Math. Proc. Camb. Phil. Soc. (1986), **99**, 135
Printed in Great Britain

On Λ-Ω-extendable distributions

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(*Received 2 January* 1985; *revised 5 June* 1985)

Abstract

Let $\Lambda \subset \Omega \subset \mathbb{R}^n$ be open subsets. We construct a natural space $\mathscr{P}(\Lambda, \Omega)$ of test functions on Λ such that the dual $\mathscr{P}(\Lambda, \Omega)'$ consists exactly of those distributions on Λ which can be extended to distributions on Ω. As an application of this representation we calculate the space $\mathscr{O}_M(\mathscr{P}(\Lambda, \Omega)')$ of multiplication operators on $\mathscr{P}(\Lambda, \Omega)'$ as well as the space $\mathscr{P}_{ar}(\Lambda, \Omega)$ of absolutely regular extendable distributions.

1. *Introduction and notations*

Let $\Lambda \subset \Omega$ be open subsets of $\mathbb{R}^n$. By $\mathfrak{T}(\Omega)$ we denote the standard LF-space topology on $\mathscr{D}(\Omega)$. Considering $\mathscr{D}(\Lambda)$ as a subspace of $\mathscr{D}(\Omega)$ we have two topologies on $\mathscr{D}(\Lambda)$: its own natural topology $\mathfrak{T}(\Lambda)$, and the relative topology $\mathfrak{T}(\Omega) \cap \mathscr{D}(\Lambda)$ induced by $(\mathscr{D}(\Omega), \mathfrak{T}(\Omega))$ on $\mathscr{D}(\Lambda)$. By the Hahn–Banach theorem we obtain

$$(\mathscr{D}(\Lambda), \mathfrak{T}(\Omega) \cap \mathscr{D}(\Lambda))' = \{T \in \mathscr{D}(\Lambda)';\ \exists S \in \mathscr{D}(\Omega)' \colon S|\Lambda = T\}.$$

However, for the purpose of calculations, the space $(\mathscr{D}(\Lambda), \mathfrak{T}(\Omega) \cap \mathscr{D}(\Lambda))$ is not suitable. For instance, this space is not complete, and does not allow the application of a closed graph theorem unless $(\partial\Lambda) \cap \Omega = \emptyset$. It is the aim of this paper to show that the completion $(\mathscr{D}(\Lambda), \mathfrak{T}(\Omega) \cap \mathscr{D}(\Lambda)^{\sim}$, i.e. the closure of $\mathscr{D}(\Lambda)$ in $(\mathscr{D}(\Omega), \mathfrak{T}(\Omega))$, is the strict inductive limit of a sequence of well known nuclear Fréchet spaces, and thus has all the nice properties we expect a space of test functions to have. These Fréchet spaces are defined as follows. Let $U \subset \mathbb{R}^n$ be an arbitrary open subset. We provide the space

$$\mathring{\mathscr{B}}(U) := \{\varphi \in \mathscr{E}(U);\ \forall \alpha \in \mathbb{N}_0^n \colon \partial^\alpha \varphi \in C_0(U)\}$$

with the natural Fréchet-space topology $\mathfrak{T}^\infty$, which is generated by the norms

$$\|\varphi\|_{\infty, m} := \sup\{|\partial^\alpha \varphi(x)|;\ |\alpha| \leqslant m, x \in U\} \quad (m \in \mathbb{N}_0).$$

This space was defined by Horváth ([8], p. 91, example 17), and was extensively studied by Valdivia [13], Vogt [14], and Dierolf, Voigt [4]. In particular, $\mathring{\mathscr{B}}(U)$ is nuclear, if U is bounded (cf. ([4], theorem (4·12), p. 79)).

For spaces of functions and distributions, as well as for general locally convex spaces we use the standard notations of Horváth [8] and Schwartz [12].

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2. *Construction of the space* $\mathscr{P}(\Lambda, \Omega)$

Inspired by a remark of Schwartz ([**11**], p. 44/45) we first prove that the property of Λ-Ω-extendability can be localized to the relative boundary $(\partial\Lambda) \cap \Omega$ of Λ in Ω.

(2·1) PROPOSITION. *For* $T \in \mathscr{D}(\Lambda)'$ *the following are equivalent:*

(*a*) $\exists S \in \mathscr{D}(\Omega)' : S|\Lambda = T$.

(*b*) $\forall x \in (\partial\Lambda) \cap \Omega \, \exists U = \mathring{U} \subset \Omega$, U *a neighbourhood of* x, $\exists C > 0, m \in \mathbb{N}_0 \, \forall \varphi \in \mathscr{D}(U \cap \Lambda)$: $|\langle T, \varphi \rangle| \leqslant C \cdot \|\varphi\|_{\infty, m}$.

Proof. (*a*) $\Rightarrow$ (*b*). Here it is sufficient to choose an open neighbourhood U of $x \in (\partial\Lambda) \cap \Omega$ such that $\overline{U}$ is a compact subset of Ω. An application of theorem XX of Schwartz ([**12**], p. 82) to S then yields (*b*).

(*b*) $\Rightarrow$ (*a*). For every $x \in (\partial\Lambda) \cap \Omega$ we choose an open neighbourhood U_x according to (*b*). Using the estimate in (*b*) and the Hahn–Banach Theorem we extend the distribution $T|U_x \cap \Lambda$ to a distribution T_x on all of U_x. We then adjoin two pairs (T_0, U_0), (T_1, U_1) to the system $((T_x, U_x); x \in (\partial\Lambda) \cap \Omega)$ by defining

$$U_0 := \Lambda, \quad T_0 := T, \quad U_1 := \Omega \backslash \overline{\Lambda}, \quad T_1 := 0.$$

Thus we obtain an open covering $(U_\iota, \iota \in I)$ of Ω and for every $\iota \in I$ a distribution T_ι on U_ι. Due to the arbitrariness of the extension T_x of $T|U_x \cap \Lambda$ to U_x the usual compatibility condition (cf. Schwartz [**12**], theorem IV, p. 27) will not be satisfied in general. Nevertheless we may use a locally finite partition of unity $(\varphi_\iota; \iota \in I)$ subordinate to $(U_\iota; \iota \in I)$ to define a distribution S on Ω by

$$\langle S, \psi \rangle := \sum_{\iota \in I} \langle T_\iota, \varphi_\iota \cdot \psi \rangle \, (\psi \in \mathscr{D}(\Omega)).$$

For all $\varphi \in \mathscr{D}(\Lambda)$ we then obtain:

$$\begin{aligned} \langle S, \varphi \rangle = \sum_{\iota \in I} \langle T_\iota, \varphi_\iota \cdot \varphi \rangle &= \sum_{\iota \in I} \langle T_\iota | U_\iota \cap \Lambda, \varphi_\iota \cdot \varphi \rangle \\ &= \sum_{\iota \in I} \langle T, \varphi_\iota \cdot \varphi \rangle = \langle T, \varphi \rangle. \end{aligned}$$

Therefore S is an extension of T to all of Ω.

Several corollaries follow from (2·1). For our purpose the most important one is the following.

(2·2) COROLLARY. *Let* $(K(j); j \in \mathbb{N})$ *be any increasing sequence of compact subsets of* Ω *such that* $K(j) \subset \mathring{K}(j+1)$ $(j \in \mathbb{N})$ *and* $\bigcup_{j \in \mathbb{N}} K(j) = \Omega$, *and let*

$$\mathscr{D}(\Omega), \mathfrak{T}(\Omega)) = \bigcup_{j \in \mathbb{N}} (\mathscr{D}_{K(j)}(\Omega), \mathfrak{T}^\infty)$$

be the corresponding representation of $(\mathscr{D}(\Omega), \mathfrak{T}(\Omega))$. *Then*

$$(\mathscr{D}(\Lambda), \mathfrak{T}(\Omega) \cap \mathscr{D}(\Lambda))' = (\bigcup_{j \in \mathbb{N}} (\mathscr{D}(\Lambda) \cap \mathscr{D}_{K(j)}(\Omega), \mathfrak{T}^\infty))',$$

i.e., $\mathscr{D}(\Lambda)$ *is a* well-located subspace *of* $(\mathscr{D}(\Omega), \mathfrak{T}(\Omega))$. (*Here the topology of uniform convergence of all derivatives is invariably denoted by* $\mathfrak{T}^\infty$, *no matter what the underlying set is.*)

Proof. Evidently all the inclusions $(\mathscr{D}(\Lambda) \cap \mathscr{D}_{K(j)}(\Omega), \mathfrak{T}^\infty) \hookrightarrow (\mathscr{D}(\Omega), \mathfrak{T}(\Omega))$ $(j \in \mathbb{N})$ are continuous. Thus we only have to prove the inclusion '$\supset$'. According to (2·1) it is

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enough to show that every continuous linear form T on $\bigcup_{j\in\mathbb{N}}(\mathscr{D}(\Lambda)\cap\mathscr{D}_{K(j)}(\Omega),\mathfrak{T}^\infty)$ satisfies (2·1*b*). For $x_j\in(\partial\Lambda)\cap\Omega$ we choose an open neighbourhood U_x such that $\overline{U}_x$ is a compact subset of Ω. Then there exists $j\in\mathbb{N}$ such that $U_x\subset K(j)$. The continuity of T on $(\mathscr{D}(\Lambda)\cap\mathscr{D}_{K(j)}(\Omega),\mathfrak{T}^\infty)$ now readily yields an estimate of the form (2·1*b*).

The next two corollaries are noted for their own interest. We don't use them further on.

(2·3) Corollary. *Let $\Lambda\subset\Omega$ be open subsets of $\mathbb{R}^n$. The following are equivalent:*

(*a*) $(\partial\Lambda)\cap\Omega=\emptyset$.

(*b*) $(\mathscr{D}(\Omega),\mathfrak{T}(\Omega))=(\mathscr{D}(\Lambda),\mathfrak{T}(\Lambda))\overset{\text{top}}{\oplus}(\mathscr{D}(\Omega\backslash\overline{\Lambda}),\mathfrak{T}(\Omega\backslash\overline{\Lambda}))$.

(*c*) $\forall T\in\mathscr{D}(\Lambda)'\,\exists S\in\mathscr{D}(\Omega)'\colon S|\Lambda=T$.

Proof. (*a*) implies $\Omega=\Lambda\cup(\Omega\backslash\overline{\Lambda})$ which in turn implies (*b*). The implication (*b*) $\Rightarrow$ (*c*) is obvious. Now let $x\in(\partial\Lambda)\cap\Omega$. Then there exists a sequence $(x_j;j\in\mathbb{N})$ in Λ such that $x_j\to x(i\to\infty)$. Let $(\alpha(j);j\in\mathbb{N})$ be a sequence in $\mathbb{N}_0^n$ such that $|\alpha(j)|\to\infty$ $(j\to\infty)$, and denote the Dirac measure concentrated in x_j by δ_j $(j\in\mathbb{N})$. Then $T:=\sum_{j\in\mathbb{N}}\partial^{\alpha(j)}\delta_j$ is a distribution on Λ which does not satisfy (2·1*b*) and thus cannot be extended to a distribution on Ω.

(2·4) Corollary. *Let $T\in\mathscr{D}(\Lambda)'$ satisfy* (2·1*b*). *Then an extension $S\in\mathscr{D}(\Omega)'$ may be chosen in such a way that* $\operatorname{supp}(S)\subset\overline{\Lambda}\cap\Omega$.

Proof. We define $\Lambda_1:=\Lambda\cup(\Omega\backslash\overline{\Lambda})$ and $T_1\in\mathscr{D}(\Lambda_1)'$ by $T_1|\Lambda:=T, T_1|(\Omega\backslash\overline{\Lambda}):=0$. Then T_1 satisfies (2·1*b*) with respect to the pair (Λ_1,Ω). Thus there exists an extension $S\in\mathscr{D}(\Omega)'$ of T_1. Evidently, $\operatorname{supp}(S)\subset\overline{\Lambda}\cap\Omega$.

Corollary (2·4) is essentially due to Łojasiewicz ([9], p. 97). By proposition IV, 3·9 of De Wilde ([1], p. 124) every well-located subspace of $(\mathscr{D}(\Omega),\mathfrak{T}(\Omega))$ is even a limit-subspace. We therefore have the following result.

(2·5) Proposition. *Let the notation be as in* (2·2). *Then*

$$(\mathscr{D}(\Lambda),\mathfrak{T}(\Omega)\cap\mathscr{D}(\Lambda))=\bigcup_{j\in\mathbb{N}}(\mathscr{D}(\Lambda)\cap\mathscr{D}_{K(j)}(\Omega),\mathfrak{T}^\infty),$$

i.e. $\mathscr{D}(\Lambda)$ *is a* limit-subspace *of* $(\mathscr{D}(\Omega),\mathfrak{T}(\Omega))$ *in the sense of* Floret ([6], p. 225).

The next lemma implies that the closure of $\mathscr{D}(\Lambda)$ in $(\mathscr{D}(\Omega),\mathfrak{T}(\Omega))$ is again a limit-subspace.

(2·6) Lemma. *Let $(E,\mathfrak{T})=\bigcup_{\iota\in I}(E_\iota,\mathfrak{T}_\iota)$ be an inductive limit, let $L\subset E$ be a limit subspace and let $M\subset E$ be a linear subspace satisfying $L\subset M\subset\bigcup_{\iota\in I}\overline{L\cap E_\iota}^{\mathfrak{T}_\iota}$. Then M is also a limit subspace.*

Proof. We put $M_\iota:=E_\iota\cap M$ $(\iota\in I)$. Let $\mathfrak{F}$ denote the inductive limit topology $\bigcup_{\iota\in I}\mathfrak{T}_\iota\cap M_\iota$ on M. Since the inclusions $(M_\iota,\mathfrak{T}_\iota\cap M_\iota)\hookrightarrow(E,\mathfrak{T})$ $(\iota\in I)$ are continuous we obtain $\mathfrak{T}\cap M\subset\mathfrak{F}$, and thus $\mathfrak{T}\cap L=\mathfrak{F}\cap L$. Therefore the comparable locally convex topologies $\mathfrak{T}\cap M$ and $\mathfrak{F}$ coincide on the subspace L which is dense in $(M,\mathfrak{F})$. By Lemma 1 of ([5], p. 349) we thus obtain $\mathfrak{T}\cap M=\mathfrak{F}$.

To apply this result to our situation we first observe the inclusions

$$\mathscr{D}(\Lambda\cap\mathring{K}(j))\subset\mathscr{D}(\Lambda)\cap\mathscr{D}_{K(j)}(\Omega)\subset\mathscr{D}(\Lambda\cap\mathring{K}(j+1))\quad(j\in\mathbb{N})$$

and then apply proposition (4·10) of ([4], p. 77) to conclude

$$\mathring{\mathscr{B}}(\Lambda\cap\mathring{K}(j))\subset\overline{\mathscr{D}(\Lambda)\cap\mathscr{D}_{K(j)}(\Omega)}^{\mathfrak{T}^\infty}\subset\mathring{\mathscr{B}}(\Lambda\cap\mathring{K}(j+1))\quad(j\in\mathbb{N}).$$

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Therefore the inductive spectra

$$\overline{(\mathscr{D}(\Lambda)\cap\mathscr{D}_{K(j)}(\Omega)}^{\mathfrak{T}^\infty},\mathfrak{T}^\infty)\quad(j\in\mathbb{N})\quad\text{and}\quad(\mathring{\mathscr{B}}(\Lambda\cap\mathring{K}(j)),\mathfrak{T}^\infty)\quad(j\in\mathbb{N})$$

are equivalent.

In the next proposition we collect the essential properties of our space of test functions.

(2·7) PROPOSITION. *Let $\Lambda\subset\Omega$ be open subsets of $\mathbb{R}^n$, and let $(K(j);j\in\mathbb{N})$ be a sequence of compact subsets of Ω such that $K(j)\subset\mathring{K}(j+1)$ $(j\in\mathbb{N})$ and $\bigcup_{j\in\mathbb{N}}K(j)=\Omega$. We define*

$$(\mathscr{P}(\Lambda,\Omega),\Re):=\bigcup_{j\in\mathbb{N}}(\mathring{\mathscr{B}}(\Lambda\cap\mathring{K}(j)),\mathfrak{T}^\infty).$$

(a) $\mathscr{D}(\Lambda)$ is dense in $(\mathscr{P}(\Lambda,\Omega),\Re)$ and $\Re=\mathfrak{T}(\Omega)\cap\mathscr{P}(\Lambda,\Omega)$ is a nuclear strict LF-space topology.

(b) $\mathscr{P}(\Lambda,\Omega)=\{\varphi\in\mathscr{D}(\Omega);\forall\alpha\in\mathbb{N}_0^n,x\in\Omega\backslash\Lambda:\partial^\alpha\varphi(x)=0\}$.

(c) $\mathscr{P}(\Lambda,\Omega)=\{T\in\mathscr{D}(\Lambda)';\exists S\in\mathscr{D}(\Omega)':S|\Lambda=T\}$.

(d) The canonical map

$$(\mathscr{D}(\Omega)',\beta(\mathscr{D}',\mathscr{D}))/\mathscr{P}(\Lambda,\Omega)^0\to(\mathscr{P}(\Lambda,\Omega)',\beta(\mathscr{P}',\mathscr{P}))$$

is a topological isomorphism, and every bounded subset of $\mathscr{P}(\Lambda,\Omega)'$ can be lifted to a bounded subset in $\mathscr{D}(\Omega)'$ (cf. De Wilde ([**1**], p. 124)).

(e) $(\mathscr{P}(\Lambda,\Omega)',\beta(\mathscr{P}',\mathscr{P}))$ is ultrabornological (cf. Schwartz ([**10**], p. 43, application)).

(2·8) *Remark.* We proved Proposition (2·5) with the help of the result of De Wilde ([**1**], p. 124) which is not applicable if we consider the subspace $\mathscr{D}^m(\Lambda)$ of $\mathscr{D}^m(\Omega)$ for $m\in\mathbb{N}_0$. Thus the question arises whether $\mathscr{D}^m(\Lambda)$ is also a limit-subspace of $\mathscr{D}^m(\Omega)$ for $m\in\mathbb{N}_0$. An affirmative answer to this question was given by S. Dierolf. Since not only the result but also the proof is interesting, we give a brief sketch. The key idea is the following: If (2·1) were available for *continuous linear maps* (into Banach spaces) instead of continuous linear functionals, the proof of (2·2) could be adapted to prove directly 'limit-subspace' instead of 'well-located'. Actually the proof of (2·1) can be modified to yield the following vector-valued version and also the corresponding result for $\mathscr{D}^m(\Lambda)$ and $\mathscr{D}^m(\Omega)$ instead of $\mathscr{D}(\Lambda)$ and $\mathscr{D}(\Omega)$.

Let $(E,\|\cdot\|)$ be a Banach space and let $j\colon(E,\|\cdot\|)\hookrightarrow(l^\infty(I),\|\cdot\|_\infty)$ be an isometric embedding. For a continuous linear map $T\colon(\mathscr{D}(\Lambda),\mathfrak{T}(\Lambda))\to(E,\|\cdot\|)$ the following are equivalent:

(a) There exists a continuous linear map $S\colon(\mathscr{D}(\Omega),\mathfrak{T}(\Omega))\to(l^\infty(I),\|\cdot\|_\infty)$ such that $S|\Lambda=T$.

(b) $\forall x\in(\partial\Lambda)\cap\Omega\,\exists U=\mathring{U}$, U a neighbourhood of x, $\exists C>0,m\in\mathbb{N}_0\,\forall\varphi\in\mathscr{D}(U\cap\Lambda)$: $\|T(\varphi)\|\leqslant C\,\|\varphi\|_{\infty,m}$.

(c) $T\colon(\mathscr{D}(\Lambda),\mathfrak{T}(\Omega)\cap\mathscr{D}(\Lambda))\to(E,\|\cdot\|)$ is continuous.

3. *Applications: The spaces $\mathscr{O}_M(\mathscr{P}(\Lambda,\Omega)')$ and $\mathscr{P}_{\mathrm{ar}}(\Lambda,\Omega)$*

Let $(E,\Re)$ and $(F,\mathfrak{S})$ be locally convex spaces of distributions on Λ such that $\mathscr{D}(\Lambda)$ is dense in $(E,\Re)$. By definition, a distribution $S\in\mathscr{D}(\Lambda)'$ is a *multiplication operator from $(E,\Re)$ into $(F,\mathfrak{S})$* if there exists a continuous linear map $[S]\colon(E,\Re)\to(F,\mathfrak{S})$ such that $[S](\varphi)=S\,.\,\varphi$ holds for all $\varphi\in\mathscr{D}(\Lambda)\subset E$ (Schwartz ([**10**], p. 69), cf. also [**2**]). We denote the space of all multiplication operators from E into F by $\mathscr{O}_M(E,F)$. For $E=F$

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we write $\mathcal{O}_M(E)$ instead of $\mathcal{O}_M(E,E)$. The multiplication operators on the space of extendable distributions play an important rôle in the solution of the problem of division of distributions (cf. Lojasiewicz [9], p. 97, proposition 3, Schwartz [11], p. 45).

(3·1) PROPOSITION. *Let $\Lambda \subset \Omega$ be open subsets of $\mathbb{R}^n$. We define $\delta: \Lambda \to \mathbb{R}$ by*

$$\delta(x) := \operatorname{dist}(x, \Omega\backslash\Lambda) \quad (x \in \Lambda).$$

Let $\mathcal{P}(\Lambda,\Omega)'$ be provided with the strong topology $\beta(\mathcal{P}', \mathcal{P})$ and let $(K(j); j \in \mathbb{N})$ denote a fundamental sequence of compact subsets of Ω (cf. (2·7)). Then

$$\begin{aligned}\mathcal{O}_M(\mathcal{P}(\Lambda,\Omega)') = \mathcal{O}_M(\mathcal{P}(\Lambda,\Omega)) &= \{\psi \in \mathcal{E}(\Lambda);\ \psi.\mathcal{P}(\Lambda,\Omega) \subset \mathcal{P}(\Lambda,\Omega)\} \\ &= \bigcup_{j\in\mathbb{N}} \mathcal{O}_M(\mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(j))) = \{\psi \in \mathcal{E}(\Lambda);\ \forall K \subset \Omega, K \text{ compact}, \\ &\qquad \forall \alpha \in \mathbb{N}_0^n\ \exists m \in \mathbb{N}_0, C > 0\ \forall x \in \Lambda \cap \mathring{K}: |\partial^\alpha \psi(x)| \leqslant C.\delta(x)^{-m}\}.\end{aligned}$$

Proof. In this proof we use the results from [2]. Since $(\mathcal{P}(\Lambda,\Omega), \mathfrak{R})$ is reflexive, we obtain $\mathcal{O}_M(\mathcal{P}(\Lambda,\Omega)') = \mathcal{O}_M(\mathcal{P}(\Lambda,\Omega)) \subset \mathcal{E}(\Lambda)$ from ([2], p. 310, (3·3)). A routine approximation argument shows that $[\psi](\varphi) = \psi.\varphi$ holds for all $\psi \in \mathcal{O}_M(\mathcal{P}(\Lambda,\Omega))$, $\varphi \in \mathcal{P}(\Lambda,\Omega)$. If $\psi \in \mathcal{E}(\Lambda)$ satisfies $\psi.\varphi \in \mathcal{P}(\Lambda,\Omega)$ for all $\varphi \in \mathcal{P}(\Lambda,\Omega)$ the closed graph theorem of Grothendieck ([7], part I, p. 17, theorem B) implies $\psi \in \mathcal{O}_M(\mathcal{P}(\Lambda,\Omega))$. The factorization theorem of Grothendieck ([7], part I, p. 16, Theorem A) shows

$$\mathcal{O}_M(\mathcal{P}(\Lambda,\Omega)) = \bigcap_{j\in\mathbb{N}} \bigcup_{m\geqslant j} \mathcal{O}_M(\mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(j)), \mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(m))).$$

We claim

$$\mathcal{O}_M(\mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(j)), \mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(m))) = \mathcal{O}_M(\mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(j)))$$

for all $m,j \in \mathbb{N}$, $m \geqslant j$. Let $[\psi]: \mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(j)) \to \mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(m))$, $[\psi](\varphi) = \psi.\varphi$ be continuous. From $[\psi](\mathcal{D}(\Lambda \cap \mathring{K}(j))) \subset \mathcal{D}(\Lambda \cap \mathring{K}(j))$ we obtain

$$[\psi](\mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(j))) \subset \mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(j))$$

and thus $\psi \in \mathcal{O}_M(\mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(j)))$. The inclusion '$\supset$' is a direct consequence of the continuity of the inclusion $\mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(j)) \hookrightarrow \mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(m))$ $(m \geqslant j)$.

To prove the inclusion '$\subset$' for the last equality in the statement of (3·1), it is sufficient to prove the estimates for the special compact sets $K(j) \subset \Omega$ $(j \in \mathbb{N})$. Let $\psi \in \mathcal{O}_M(\mathcal{P}(\Lambda,\Omega))$, $\alpha \in \mathbb{N}_0^n$, and $j \in \mathbb{N}$ be given. Then $\psi \in \mathcal{O}_M(\mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(j+1)))$ and the description of $\mathcal{O}_M(\mathring{\mathcal{B}}(\Lambda \cap \mathring{K}(j+1)))$ proved in ([2], p. 321, (4.4)) shows the existence of $m \in \mathbb{N}_0$, $C > 0$ such that $|\partial^\alpha \psi(x)| \leqslant C.r(x)^{-m}$ holds for all $x \in \Lambda \cap \mathring{K}(j+1)$, where $r(x) = \operatorname{dist}(x, \complement(\Lambda \cap \mathring{K}(j+1)))$. (Since $\Lambda \cap \mathring{K}(j+1)$ is bounded, it is not necessary to use the modified boundary distance $\rho = \min\{r, 1\}$ as it is done in loc. cit.) From $K(j) \subset \mathring{K}(j+1)$ we infer $\operatorname{dist}(\mathring{K}(j), \complement\mathring{K}(j+1)) =: 2\eta > 0$. We define

$$M := \{x \in \Lambda \cap \mathring{K}(j);\ r(x) \geqslant \eta\} \quad \text{and} \quad N := \{x \in \Lambda \cap \mathring{K}(j);\ r(x) < \eta\}.$$

Since M is bounded, we have $\beta := \sup\{\delta(x);\ x \in M\} < \infty$ and thus obtain

$$|\partial^\alpha \psi(x)| \leqslant C.\eta^{-m} \leqslant C.\eta^{-m}.\beta^m.\delta(x)^{-m} \quad (x \in M).$$

If for $x \in \Lambda \cap \mathring{K}(j)$ the boundary distance $r(x)$ is attained at a point $z \in \complement\mathring{K}(j+1)$, we obtain $r(x) = |x - z| \geqslant 2\eta$. For every $x \in N$ the boundary distance $r(x)$ is therefore

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attained at a point $z \in \Omega \backslash \Lambda$ and this implies $r(x) = |x-z| \geqslant \delta(x)\,(x \in N)$. We thus conclude $|\partial^\alpha \psi(x)| \leqslant C.r(x)^{-m} \leqslant C.\delta(x)^{-m}\,(x \in N)$. Taking the two estimates together we obtain

$$|\partial^\alpha \psi(x)| \leqslant C(1+(\beta/\eta)^m).\delta(x)^{-m} = C'.\delta(x)^{-m}\,(x \in \Lambda \cap \mathring{K}(j)).$$

If $\psi \in \mathscr{E}(\Lambda)$ satisfies the estimates specified in the statement of (3·1) we use the inequality $\operatorname{dist}(x, \complement(\Delta \cap \mathring{K})) \leqslant \operatorname{dist}(x, \Omega\backslash\Lambda) = \delta(x)\,(x \in \Lambda \cap \mathring{K})$ and the description of $\mathscr{O}_M(\mathring{\mathscr{B}}(\Lambda \cap \mathring{K}))$ given in ([2], p. 321, (4·4)) to conclude $\psi \in \bigcap_{j \in \mathbb{N}} \mathscr{O}_M(\mathring{\mathscr{B}}(\Lambda \cap \mathring{K}(j)))$.

(3·2) *Remark.* If $\Omega \subset \mathbb{R}^n$ is an open bounded set, $\mathscr{O}_M(\mathscr{P}(\Omega, \mathbb{R}^n))$ coincides with the space $\mathscr{H}_\Omega$ introduced by Łojasiewicz ([9], p. 89, 2.) (cf. also loc. cit., p. 97, proposition 3).

Since $\mathscr{P}(\Lambda, \Omega)$ is i.g. not a subspace of $L^\infty_{\mathrm{comp}}(\Lambda) = \{f \in L^\infty(\Lambda);\ \mathrm{supp}\,(f) \text{ compact}\}$, it is not evident how a distribution $T \in \mathscr{P}(\Lambda, \Omega)'$ may be represented by a locally integrable function $g \in L^1_{\mathrm{loc}}(\Lambda)$. Following ([3] p. 396, remark (1·3)) we call $T \in \mathscr{P}(\Lambda, \Omega)'$ *absolutely regular* if there exists $f \in L^1_{\mathrm{loc}}(\Lambda)$ such that for every $h \in L^\infty(\Lambda)$ the linear form

$$\varphi \mapsto \int_\Lambda h(x).f(x).\varphi(x)\,dx$$

is continuous on $(\mathscr{D}(\Lambda), \mathfrak{R} \cap \mathscr{D}(\Lambda))$ and

$$\langle T, \varphi \rangle = \int_\Lambda f(x)\,\varphi(x)\,dx (\varphi \in \mathscr{D}(\Lambda))$$

holds. By $\mathscr{P}_{\mathrm{ar}}(\Lambda, \Omega)$ we denote the space of all absolutely regular elements in $\mathscr{P}(\Lambda, \Omega)'$.

(3·3) PROPOSITION. *Let the notation be as above and let χ_A denote the characteristic function of $A \subset \mathbb{R}^n$. Then*

$$\begin{aligned}\mathscr{P}_{\mathrm{ar}}(\Lambda, \Omega) &= \mathscr{O}_M(\mathscr{P}(\Lambda, \Omega), L^1(\Lambda)) = \{f \in L^1_{\mathrm{loc}}(\Lambda);\ f.\mathscr{P}(\Lambda, \Omega) \subset L^1(\Lambda)\} \\ &= \bigcap_{j \in \mathbb{N}} \mathscr{O}_M(\mathring{\mathscr{B}}(\Lambda \cap \mathring{K}(j)), L^1(\Lambda)) = \{f \in L^1_{\mathrm{loc}}(\Lambda);\ \forall K \subset \Omega, K \text{ compact} \\ &\qquad \exists m \in \mathbb{N}_0 \colon \chi_{\Lambda \cap K}.f.\delta^m \in L^1(\Lambda)\},\end{aligned}$$

and $\langle f, \varphi \rangle = \int_\Lambda f(x).\varphi(x)\,dx$ holds for all $f \in \mathscr{P}_{\mathrm{ar}}(\Lambda, \Omega)$, $\varphi \in \mathscr{P}(\Lambda, \Omega)$.

Proof. Let $f \in \mathscr{P}_{\mathrm{ar}}(\Lambda, \Omega)$. It follows from (2·5) that for each $j \in \mathbb{N}$ the linear map $[f] \colon \mathscr{D}(\Lambda) \cap \mathscr{D}_{K(j)}(\Omega) \to L^1(\Lambda)$, $[f](\varphi) = f.\varphi$, is $\mathfrak{T}^\infty - \sigma(L^1, L^\infty)$-continuous and thus $\mathfrak{T}^\infty - \|.\|_1$-continuous. An easy approximation argument shows that $f.\varphi \in L^1(\Lambda)$ holds for all $\varphi \in \mathring{\mathscr{B}}(\Lambda \cap \mathring{K}(j))$. By the closed graph theorem the latter fact implies $f \in \mathscr{O}_M((\mathring{\mathscr{B}}(\Lambda \cap \mathring{K}(j)), L^1(\Lambda))$. Since the equality

$$\mathscr{O}_M(\mathscr{P}(\Lambda, \Omega), L^1(\Lambda)) = \bigcap_{j \in \mathbb{N}} \mathscr{O}_M(\mathring{\mathscr{B}}(\Lambda \cap \mathring{K}(j)), L^1(\Lambda))$$

follows from (2·7) we have already proved

$$\begin{aligned}\mathscr{P}_{\mathrm{ar}}(\Lambda, \Omega) &\subset \{f \in L^1_{\mathrm{loc}}(\Lambda);\ f.\mathscr{P}(\Lambda, \Omega) \subset L^1(\Lambda)\} \\ &\subset \bigcap_{j \in \mathbb{N}} \mathscr{O}_M(\mathring{\mathscr{B}}(\Lambda \cap \mathring{K}(j)), L^1(\Lambda)) = \mathscr{O}_M(\mathscr{P}(\Lambda, \Omega), L^1(\Lambda)).\end{aligned}$$

From ([2], p. 310, (3·3)) we infer $\mathscr{O}_M(\mathscr{P}(\Lambda, \Omega), L^1(\Lambda)) \subset L^1_{\mathrm{loc}}(\Lambda)$. If $f \in \mathscr{O}_M(\mathscr{P}(\Lambda, \Omega), L^1(\Lambda))$ the linear map $[f] \colon \mathscr{D}(\Lambda) \to L^1(\Lambda)$, $[f](\varphi) = f.\varphi$, is $\mathfrak{R} \cap \mathscr{D}(\Lambda) - \|\cdot\|_1$-continuous which implies $f \in \mathscr{P}_{\mathrm{ar}}(\Lambda, \Omega)$. Thus all the above inclusions are in fact equalities.

The proof of the last equality in the statement of (3·3) is similar to the proof given in

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(3·1). To show the inclusion ' $\subset$ ' it is again sufficient to prove the estimates for the special compact sets $K(j)$ $(j \in \mathbb{N})$. Let $f \in \mathscr{P}_{ar}(\Lambda, \Omega)$ and $j \in \mathbb{N}$ be given. Then

$$f \in \mathscr{O}_M(\mathring{\mathscr{B}}(\Lambda \cap \mathring{K}(j+1)), L^1(\Lambda))$$

and the description of that space given in [(**2**], p. 321, (4·4)) shows the existence of $m \in \mathbb{N}_0$ and $g \in L^1(\Lambda)$ such that $\chi_{\Lambda \cap \mathring{K}(j+1)} \cdot f \cdot r^m = g$ holds, where

$$r(x) = \operatorname{dist}(x, \complement(\Lambda \cap \mathring{K}(j+1))).$$

As in the proof of (3·1) we now define $2\eta := \operatorname{dist}(\mathring{K}(j), \complement \mathring{K}(j+1))$, $M := \{x \in \Lambda \cap K(j);\ r(x) \geqslant \eta\}$, $N := \{x \in \Lambda \cap K(j);\ r(x) < \eta\}$, and $\beta := \sup\{\delta(x);\ x \in M\}$. The above representation then shows

$$\chi_M \cdot |f| = \chi_M \cdot |g| \cdot r^{-m} \leqslant \chi_M \cdot |g| \cdot \eta^{-m} \leqslant \chi_M \cdot |g| \cdot \left(\frac{\beta}{\eta}\right)^m \cdot \delta^{-m},$$

$$\chi_N \cdot |f| = \chi_N \cdot |g| \cdot r^{-m} \leqslant \chi_N \cdot |g| \, \delta^{-m},$$

and thus

$$\chi_{\Lambda \cap K(j)} \cdot |f| \leqslant \chi_{\Lambda \cap K(j)} \cdot |g| \cdot \left(1 + \left(\frac{\beta}{\eta}\right)^m\right) \cdot \delta^{-m}.$$

Since the verification of the remaining inclusion ' $\supset$ ' consists of an easy modification of the corresponding proof given in (3·1), we omit that part of the proof.

(3·4) *Remarks.* (*a*) In a similar way the measures $\mu \in \mathscr{M}(\Lambda)$ satisfying

$$\varphi \cdot \mu \in \mathscr{M}^1(\Lambda) \, (\varphi \in \mathscr{P}(\Lambda, \Omega))$$

may be characterized (cf. Schwartz [**12**], p. 243, Theorem VIII).

(*b*) It follows from the proof of (2·1) that a measure μ on Λ can be extended to a *measure* ν on Ω if and only if the estimates in (2·1) are true with $m = 0$. Similarly, a function $f \in \mathscr{P}_{ar}(\Lambda, \Omega)$ can be extended to a locally integrable function $g \in L^1_{loc}(\Omega)$ if and only if $\chi_{\Lambda \cap K} \cdot f \in L^1(\Lambda)$ holds for all compact subsets K of Ω.

(*c*) We take the opportunity to mention that the proof of (4·4) in ([**2**], p. 321) can be simplified considerably if we use the weight function γ of Voigt ([**15**], p. 345) instead of the modified boundary distance ρ.

I am grateful to Dr S. Dierolf for several helpful conversations on the subject of this paper. Special thanks are due to Professor Dr K. Floret, who called my attention to a result of De Wilde [**1**] which turned out to be very useful. Finally, I would like to thank the referee for pointing out that there had been an error in the application of Lemma (2·6).

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