Euro. Jnl of Applied Mathematics (2006), *vol.* 17, *pp.* 705–733. © 2007 Cambridge University Press
doi:10.1017/S095679250600670X Printed in the United Kingdom

A singularly perturbed boundary-value problem arising in phase transitions

R. WONG and Y. ZHAO

Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong
email: mawong@cityu.edu.hk

(*Received 25 October 2005; revised 4 July 2006; first published online 30 January 2007*)

This paper is concerned with the positive solutions of the boundary-value problem

$$\begin{cases} \varepsilon u'' - \sigma(u) = -\gamma, \\ u(0) = u(1) = 0, \end{cases}$$

where ε is a small positive parameter and γ is a positive constant. The nonlinear term $\sigma(u)$ behaves like a cubic; it vanishes only at $u = 0$, where $\sigma'(0) > 0$ and $\sigma''(0) < 0$. This problem arises in a study of phase transitions in a slender circular cylinder composed of an incompressible phase-transforming material. Here, we determine the number of solutions to the problem for any given γ, derive asymptotic formulas for these solutions, and show that the error terms associated with these formulas are exponentially small, except for one critical value of γ. Our approach is again based on the shooting method used previously by Ou & Wong (*Stud. Appl. Math.* **112** (2004), 161-200).

1 Introduction

In a recent article, Dai [4] established a model equation for a slender circular cylinder composed of an incompressible phase-transforming material. He used the equation to resolve a long standing issue concerning the modeling of dynamical phase transitions. The model equation is

$$w_{tt} - \rho^{-1}\sigma_Z + \frac{3}{4}a^2c_T^2 w_{ZZZZ} - \frac{3}{4}a^2 w_{ZZtt} + \frac{1}{8}a^2c_T^{-2}w_{tttt} = 0, \tag{1.1}$$

where w is the axial displacement, ρ is the density, σ is the stress, a is the radius of the cylinder, Z is the vertical component in the cylindrical coordinate and c_T is the shear-wave speed. For small a, equation (1.1) represents a singular perturbation problem. Here, we shall consider only the time independent one-dimensional case. Upon an integration with respect to Z, we obtain from (1.1)

$$-\rho^{-1}\sigma(E) + \frac{3}{4}a^2c_T^2 E_{ZZ} = D, \tag{1.2}$$

where $E = w_Z$ and D is a constant. Physically, E stands for the strain, and $-D$ is the given external force which is positive for a tension problem. In fact, equation (1.2) appears frequently in papers concerning phase transitions. For instance, it can be found in Van

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

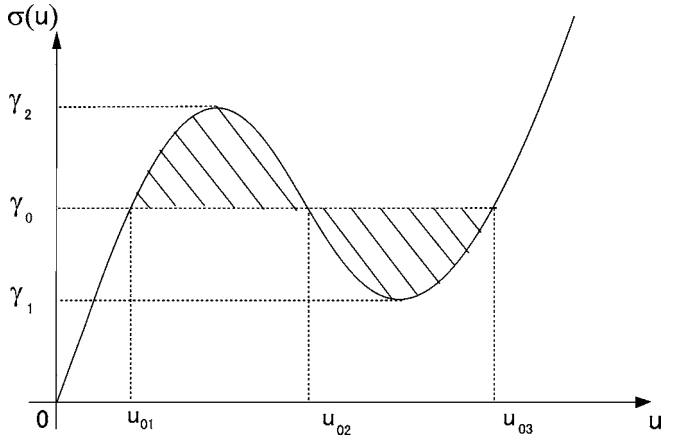


FIGURE 1. The shape of $\sigma(u)$.

der Waals' classic paper [13] on thermodynamic theory of capillarity. It can also be considered as the equation for criterion of viscosity-strain gradient admissibility about elastic material [1, p. 23] [2, 3, 8]. Mention should also be made of the well-known Cahn-Hilliard equation. Its one dimensional form can be written as

$$\varepsilon u'' + g(u) - a = 0, \qquad 0 < x < 1, \tag{1.3}$$

where $a = \int_0^1 g(u(x))dx$, subject to $\int_0^1 u(x)dx = m$ and $u_x(0) = u_x(1) = 0$. It corresponds to (1.2) with a displacement-controlled condition [7].

In this paper, we consider (1.2) with boundary conditions $E = 0$ at $Z = 0$ and 1. A typical shape of the stress-strain function $\sigma(u)$ is shown in Figure 1. This boundary value problem describes the physical model of a slender circular cylinder, composed of an incompressible elastic material whose two ends are fixed to rigid bodies, subject to an axial external force. These boundary conditions are very important for this type of problem. They imply that all the solutions have to be nontrivial; moreover, they can result in different bifurcation phenomena, and reduce the number of solutions. Recently, Dai and Bi [5] have used them to construct the unique solution for the necking in a hyperelastic rod. Our results give mathematical explanations of these physical phenomena.

By non-dimensionalizing (1.1), we can rewrite it as

$$\begin{cases} \varepsilon u'' - \sigma(u) = -\gamma, & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.4}$$

where ε is a small positive parameter and γ is a positive constant. The nonlinear term $\sigma(u)$ behaves like a cubic; it vanishes at $u = 0$, where $\sigma'(0) > 0$ and $\sigma''(0) < 0$. Thus, $\sigma(u)$ is of the form specified by the Van der Waals potential; see Figure 1, where γ_1 and γ_2 are, respectively, the local minimum and local maximum of $\sigma(u)$. The value γ_0 is determined by

$$\int_{u_{01}}^{u_{03}} (\sigma(s) - \gamma_0)ds = 0, \tag{1.5}$$

where $\sigma(u_{01}) = \sigma(u_{02}) = \sigma(u_{03}) = \gamma_0$, and $u_{01} < u_{02} < u_{03}$.

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

In this paper, we shall determine the number of solutions to problem (1.4) for each given value of γ, and derive asymptotic formulas for these solutions as $\varepsilon \to 0$, which hold uniformly for x in the entire interval $[0, 1]$. Furthermore, we shall demonstrate that the error terms associated with these asymptotic approximations are exponentially small, except for the case $\gamma = \gamma_2$. It is found that as the external stress increases there are bifurcations from one solution to three solutions and then to one solution. In the region where there are three solutions, one solution has a sudden convexness in the middle, which indicates the appearance of a neck. In a related paper by García-Melián & Sabina de Lis [8], the authors have also considered positive solutions to the problem: $\varepsilon u'' + f(u) = 0$, $u(0) = u(1) = 0$. They allow f to be a rather general function, not just a cubic, but they provide asymptotic solutions only in the neighborhoods of points where the maximum values of the solutions are attained, i.e., the approximations are not valid in the entire interval $[0, 1]$.

Our approach here is again based on the shooting method used by Ou & Wong [11, 12]. In this method, one first converts the boundary-value problem (1.4) into an initial-value problem, and then chooses an initial value so that the solution of the initial-value problem also satisfies the boundary conditions. Attention should be called to a highly cited article [10] on the study of the number of solutions of equation (1.3) by transversality arguments. Their main idea is to take the problem of counting solutions as an inverse problem and then use Picard-Fuchs equations and transversality of level curves to get the number of solutions. Compared with their approach, our method is easier to follow, and provides explicit formulas for asymptotic solutions.

Since drawings can often illustrate much better than tedious calculations, we have included in the following sections many graphs to explain our arguments.

2 The case $0 < \gamma \leqslant \gamma_1$

Suggested by the shooting method, we begin with the initial-value problem (IVP)

$$\begin{cases} \varepsilon u'' - \sigma(u) = -\gamma, \\ u(0) = 0, \ u'(0) = k, \end{cases} \tag{2.1}$$

where k is a positive number since u is nonnegative. Except for the case when $\gamma = \gamma_1$, the equation $\sigma(u) = \gamma$ has one and only one solution, which we denote by u_1; i.e., $\sigma(u_1) = \gamma$. For convenience, we introduce the notations

$$F(u) = \int_0^u (\sigma(s) - \gamma) ds \qquad \textit{and} \qquad k_1 = \sqrt{\frac{2}{\varepsilon} \int_{u_1}^0 (\sigma(s) - \gamma)\, ds}. \tag{2.2}$$

(The argument given in this section works even in the case when $\gamma = \gamma_1$.) Obviously, $F(0) = 0$ and $F'(u) = \sigma(u) - \gamma$. Furthermore, with the properties of $\sigma(u)$, we have $F'(u_1) = 0$, $F'(u) < 0$ if $0 < u < u_1$ and $F'(u) > 0$ if $u > u_1$. Therefore, $F(u_1)$ is negative, and is a minimum of $F(u)$. A typical shape of $F(u)$ is shown in Figure 2. Multiplying both sides of the differential equation in (2.1) by u', and integrating from 0 to x, we obtain

$$\frac{\varepsilon}{2}(u')^2 = \int_0^u (\sigma(s) - \gamma) ds + \frac{\varepsilon}{2} k^2 = F(u) + \frac{\varepsilon}{2} k^2. \tag{2.3}$$

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

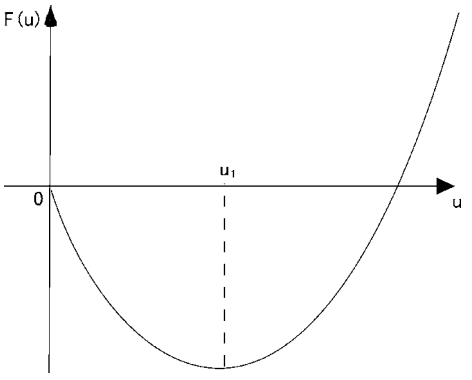


FIGURE 2. Sketch of $F(u)$.

This is a crucial equation, and we will use it throughout our analysis. First of all, we give an interval of k which ensures the existence of the solution to IVP (2.1).

Lemma 2.1 *Let us denote the solution to the IVP* (2.1) *by* $u(x,k)$.

(*a*) *If* $k > k_1$, *i.e.*, $F(u_1) > -\frac{\varepsilon}{2}k^2$, *then* $u(x,k)$ *is increasing for* $x > 0$ *and* $u(1,k) \neq 0$.

(*b*) *If* $0 \leqslant k < k_1$, *i.e.*, $F(u_1) < -\frac{\varepsilon}{2}k^2$, *then* $u(x,k)$ *is convex and has another intersection with the* $x-$*axis other than at* $x = 0$.

(*c*) *In the critical case when* $k = k_1$, *we have* $F(u_1) = -\frac{\varepsilon}{2}k_1^2$. *Let* $\xi = x/\sqrt{\varepsilon}$ *for* $x \in [0, 1/2]$, *and write* $f(\xi) = u(x,k)$. *Then*

$$f''(\xi) - \sigma(f) = -\gamma, \qquad 0 < \xi < \infty, \tag{2.4}$$

and

$$f(0) = 0, \qquad f'(0) = \sqrt{\varepsilon}k_1.$$

(From (2.2), *it is clear that* $\sqrt{\varepsilon}k_1$ *is a constant, independent of* ε.*) Furthermore,*

$$f(\xi) \sim u_1 - Ae^{-\lambda\xi} \qquad \text{as } \xi \to \infty, \tag{2.5}$$

where $\lambda = \sqrt{\sigma'(u_1)}$ *and* A *is an explicitly given positive constant. (Note:* $\sigma'(u_1) > 0$.*)*

Proof (*a*) If $F(u_1) > -\frac{\varepsilon}{2}k^2$, then it is readily seen from (2.3) that $\frac{\varepsilon}{2}(u')^2 > 0$. Since $u'(0,k) = k > 0$ and $u'(x,k)$ never vanishes, it follows that $u'(x,k) > 0$ for $x > 0$. Therefore, $u(x,k)$ is increasing for $x > 0$. For such k, $u(1,k) \neq 0$.

(*b*) If $F(u_1) < -\frac{\varepsilon}{2}k^2$, then it is obvious that the equation $F(u) = -\frac{\varepsilon}{2}k^2$ has two roots $u_0(k)$ and $u_0^*(k)$, i.e.,

$$F(u_0(k)) = F(u_0^*(k)) = -\frac{\varepsilon}{2}k^2. \tag{2.6}$$

It is easy to see that $u_0(k) \in (0, u_1)$ and $u_0^*(k) \in (u_1, \infty)$; cf. Figure 3. Since $u(x,k)$ is continuous and $(u'(x,k))^2$ is nonnegative, the quantity on the right-hand side of (2.3) is

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

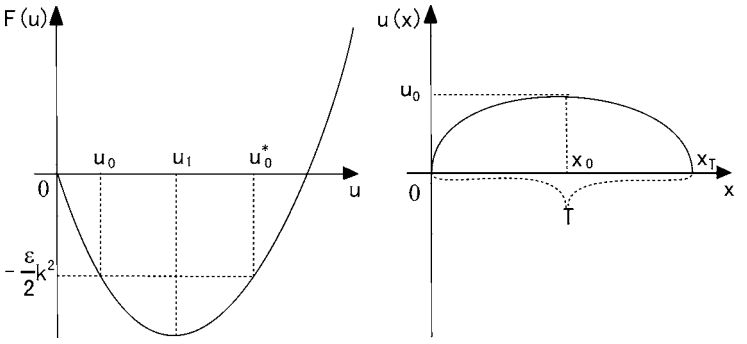


FIGURE 3. Case (b); graphs of $F(u)$ and $u(x)$.

also nonnegative, i.e., $F(u) \geqslant -\frac{\varepsilon}{2}k^2$. This means $u(x,k)$ must be in $[0, u_0(k)]$; for otherwise, this quantity would be negative, which is impossible. This also implies that in this case, the value $u_0^*(k)$ is never attained by the solution $u(x,k)$. From the initial conditions in (2.1) and the equation (2.3), there exist two points $x = x_0$ and $x = x_T$ with $x_0 < x_T$, such that

$$u'(x_0,k) = 0, \qquad u(x_0,k) = u_0(k) \in (0,u_1) \tag{2.7}$$

and

$$u(x_T,k) = 0, \qquad u'(x_T,k) = -k. \tag{2.8}$$

Therefore, the positive solution $u(x,k)$ exists only in the interval $[0,x_T]$, and is symmetric with respect to x_0; again see Figure 3.

(*c*) When $F(u_1) = -\frac{\varepsilon}{2}k_1^2$, with $\xi = x/\sqrt{\varepsilon}$ and $u(x,k) = f(\xi)$, it is easily verified that f satisfies (2.4). From (2.3), it can also be shown that

$$\frac{1}{2}\left(\frac{df}{d\xi}\right)^2 = \int_{u_1}^{f} (\sigma(s) - \gamma)ds. \tag{2.9}$$

We first prove that

$$0 \leqslant f < u_1 \tag{2.10}$$

and

$$\lim_{\xi\to\infty} f(\xi) = u_1. \tag{2.11}$$

Since $f(0) = u(0,k_1) = 0$ and $f'(0) = \sqrt{\varepsilon}k_1 > 0$, one readily sees that $f(\xi) < u_1$ and $f'(\xi) > 0$ for ξ in a right neighborhood of 0. Assume to the contrary that there exists a point η such that $f(\eta) = u_1$ and $0 < f(\xi) < u_1$ for $0 < \xi < \eta$. Define

$$V(f) := \int_{u_1}^{f} (\sigma(s) - \gamma)ds = F(f) - F(u_1). \tag{2.12}$$

Clearly, $V(u_1) = V'(u_1) = 0$ and $V''(u_1) = \sigma'(u_1) > 0$. Since $V(f)$ is sufficiently smooth, we may write

$$V(f) = (u_1 - f)^2 H(f). \tag{2.13}$$

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

Furthermore, since $f < u_1$ and $F(u)$ is decreasing in $(0, u_1)$, by (2.12) $V(f) > 0$ and $H(f)$ is a positive function. From (2.9), we obtain

$$\frac{df}{(u_1 - f)\sqrt{2H(f)}} = d\xi. \tag{2.14}$$

Integrating both sides from $\xi = 0$ to $\xi = \eta$ gives

$$\int_0^{u_1} \frac{df}{(u_1 - f)\sqrt{2H(f)}} = \int_0^{\eta} d\xi, \tag{2.15}$$

which is not possible since the left-hand side is infinite while the right-hand side is finite. Thus, we get (2.10). Now, assume $\lim_{\xi \to \infty} f(\xi) = l \leqslant u_1$. Letting $\eta \to \infty$ in (2.15), we obtain

$$\int_0^{l} \frac{df}{(u_1 - f)\sqrt{2H(f)}} = \infty,$$

which leads to the conclusion that $l = u_1$, thus proving (2.11).

To derive the asymptotic behavior of $f(\xi)$, we return to (2.14) and obtain

$$\int_0^{f} \frac{ds}{(u_1 - s)\sqrt{2H(s)}} = \xi. \tag{2.16}$$

The integral on the left-side of (2.16) can be written as

$$\int_0^{f} \frac{ds}{(u_1 - s)\sqrt{2H(u_1)}} + \int_0^{u_1} \frac{1}{\sqrt{2}(u_1 - s)} \left(\frac{1}{\sqrt{H(s)}} - \frac{1}{\sqrt{H(u_1)}} \right) ds$$
$$- \int_f^{u_1} \frac{1}{\sqrt{2}(u_1 - s)} \left(\frac{1}{\sqrt{H(s)}} - \frac{1}{\sqrt{H(u_1)}} \right) ds.$$

Thus, we have

$$\int_0^{f} \frac{ds}{(u_1 - s)\sqrt{2H(s)}} = \frac{1}{\lambda} [\ln(u_1) - \ln(u_1 - f)] + c - d(f), \tag{2.17}$$

where

$$c = \int_0^{u_1} \left(\frac{1}{\sqrt{2V(s)}} - \frac{1}{(u_1 - s)\lambda} \right) ds, \tag{2.18}$$

$$d(f) = \int_f^{u_1} \frac{1}{\sqrt{2}(u_1 - s)} \left(\frac{1}{\sqrt{H(s)}} - \frac{1}{\sqrt{H(u_1)}} \right) ds,$$

and

$$\lambda = \sqrt{\sigma'(u_1)} = \sqrt{2H(u_1)}.$$

Note that the last two integrals are well defined. Coupling (2.16) and (2.17), we obtain

$$f = u_1 - u_1 e^{\lambda c} e^{-\lambda \xi} e^{-d(f)\lambda}.$$

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

Since $d(f) \to 0$ as $\xi \to \infty$, the result in (2.5) follows with $A = u_1 e^{\lambda c}$. This completes the proof of the lemma. □

From Lemma 2.1, it is clear that in order to make $u(1,k) = 0$, we have to choose k in the interval $(0, k_1)$. The question that we face now is how to choose the initial value k so that $u(1,k) = 0$. Recall that in Lemma 2.1, we have used x_T to denote the first point on the positive x-axis where $u(x,k)$ vanishes, i.e., $u(x_T,k) = 0$; see Figure 3. If $x_T = 1$, then our job is done. Let $T(k)$ stand for the length of the interval $[0, x_T]$. In Figure 3, we also have the point $u_0(k) \in [0, u_1]$ satisfying

$$\int_0^{u_0(k)} (\sigma(s) - \gamma)\, ds = -\frac{\varepsilon}{2} k^2. \tag{2.19}$$

It is readily seen that $u_0(k) \to 0$ as $k \to 0$, and that $u_0(k) \to u_1$ as $k \to k_1$. Differentiating both sides of (2.19) with respect to k shows that $u_0'(k) > 0$. From (2.3) and (2.19), we obtain

$$\sqrt{\varepsilon} \frac{du}{\sqrt{2 \int_{u_0(k)}^{u} (\sigma(s) - \gamma)\, ds}} = ds.$$

In view of the symmetry of the graph of $u(x)$, integrating the above equation from 0 to $u_0(k)$ yields

$$T(k) = \sqrt{2\varepsilon} \int_0^{u_0(k)} \frac{du}{\sqrt{\int_{u_0(k)}^{u} (\sigma(s) - \gamma)\, ds}}. \tag{2.20}$$

The remaining work is to find the conditions on k such that $T(k) = 1$.

Lemma 2.2

$$\lim_{k \to 0^+} T(k) = 0 \tag{2.21}$$

$$\lim_{k \to k_1} T(k) = \infty \tag{2.22}$$

Proof Since $F(0) = 0$ and $F'(0) \neq 0$, we can write $F(u) = uG(u)$, where $G(u)$ is a sufficiently smooth function with $G(0) = -\gamma < 0$. From (2.20), we have

$$\begin{aligned} T(k) &= \sqrt{2\varepsilon} \int_0^{u_0(k)} \frac{du}{\sqrt{F(u) - F(u_0(k))}} = \sqrt{2\varepsilon} \int_0^{u_0(k)} \frac{du}{\sqrt{uG(u) - u_0(k) G(u_0(k))}} \\ &= \sqrt{2\varepsilon} \int_0^{u_0(k)} \frac{du}{\sqrt{G(0)(u - u_0(k))}} + \sqrt{2\varepsilon} \int_0^{u_0(k)} \frac{du}{\sqrt{uG(u) - u_0(k) G(u_0(k))}} \\ &\qquad - \sqrt{2\varepsilon} \int_0^{u_0(k)} \frac{du}{\sqrt{G(0)(u - u_0(k))}}. \end{aligned}$$

Since $u_0(k) \to 0$ as $k \to 0$, it follows that

$$\sqrt{2\varepsilon} \int_0^{u_0(k)} \frac{du}{\sqrt{G(0)(u - u_0(k))}} = \frac{2\sqrt{2\varepsilon}}{\sqrt{\gamma}} \sqrt{u_0(k)} \to 0 \qquad \text{as } k \to 0.$$

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

Note that the integral

$$\sqrt{2\varepsilon}\int_0^{u_0(k)}\left(\frac{1}{\sqrt{F(u)-F(u_0(k))}}-\frac{1}{\sqrt{G(0)(u-u_0(k))}}\right)du$$

exists and the length of the integration interval tends to zero as $k\to 0$, we have $T(k)\to 0$ as $k\to 0$, thus proving (2.21).

In the proof of Lemma 2.1(c), we have already shown that when $k=k_1$, $T(k)=\infty$. Since $u_0(k)\to u_1$ as $k\to k_1$ and $T(k)$ is continuous with respect to k, the result in (2.22) follows. □

From Lemma 2.2, we know that $T(k)$ varies between 0 and ∞ as k varies from 0 to k_1. Our question is how many times $T(k)$ can equal to 1 with different values of k.

Lemma 2.3 *When $k\in(0,k_1)$, $T'(k)>0$.*

Proof We set $u=u_0(k)t$ in (2.20), where $t\in[0,1]$, and obtain

$$T(k)=\sqrt{2\varepsilon}\int_0^1\frac{u_0(k)}{\sqrt{\int_{u_0(k)}^{u_0(k)t}(\sigma(s)-\gamma)\,ds}}dt. \tag{2.23}$$

Differentiating (2.23) with respect to $u_0(k)$, we get

$$\frac{dT}{du_0}=\sqrt{2\varepsilon}\int_0^1\frac{W(u_0(k))-W(u_0(k)t)}{2\left(\int_{u_0(k)}^{u_0(k)t}(\sigma(s)-\gamma)\,ds\right)^{3/2}}dt,$$

where

$$W(u)=u(\sigma(u)-\gamma)-2\int_0^u(\sigma(s)-\gamma)\,ds.$$

Straightforward differentiation of $W(u)$ also gives

$$\frac{dW(u)}{du}=u\sigma'(u)-(\sigma(u)-\gamma).$$

Obviously, when $u\in(0,u_1)$, $\sigma'(u)>0$ and $\sigma(u)-\gamma<0$. Here, $W'(u)>0$, $W(u_0(k))-W(u_0(k)t)>0$ for $u_0(k)>u_0(k)t$, and $dT/du_0>0$. Since $du_0/dk>0$, by the chain rule we get $dT(k)/dk>0$. □

The above three lemmas lead to the following conclusion.

Proposition 1 *When $0<\gamma\leqslant\gamma_1$, the boundary-value problem* (1.4) *has one and only one solution.*

We denote by k_0 the value of k that satisfies $T(k_0)=1$. From (2.20), we have

$$T(k_0)=\sqrt{2\varepsilon}\int_0^{u_0(k_0)}\frac{du}{\sqrt{\int_{u_0(k_0)}^{u}(\sigma(s)-\gamma)\,ds}}=1. \tag{2.24}$$

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

Now, we would like to derive an asymptotic formula for the solution to the boundary-value problem (1.4) when $0 < \gamma \leqslant \gamma_1$.

Proposition 2 *When $0 < \gamma \leqslant \gamma_1$, the asymptotic formula for the solution of the boundary-value problem* (1.4) *is*

$$u(x) = f\left(\frac{x}{\sqrt{\varepsilon}}\right) + f\left(\frac{1-x}{\sqrt{\varepsilon}}\right) - u_1 + O\left(e^{-\lambda/(2\sqrt{\varepsilon})}\right). \tag{2.25}$$

where f is given in (2.4) with initial conditions $f(0) = 0$, $f'(0) = \sqrt{\varepsilon}k_1$, and $\lambda = \sqrt{\sigma'(u_1)}$.

Proof Using (2.13), we can rewrite (2.24) as

$$\int_0^{u_0(k_0)} \frac{du}{\sqrt{(u_1-u)^2H(u) - (u_1-u_0)^2H(u_0(k_0))}} = \frac{1}{\sqrt{2\varepsilon}}.$$

Let $t = u_1 - u$ and $a = u_1 - u_0(k_0)$. Then we have

$$\int_a^{u_1} \frac{dt}{\sqrt{t^2h(t) - a^2h(a)}} = \sqrt{\frac{1}{2\varepsilon}}, \tag{2.26}$$

where $h(t) = H(u_1 - t) > 0$ for any $t \in [0, u_1]$. Note that the quantity inside the square root in (2.24) is positive. Therefore,

$$t^2h(t) - a^2h(a) > 0$$

for $t \in (a, u_1)$. Also note that

$$h(0) = H(u_1) = \frac{\sigma'(u_1)}{2} = \frac{\lambda^2}{2}, \qquad h'(0) = -\frac{\sigma''(u_1)}{6}. \tag{2.27}$$

From (2.20), it is clear that T is actually a function of $u_0(k)$. For any given $u_0(k)$ in the interval $(0, u_1)$, $T(u_0(k)) = O(\sqrt{\varepsilon})$. Thus, if $T(u_0(k_0)) = 1$, a must tend to zero as $\varepsilon \to 0$.

Let us now consider the function

$$Y(t,a) = \frac{t^2h(t) - a^2h(a)}{t^2 - a^2},$$

which is positive and continuous in the interval (a, u_1). Using l'Hospital's rule, we have

$$\lim_{t\to a} Y(t,a) = h(a) + \frac{a}{2}h'(a)$$

and

$$\lim_{t\to a} \frac{\partial Y(t,a)}{\partial t} = \frac{3}{4}h'(a) + \frac{1}{4}ah''(a).$$

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

The left-hand side of (2.26) can now be written as

$$\begin{aligned}\int_a^{u_1} \frac{dt}{\sqrt{t^2h(t)-a^2h(a)}} &= \int_a^{u_1} \frac{dt}{\sqrt{(t^2-a^2)Y(t,a)}} \\ &= \frac{1}{\sqrt{h(a)+\frac{a}{2}h'(a)}} \int_a^{u_1} \frac{dt}{\sqrt{t^2-a^2}} + G_1(a) \\ &= \frac{1}{\sqrt{h(a)+\frac{a}{2}h'(a)}} \cosh^{-1}\frac{u_1}{a} + G_1(a),\end{aligned}$$

where

$$\begin{aligned}G_1(a) &= \int_a^{u_1} \frac{1}{\sqrt{t^2-a^2}} \left(\frac{1}{\sqrt{Y(t,a)}} - \frac{1}{\sqrt{h(a)+\frac{a}{2}h'(a)}} \right) dt \\ &= \int_0^{u_1} \frac{1}{\sqrt{t^2}} \left(\frac{1}{\sqrt{Y(t,0)}} - \frac{1}{\sqrt{h(0)}} \right) dt + o(1)\end{aligned} \tag{2.28}$$

as $a \to 0$. Note that

$$\frac{1}{\sqrt{h(a)+\frac{a}{2}h'(a)}} \cosh^{-1}\frac{u_1}{a} = \frac{1}{\sqrt{h(0)}} \left(\ln\frac{u_1}{a} + \ln 2 \right) + o(1).$$

By (2.18), the last integral in (2.28) is equal to $\sqrt{2}c$. Thus

$$\int_a^{u_1} \frac{dt}{\sqrt{t^2h(t)-a^2h(a)}} = \frac{\sqrt{2}}{\lambda} \left(\ln\frac{u_1}{a} + \ln 2 \right) + \sqrt{2}c + o(1) \tag{2.29}$$

Coupling this equation with (2.26) yields

$$u_1 - u_0(k_0) = a = O(e^{-\lambda/(2\sqrt{\varepsilon})}) \qquad as\ \varepsilon \to 0. \tag{2.30}$$

The argument above gives an estimate for a. Next, we are going to approximate the error between $u(x,k_0)$ and $u(x,k_1)$.

First of all, we restrict x in the interval $[0, \frac{1}{2}]$. It is readily seen that at the boundary $x = 0$, $u(x,k_1)$ and $u(x,k_0)$ satisfy

$$u'(0,k_1) > u'(0,k_0).$$

We assume that there exists a point $\eta \in (0, \frac{1}{2})$ such that $u(\eta,k_1) = u(\eta,k_0)$. Obviously, $u(x,k_1) > u(x,k_0)$ for $x \in (0,\eta)$ and

$$u'(\eta,k_1) \leqslant u'(\eta,k_0). \tag{2.31}$$

Using the energy equation (2.3) associated with $u(x,k_1)$ and $u(x,k_0)$, we have

$$\frac{\varepsilon}{2} \left(u'(x,k_1) \right)^2 = \int_0^{u(x,k_1)} (\sigma(s)-\gamma)\, ds + \frac{\varepsilon}{2}k_1^2 \tag{2.32}$$

and

$$\frac{\varepsilon}{2} \left(u'(x,k_0) \right)^2 = \int_0^{u(x,k_0)} (\sigma(s)-\gamma)\, ds + \frac{\varepsilon}{2}k_0^2. \tag{2.33}$$

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

Since $k_1 > k_0$, $u'(\eta, k_1) > 0$ and $u'(\eta, k_0) > 0$, we conclude from (2.32) and (2.33) that

$$u'(\eta, k_1) > u'(\eta, k_0),$$

which contradicts (2.31), thus

$$u(x, k_1) > u(x, k_0) \qquad for\ x \in \left(0, \frac{1}{2}\right]. \tag{2.34}$$

Let b be a point in the interval $[0, u_1)$. Correspondingly, we denote by $x_{k_0,b}$ and $x_{k_1,b}$ the points in the interval $[0, \frac{1}{2}]$ such that $u(x_{k_0,b}, k_0) = b$ and $u(x_{k_1,b}, k_1) = b$. From (2.32) and (2.2), we get

$$x = \sqrt{\frac{\varepsilon}{2}} \int_{u_1 - u(x,k_1)}^{u_1} \frac{dt}{\sqrt{t^2 h(t)}}, \tag{2.35}$$

where t and $h(t)$ are defined in (2.26). Similarly, we obtain from (2.33)

$$x = \sqrt{\frac{\varepsilon}{2}} \int_{u_1 - u(x,k_0)}^{u_1} \frac{dt}{\sqrt{t^2 h(t)}} + O\left(\sqrt{\frac{\varepsilon}{2}} a^2\right) \tag{2.36}$$

for $0 \leqslant x \leqslant x_{k_0,b}$. With $x = x_{k_1,b}$ in (2.35) and $x = x_{k_0,b}$ in (2.36), we have upon subtracting (2.35) from (2.36)

$$s := x_{k_0,b} - x_{k_1,b} = O\left(\sqrt{\frac{\varepsilon}{2}} a^2\right). \tag{2.37}$$

Since $t^2 h(t)$ is bounded away from zero for $t \in [u_1 - b, u_1]$, subtracting (2.35) from (2.36) gives

$$u(x, k_1) - u(x, k_0) = O(a^2) \tag{2.38}$$

for $0 \leqslant x \leqslant x_{k_0,b}$. Note that

$$u(x_{k_0,b} - s, k_1) = b = u(x_{k_0,b}, k_0).$$

Since $k_1 > k_0$, we obtain again from (2.32) and (2.33)

$$u'(x_{k_0,b} - s, k_1) > u'(x_{k_0,b}, k_0). \tag{2.39}$$

Based on the argument similar to that for (2.34), one can readily show by using contradiction

$$u(x - s, k_1) \geqslant u(x, k_0) \geqslant b.$$

for $x_{k_0,b} \leqslant x \leqslant \frac{1}{2}$ since $u(x, k_0)$ is increasing in $x \in [0, \frac{1}{2}]$.

Now let $R(x)$ denote the difference between $u(x - s, k_1)$ and $u(x, k_0)$; i.e.,

$$R(x) := u(x - s, k_1) - u(x, k_0).$$

From the differential equation in (1.4), we have

$$\begin{aligned} R''(x) &= u''(x - s, k_1) - u''(x, k_0) \\ &= \frac{1}{\varepsilon}[\sigma(u(x - s, k_1)) - \sigma(u(x, k_0))] \geqslant 0. \end{aligned}$$

The last inequality follows from the fact that $\sigma(u)$ is increasing for $u \in [0, u_1]$.

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

The convexity of $R(x)$ implies that $R'(x)$ is increasing, i.e., $R'(x) \geqslant R'(x_{k_0,b}) > 0$ from (2.39). Since $R(x_{k_0,b}) = 0$,

$$0 \leqslant R(x) \leqslant R\left(\frac{1}{2}\right) = u\left(\frac{1}{2} - s, k_1\right) - u\left(\frac{1}{2}, k_0\right)$$

for $x_{k_0,b} \leqslant x \leqslant \frac{1}{2}$. From(2.37), we have $u(\frac{1}{2} - s, k_1) = u(\frac{1}{2}, k_1) + O(a^2)$. Recall from (2.7) that $u_0(k)$ denotes the maximum value of $u(x,k)$. From (2.24), it follows that $u_0(k_0) = u(\frac{1}{2}, k_0)$; see Figure 3. Since $f(\xi) = u(x,k)$, by Lemma 2.1(c) we have $u(\frac{1}{2}, k_1) = u_1 + O(e^{-\lambda/(2\sqrt{\varepsilon})})$. By (2.30), we also have $u(\frac{1}{2}, k_1) - u(\frac{1}{2}, k_0) = O(e^{-\lambda/(2\sqrt{\varepsilon})})$. Thus,

$$R(x) < \left[u\left(\frac{1}{2} - s, k_1\right) - u\left(\frac{1}{2}, k_1\right)\right] + \left[u\left(\frac{1}{2}, k_1\right) - u\left(\frac{1}{2}, k_0\right)\right] = O\left(e^{-\lambda/(2\sqrt{\varepsilon})}\right).$$

Again by (2.37), $u(x - s, k_1) = u(x, k_1) + O(a^2)$. On account of the definition of $R(x)$, it follows that

$$u(x, k_1) - u(x, k_0) = O\left(e^{-\lambda/(2\sqrt{\varepsilon})}\right) \tag{2.40}$$

for $x_{k_0,b} \leqslant x \leqslant \frac{1}{2}$. Coupling (2.38) and (2.40), we have

$$u(x, k_1) - u(x, k_0) = O(a) = O\left(e^{-\lambda/(2\sqrt{\varepsilon})}\right)$$

for all $x \in [0, \frac{1}{2}]$. From (2.5), we already have

$$f\left(\frac{1-x}{\sqrt{\varepsilon}}\right) = u_1 + O\left(e^{-\lambda/(2\sqrt{\varepsilon})}\right)$$

when $x \in [0, \frac{1}{2}]$; therefore, (2.25) is proved for $x \in [0, \frac{1}{2}]$. In view of the symmetry of $u(x)$, (2.25) holds in the whole interval $x \in [0, 1]$. □

3 The case $\gamma > \gamma_2$

When $\gamma > \gamma_2$, the equation $\sigma(u) = \gamma$ still has only one solution. We denote it again by u_1, i.e., $\sigma(u_1) = \gamma$. In this case, the functions $\sigma(u)$ and $F(u)$ have almost the same properties as those in the case when $0 < \gamma \leqslant \gamma_1$; see Figure 4 below. By following the same argument as in Section 2, we obtain the following conclusion.

Proposition 3 *When $\gamma > \gamma_2$, the boundary-value problem* (1.4) *has one and only one solution, and the solution has the asymptotic formula*

$$u(x) = f\left(\frac{x}{\sqrt{\varepsilon}}\right) + f\left(\frac{1-x}{\sqrt{\varepsilon}}\right) - u_1 + O\left(e^{-\lambda/(2\sqrt{\varepsilon})}\right),$$

where $f(\xi)$ is given in (2.4) with initial conditions $f(0) = 0$, $f'(0) = \sqrt{\varepsilon}k_1$, and $\lambda = \sqrt{\sigma'(u_1)}$.

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

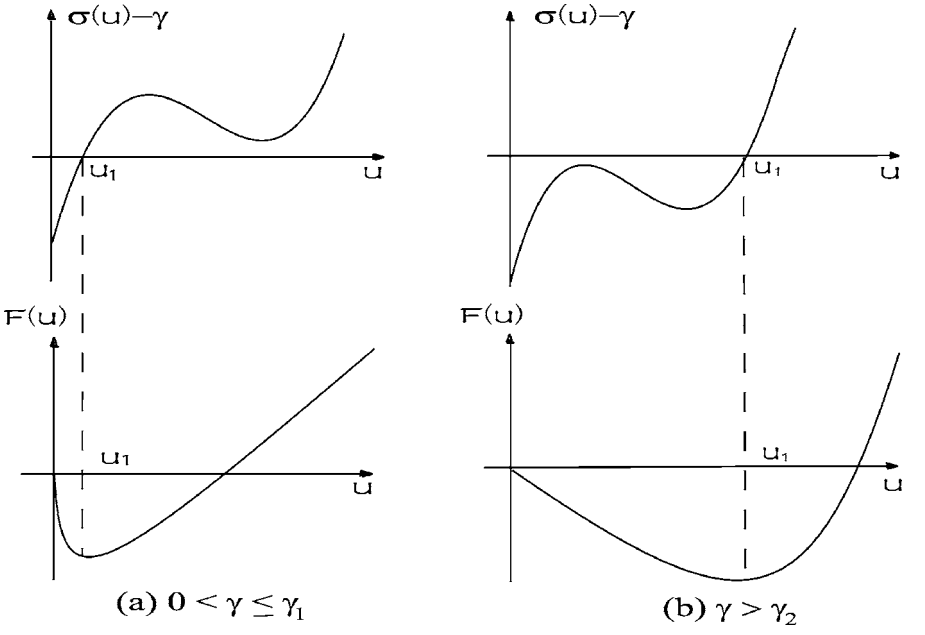


FIGURE 4. Shape of $F(u)$.

4 The case $\gamma_1 < \gamma \leqslant \gamma_0$

In this case, the equation $\sigma(u) = \gamma$ has three solutions; see Figure 1. We denote them by u_1, u_2 and u_3; i.e., $\sigma(u_1) = \sigma(u_2) = \sigma(u_3) = \gamma$. Because of the definition of the γ_0, when $\gamma_1 < \gamma \leqslant \gamma_0$ it can be readily seen that

$$\int_{u_1}^{u_2} (\sigma(s) - \gamma) ds \geqslant \int_{u_3}^{u_2} (\sigma(s) - \gamma) ds; \tag{4.1}$$

see Figure 5. We let $F(u)$ be defined as in (2.2). Obviously, $F(0) = 0$ and $F'(u) = \sigma(u) - \gamma$. However, $F'(u)$ now has three zeros, namely, u_1, u_2 and u_3. Since $\int_{u_1}^{u_3} (\sigma(s) - \gamma_0) ds \geqslant 0$ from (4.1), it follows that

$$\begin{aligned} F(u_1) &= \int_0^{u_1} (\sigma(s) - \gamma) ds \\ &\leqslant \int_0^{u_1} (\sigma(s) - \gamma) ds + \int_{u_1}^{u_3} (\sigma(s) - \gamma) ds = F(u_3). \end{aligned}$$

Therefore, $F(u)$ has the typical shape shown in Figure 5. We can easily see from this figure that the equation $F(u) = -\frac{\varepsilon}{2} k^2$ has more than one root. As demonstrated in the proof of the Lemma 2.1(b), here too only the first root $u_0(k)$ makes sense to have a positive solution $u(x) > 0$. By following strictly the argument used in Section 2, one again reaches the conclusion stated below.

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

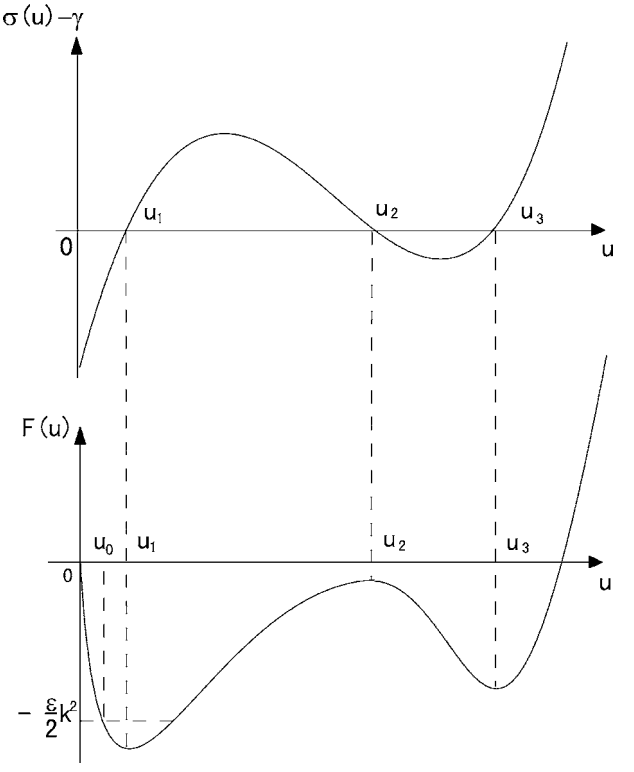


FIGURE 5. Graph of $\sigma(u) - \gamma$ and $F(u)$ for $\gamma_1 < \gamma \leqslant \gamma_0$.

Proposition 4 *When $\gamma_1 < \gamma \leqslant \gamma_0$, the boundary-value problem (1.4) has one and only one solution, and the solution has the asymptotic formula*

$$u(x) = f\left(\frac{x}{\sqrt{\varepsilon}}\right) + f\left(\frac{1-x}{\sqrt{\varepsilon}}\right) - u_1 + O\left(e^{-\lambda/(2\sqrt{\varepsilon})}\right), \tag{4.2}$$

where f is given in (2.4) *with initial conditions $f(0) = 0$, $f'(0) = \sqrt{\varepsilon}k_1$, and $\lambda = \sqrt{\sigma'(u_1)}$.*

5 The case $\gamma_0 < \gamma < \gamma_2$

In this case, the equation $\sigma(u) = \gamma$ still has three solutions u_1, u_2 and u_3. However, when $\gamma_0 < \gamma < \gamma_2$, the inequality corresponding to (4.1) becomes

$$\int_{u_1}^{u_2} (\sigma(s) - \gamma_0)ds < \int_{u_3}^{u_2} (\sigma(s) - \gamma_0)ds. \tag{5.1}$$

Using (5.1), we have $\int_{u_1}^{u_3} (\sigma(s) - \gamma_0)ds \leqslant 0$, and the relationship between $F(u_1)$ and $F(u_3)$ becomes

$$\begin{aligned} F(u_1) &= \int_0^{u_1} (\sigma(s) - \gamma)ds \\ &> \int_0^{u_1} (\sigma(s) - \gamma)ds + \int_{u_1}^{u_3} (\sigma(s) - \gamma)ds = F(u_3). \end{aligned}$$

Therefore, $F(u)$ has the typical shape shown in Figure 6.

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

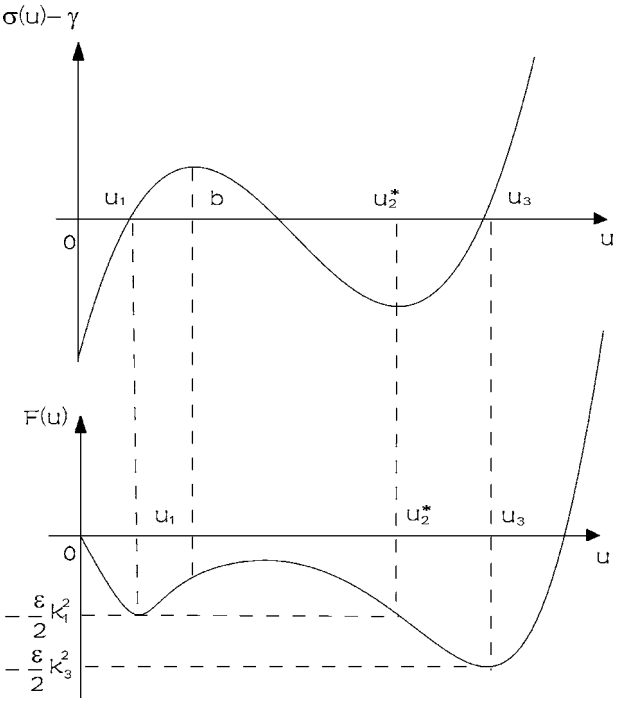


FIGURE 6. Graphs of $\sigma(u) - \gamma$ and $F(u)$ for $\gamma_0 < \gamma < \gamma_2$

By the argument given in the proof of Lemma 2.1(a) and (b), it can be shown that in order to ensure the existence of a solution to (1.4), the initial value k must satisfy $-\frac{1}{2}\varepsilon k^2 \geqslant F(u_3)$. Under this condition, the equation $F(u) = -\frac{1}{2}\varepsilon k^2$ has more than one solution. Again by using the argument in the proof of Lemma 2.1(b), only the smallest solution to the polynomial equation makes sense to have a nonnegative solution to the boundary-value problem (1.4). As before, we denote it by $u_0(k)$, i.e., $F(u_0(k)) = -\frac{1}{2}\varepsilon k^2$. In this case, we can readily see from Figure 6 that $u_0(k) \in [0, u_1] \bigcup (u_2^*, u_3]$, where u_2^* is the point that satisfies

$$\int_{u_1}^{u_2^*} (\sigma(s) - \gamma) ds = 0, \tag{5.2}$$

which can also be written as $F(u_1) = F(u_2^*)$.

To determine the number of solutions to (1.4), we need know the behavior of $T(k)$.

Lemma 5.1 *When* $\gamma_0 < \gamma < \gamma_2$,

$$\lim_{k \to 0} T(k) = 0, \tag{5.3}$$

$$\lim_{k \to k_1} T(k) = \infty, \tag{5.4}$$

and

$$\lim_{k \to k_3} T(k) = \infty, \tag{5.5}$$

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

where k_1 *satisfies*

$$-\frac{\varepsilon}{2}k_1^2 = \int_0^{u_1} (\sigma(s)-\gamma)ds = \int_0^{u_2^*} (\sigma(s)-\gamma)ds = F(u_1) = F(u_2^*), \tag{5.6}$$

and k_3 *satisfies*

$$-\frac{\varepsilon}{2}k_3^2 = \int_0^{u_3} (\sigma(s)-\gamma)ds. \tag{5.7}$$

Proof As before, $T(k)$ is defined in (2.20). When $k \to 0$, we can prove (5.3) as we did in Lemma 2.2. When $k \to k_1^-$, $u_0(k) \to u_1$ by (5.6), and

$$\lim_{k\to k_1^-} T(k) = \sqrt{2\varepsilon} \int_0^{u_1} \frac{du}{\sqrt{\int_{u_1}^u (\sigma(s)-\gamma)\,ds}}. \tag{5.8}$$

Put $V(u) = \int_{u_1}^u (\sigma(s)-\gamma)ds = F(u) - F(u_1)$. Obviously, $V(u_1) = V'(u_1) = 0$ and $V''(u_1) = \sigma'(u_1) > 0$. Thus, we can write $V(u)$ as

$$V(u) = (u_1 - u)^2 H(u), \tag{5.9}$$

where $H(u)$ is a positive function on $[0, u_1]$. Inserting (5.9) into (5.8) gives

$$\lim_{k\to k_1^-} T(k) = \sqrt{2\varepsilon} \int_0^{u_1} \frac{du}{(u_1-u)\sqrt{H(u)}} = \infty. \tag{5.10}$$

When $k \to k_1^+$, $u_0(k) \to u_2^*$ again by (5.6), and

$$\begin{aligned}\lim_{k\to k_1^+} T(k) &= \sqrt{2\varepsilon} \int_0^{u_2^*} \frac{du}{\sqrt{\int_{u_2^*}^u (\sigma(s)-\gamma)\,ds}} \\ &= \sqrt{2\varepsilon} \int_0^{u_1} \frac{du}{\sqrt{\int_{u_1}^u (\sigma(s)-\gamma)\,ds}} + \sqrt{2\varepsilon} \int_{u_1}^{u_2^*} \frac{du}{\sqrt{\int_{u_1}^u (\sigma(s)-\gamma)\,ds}}.\end{aligned}$$

In the last equality, we have made use of the second equality in (5.6). Based on the argument given above, we again have

$$\sqrt{2\varepsilon} \int_{u_1}^{u_2^*} \frac{du}{\sqrt{\int_{u_1}^u (\sigma(s)-\gamma)\,ds}} = \sqrt{2\varepsilon} \int_{u_1}^{u_2^*} \frac{du}{(u-u_1)\sqrt{H(u)}} = \infty$$

and

$$\sqrt{2\varepsilon} \int_0^{u_1} \frac{du}{\sqrt{\int_{u_1}^u (\sigma(s)-\gamma)\,ds}} = \sqrt{2\varepsilon} \int_0^{u_1} \frac{du}{(u_1-u)\sqrt{H(u)}} = \infty.$$

Therefore,

$$\lim_{k\to k_1^+} T(k) = \infty. \tag{5.11}$$

Coupling (5.10) and (5.11) completes the proof of (5.4).

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

Similarly, we can write $\int_{u_3}^{u} (\sigma(s)-\gamma)\,ds = (u_3-u)^2 H^*(u) > 0$. When $k \to k_3$, $u_0(k) \to u_3$ by (5.7), and

$$\begin{aligned}\lim_{k\to k_3} T(k) &= \sqrt{2\varepsilon}\int_0^{u_3} \frac{du}{\sqrt{\int_{u_3}^{u}(\sigma(s)-\gamma)\,ds}} \\ &= \sqrt{2\varepsilon}\int_0^{u_3} \frac{du}{(u_3-u)\sqrt{H^*(u)}} = \infty.\end{aligned}$$

□

When $k \in (0, k_1)$, we can also verify $T'(k) > 0$ by following the argument in Lemma 2.3. Thus, the boundary-value problem (1.4) has only one solution when $k \in (0, k_1)$, and the approximate solution is given by

$$u(x) = f\left(\frac{x}{\sqrt{\varepsilon}}\right) + f\left(\frac{1-x}{\sqrt{\varepsilon}}\right) - u_1 + O\left(e^{-\lambda_1/(2\sqrt{\varepsilon})}\right), \tag{5.12}$$

where f satisfies (2.4) with initial conditions $f(0) = 0$, $f'(0) = \sqrt{\varepsilon}k_1$, and $\lambda_1 = \sqrt{\sigma'(u_1)}$; see Propositions 1 and 2.

Remark 1 *When $k \in (k_1, k_3)$, $u_0(k) \in (u_2^*, u_3)$. From the definition of $T(k)$ in* (2.20), *we can write $T(k)$ as $T(u_0(k))$. Indeed, (2.20) can be written as*

$$T(u_0(k)) = \sqrt{2\varepsilon}\int_0^{u_0(k)} \frac{du}{\sqrt{\int_{u_0(k)}^{u}(\sigma(s)-\gamma)\,ds}}. \tag{5.13}$$

When $k \to k_1^+$, $u_0(k) \to u_2^*$ by (5.6). *Thus, in view of* (5.4), *we have $T(u_0(k)) \to \infty$ as $u_0(k) \to u_2^*$. Similarly,* (5.5) *gives $T(u_0(k)) \to \infty$ as $u_0(k) \to u_3$. Note that $T(u_0(k))$ is finite and continuous when $u_0(k) \in (u_2^*, u_3)$. From* (2.20), *we also have $T(u_0(k)) = O(\sqrt{\varepsilon})$ when $u_0(k) \in (u_2^*, u_3)$. Therefore, there exists $u_0(k_0)$ in the interval (u_2^*, u_3) satisfying $T(u_0(k_0)) = 1$. From* (5.13), *it follows that*

$$\int_0^{u_0(k_0)} \frac{du}{\sqrt{\int_{u_0(k_0)}^{u}(\sigma(s)-\gamma)\,ds}} = \frac{1}{\sqrt{2\varepsilon}}.$$

Hence, $u_0(k_0) \to u_2^*$ or $u_0(k_0) \to u_3$ as $\varepsilon \to 0$.

Next, we give an estimate for $u_0(k_0) - u_2^*$ when $u_0(k_0)$ is in the right neighborhood of u_2^*.

Lemma 5.2 *When $\gamma_0 < \gamma < \gamma_2$ and $u_0(k_0)$ is in the right neighborhood of u_2^*,*

$$u_0(k_0) - u_2^* = O(e^{-\lambda_1/(2\sqrt{\varepsilon})}) \qquad \text{as } \varepsilon \to 0, \tag{5.14}$$

where $\lambda_1 = \sqrt{\sigma'(u_1)}$.

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

Proof As before we denote by k_0 the value of k that satisfies $T(k) = 1$. Note that

$$\int_{u_0(k_0)}^{u} (\sigma(s) - \gamma)ds = \int_{u_1}^{u} (\sigma(s) - \gamma)ds + \int_{u_0(k_0)}^{u_1} (\sigma(s) - \gamma)ds.$$

Since $\int_{u_2^*}^{u_1} (\sigma(s) - \gamma)ds = 0$, we can write $T(k_0)$ as

$$T(k_0) = \sqrt{2\varepsilon} \int_0^{u_0(k_0)} \frac{du}{\sqrt{\int_{u_1}^{u} (\sigma(s) - \gamma)ds + \int_{u_0(k_0)}^{u_2^*} (\sigma(s) - \gamma)ds}} = 1. \tag{5.15}$$

As in (5.9), we have

$$\int_{u_1}^{u} (\sigma(s) - \gamma)ds = (u - u_1)^2 H(u),$$

where $H(u)$ is a smooth function. Again let $t = u - u_1$ and $a = u_0(k_0) - u_2^*$. By the mean-value theorem for integrals, we get

$$\int_{u_0(k_0)}^{u_2^*} (\sigma(s) - \gamma)ds = aN(a),$$

where $N(a)$ is a positive smooth function. Then, from (5.15) we obtain

$$\int_{-u_1}^{a+u_2^*-u_1} \frac{dt}{\sqrt{t^2 h(t) + aN(a)}} = \frac{1}{\sqrt{2\varepsilon}}, \tag{5.16}$$

where $h(t) = H(t + u_1)$. Note that

$$h(0) = H(u_1) = \frac{\sigma'(u_1)}{2} = \frac{\lambda_1^2}{2} > 0.$$

Let

$$m = \int_{-u_1}^{u_2^*-u_1} \left(\frac{1}{\sqrt{t^2 h(t)}} - \frac{1}{\sqrt{t^2 h(0)}} \right) dt.$$

Recall $a \to 0$ as $\varepsilon \to 0$. Clearly,

$$\int_{-u_1}^{a+u_2^*-u_1} \left(\frac{1}{\sqrt{t^2 h(t) + aN(a)}} - \frac{1}{\sqrt{t^2 h(0) + aN(a)}} \right) dt = m + o(1).$$

The left side of (5.16) can then be written as

$$\int_{-u_1}^{a+u_2^*-u_1} \frac{dt}{\sqrt{t^2 h(0) + aN(a)}} + m + o(1). \tag{5.17}$$

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

Now we concentrate on the first integral in (5.17), and obtain

$$\begin{aligned}\int_{-u_1}^{a+u_2^*-u_1} \frac{dt}{\sqrt{t^2h(0)+aN(a)}} &= \frac{1}{\sqrt{h(0)}}\left[\sinh^{-1}\frac{a+u_2^*-u_1}{\sqrt{aN(a)/h(0)}}+\sinh^{-1}\frac{u_1}{\sqrt{aN(a)/h(0)}}\right]\\ &= \frac{1}{\sqrt{h(0)}}\ln\frac{4(a+u_2^*-u_1)u_1h(0)}{aN(0)}+o(1).\end{aligned} \tag{5.18}$$

Inserting (5.18) into (5.17) yields

$$\frac{1}{\sqrt{h(0)}}\ln\frac{4(a+u_2^*-u_1)u_1h(0)}{aN(0)}+m+o(1)=\frac{1}{\sqrt{2\varepsilon}}. \tag{5.19}$$

Since the numerator in the logarithm tends to a finite number as $\varepsilon\to 0$, it follows that

$$u_0(k_0)-u_2^*=a=O(e^{-\lambda_1/(2\sqrt{\varepsilon})}) \qquad \text{as } \varepsilon\to 0. \qquad \square$$

Remark 2 *By following the above argument leading to* (5.18) *with* $a=u_0(k)-u_2^*$, *one can show from* (5.13) *that for* $u_0(k)$ *in the right neighborhood of* u_2^*,

$$T(u_0(k))=\sqrt{\frac{2\varepsilon}{h(0)}}\ln\frac{u_0(k)-u_1}{u_0(k)-u_2^*}+o(1).$$

For $u_0(k)$ *in the left neighborhood of* u_3, *we follow the argument given in the proof of Proposition 2. With the aid of* (2.29), *we also get*

$$T(u_0(k))=-\sqrt{\frac{2\varepsilon}{\tilde{h}(0)}}\ln(u_3-u_0(k))+o(1),$$

where $\tilde{h}(0)=\frac{1}{2}\sigma'(u_3)$. *Recall that*

$$\lim_{u_0(k)\to u_2^*}T(u_0(k))=\infty, \qquad \lim_{u_0(k)\to u_3}T(u_0(k))=\infty$$

and $T(u_0(k))$ *is finite for any given* $u_0(k)\in(u_2^*,u_3)$. *Therefore, in view of its behavior near the endpoints* u_2^* *and* u_3, $T(u_0(k))$ *is decreasing in* $(u_2^*,u_2^*+\delta)$ *and increasing in* $(u_3-\delta,u_3)$, *where* δ *is independent of* ε. *Furthermore, there exist two values of* $u_0(k)$ *satisfying* $T(u_0(k))=1$. *Since* $u_0(k)$ *is monotonic in* k, *there exist two values of* k *satisfying* $T(k)=1$. *Thus, the boundary-value problem* (1.4) *has two solutions when* $k\in(k_1,k_3)$; *see Figure 7.*

To derive the asymptotic solution to the boundary-value problem (1.4) when $u_0(k_0)\in(u_2^*,u_2^*+\delta)$, we have to introduce a function g.

Lemma 5.3 *There exists a function* g *that satisfies*

$$g''(\xi)-\sigma(g)=-\gamma, \qquad 0<\xi<\infty, \tag{5.20}$$

$$g(0)=u_2^*, \qquad g'(0)=0.$$

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

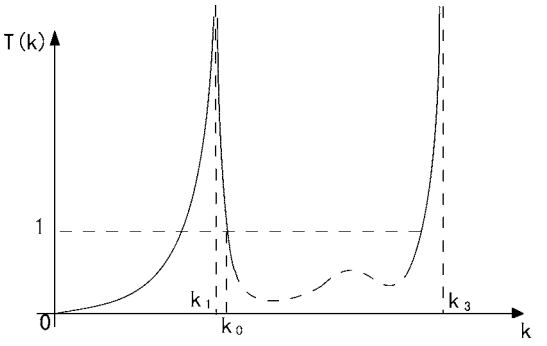


FIGURE 7. Graph of $T(k)$ when $\gamma_0 < \gamma \leqslant \gamma_2$.

Moreover,

$$g(\xi) \sim u_1 + Ce^{-\lambda_1 \xi} \qquad as\ \xi \to \infty, \tag{5.21}$$

where C is a constant and, as in (5.14), $\lambda_1 = \sqrt{\sigma'(u_1)}$.

Proof Let $\xi = \left|\frac{1}{2} - x\right|/\sqrt{\varepsilon}$ and $g(\xi) = u(x, \varepsilon)$. It is readily seen that $g(\xi)$ satisfies (5.20) and $g(0) = u_2^*$. The energy equation associated with (5.20) is

$$\frac{1}{2}\left(\frac{dg}{d\xi}\right)^2 = \int_{u_2^*}^{g} (\sigma(s) - \gamma)ds = \int_{u_1}^{g} (\sigma(s) - \gamma)ds; \tag{5.22}$$

cf. (5.2). First, we show that $u_1 < g \leqslant u_2^*$, $g(\infty) = u_1$ and $g'(\xi) \leqslant 0$ in the whole interval $0 < \xi < \infty$. Assume on the contrary that $g'(\xi) > 0$ in the right neighborhood of zero. Then, there exists a point η such that $u_2^* < g(\eta) < u_3$ and $\int_{u_2^*}^{g(\eta)} (\sigma(s) - \gamma)ds < 0$, which contradicts (5.22). Therefore, $g'(\xi) \leqslant 0$ and $u_1 < g < u_2^*$ for ξ in a right neighborhood of 0.

We assume there exists a point η^* that $g(\eta^*) = u_1$ and $u_1 < g(\xi) < u_2^*$ for $0 < \xi < \eta^*$. For convenience, we define $V(g) = \int_{u_1}^{g} (\sigma(s) - \gamma)ds$ as before. Then $V(g) = (g - u_1)^2 H(g)$, where $H(g)$ is a positive function; see (2.12)-(2.14). From (5.22), we obtain

$$\frac{dg}{(u_1 - g)\sqrt{2H(g)}} = d\xi. \tag{5.23}$$

To show that $u_1 < g(\xi) < u_2^*$ for all points ξ, we integrate both sides of (5.23) from $\xi = 0$ to $\xi = \eta^*$ to obtain

$$\int_{u_2^*}^{u_1} \frac{dg}{(u_1 - g)\sqrt{2H(g)}} = \int_0^{\eta^*} d\xi,$$

which is not possible since the left-hand side is infinite while the right-hand side is finite. Thus, $u_1 < g(\xi) < u_2^*$ on the whole interval $0 < \xi < \infty$. By (5.22), $g'(\xi)$ never vanishes when $0 < \xi < \infty$; that is, $g'(\xi) < 0$ when $0 < \xi < \infty$. Now we assume $\lim_{\xi \to \infty} g(\xi) = n \geqslant u_1$. From (5.23), we have

$$\int_{u_2^*}^{n} \frac{dg}{(u_1 - g)\sqrt{2H(g)}} = \infty,$$

which implies that $n = u_1$.

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

In order to determine the asymptotic behavior of $g(\xi)$, we return to (5.23). Integration on both sides of (5.23) gives

$$\int_{u_2^*}^{g} \frac{ds}{(u_1 - s)\sqrt{2H(s)}} = \xi. \tag{5.24}$$

The argument following (2.16) then shows that the integral on the left-hand side is equal to

$$\frac{1}{\lambda_1}[\ln(u_2^* - u_1) - \ln(g - u_1)] + p - q(g), \tag{5.25}$$

where

$$p = \int_{u_2^*}^{u_1} \left(\frac{1}{\sqrt{2V(s)}} - \frac{1}{(u_1 - s)\lambda_1} \right) ds,$$

$$q(g) = \int_{g}^{u_1} \frac{1}{\sqrt{2}(u_1 - s)} \left(\frac{1}{\sqrt{H(s)}} - \frac{1}{\sqrt{H(u_1)}} \right) ds,$$

and $\lambda_1 = \sqrt{\sigma'(u_1)}$. The last two integrals are well defined. Replacing the left-hand side of (5.24) by (5.25), we obtain

$$g = u_1 + (u_2^* - u_1)e^{p\lambda_1} e^{-\lambda_1 \xi} e^{-q(g)\lambda_1}.$$

Since $q(g) \to 0$ as $\xi \to 0$, (5.21) follows with $C = (u_2^* - u_1)e^{\lambda_1 p}$. □

For convenience, we put

$$u_L(x) := f\left(\frac{x}{\sqrt{\varepsilon}}\right) + f\left(\frac{1-x}{\sqrt{\varepsilon}}\right) + g\left(\frac{\left|\frac{1}{2} - x\right|}{\sqrt{\varepsilon}}\right) - 2u_1. \tag{5.26}$$

We now investigate the difference between $u_L(x)$ and $u(x, k_0)$ when $u_0(k_0) \in (u_2^*, u_2^* + \delta)$.

Lemma 5.4 *When $\gamma_0 < \gamma < \gamma_2$ and $u_0(k_0) \in (u_2^*, u_2^* + \delta)$,*

$$u_L(x) - u(x, k_0) = O(e^{-\lambda_1/(4\sqrt{\varepsilon})}). \tag{5.27}$$

Proof First, let us restrict x to the interval $[0, \frac{1}{2}]$. Since $u_0(k_0)$ is the maximum value of $u(x, k_0)$, we have $u(x, k_0) \in [0, u_0(k_0)]$. Also, since $u_1 \leqslant g \leqslant u_2^*$ (see the proof of Lemma 5.3), by (2.5) and (2.10) we have $u_L(x) \in [0, u_2^*]$. Let x_1 denote the point that satisfies $u(x_1, k_0) = u_1$. Then, as in (5.16),

$$x_1 = \sqrt{\frac{\varepsilon}{2}} \int_{-u_1}^{0} \frac{dt}{\sqrt{t^2 h(t) + aN(a)}}$$

and

$$\frac{1}{2} - x_1 = \sqrt{\frac{\varepsilon}{2}} \int_{0}^{u_0(k_0) - u_1} \frac{dt}{\sqrt{t^2 h(t) + aN(a)}}.$$

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

From (5.17), it is readily seen that

$$\int_{-u_1}^{0} \frac{dt}{\sqrt{t^2h(t)+aN(a)}} \sim \int_{0}^{u_0(k_0)-u_1} \frac{dt}{\sqrt{t^2h(t)+aN(a)}}.$$

Thus, $x_1 \sim \frac{1}{4}$. Since $a \to 0$ as $\varepsilon \to 0$, from the two equations above, we have

$$\frac{x_1}{\sqrt{\varepsilon}} \to \infty \qquad as\ \varepsilon \to 0$$

and

$$\frac{1/2 - x_1}{\sqrt{\varepsilon}} \to \infty \qquad as\ \varepsilon \to 0.$$

Since $u(x,k_0)$ is increasing for $x \in [0, \frac{1}{2}]$, $u(x,k_0) \in [0,u_1]$ when $x \in [0,x_1]$ and $u(x,k_0) \in [u_1, u_0(k_0)]$ when $x \in [x_1, \frac{1}{2}]$.

When $x \in [0,x_1]$, by strictly following the argument in Proposition 2 we obtain

$$u(x,k_0) = f\left(\frac{x}{\sqrt{\varepsilon}}\right) + f\left(\frac{1-x}{\sqrt{\varepsilon}}\right) - u_1 + O(e^{-\lambda_1/(4\sqrt{\varepsilon})}). \tag{5.28}$$

On account of (5.21), we also have $g(|\frac{1}{2} - x|/\sqrt{\varepsilon}) = u_1 + O(e^{-\lambda_1/(4\sqrt{\varepsilon})})$ when $x \in [0,x_1]$. Thus, (5.27) holds in the interval $x \in [0,x_1]$.

When $x \in [x_1, \frac{1}{2}]$, we show that

$$u(x,k_0) - g\left(\frac{|\frac{1}{2} - x|}{\sqrt{\varepsilon}}\right) = O(e^{-\lambda_1/(4\sqrt{\varepsilon})}). \tag{5.29}$$

Put $u_g(x) := g(|\frac{1}{2} - x|/\sqrt{\varepsilon})$. The energy equations for $u(x,k_0)$ and $u_g(x)$ are given by (2.33) and (5.22). Let b be the point in the interval $[u_1, u_0(k_0)]$ such that b satisfies $\sigma'(b) = 0$ and $\sigma''(b) < 0$; cf. Figure 6. Correspondingly, we let $x_{k_0,b}$ and $x_{g,b}$ denote the points in the interval $[x_1, \frac{1}{2}]$ such that $u(x_{k_0,b}, k_0) = u_g(x_{g,b}) = b$. From (2.33), we get

$$\frac{1}{2} - x = \sqrt{\frac{\varepsilon}{2}} \int_{u(x,k_0)-u_1}^{u_0(k_0)-u_1} \frac{du}{\sqrt{t^2h(t)}} + O(\sqrt{\varepsilon}a), \tag{5.30}$$

where t and $h(t)$ are defined in (5.16). Similarly, from (5.22) we have

$$\frac{1}{2} - x = \sqrt{\frac{\varepsilon}{2}} \int_{u_g(x)-u_1}^{u_2^*-u_1} \frac{dt}{\sqrt{t^2h(t)}}. \tag{5.31}$$

Subtracting (5.31) from (5.30) yields

$$\begin{aligned} &\int_{u(x,k_0)-u_1}^{u_0(k_0)-u_1} \frac{du}{\sqrt{t^2h(t)}} - \int_{u_g(x)-u_1}^{u_2^*-u_1} \frac{dt}{\sqrt{t^2h(t)}} + O(a) \\ &= \int_{u(x,k_0)-u_1}^{u_g(x)-u_1} \frac{dt}{\sqrt{t^2h(t)}} + \int_{u_2^*-u_1}^{u_0(k_0)-u_1} \frac{dt}{\sqrt{t^2h(t)}} + O(a) = 0. \end{aligned} \tag{5.32}$$

Since $t^2h(t)$ is bounded away from zero for $t \in [b-u_1, u_0(k_0)-u_1]$ and $u_0(k_0) - u_2^* = a$,

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

the second integral on the right-hand side of (5.32) is $O(a)$ and the first integral gives

$$u(x,k_0)-u_g(x)=O(a) \tag{5.33}$$

for $x\in[x_{k_0,b},\frac{1}{2}]$. Let $x=x_{k_0,b}$ in (5.30) and $x=x_{g,b}$ in (5.31). Then, subtracting (5.31) from (5.30) gives

$$s:=x_{k_0,b}-x_{g,b}=O(\sqrt{\varepsilon}a). \tag{5.34}$$

Note that $u_g(x_{k_0,b}-s)=b=u(x_{k_0,b},k_0)$. Since $u_0(k_0)>u_2^*$, from (2.33) and (5.22) we have

$$u'(x_{k_0,b},k_0)>u_g'(x_{k_0,b}-s).$$

Thus, using the argument following (2.39), we obtain $u(x,k_0)<u_g(x-s)$ for $x\in[x_1,x_{k_0,b}]$. Define $R(x)$ as the difference between $u(x,k_0)$ and $u_g(x-s)$; i.e.,

$$R(x):=u_g(x-s)-u(x,k_0).$$

From the differential equation in (1.4), we have

$$\begin{aligned} R'' &= u_g''(x-s)-u''(x,k_0)\\ &= \frac{1}{\varepsilon}[\sigma(u_g(x-s))-\sigma(u(x,k_0))]\geqslant 0. \end{aligned}$$

The last inequality follows from the fact that $\sigma(u)$ is increasing for $u\in[u_1,b]$. Thus, $R'(x)$ is increasing and $R'(x)\leqslant R'(x_{k_0,b})<0$. Since $R(x_{k_0,b})=0$ and $R'(x)<0$,

$$0=R(x_{k_0,b})<R(x)<R(x_1)=u_g(x_1-s)-u_1=O(e^{-\lambda_1/(4\sqrt{\varepsilon})}) \tag{5.35}$$

for $x_1\leqslant x\leqslant x_{k_0,b}$. To demonstrate the O-estimate in (5.35), we note that $u_g(x_1-s)=u_g(x_1)+O(a)$ from (5.34), and that $u_g(x_1)=u_1+O(e^{-\lambda_1/(4\sqrt{\varepsilon})})$ from (5.21). Again, by (5.34), $u_g(x-s)=u_g(x)+O(a)$. Therefore, by coupling (5.33) and (5.35), we obtain (5.29). Note that by (2.5), $f(x/\sqrt{\varepsilon})=u_1+O(a)$ and $f((1-x)/\sqrt{\varepsilon})=u_1+O(a)$ when $x\in[x_1,\frac{1}{2}]$. Hence, (5.27) holds when $x\in[x_1,\frac{1}{2}]$.

We have now proved (5.27) when $x\in[0,\frac{1}{2}]$. Because of the symmetry of $u(x)$, (5.27) in fact holds in the whole interval $x\in[0,1]$, and we can write the asymptotic formula for the $u(x,k_0)$ when $u_0(k_0)\in(u_2^*,u_2^*+\delta)$ as

$$u(x,k_0)=f\left(\frac{x}{\sqrt{\varepsilon}}\right)+f\left(\frac{1-x}{\sqrt{\varepsilon}}\right)+g\left(\frac{\left|\frac{1}{2}-x\right|}{\sqrt{\varepsilon}}\right)-2u_1+O\left(e^{-\lambda_1/(4\sqrt{\varepsilon})}\right), \tag{5.36}$$

where f is defined in (2.4) with initial conditions $f(0)=0$ and $f'(0)=\sqrt{\varepsilon}k_1$, g is defined in (5.21) with initial conditions $g(0)=u_2^*$ and $g'(0)=0$, and $\lambda_1=\sqrt{\sigma'(u_1)}$. □

When $u_0(k_0)\in(u_3-\delta,u_3)$, $F(u_3)<F(u_0(k_0))<0$ and $F'(u_3)=0$. The rest of work is easy, and we just follow what we have done in the proof of Proposition 2 and get the

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

asymptotic formula for the solution to (1.4)

$$u(x) = f\left(\frac{x}{\sqrt{\varepsilon}}\right) + f\left(\frac{1-x}{\sqrt{\varepsilon}}\right) - u_3 + O\left(e^{-\lambda_2/(2\sqrt{\varepsilon})}\right), \tag{5.37}$$

where f is given in (2.4) with initial conditions $f(0) = 0$ and $f'(0) = \sqrt{\varepsilon}k_3$, and where $\lambda_2 = \sqrt{\sigma'(u_3)}$. Note that, as in Lemma 2.1(c), by (5.7) $\sqrt{\varepsilon}k_3$ is a constant, independent of ε.

From the argument above, we can make the following conclusion.

Proposition 5 *When $\gamma_0 < \gamma < \gamma_2$, the boundary-value problem* (1.4) *has three different solutions, and the asymptotic formulas for the three solutions are given by* (5.12), (5.36) *and* (5.37).

6 The case $\gamma = \gamma_2$

In this case, the equation $\sigma(u) = \gamma$ has two solutions. We denote them by u_1 and u_3 in order to be consistent with the notations in Section 5. We still use $F(u)$ for the integral $\int_0^u (\sigma(s) - \gamma)ds$; $F(u)$ has the typical shape shown in Figure 8 below. Again, we define $V(u) = \int_{u_1}^u (\sigma(s) - \gamma)ds$. Note that $V(u_1) = V'(u_1) = V''(u_1) = 0$. Thus, we can write $V(u)$ as

$$V(u) = (u_1 - u)^3 H(u), \tag{6.1}$$

where $H(u)$ is a positive smooth function. We also note that $\sigma'(u_1) = 0$. Hence, $\lambda_1 = 0$ and we will no longer have the exponentially small error terms as in the previous cases; see, e.g., (5.12) and (5.36). Since most of the arguments in this section are similar to those used in Section 5, we just outline our proofs briefly and omit the details.

By the argument in the proof of Lemma 2.2, we have $\lim_{k\to 0} T(k) = 0$. Moreover, we know that $T'(k) > 0$ when $k \in (0, k_1)$. Therefore , there exits only one k_0 satisfying $T(k_0) = 1$ and $u_0(k_0) \to u_1$ as $\varepsilon \to 0$; see the statement following (2.27). Again, we wish to use the solution to the problem (2.1) when $k = k_1$ to approximate the solution to (1.4) when $k_0 \in (0, k_1)$. First, we investigate the behavior of the solution to (2.1) when $k = k_1$.

Lemma 6.1 *When $\gamma = \gamma_2$ and $k = k_1$, we let $\xi = x/\sqrt{\varepsilon}$ and $f(\xi) = u(x, k)$. Then*

$$f''(\xi) - \sigma(f) = -\gamma, \qquad 0 < \xi < \infty, \tag{6.2}$$

and

$$f(0) = 0, \qquad f'(0) = \sqrt{\varepsilon}k_1.$$

(Note: by (5.6), *$\sqrt{\varepsilon}k_1$ is a constant.) Furthermore,*

$$f(\xi) = u_1 + O\left(\frac{1}{\xi^2}\right) \qquad \text{as } \xi \to \infty. \tag{6.3}$$

Proof We use the same argument as that presented in the proof of Lemma 2.1(c). However, instead of (2.13), $V(x)$ here is given by (6.1). Thus, the formula corresponding to (2.17)

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

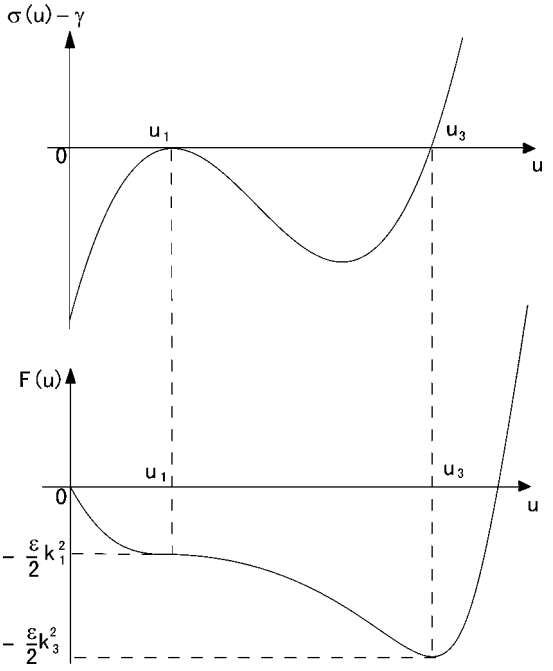


FIGURE 8. Graphs of $\sigma(u)-\gamma$ and $F(u)$ for $\gamma=\gamma_2$.

becomes

$$\int_0^f \frac{ds}{(u_1-s)^{3/2}\sqrt{2H(s)}} = \frac{\sqrt{2}}{\sqrt{H(u_1)}}\left(\frac{1}{\sqrt{u_1-f}}-\frac{1}{\sqrt{u_1}}\right)+c-d(f),$$

where

$$c=\int_0^{u_1}\left(\frac{1}{\sqrt{2V(s)}}-\frac{1}{(u_1-s)^{3/2}\sqrt{2H(u_1)}}\right)ds$$

and

$$d(f)=\int_f^{u_1}\frac{1}{\sqrt{2}(u_1-s)^{3/2}}\left(\frac{1}{\sqrt{H(s)}}-\frac{1}{\sqrt{H(u_1)}}\right)ds.$$

Note that both c and $d(f)$ are well defined. Thus, (6.3) is proved. □

Next, we derive the asymptotic formula for the solution to (1.4) when $k_0\in(0,k_1)$.

Lemma 6.2 *When $\gamma=\gamma_2$ and $k_0\in(0,k_1)$, the asymptotic formula for the solution to* (1.4) *is*

$$u(x)=f\left(\frac{x}{\sqrt{\varepsilon}}\right)+f\left(\frac{1-x}{\sqrt{\varepsilon}}\right)-u_1+O(\varepsilon), \tag{6.4}$$

where f is given in (6.2) with initial conditions $f(0)=0$ and $f'(0)=\sqrt{\varepsilon}k_1$.

Proof As in the proof of Proposition 2, we let $t=u_1-u$ and $a=u_1-u_0(k_0)$. Using (6.1),

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

we have from (2.24)

$$\int_a^{u_1} \frac{dt}{\sqrt{t^3h(t)-a^3h(a)}} = \sqrt{\frac{1}{2\varepsilon}}, \tag{6.5}$$

instead of (2.26), where $h(t) = H(u_1 - t) > 0$ and $h(0) = -\sigma''(u_1)/3 > 0$.

The left-hand side of (6.5) can be written as

$$\int_a^{u_1} \frac{dt}{\sqrt{t^3h(a)-a^3h(a)}} + \int_a^{u_1} \left(\frac{1}{\sqrt{t^3h(t)-a^3h(a)}} - \frac{1}{\sqrt{t^3h(a)-a^3h(a)}} \right) dt, \tag{6.6}$$

As $a \to 0$, the first term in (6.6) is equal to

$$\begin{aligned} \frac{1}{\sqrt{h(a)}} \int_a^{u_1} \frac{dt}{\sqrt{t^3-a^3}} &= \frac{1}{\sqrt{ah(a)}} \int_1^{\infty} \frac{dt}{\sqrt{t^3-1}} + o(1) \\ &= \frac{2\Gamma(7/6)\sqrt{\pi}}{\Gamma(2/3)\sqrt{ah(a)}} + o(1), \end{aligned} \tag{6.7}$$

and the second term is equal to

$$\int_a^{u_1} \left(\frac{1}{\sqrt{t^3h(t)-a^3h(a)}} - \frac{1}{\sqrt{t^3h(a)-a^3h(a)}} \right) dt = m + o(1), \tag{6.8}$$

where

$$m = \int_0^{u_1} \left(\frac{1}{\sqrt{t^3h(t)}} - \frac{1}{\sqrt{t^3h(0)}} \right) dt.$$

A combination of (6.5), (6.6), (6.7) and (6.8) gives

$$\frac{2\Gamma(7/6)\sqrt{\pi}}{\Gamma(2/3)\sqrt{ah(0)}} + m + o(1) = \sqrt{\frac{1}{2\varepsilon}};$$

see the sentence following (6.5). Therefore,

$$a = O(\varepsilon) \qquad as\ \varepsilon \to 0.$$

From here on, we follow the proof of Proposition 2 to reach the conclusion (6.4). □

When $k \in (k_1, k_3)$, $u_0(k) \in (u_1, u_3)$. The argument in Remark 1 shows that there exists $u_0(k_0)$ satisfying $T(u_0(k_0)) = 1$ in the right neighborhood of u_1 and in the left neighborhood of u_3. Furthermore, $u_0(k_0) \to u_1$ or $u_0(k_0) \to u_3$ as $\varepsilon \to 0$.

Now we wish to give an estimate for $u_0(k_0) - u_1$ when $u_0(k_0)$ is in the right neighborhood of u_1.

Lemma 6.3 *When $\gamma = \gamma_2$ and $u_0(k_0)$ is in the right neighborhood of u_1,*

$$u_0(k_0) - u_1 = O(\varepsilon) \qquad as\ \varepsilon \to 0. \tag{6.9}$$

Proof As in the proof of Lemma 6.2, we first let $t = u_1 - u$ and $a = u_1 - u_0(k_0)$ to get

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

(6.5). Then, we again write the left-hand side of (6.5) in the form given in (6.6). However, since a is negative here, the first term in (6.6) becomes

$$\begin{aligned} \frac{1}{\sqrt{h(a)}}\int_a^{u_1}\frac{dt}{\sqrt{t^3-a^3}} &= \frac{1}{\sqrt{-ah(a)}}\int_{-\infty}^{1}\frac{dt}{\sqrt{1-t^3}}+o(1) \\ &= \frac{\pi\Gamma(1/3)}{\Gamma(5/6)\sqrt{-ah(a)}}+o(1). \end{aligned} \tag{6.10}$$

The second term in (6.6) is still given by (6.8). From (6.10) and (6.8), we obtain

$$\frac{\pi\Gamma(1/3)}{\Gamma(5/6)\sqrt{-ah(0)}}+m+o(1)=\sqrt{\frac{1}{2\varepsilon}}. \tag{6.11}$$

Thus, $a=O(\varepsilon)$ as $\varepsilon\to 0$ and (6.9) is proved. □

From (6.11), we note that in the right neighborhood of u_1,

$$T(u_0(k))=\frac{\sqrt{2\varepsilon}\pi\Gamma(1/3)}{\Gamma(5/6)\sqrt{(u_0(k)-u_1)h(0)}}+\sqrt{2\varepsilon}m+o(\sqrt{\varepsilon}).$$

By following the argument in the proof of Proposition 2, we can also get the behavior of $T(u_0(k))$ in the left neighborhood of u_3 given by

$$T(u_0(k))=-\sqrt{\frac{2\varepsilon}{\tilde{h}(0)}}\ln(u_3-u_0(k))+o(1).$$

The same argument in Remark 2 then shows that there exist two values of k satisfying $T(k)=1$, and the boundary-value problem (1.4) has two solutions when $k\in(k_1,k_3)$.

Finally, we derive the asymptotic formula for the boundary-value problem (1.4) when $u_0(k_0)$ is in the right neighborhood of u_1.

Lemma 6.4 *When $\gamma=\gamma_2$ and $u_0(k_0)$ is in the right neighborhood of u_1, the asymptotic formula for the solution to* (1.4) *is*

$$u(x)=f\left(\frac{x}{\sqrt{\varepsilon}}\right)+f\left(\frac{1-x}{\sqrt{\varepsilon}}\right)-u_1+O(\varepsilon), \tag{6.12}$$

where f is given in (6.2) *with initial conditions $f(0)=0$ and $f'(0)=\sqrt{\varepsilon}k_1$.*

Proof As in the proof of Lemma 5.4, we again denote by x_1 the point satisfying $u(x_1,k_0)=u_1$. For $x\in[0,x_1]$, we have $u(x,k_0)\in[0,u_1]$. By following strictly the argument in Proposition 2, one can show that (6.12) holds in the interval $[0,x_1]$; see also (5.28). For $x\in[x_1,\frac{1}{2}]$, we have $u(x,k_0)\in[u_1,u_0(k_0)]$. From (6.9), $0\leqslant u(x,k_0)-u_1\leqslant u_0(k_0)-u_1=O(\varepsilon)$. Moreover, for any given x in the interval $[x_1,\frac{1}{2}]$, $f(x/\sqrt{\varepsilon})=u_1+O(\varepsilon)$ and $f((1-x)/\sqrt{\varepsilon})=u_1+O(\varepsilon)$. Thus, (6.12) holds for $x\in[0,\frac{1}{2}]$. By the symmetry of $u(x,k_0)$, (6.12) holds for all $x\in[0,1]$. □

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

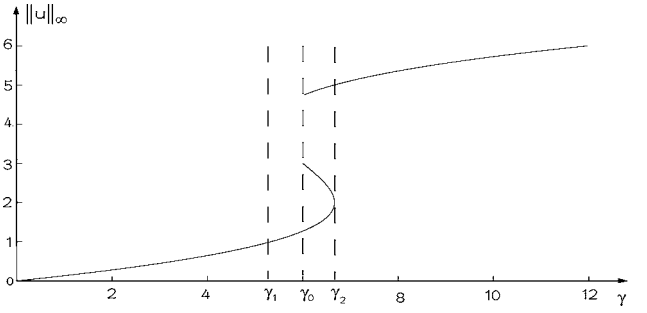


FIGURE 9. Bifurcations when $\sigma(u) = \frac{1}{3}u^3 - 3u^2 + 8u$.

When $u_0(k_0)$ is in the left neighborhood of u_3, the argument in Section 5 gives the asymptotic formula

$$u(x) = f\left(\frac{x}{\sqrt{\varepsilon}}\right) + f\left(\frac{1-x}{\sqrt{\varepsilon}}\right) - u_3 + O\left(e^{-\lambda_2/(2\sqrt{\varepsilon})}\right), \tag{6.13}$$

where f is given in (2.4) with initial conditions $f(0) = 0$ and $f'(0) = \sqrt{\varepsilon}k_3$, and where $\lambda_2 = \sqrt{\sigma'(u_3)}$; see also (5.37).

Summarizing the results above, we have the following conclusion

Proposition 6 *When $\gamma = \gamma_2$, the boundary-value problem* (1.4) *has three different solutions, and the asymptotic formulas for the three solutions are given by* (6.4), (6.12) *and* (6.13).

7 Conclusions

From the results presented above with different values of the external force γ, the whole deformation process of elastic rod with two fixed ends subjected to a tensile stress can be described. In summary, there are four states: a low-strain state, a transition state, a two-strain state and a high-strain state; see Figure 9. Our results agree with the physical phenomena observed in experiments; see Dai & Bi [5] and Dai & Cai [6] and the references therein.

In this paper, the material which we have considered is incompressible elastic. Actually, our method can be applied to other incompressible phase-transforming materials once some description of stress-strain function $\sigma(u)$ in (1.4) is given. Moreover, besides Dirichlet conditions, Neumann conditions which correspond to "free ends" in physics can also be dealt with by using our method [12, p. 190].

References

[1] ABEYARATNE, R. & KNOWLES, J. (1993) Nucleation, kinetics and admissibility criteria for propagating phase boundaries. In: Dunn, J. R., Fosdick, L. and Slemrod, M. (eds.), *Shock Induced Transitions and Phase Structures in General Media, IMA Vol. Math. Appl.*. Springer-Verlag, London **52**, 1–33.

[2] CAHN, J. W. & HILLIARD, J. E. (1958) Free energy of a nonuniform system. I. Interfacial free energy. *J. Chem. Physics* **28**, 258–264.

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press

[3] Carr, J., Gurtin, M. E. & Slemrod, M. (1984) Structured phase transitions on a finite interval. *Arch. Rational Mech. Anal.* **86**(4), 317–351.
[4] Dai, H. H. (2004) Non-existence of one-dimensional stress problems in solid-solid phase transitions and uniqueness conditions for incompressible phase-transforming materials. *C. R. Acad. Sci. Paris, Ser.* **338**(12), 981–984.
[5] Dai, H. H. & Bi, Q. (2006) On constructing the unique solution for the necking in a hyperelastic rod. *J. Elasticity*, **82**, 215–241.
[6] Dai, H. H. & Cai, Z. (2006) Phase transitions in a slender cylinder composed of an incompressible elastic material. I. Asymptotic model equation. *Proc. R. Soc. A.* **462**(2065), 75–95.
[7] Dai, H. H. & Cai, Z. Phase transitions in a slender cylinder composed of an incompressible elastic material. II. Analytical solutions for two boundary-value problems. To be published.
[8] Davis, H. T. & Scriven, L. E. (1982) Stress and structure in fluid interfaces. *Advances Chem. Phys.* **49**, 681–706.
[9] García-Melián, J. & Sabina de Lis, J. (2000) Stationary patterns to diffusion problems. *Math. Meth. Appl. Sci.* **23**, 1467–1489.
[10] Grinfeld, M. & Novick-Cohen, A. (1995) Counting stationary solutions of the Cahn-Hilliard equation by transversality arguments. *Proc. Roy. Soc. Edinburgh Sect. A.* **125**(2), 351–370.
[11] Ou, C. H. & Wong, R. (2003) On a two-point boundary-value problem with spurious solutions. *Stud. Appl. Math.* **111**(4), 377–408.
[12] Ou, C. H. & Wong, R. (2004) Shooting method for nonlinear singularly perturbed boundary-value problems. *Stud. Appl. Math.* **112**(2), 161–200.
[13] Rowlinson, J. S. (1979) Translation of J. D. Van der Waals' "The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density". *J. Stat. Phys.* **49**, 197–244.

https://doi.org/10.1017/S095679250600670X Published online by Cambridge University Press