Bull. Aust. Math. Soc. **113** (2026), 515–522
doi:10.1017/S0004972725100579

A NOTE ON THE OPEN MAPPINGS OF LOCALLY COMPACT GROUPS

ZOUHOUR JLALI

(Received 30 July 2025; accepted 18 August 2025; first published online 11 November 2025)

Abstract

Let G be a locally compact topological group and $\mathcal{L}(G)$ the space of all its closed subgroups endowed with the Vietoris topology. Let $\mathcal{L}_c(G)$ be the subspace of all compact subgroups of G. Any continuous morphism $\varphi\colon G \to H$ between locally compact groups G and H functorially induces a continuous map $\varphi_*\colon \mathcal{L}_c(G) \to \mathcal{L}_c(H)$ given by $\varphi_*(L) = \varphi(L)$. The main problem addressed in this paper is that of determining the relationship between the openness of φ and the openness of φ_*. For example, we show that if G is locally compact with compact identity component and H is locally compact and totally disconnected, then φ is open if and only if φ_* is open.

2020 *Mathematics subject classification*: primary 22D05.

Keywords and phrases: locally compact group, identity component, compactly ruled group, totally disconnected group, compact subgroup, open mapping theorem, Vietoris topology.

1. Introduction and main results

Let G be a locally compact topological group and $\mathcal{L}(G)$ the space of all its closed subgroups endowed with the Vietoris topology. An open prebase of the Vietoris topology on $\mathcal{L}(G)$ is formed by the sets

$$\mathrm{D}_1(U) = \{L \in \mathcal{L}(G) \mid L \subseteq U\},$$
$$\mathrm{D}_2(V) = \{L \in \mathcal{L}(G) \mid L \cap V \neq \emptyset\},$$

where U, V range over all open subsets of G.

A locally compact group G is said to be *compactly ruled* if any compact subset of G is contained in a compact subgroup. The subspace of all compact subgroups, open subgroups, compact open subgroups and compactly ruled subgroups of G are denoted respectively by $\mathcal{L}_c(G)$, $\mathcal{L}_o(G)$, $\mathcal{L}_{oc}(G)$ and $\mathcal{L}_{cr}(G)$. Clearly,

$$\mathcal{L}_{oc}(G) \subseteq \mathcal{L}_c(G) \subseteq \mathcal{L}_{cr}(G).$$

We denote the closure of a subset A of a topological space X by $\overline{A}$. Our first result gives a characterisation of compactly ruled groups in terms of the Vietoris space $\mathcal{L}(G)$.

© The Author(s), 2025. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.

https://doi.org/10.1017/S0004972725100579 Published online by Cambridge University Press

PROPOSITION (PROPOSITION 2.6 BELOW). *Let G be a locally compact group. Then, the following conditions are equivalent:*

(1) *G is compactly ruled;*
(2) *$\overline{\mathcal{L}_c(G)} = \mathcal{L}(G)$;*
(3) *$\overline{\mathcal{L}_{oc}(G)} = \{L \in \mathcal{L}(G) \mid G_0 \subseteq L\}$;*
(4) *$G \in \overline{\mathcal{L}_{oc}(G)}$;*
(5) *$G \in \mathcal{L}_c(G)$.*

REMARK 1.1. The Vietoris topology τ_v on the set of closed subgroups $\mathrm{Sub}(G)$ of a locally compact group G is finer than the Chabauty topology τ_c on $\mathrm{Sub}(G)$. Moreover, if F is a subset of $\mathrm{Sub}(G)$, then the group G belongs to the closure of F in $(\mathrm{Sub}(G), \tau_v)$ if and only if G belongs to the closure of F in $(\mathrm{Sub}(G), \tau_c)$. However, the above proposition remains true if the Vietoris topology is replaced with the Chabauty topology on $\mathrm{Sub}(G)$ (compare with [11, Theorem 3.18]).

NOTATION 1.2. The identity component of a topological group G, denoted by G_0, is the connected component of the identity element in G.

Let G and H be two locally compact groups. For every topological morphism $\varphi\colon G \to H$, let us denote by φ_* the induced function $\varphi_*\colon \mathcal{L}_c(G) \to \mathcal{L}_c(H)$ defined by $L \mapsto \varphi(L)$. Clearly, φ_* is continuous [4]. We are interested in the relationship between the openness of φ and the openness of φ_* (see [2, Question 1.3] and [5, page 859]). The main results are the following.

THEOREM A (THEOREM 3.10 BELOW). *Let G be a locally compact group and H a locally compact group with compact identity component (for example, a totally disconnected group or a compactly ruled group). Let $\varphi\colon G \to H$ be a continuous morphism. Then, φ is open if the following conditions are satisfied:*

(1) *$\varphi_*\colon \mathcal{L}_c(G) \to \mathcal{L}_c(H)$ is open;*
(2) *there is a compact open subgroup A of G such that $H_0 \subseteq \varphi(A)$.*

THEOREM B (THEOREM 3.11 BELOW). *Let G, H be locally compact groups with compact identity components and $\varphi\colon G \to H$ a continuous morphism. Then, the following conditions are equivalent:*

(1) *φ is open;*
(2) *φ_* is open and $H_0 \subseteq \bigcap_{L \in \mathcal{L}_{oc}(G)} \varphi(L)$;*
(3) *φ_* is open and there is $L \in \mathcal{L}_{oc}(G)$ such that $H_0 \subseteq \varphi(L)$.*

In particular, the following corollary holds.

COROLLARY. *Let G be a locally compact group with compact identity component, H a locally compact totally disconnected group and $\varphi\colon G \to H$ a continuous morphism. Then, φ is open if and only if φ_* is open.*

https://doi.org/10.1017/S0004972725100579 Published online by Cambridge University Press

2. Space of compact subgroups

If A is a closed subgroup of a topological group G, then $[A, G]$ denotes the lattice of all closed subgroups of G containing A: $[A, G] = \{L \in \mathcal{L}(G) \mid A \subseteq L\}$. We begin with the following elementary (and probably well-known) lemma which will be very useful.

LEMMA 2.1. *Let G be a locally compact group and N a closed normal subgroup of G. Then, the canonical projection $\pi\colon G \to G/N$ induces a bijective continuous map $\pi_\bullet\colon [N, G] \to \mathcal{L}(G/N)$ defined by $H \mapsto \pi(H)$. Moreover, if N is compact, then $\pi_\bullet$ is a homeomorphism with continuous inverse $\pi^\bullet\colon \mathcal{L}(G/N) \to [N, G]$ given by $L \mapsto \pi^{-1}(L)$.*

PROOF. The continuity of $\pi_\bullet$ follows from the fact that, for every open subset V of G/N, we observe that $\pi_\bullet^{-1}(\mathrm{D}_1(V)) = \mathrm{D}_1(\pi^{-1}(V)) \cap [N, G]$ and $\pi_\bullet^{-1}(\mathrm{D}_2(V)) = \mathrm{D}_2(\pi^{-1}(V)) \cap [N, G]$. However, if N is compact, then π is a closed map and the continuity of $\pi^\bullet$ follows from the fact that $(\pi^\bullet)^{-1}(\mathrm{D}_1(U)) = \mathrm{D}_1(G/N \setminus \pi(G \setminus U))$ and $(\pi^\bullet)^{-1}(\mathrm{D}_2(U)) = \mathrm{D}_2(\pi(U))$ for every open subset U of G. □

REMARK 2.2. In general, the map $\pi^\bullet$ is not continuous (see [12, Example 1]).

PROPOSITION 2.3 [8, Proposition 1.3]. *For a locally compact group G, the following properties are equivalent:*

(1) *G is compactly ruled;*
(2) *every finite subset of G is contained in a compact subgroup;*
(3) *every compact subset of G is contained in a compact open subgroup;*
(4) *every open compactly generated subgroup of G is compact;*
(5) *G is a directed union of compact open subgroups;*
(6) *G_0 is compact and G/G_0 is compactly ruled.*

PROPOSITION 2.4 (Protasov, [13]; see also [15, Theorem 15]). *For every locally compact group G, $\mathcal{L}_{cr}(G)$ is the closure of $\mathcal{L}_c(G)$ in $\mathcal{L}(G)$.*

Combining [3, Theorem 3] and the above proposition gives the next result.

COROLLARY 2.5. *For any connected locally compact group G, $\mathcal{L}_c(G)$ is closed in $\mathcal{L}(G)$.*

Let G be a compactly ruled group and $(G_\alpha)_{\alpha\in I}$ a directed family of compact open subgroups of G covering G. Then,

$$\mathcal{L}_c(G) = \bigcup_{\alpha\in I} \mathcal{L}(G_\alpha),$$
$$\mathcal{L}_{oc}(G) = \bigcup_{\alpha\in I} \mathcal{L}_o(G_\alpha),$$
$$\mathcal{L}(G) = \overline{\mathcal{L}_c(G)} = \overline{\bigcup_{\alpha\in I} \mathcal{L}(G_\alpha)}.$$

Compactly ruled groups admit the following characterisation in terms of the Vietoris space $\mathcal{L}(G)$.

https://doi.org/10.1017/S0004972725100579 Published online by Cambridge University Press

PROPOSITION 2.6. *Let G be a locally compact group. Then, the following conditions are equivalent:*

(1) *G is compactly ruled;*
(2) *$\overline{\mathcal{L}_c(G)} = \mathcal{L}(G)$;*
(3) *$\overline{\mathcal{L}_{oc}(G)} = [G_0, G]$;*
(4) *$G \in \overline{\mathcal{L}_{oc}(G)}$;*
(5) *$G \in \overline{\mathcal{L}_c(G)}$.*

PROOF. (1) $\Rightarrow$ (2). As G is compactly ruled, any closed subgroup of G is also compactly ruled [8, Proposition 1.9(i)] and therefore, $\mathcal{L}_{cr}(G) = \mathcal{L}(G)$. Then, $\overline{\mathcal{L}_c(G)} = \mathcal{L}_{cr}(G) = \mathcal{L}(G)$.

(2) $\Rightarrow$ (1). This follows from Proposition 2.4.

(1) $\Rightarrow$ (3). Assume that G is compactly ruled and consider first the case when G is totally disconnected. Let $(G_\alpha)_{\alpha\in I}$ be a directed family of compact open subgroups of G covering G. Then, $\bigcup_{\alpha\in I}\mathcal{L}(G_\alpha)$ is dense in $\mathcal{L}(G)$ and, for every $\alpha \in I$, $\mathcal{L}_{oc}(G_\alpha)$ is dense in $\mathcal{L}(G_\alpha)$ (see [6, Lemma 3.2] or [7, Theorem 3.12]) and therefore, $\bigcup_{\alpha\in I}\mathcal{L}_{oc}(G_\alpha)$ is dense in $\mathcal{L}(G)$. Thus, $\mathcal{L}_{oc}(G)$ is dense in $\mathcal{L}(G)$. In the general case, consider the canonical projection $\pi\colon G \to G/G_0$. Since G_0 is compact, by Lemma 2.1, $\pi^\bullet$ is a homeomorphism of $\mathcal{L}(G/G_0)$ onto $[G_0, G]$. However, also since G_0 is compact, π is perfect, that is, for every compact subset $C \subseteq G/G_0$, the preimage $\pi^{-1}(C)$ is also compact [16, Lemma 32.8] and, therefore, $\pi^\bullet(\mathcal{L}_{oc}(G/G_0)) = \mathcal{L}_{oc}(G)$. Hence, $\overline{\mathcal{L}_{oc}(G)} = [G_0, G]$.

The implications (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial.

(5) $\Rightarrow$ (1). Let B be an open compactly generated subgroup of G. Let $(H_i)_{i\in I}$ be a net of compact subgroups with limit G. As B is open, the net $(H_i \cap B)_{i\in I}$ converges to B. Since the Vietoris topology is finer than the Chabauty topology, [17, Lemma 5.7] implies that B is compact. Consequently, any open compactly generated subgroup of G is compact, which proves that G is compactly ruled. □

3. The relationship between openness of φ and φ_*

The open mapping theorem for topological groups states that a surjective continuous morphism of a locally compact group G onto a locally compact group H is an open mapping if G is σ-compact [10, Theorem 1.6]. Recently, Cowling, Hofmann and Morris proved the following theorem [1, Theorem 2, Corollaries 7, 4].

THEOREM 3.1. *Let $\varphi\colon G \to H$ be a morphism of locally compact groups. Then, the following conditions are equivalent:*

(1) *φ is open;*
(2) *$\varphi(L)$ is open in H for all open subgroups L of G;*
(3) *$\varphi(L)$ is open in H for all compactly generated open subgroups L of G;*
(4) *there is a compactly generated open subgroup L of G such that $\varphi(L)$ is open in H;*
(5) *$\gamma\colon G/G_0 \to H/H_0$, given by $gG_0 \mapsto \varphi(g)H_0$, is open and $H_0 = \bigcap_{L\in\mathcal{L}_o(G)}\varphi(L)$.*

https://doi.org/10.1017/S0004972725100579 Published online by Cambridge University Press

PROPOSITION 3.2 [14, Lemma 2]. *Let φ be a continuous morphism of a compact group G onto a compact group H. Then, the induced map $\varphi_*\colon \mathcal{L}(G) \to \mathcal{L}(H)$ is open.*

REMARK 3.3. Another proof of this result in the case of profinite groups can be found in [5, Proposition 5].

In what follows, we often use the next obvious statement.

LEMMA 3.4. *Let $\varphi\colon G \to H$ be a topological morphism of locally compact groups and L a compact subgroup of G. Then, $\varphi_*(\mathcal{L}(L)) = \mathcal{L}(\varphi(L))$.*

PROOF. Let $B \in \varphi_*(\mathcal{L}(L))$ and let $A \in \mathcal{L}(L)$ such that $\varphi_*(A) = B$. As $A \subseteq L$, $\varphi(A) \subseteq \varphi(L)$ and so $B \in \mathcal{L}(\varphi(L))$. Hence, $\varphi_*(\mathcal{L}(L)) \subseteq \mathcal{L}(\varphi(L))$. Conversely, let $B \in \mathcal{L}(\varphi(L))$. We have $B = B \cap \varphi(L) = \varphi(\varphi^{-1}(B) \cap L) = \varphi_*(A)$, where $A = \varphi^{-1}(B) \cap L \in \mathcal{L}(L)$. Hence, $B \in \varphi_*(\mathcal{L}(L))$ and, therefore, $\mathcal{L}(\varphi(L)) \subseteq \varphi_*(\mathcal{L}(L))$. □

A locally compact group G is *periodic* if any element belongs to a compact subgroup.

PROPOSITION 3.5 (Subgroup with open Vietoris subspace). *Let G be a locally compact group and H a closed subgroup of G.*

(1) *$\mathcal{L}(H)$ is homeomorphic to the closed subspace $\{L \in \mathcal{L}(G) \mid L \subseteq H\}$ of $\mathcal{L}(G)$.*
(2) *If $\mathcal{L}(H)$ is open in $\mathcal{L}(G)$, then $\mathcal{L}(H \cap A)$ is open in $\mathcal{L}(A)$ for every closed subgroup A of G.*
(3) *If H is open in G, then $\mathcal{L}(H)$ is open in $\mathcal{L}(G)$.*
(4) *If G is totally disconnected, then H is open in G if and only if $\mathcal{L}(H)$ is open in $\mathcal{L}(G)$.*
(5) *If G is periodic and if $\mathcal{L}(H)$ is open in $\mathcal{L}(G)$, then HG_0 is open in G.*

PROOF. (1) See [12, Lemma 1].

(2) This follows from the continuity of the map $\mathcal{L}(A) \to \mathcal{L}(G)$ defined by $L \mapsto L$.

(3) If H is open in G, then $\mathrm{D}_1(H) = \mathcal{L}(H)$ and so $\mathcal{L}(H)$ is open in $\mathcal{L}(G)$.

(4) Suppose $\mathcal{L}(H)$ is open in $\mathcal{L}(G)$. As the trivial subgroup $\{e\} \in \mathcal{L}(H)$, there exists a compact subgroup U of G such that $\{e\} \in \mathrm{D}_1(U) \subseteq \mathcal{L}(H)$. Then, $U \subseteq H$ and so H is open in G.

(5) We first consider the situation when G is compact. Let $\pi\colon G \to G/G_0$ be the canonical projection. Since the induced map π_* is open, $\pi_*(\mathcal{L}(H)) = \mathcal{L}(\pi(H))$ is open in $\mathcal{L}(G/G_0)$ and therefore, by condition (4), $\pi(H)$ is open in G/G_0. Hence, HG_0 is open in G. Consider now the case when G is periodic. Let $(G_i)_{i\in I}$ be a family of compact open subgroups of G covering G. For any $i \in I$, by condition (2), $\mathcal{L}(H \cap G_i)$ is open in $\mathcal{L}(G_i)$ and hence, $(H \cap G_i)G_0$ is open in G_i and so in G. Then, $HG_0 = \bigcup_{i\in I}(H \cap G_i)G_0$ is open in G. □

REMARK 3.6. As is well known, the Lie groups are precisely the locally compact groups without arbitrary small subgroups, and they admit the following characterisation in terms of the Vietoris space $\mathcal{L}(G)$: G is a Lie group if and only if the trivial

https://doi.org/10.1017/S0004972725100579 Published online by Cambridge University Press

subgroup $\{e\} = \mathcal{L}(\{e\})$ is open in $\mathcal{L}(G)$. In general, $\mathcal{L}(H)$ open in $\mathcal{L}(G)$ does not imply that H is open in G.

Proposition 3.2 can be improved slightly as follows.

PROPOSITION 3.7. *Let φ be an open continuous morphism of a compact group G into a locally compact group H. Then, the induced map $\varphi_*\colon \mathcal{L}(G) \to \mathcal{L}_c(H)$ is open.*

PROOF. Since $\varphi(G)$ is open in H, by Proposition 3.5(3), $\varphi_*(\mathcal{L}(G)) = \mathcal{L}(\varphi(G))$ is open in $\mathcal{L}(H)$ and so in $\mathcal{L}_c(H)$. As $\varphi_*\colon \mathcal{L}(G) \to \varphi_*(\mathcal{L}(G))$ is open, $\varphi_*\colon \mathcal{L}(G) \to \mathcal{L}_c(H)$ is open. □

PROPOSITION 3.8. *In a totally disconnected locally compact group, every compact subgroup is contained in a compact open subgroup.*

PROOF. See [9, proof of Lemma 1.16] or [18, Lemma 7]. □

PROPOSITION 3.9. *Let G be a locally compact group with compact identity component G_0. Then, the following hold.*

(1) *Every compact subgroup of G is contained in a compact open subgroup.*
(2) $\mathcal{L}_c(G) = \bigcup_{H \in \mathcal{L}_{oc}(G)} \mathcal{L}(H)$.
(3) *$\mathcal{L}_c(G)$ is open in $\mathcal{L}(G)$.*
(4) *Let H be a compact subgroup of G. If $\mathcal{L}(H)$ is open in $\mathcal{L}(G)$, then HG_0 is open in G.*

PROOF. (1) As G_0 is compact, the canonical projection $\pi\colon G \to G/G_0$ is perfect [16, Lemma 32.8]. The result follows from the fact that G/G_0 is a totally disconnected locally compact group and Proposition 3.8.

(2) Trivial.

(3) Follows from condition (2) and Proposition 3.5(3).

(4) Let O be a compact open subgroup of G containing the compact subgroup HG_0. By condition (2), $\mathcal{L}(H \cap O)$ is open in $\mathcal{L}(O)$ and hence, $(H \cap O)G_0$ is open in O and so in G. As $(H \cap O)G_0 = HG_0 \cap O = HG_0$, then HG_0 is open in G. □

THEOREM 3.10. *Let G be a locally compact group and H a locally compact group with compact identity component. Let $\varphi\colon G \to H$ be a continuous morphism. Then, φ is open if the following conditions are satisfied:*

(1) *$\varphi_*\colon \mathcal{L}_c(G) \to \mathcal{L}_c(H)$ is open;*
(2) *there is a compact open subgroup A of G such that $H_0 \subseteq \varphi(A)$.*

PROOF. By Proposition 3.5(3), $\mathcal{L}(A)$ is open in $\mathcal{L}(G)$ and so $\varphi_*(\mathcal{L}(A)) = \mathcal{L}(\varphi(A))$ is open in $\mathcal{L}_c(H)$. By Proposition 3.9(3), we deduce that $\mathcal{L}(\varphi(A))$ is open in $\mathcal{L}(H)$. As $H_0 \subseteq \varphi(A)$, by Proposition 3.9(4), $\varphi(A)$ is open in H and therefore, by Theorem 3.1, φ is open. □

https://doi.org/10.1017/S0004972725100579 Published online by Cambridge University Press

THEOREM 3.11. *Let G, H be locally compact groups with compact identity components and $\varphi\colon G \to H$ a continuous morphism. Then, the following conditions are equivalent:*

(1) *φ is open;*
(2) *φ_* is open and $H_0 \subseteq \bigcap_{L\in\mathcal{L}_{oc}(G)} \varphi(L)$;*
(3) *φ_* is open and there is $L \in \mathcal{L}_{oc}(G)$ such that $H_0 \subseteq \varphi(L)$.*

PROOF. (1) $\Rightarrow$ (2). For every $L \in \mathcal{L}_{oc}(G)$, the subgroup $\varphi(L)$ is open in H and so $H_0 \subseteq \varphi(L)$. Then, $H_0 \subseteq \bigcap_{L\in\mathcal{L}_{oc}(G)} \varphi(L)$. Let now Ω be an open subset of $\mathcal{L}_c(G)$. For $L \in \mathcal{L}_{oc}(G)$, in view of Proposition 3.2, the map $\varphi_*\colon \mathcal{L}(L) \to \mathcal{L}(\varphi(L)) = \varphi_*(\mathcal{L}(L))$ is open and therefore, $\varphi_*(\mathcal{L}(L) \cap \Omega)$ is open in $\mathcal{L}(\varphi(L))$ and so, by Proposition 3.5(3), in $\mathcal{L}(H)$ since $\varphi(L)$ is open in H. Thus, $\mathcal{L}(\varphi(L))$ is open in $\mathcal{L}_c(H)$. As $\Omega = \bigcup_{L\in\mathcal{L}_{oc}(G)}(\mathcal{L}(L) \cap \Omega)$, it follows that $\varphi_*(\Omega) = \bigcup_{L\in\mathcal{L}_{oc}(G)} \varphi_*(\mathcal{L}(L) \cap \Omega)$ is open in $\mathcal{L}_c(H)$, which proves the openness of φ_*.

Since $\mathcal{L}_{oc}(G) \neq \emptyset$, the implication (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Since $\mathcal{L}(L)$ is open in $\mathcal{L}_c(G)$, $\varphi_*(\mathcal{L}(L)) = \mathcal{L}(\varphi(L))$ is open in $\mathcal{L}_c(H)$ and so in $\mathcal{L}(H)$ (Proposition 3.9(3)). By Proposition 3.9(4), $\varphi(L)$ is open in H and the result follows from Theorem 3.1. □

In particular, we deduce the following corollary.

COROLLARY 3.12. *Let G be a locally compact group with compact identity component, H a locally compact totally disconnected group and $\varphi\colon G \to H$ a continuous morphism. Then, φ is open if and only if φ_* is open.*

REMARK 3.13. This result was proved by Cowling *et al*. for the case of profinite groups [1, Corollary 9].

References

[1] M. G. Cowling, K. H. Hofmann and S. A. Morris, 'Open mappings of locally compact groups', *J. Group Theory* **27** (2024), 1143–1149.
[2] B. Csikós, T. Kátay, A. Kocsis and M. Pálfy, 'Compact Lie groups isolated up to conjugacy', Preprint, 2022, arXiv:2209.15389.
[3] D. Z. Djocovic, 'The union of compact subgroups of a connected locally compact group', *Math. Z.* **158** (1978), 99–105.
[4] V. Donjuán, N. Jonard-Pérez and A. López-Poo, 'Some notes on induced functions and group actions on hyperspaces', *Topol. Appl.* **311** (2022), Article no. 107954.
[5] S. Fisher and P. Gartside, 'On the space of subgroups of a compact group II', *Topol. Appl.* **156** (2009), 855–861.
[6] P. Gartside and M. Smith, 'Classifying spaces of subgroups of profinite groups', *J. Group Theory* **13** (2010), 315–336.
[7] H. Hamrouni and Z. Jlali, 'A characterization of totally disconnected compactly ruled groups', *Proc. Math. Sci.* **131** (2021), 1–10.
[8] W. Herfort, K. H. Hofmann and F. G. Russo, *Periodic Locally Compact Groups*, De Gruyter Studies in Mathematics, 71 (De Gruyter, Berlin, 2019).
[9] K. H. Hofmann, 'Locally compact semigroups in which a subgroup with compact complement is dense', *Trans. Amer. Math. Soc.* **106** (1963), 19–51.

https://doi.org/10.1017/S0004972725100579 Published online by Cambridge University Press

[10] K. H. Hofmann and S. A. Morris, 'Open mapping theorem for topological groups', *Topology Proc.* **31** (2007), 533–551.
[11] B. Kadri, 'A characterization of compactly ruled groups', *Monatsh. Math.* **207** (2025), 293–300.
[12] I. V. Protasov, 'Topological groups with compact lattice of closed subgroups', *Sib. Math. J.* **20** (1979), 270–275.
[13] I. V. Protasov, 'Limits of compact subgroups of a topological groups', *Dopov. AN Ukrain. SSR. Ser. A* **N5** (1986), 64–66.
[14] I. V. Protasov, 'Souslin number of the space of subgroups of a locally compact group', *Ukr. Math. J.* **40** (1988), 559–562.
[15] I. V. Protasov, 'Selective survey on spaces of closed subgroups of topological groups', *Axioms* **7**(4) (2018), Article no. 75.
[16] M. Stroppel, *Locally Compact Groups*, EMS Textbooks in Mathematics (European Mathematical Society (EMS), Zürich, 2006).
[17] S. P. Wang, 'On the Limit of subgroups in a group', *Amer. J. Math.* **92** (1970), 708–724.
[18] G. Willis, 'The structure of totally disconnected locally compact groups', *Math. Ann.* **300** (1994), 341–363.

ZOUHOUR JLALI, Department of Mathematics,
Faculty of Sciences at Sfax, Sfax University, B.P. 1171. 3000 Sfax, Tunisia
e-mail: zouhourjlali2@gmail.com

https://doi.org/10.1017/S0004972725100579 Published online by Cambridge University Press