(1-\lambda_2)x^2 + (\lambda_1 + \lambda_2 - \lambda_3 - 1)x + (1-\lambda_1)$$

*has distinct real roots, then the configuration space*

$$\mathsf{Conf}^{(6)}_{\lambda_1 I, \lambda_2 I, \lambda_3 I}(\mathsf{Lag}(2n))$$

*is finite.*

**Proof** The proof is a slight variation of the $\mathsf{PGL}(2n,\mathbb{R})$ case presented in Section 4. Let $L_1^{\pm}, L_2^{\pm}, L_3^{\pm}$ be six pairwise transverse Lagrangian subspaces such that

$$\begin{aligned} [L_2^+, L_2^-; L_3^+, L_3^-] &= \lambda_1 I, \\ [L_1^+, L_1^-; L_3^+, L_3^-] &= \lambda_2 I, \end{aligned}$$

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and

$$[L_1^+, L_1^-; L_2^+, L_2^-] = \lambda_3 I.$$

We take as our fixed pair of Lagrangians $L_1^+, L_2^+$ and write all the others as graphs of symmetric linear maps from $L_1^+$ to $L_2^+$, so

$$L_3^+ = \mathsf{Graph}(N) \quad \text{and} \quad L_i^- = \mathsf{Graph}(M_i).$$

By Lemma 4.4, we know that the map $A_1 = N^{-1}M_1$ must satisfy the polynomial equation

$$\lambda_3(1-\lambda_2)A_1^2 + (\lambda_1+\lambda_2-\lambda_3-1)A_1 + (1-\lambda_1). \tag{6.1}$$

In particular, the map $N^{-1}M_1$ is diagonalizable, and therefore, $N, M_1$ are simultaneously diagonalizable by Theorem 6.5. The stabilizer $\mathrm{Stab}(L_1^+, L_2^+)$ in $\mathsf{PSp}^{\pm}(2n,\mathbb{R})$ is isomorphic to $\mathsf{GL}(n,\mathbb{R})$, and it acts on the Lagrangians defined by quadratic forms (or graphs) by change of basis. We can, therefore, by applying an element of $\mathrm{Stab}(L_1^+, L_2^+)$, assume that $N, M_1$ are diagonal, and so there are finitely many solutions to (6.1). Hence, there are only finitely many possibilities for $M_1, M_2, M_3$, finishing the proof. ∎

The orthogonal diagonalization theorem for quadratic forms allows us to prove a version of Proposition 6.11 with a hypothesis on the Maslov index instead of the roots of a polynomial.

***Proposition 6.12*** *Let $(L_1^{\pm}, L_2^{\pm}, L_3^{\pm})$ be a 6-tuple of pairwise transverse Lagrangians such that $\mathsf{M}(L_1^+, L_2^+, L_3^+) = (n, 0)$. Then this tuple is isolated in the configuration space $\mathsf{Conf}^{(6)}_{\lambda_1 I, \lambda_2 I, \lambda_3 I}(\mathsf{Lag}(2n))$.*

**Proof** The statement follows from the same argument as Proposition 6.11, since if $\mathsf{M}(L_1^+, L_2^+, L_3^+) = (n, 0)$, we can normalize $N$ to the identity matrix and orthogonally diagonalize $M_1$ by applying an element of $\mathrm{Stab}(L_1^+, L_2^+, L_3^+) \cong \mathrm{O}(n)$. ∎

We deduce the local rigidity theorems for $\mathsf{PSp}^{\pm}(2n,\mathbb{R})$ (Theorems 2 and 3) from the two previous propositions, in the same way as in $\mathsf{PGL}(2n,\mathbb{R})$ (Theorem 5.5).

***Theorem 6.13*** *Let $\Gamma = \Delta(k_1, k_2, k_3)$ be a hyperbolic triangle group. Then the representations obtained by composing the geometric representation with a $(p, q)$ diagonal representation*

$$i_{(p,q)} : \mathsf{PSp}^{\pm}(2,\mathbb{R}) \longrightarrow \mathsf{PSp}^{\pm}(2n,\mathbb{R})$$

*are locally rigid.*

**Proof** Let $\rho_t : \Gamma \to \mathsf{PSp}^{\pm}(2n,\mathbb{R})$ be a continuous family of representations such that $\rho_0 = i_{2n} \circ \rho_{\mathrm{geom}}$. Consider the configuration of six Lagrangians $L_1^{\pm}, L_2^{\pm}, L_3^{\pm}$ of $\mathbb{R}^{2n}$ corresponding to the eigenspaces of the generators. By Lemma 5.4 and Proposition

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5.2, the three cross-ratios

$$C_1 = [U_2^+, U_2^-; U_3^+, U_3^-],$$
$$C_2 = [U_1^+, U_1^-; U_3^+, U_3^-],$$
$$C_3 = [U_1^+, U_1^-; U_2^+, U_2^-],$$

satisfy $C_i = \sin^2(\frac{\pi}{2k_i})I$.

Denote by $L_i^\pm(t)$ the corresponding paths of eigenspaces, and by $C_i(t)$ the cross-ratios as above.

As in the proof of Theorem 5.5, the cross-ratios $C_i$ must remain fixed throughout the deformation $\rho_t$, and so the path of tuples $(L_1^\pm(t), L_2^\pm(t), L_3^\pm(t))$ lies in $\mathsf{Conf}^{(6)}_{C_1,C_2,C_3}(\mathsf{Lag}(V))$. By Lemma 5.1, the eigenvalues of the polynomial

$$\lambda_3(1-\lambda_2)x^2 + (\lambda_1+\lambda_2-\lambda_3-1)x + (1-\lambda_1)$$

are real and distinct, and so Proposition 6.11 applies, and the configuration $L_1^\pm(t)$, $L_2^\pm(t)$, $L_3^\pm(t)$ remains fixed up to the diagonal action of $\mathsf{PSp}^\pm(2n,\mathbb{R})$. We conclude that $\rho_t$ is conjugate to $i_{(p,q)} \circ \rho_{\text{geom}}$. ∎

Theorem 6.13 gives $\lceil \frac{n+1}{2} \rceil$ isolated points in the character variety $\chi(\Gamma, \mathsf{PSp}^\pm(2n,\mathbb{R}))$, since this is the number of possible signatures $(p,q)$ with $p+q=n$, up to reversing $p$ and $q$.

***Theorem 6.14*** *Let $\Gamma = \Delta(k_1,k_2,k_3)$ be a hyperbolic triangle group. Let $\rho : \mathsf{PSp}^\pm(2,\mathbb{R}) \to \mathsf{PSp}^\pm(2n,\mathbb{R})$ be any representation such that the images of the generators are Lagrangian reflections, and denote their eigenspaces by $L_1^\pm$, $L_2^\pm$, $L_3^\pm$, respectively. Assume that $\mathsf{M}(L_1^+, L_2^+, L_3^+) = (n,0)$. Then $\rho$ is locally rigid.*

**Proof** The proof is identical to the proof of the previous theorem, except for the last part where we deduce that the path $(L_1^\pm(t), L_2^\pm(t), L_3^\pm(t))$ in $\mathsf{Conf}^{(6)}_{C_1,C_2,C_3}(\mathsf{Lag}(V))$ is constant by applying Proposition 6.12 instead. ∎

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