Proc. Camb. Phil. Soc. (1966), **62**, 33
PCPS 62–9
Printed in Great Britain

On convex sets and measures

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(*Received* 25 *March* 1965)

Introduction. If α is a positive real number then, for each set E in R_3, we define

$$\Lambda_\rho^\alpha(E) = \inf_{U(\rho,E)} \sum_{U \in U(\rho,E)} A_\alpha (d(U))^\alpha,$$

where $U(\rho, E)$ is any countable collection of convex sets, each of diameter at most ρ, whose union covers E, and A_α is a positive real number. Then the convex α-dimensional measure $\Lambda^\alpha(E)$, of E, is $\lim_{\rho \to 0} \Lambda_\rho^\alpha(E)$. In this paper we shall only be considering the cases $\alpha = 1, 2$ where, as is usual, we take $A_1 = 1$ and $A_2 = \frac{1}{4}\pi$. The symbols $\Lambda_s^2(E)$, $\Lambda_s(E)$ will denote the spherical (circular) measure of E, i.e. when the coverings are restricted to being collections of spheres (circles), and $E(z)$ is the intersection of a set E (in R_3) with the plane (x, y) at z.

Besicovitch has shown ((1), (2)) that

$$(a) \quad \Lambda_s^2(E) \geqslant \int \Lambda_s E(z)\, dz,$$

and

$$(b) \quad \Lambda^2(E) \geqslant \tfrac{1}{4}\pi \int \Lambda E(z)\, dz,$$

where the indefinite integral is used, as throughout this paper, to indicate that the integration is taken over all z for which the integrand is non-zero. In recent work ((3)) one of us has shown that (a) is false, in general, for convex measures and in this paper we extend this result by proving

$$(c) \quad \Lambda^2(E) \geqslant \phi \int \Lambda E(z)\, dz,$$

where $\phi = \frac{1}{4}\pi((\frac{1}{4}\pi + \frac{1}{2})/\sqrt{2})^{-1} \doteq 0{\cdot}8640$, and this result is the best possible.

We begin by proving a result for convex sets (Theorem 1) which is of some interest independent of the main result.

2. *A result for convex sets in* R_3.

Lemma 1. *Suppose that* AOB, CQD *are line segments, centres* O *and* Q *which lie entirely in the planes* $z = z_1$, $z = z_2$ *respectively. Then* $\sup RS$, *where the supremum is taken over all pairs of points* R, S, *with* $R \in AOB$, $S \in CQD$, *has its minimum value when* Q *is directly above* O *and* CQD *is perpendicular to* AOB. *Further, in this position,* $(\sup RS)^2 = a^2 + b^2 + (z_1 - z_2)^2$, *where* $AB = 2a$ *and* $CD = 2b$.

Proof. It is evident that the suggested minimum configuration does indeed yield $(\sup RS)^2 = a^2 + b^2 + (z_1 - z_2)^2$ and thus it suffices to show that in a general position

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one of AC^2, AD^2, BC^2, BD^2 is not less than $a^2+b^2+(z_1-z_2)^2$. The details are trivial and will be omitted.

THEOREM 1. *Given any convex set E in R_3,*

(i) $(\frac{1}{4}\pi+\frac{1}{2})/\sqrt{2}\,(d(E))^2 \geqslant \int d(E(z))\,dz$, *and*

(ii) *this bound is attained.*

Proof. (i) We can suppose without loss of generality that the set E has diameter one and lies between the planes $z=0$ and $z=1$. Consider the graph of $(d(E(z)), z)$. Let E^* denote the set of all ordered pairs (w,u) which satisfy $0 \leqslant w \leqslant d(E(z))$, $0 \leqslant u \leqslant 1$. Then E^* is contained in the unit square $0 \leqslant w \leqslant 1$, $0 \leqslant u \leqslant 1$. If λ is any real number in the unit interval, then we use β to denote the supremum of all δ such that the rectangle $(1-\lambda, 1-\delta), (1-\lambda, 1), (1, 1-\delta), (1,1)$ does not meet E^*. Similarly γ denotes the supremum of all δ such that the rectangle $(1-\lambda, \delta), (1-\lambda, 0), (1, \delta), (1, 0)$ does not meet E^*. By the definition of β, given any positive ϵ, there exists z_1, such that

$$1-\beta-\epsilon \leqslant z_1 \leqslant 1-\beta,$$

and points P_1, Q_1, of $E(z_1)$, such that $P_1Q_1 \geqslant 1-\lambda$. Similarly, there exists z_2, so that $\gamma \leqslant z_2 \leqslant \gamma+\epsilon$, and points P_2, Q_2, of $E(z_2)$, so that $P_2Q_2 \geqslant 1-\lambda$. As the distance between any two of the points P_1, Q_1, P_2, Q_2 is at most one, we can use Lemma 1 to deduce that

$$\begin{aligned} 1 &\geqslant (1-2\epsilon-(\beta+\gamma))^2+\tfrac{1}{4}((P_1Q_1)^2+(P_2Q_2)^2) \\ &\geqslant (1-2\epsilon-(\beta+\gamma))^2+\tfrac{1}{2}(1-\lambda)^2. \end{aligned}$$

Hence
$$\beta+\gamma \geqslant 1-2\epsilon-\sqrt{(1-\tfrac{1}{2}(1-\lambda)^2)}$$

and, as this is true for all positive ϵ, we deduce that

$$\beta+\gamma \geqslant 1-\sqrt{(1-\tfrac{1}{2}(1-\lambda)^2)}. \tag{1}$$

Now
$$\begin{aligned} \int d(E(z))\,dz &\leqslant 1-\int_0^1 (\beta+\gamma)\,d\lambda \\ &\leqslant 1-\int_0^1 (1-\sqrt{(1-\tfrac{1}{2}(1-\lambda)^2)})\,d\lambda \quad \text{(by (1))} \\ &= \left(\frac{\pi}{4}+\frac{1}{2}\right)\Big/\sqrt{2}. \end{aligned}$$

This completes the proof of (i).

Proof. (ii) We need to construct a closed convex set E, of diameter one, so that

$$\int d(E(z))\,dz = \left(\frac{\pi}{4}+\frac{1}{2}\right)\Big/\sqrt{2}. \tag{2}$$

In order to obtain an extremal figure we need such a figure to give equality in (1) for all λ in the unit interval. And so it is not surprising that we can take E to be the convex hull of the continuous curve E', which is defined by the equations

$$x=-y, \quad z=-\tfrac{1}{2}\sqrt{(1-2(x^2+y^2))} \quad \text{for} \quad -\tfrac{1}{2} \leqslant z \leqslant -\frac{1}{2\sqrt{2}};$$

$$x^2+y^2=\tfrac{1}{4}, \quad 2|x| = 1-\alpha z^2 \quad \text{for} \quad -\frac{1}{2\sqrt{2}} \leqslant z \leqslant \frac{1}{2\sqrt{2}},$$

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where $xy \geqslant 0$ for $z \geqslant 0$ and $xy \leqslant 0$ for $z \leqslant 0$;

$$x = y, \quad z = \tfrac{1}{2}\sqrt{(1-2(x^2+y^2))} \quad \text{for} \quad \frac{1}{2\sqrt{2}} \leqslant z \leqslant \tfrac{1}{2};$$

where, for symmetry about the plane $z = 0$, we take $\alpha = 8-4\sqrt{2}$. It is easy to verify that E' satisfies (2) and so it only remains to show that $d(E') = 1$. There are, essentially, six possibilities for the relative positions of two points of E'. We need to show that

(a) two points of the spiral are at most one apart when (i) the z-coordinates of the points have the same sign, and (ii) the z-coordinates of the points have different signs;

(b) (i) two points on the same cap are at most one apart, and (ii) two points on different caps are at most one apart;

(c) a point on the spiral and a point on a cap are at most one part (i) if the z-coordinates of the points have the same sign, and (ii) if the z-coordinates are of different signs.

Proof of a(i). Let z_1, z_2 have the same sign and satisfy $|z_i| \leqslant \dfrac{1}{2\sqrt{2}}$ for $i = 1, 2$. If the diameter of $E'(z_i)$ makes an angle θ_i with the plane $y = 0$, then, for L in $E'(z_1)$ and M in $E'(z_2)$, we have

$$(LM)^2 \leqslant \tfrac{1}{2} + \tfrac{1}{2}\cos(\theta_1 - \theta_1) + (z_1 - z_2)^2.$$

And so we need to show that

$$1 - 2(z_1 - z_2)^2 \geqslant \cos(\theta_1 - \theta_2).$$

However, since $\alpha > 2$, it is sufficient to show that

$$\begin{aligned} 1 - \alpha(z_1 - z_2)^2 &\geqslant \cos(\theta_1 - \theta_2) \\ &= (1 - \alpha z_1^2)(1 - \alpha z_2^2) + \alpha z_1 z_2 (2 - \alpha z_1^2)^{\frac{1}{2}} (2 - \alpha z_2^2)^{\frac{1}{2}}, \end{aligned}$$

which reduces to $(z_1 - z_2)^2 \geqslant 0$.

Proof of a(ii). If, in the proof of a(i), z_1 and z_2 have opposite signs, then

$$(LM)^2 \leqslant \tfrac{1}{2} + \tfrac{1}{2}\cos(\theta_1 + \theta_2) + (z_1 - z_2)^2.$$

We can suppose without loss of generality that $z_2 = -z_1 t$, where $0 < t \leqslant 1$, and so we need to show that

$$1 \geqslant 2z_1^2(1+t)^2 + (1 - \alpha z_1^2)(1 - \alpha z_1^2 t^2) - \alpha z_1^2 t (2 - \alpha z_1^2)^{\frac{1}{2}} (2 - \alpha z_1^2 t^2)^{\frac{1}{2}}. \tag{3}$$

Let $g(t)$ denote the right-hand side of (3). Then $g(1) = 1 + (8 - 4\alpha + 2\alpha^2 z_1^2) z_1^2$ and we need to show that

$$g(1) - g(t) + (4\alpha - 8 - 2\alpha^2 z_1^2) z_1^2 \geqslant 0,$$

which reduces on noting that $(2 - \alpha z_1^2 t^2)^{\frac{1}{2}} (2 - \alpha z_1^2)^{-\frac{1}{2}} \geqslant 1$, to showing that

$$(\alpha - 2) z_1^2 (1-t)^2 + z_1^2 t (4\alpha - 8 - (1+t)\alpha^2 z_1^2) \geqslant 0.$$

Which, since $2\alpha - 4 = \frac{1}{8}\alpha^2 \geqslant \alpha^2 z_1^2$, is trivial. The proof of b(i) is trivial.

Proof of b(ii). Suppose that L and M are points of different caps and are at distances a and b respectively from the z axis. Then

$$\begin{aligned} (LM)^2 &\leqslant a^2 + b^2 + \tfrac{1}{4}((1 - 2a^2)^{\frac{1}{2}} + (1 - 2b^2)^{\frac{1}{2}})^2 \\ &\leqslant 1, \text{ since } (a^2 - b^2)^2 \geqslant 0. \end{aligned}$$

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Proof of c(i). Suppose that L is a point of the cap which lies in the half space $z \geqslant 0$ and that L is a distance a from the z-axis. If $0 \leqslant z \leqslant \dfrac{1}{2\sqrt{2}}$, let M be a point of $E'(z)$ and suppose that the diameter of $E'(z)$ makes an acute angle θ with the plane $y = 0$. Then

$$(LM)^2 \leqslant \tfrac{1}{4} + a^2 + a\cos(\tfrac{1}{4}\pi - \theta) + (\tfrac{1}{2}(1-2a^2)^{\frac{1}{2}} - z)^2. \tag{4}$$

If $h(a)$ denotes the right-hand side of (4), then $(d/da)\,h(a) \geqslant 0$, for $a \geqslant 0$ and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$. Hence, we need only consider the case $a = \frac{1}{2}$ which has been dealt with in a(i).

Proof of c(ii). Suppose, with the same notation as in the proof of c(i), that L lies in the half space $z \geqslant 0$ and M lies in the half space $z \geqslant 0$. Then

$$(LM)^2 \leqslant \tfrac{1}{4} + a^2 + a\cos(\tfrac{1}{4}\pi + \theta) + (\tfrac{1}{2}(1-2a^2)^{\frac{1}{2}} + |z|)^2. \tag{5}$$

Then, if the right-hand side of (5) is denoted by $j(a)$, $(d/da)\,j(a) \geqslant 0$ for $z \leqslant \dfrac{1}{2\sqrt{2}}$, $0 \leqslant a \leqslant \frac{1}{2}$ and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$. Hence again it is sufficient to consider the case $a = \frac{1}{2}$ which has been dealt with in a(ii).

This completes the proof of Theorem 1. However, it is of interest to note the following characterizations of the extremal figures of Theorem 1. Suppose that E has diameter one and is an extremal figure for Theorem 1. Then

(i) there exist real numbers z_1 and z_2, with $z_1 - z_2 = \frac{1}{2}$, so that, if $z_2 \leqslant z \leqslant z_1$, then $d(E(z)) = 1$. Further, the diameters of $E(z_1)$ and $E(z_2)$ are unique and take the minimal configuration described in Lemma 1;

(ii) if, for $0 \leqslant r \leqslant 1$, $z_3 = \inf\limits_{z \leqslant z_3} (z\colon d(E(z)) \geqslant r)$ and

$$z_4 = \sup_{z \geqslant z_1} (z\colon d(E(z)) \geqslant r), \quad \text{then} \quad z_4 - z_3 = \sqrt{(1 - \tfrac{1}{2}r^2)}.$$

3. THEOREM 2. *Let E be a bounded set of positive finite two-dimensional convex measure, then*

$$\Lambda^2 E \geqslant \phi \int \Lambda E(z)\,dz.$$

Proof. Given any $\rho > 0$ let u be an element of a $U(\rho, E)$. Then by Theorem 1

$$\tfrac{1}{4}\pi(du)^2 \geqslant \phi \int d(u(z))\,dz$$

and hence

$$\sum_{U(\rho,E)} \tfrac{1}{4}\pi(du)^2 \geqslant \phi \int \sum_{U(\rho,E)} d(u(z))\,dz$$

$$\geqslant \phi \int \Lambda_\rho E(z)\,dz.$$

Hence $\Lambda_\rho^2 E \geqslant \phi \int \Lambda_\rho E(z)\,dz$. Letting $\rho \to 0$ we have the required result.

THEOREM 3. *Given any $\mu > 0$ there exists a set (E^μ say) of finite positive two-dimensional measure such that*

$$\Lambda^2 E^\mu < (\phi + \mu) \int \Lambda E^\mu(z)\,dz.$$

Proof. We construct such a set.

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We take a set of the type constructed in Theorem 1, place in it a large number of smaller similar shapes and so on in a way analogous to the techniques developed by Besicovitch and others for plane sets. Let E_0 be the set E constructed in Theorem 1.

E_0^μ is that part of E_0 contained between the planes $z = f(\mu) - \frac{1}{2}$ and $z = \frac{1}{2} - f(\mu)$, where $f(\mu)$ is a positive function of μ satisfying

$$\int_{z=-\frac{1}{2}}^{z=\frac{1}{2}} d(E_0(z))\,dz = \int_{z=f(\mu)-\frac{1}{2}}^{z=\frac{1}{2}-f(\mu)} d(E_0(z))\,dz + \tfrac{1}{2}\mu. \tag{6}$$

We will put $z_0' = (0, 0, f(\mu) - \frac{1}{2})$, $z_0'' = (0, 0, \frac{1}{2} - f(\mu))$ and call the line $z_0' z_0''$ the axis of E_0^μ.

We now show that for each z, $E_0^\mu(z)$ contains a rhombus whose major diagonal equals $d(E_0^\mu(z))$ and whose minor diagonal is $k(\mu)$ times the length of the major diagonal where $k(\mu)$ is some positive number independent of z. Let $P(z), Q(z)$ be points on the boundary of $E_0^\mu(z)$ such that $P(z)\,Q(z) = d(E_0^\mu(z))$. $p(z), q(z)$ are points on the boundary of $E_0^\mu(z)$ such that $p(z)\,q(z)$ is the perpendicular bisector of $P(z)\,Q(z)$.

LEMMA 2. *There is a positive number $k = k(\mu)$ such that*

$$\frac{p(z)\,q(z)}{P(z)\,Q(z)} \geqslant k(\mu) > 0 \quad \textit{for all} \quad z_0' \leqslant z \leqslant z_0''.$$

Proof. It is evident from the construction that this ratio has its minimum value at $z = z_0'$ and $z = z_0''$, and that it is positive at these points provided $\mu > 0$. We put this minimum value equal to $k(\mu)$.

We now subdivide the axis of E_0^μ and for simplicity will consider only $z \geqslant 0$. The fineness of the subdivision will depend ultimately on μ: for the time being, however, we will use any positive constant ϵ such that

$$\epsilon < \frac{\sqrt{(1+k^2)} - 1}{6}\, d(E_0^\mu(z_0'')). \tag{7}$$

For $0 \leqslant z \leqslant \dfrac{1}{2\sqrt{2}}$, $P(z)\,Q(z) = 1$ and we take an equal subdivision,

$$0 = z^1 < z^2 < \ldots < z^r = \frac{1}{2\sqrt{2}},$$

where r is chosen as follows: Let L^i, K^i be points on $P(z^i)\,Q(z^i)$ such that

$$K^i P(z^i) = L^i Q(z^i) = \tfrac{1}{2}\epsilon,$$

let R^i be a rectangle base $L^i K^i$ height $z^{i+1} z^i$, choose r^i so that all R^i $(0 \leqslant i < r)$ are contained in E_0^μ. For $\dfrac{1}{2\sqrt{2}} \leqslant z \leqslant z_0''$ we take an equal subdivision,

$$\frac{1}{2\sqrt{2}} = z^r < \ldots < z^n = z_0'',$$

where n is chosen so that

$$\max_{(z^i \leqslant z \leqslant z^{i+1})} (\text{variation } P(z)\,Q(z), \quad \text{variation}_{(z^i \leqslant z \leqslant z^{i+1})} p(z)\,q(z)) \leqslant \epsilon.$$

For each $r \leqslant i < n$, R^i will be the rectangle constructed on $P(z^i)\,Q(z^i)$, base

$$P(z^{i+1})\,Q(z^{i+1}),$$

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height $z^{i+1}-z^i$. By the construction we have for all $1 \leqslant i < n$, and each z in

$$z^i \leqslant z \leqslant z^{i+1},$$

$$d(R^i(z)) \geqslant d(E_0^\mu(z)) - \epsilon. \tag{8}$$

We bisect each R^i with a rectangle r^i perpendicular to R^i and base $k = k(\mu)$ times the base of R^i. *Let P^i be the convex hull of R^i and r^i and let the four rectangular faces of P^i be F_j^i $(1 \leqslant j \leqslant 4)$.*

LEMMA 3. (1) *Each F_j^i is contained in E_0^μ,*

(2) $d(F_j^i(z)) > \frac{1}{2}(d(E_0^\mu(z))) + 2\epsilon$.

Proof. (1) follows from Lemma 2 and the construction of the F_j^i. We prove (2).

$$\begin{aligned}
d(F_j^i(z)) &= \sqrt{\left(\left(\frac{d(R^i(z))}{2}\right)^2 + \left(\frac{d(r^i(z))}{2}\right)^2\right)} \\
&= \frac{d(R^i(z))}{2}\sqrt{(1+k^2)} \\
&\geqslant \frac{d(E_0^\mu(z)) - \epsilon}{2}\sqrt{(1+k^2)} \quad \text{(by (8))} \\
&= \frac{d(E_0^\mu(z))}{2} + \frac{\sqrt{(1+k^2)}-1}{2} d(E_0^\mu(z)) - \frac{\epsilon}{2}\sqrt{(1+k^2)} \\
&< \tfrac{1}{2} d(E_0^\mu(z)) + 3\epsilon \frac{d(E_0^\mu(z))}{d(E_0^\mu(z_0''))} - \epsilon \quad \text{(by (7))} \\
&\geqslant \tfrac{1}{2} d(E_0^\mu(z)) + 2\epsilon.
\end{aligned}$$

For each z in $z^i \leqslant z \leqslant z^{i+1}$, let $V_j^i(z)$ be the segment of $F_j^i(z)$ distant ϵ from the endpoint of $F_j^i(z)$ that is nearer to one of $P(z)$, $Q(z)$ and of length $\frac{1}{2}(dE_0^\mu(z))$. *Let*

$$V_j^i = V_j^i(z) \times (z^i \leqslant z \leqslant z^{i+1}) \quad \textit{for} \quad 1 \leqslant j \leqslant 4 \quad \textit{and all} \quad 1 \leqslant i < n.$$

We construct a symmetric set of $4n$ V_j^i's in the lower half of E_0^μ. These $8n$ plane figures V_j^i will be called the vanes of E_0^μ and it is these vanes that form the 'scaffold' on which we construct a set of non-overlapping sets each similar to E_0^μ.

Properties of the vanes of E_0^μ. It must be emphasized that throughout this construction many details have been omitted. The justification for this lies in (3) where a completely specified construction and full proofs are given for an analogous construction.

Property v 1. For each i the four vanes V_j^i are some positive distance apart and a positive distance from the curved surface of E_0^μ. *Let the least of all these distances (for all i and j) be β_1.*

Property v 2. $d(V_j^i(z)) = \frac{1}{2} d(E_0^\mu(z))$ for $z^i \leqslant z \leqslant z^{i-1}$, all i and j.

Property v 3. The total surface area of all vanes

$$= 2\int_{z_0'}^{z''} d(E_0^\mu(z))\, dz.$$

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The term 'E-shape' will mean a set similar to E_0^μ with its axis parallel to the axis of E_0^μ. Given any E-shape, E^* say, the associated H-shape (H^*) is a plane figure symmetric about and containing the axis of E^* such that for all z, $d(H^*(z)) = 2d(E^*(z))$. We pack the vanes with H-shapes and construct on each the corresponding E-shape. Consider then a vane V_j^i. By the Vitali covering theorem we can find a finite set of non-overlapping H-shapes contained in V_j^i whose total area is as near as we please to the area of V_j^i and such that $d(H) \leqslant 2\beta_1$, i.e. we can find a finite set of non-overlapping H-shapes, $\mathscr{H}$ say, with the following properties:

Property $h1$. $d(H) \leqslant 2\beta_1$.

Property $h2$. $H \subset V_j^i$.

Property $h3$. There exists a set Z^i in $[z^i, z^{i+1}]$ such that $\Lambda Z^i > z^{i+1} - z^i - \epsilon/4n$ and for each z in Z^i

$$\sum_{\mathscr{H}} d(H(z)) \geqslant \tfrac{1}{2} d(E_0^\mu(z)) - \tfrac{1}{4}\epsilon.$$

On each H-shape so constructed in each vane of E_0^μ we construct the corresponding E-shape. The set of all these E-shapes (a typical element will be referred to as e_1^μ) is E_1^μ. From the 'v' and 'h' properties above we have the following:

Property $e1$. No two e_1^μ's overlap and all e_1^μ are contained in E_0^μ.

Property $e2$. There exists a set Z_1 in $[z_0', z_0'']$ such that $\Lambda Z_1 > z_0'' - z_0' - \tfrac{1}{2}\epsilon$ and for all z in Z_1,

$$\sum_{E_1^\mu} d(e_1^\mu(z)) \geqslant d(E_0^\mu(z)) - \tfrac{1}{2}\epsilon.$$

Property $e3$. $\sum_{E_1^\mu} (d(e_1^\mu))^2 \leqslant (d(E_0^\mu))^2$.

Property $e4$. The number of elements of E_1^μ is finite, $= N_1$, say.

We can now define by induction the sequence of sets E_r^μ. We can construct vanes in each e_r^μ of E_r^μ and construct H-shapes on these vanes (replacing β_1 by β_r in property $h1$ and ϵ in property $h3$, etc., by an extremely small constant depending on the size of the smallest e_r^μ and their total number, N_r) in such a way that the 'e' properties above generalize to the following:

Property $E1$. No two e_r^μ meet and each e_r^μ is contained in some element of E_{r-1}^μ.

Property $E2$. There exists a set $Z\,(= Z_1 \cap Z_2 \cap \ldots)$ in $[z_0', z_0'']$ such that

$$\Lambda Z \geqslant z_0'' - z_0' - \epsilon,$$

and, for all z in Z,

$$\sum_{E_r^\mu} d(e_r^\mu(z)) \geqslant \sum_{E_{r-1}^\mu} d(e_{r-1}^\mu(z)) - \frac{\epsilon}{2^r}$$

$$\geqslant d(E_0^\mu(z)) - \epsilon.$$

Property $E3$. $\sum_{E_r^\mu} (d(e_r^\mu))^2 \leqslant \sum_{E_{r-1}^\mu} (d(e_{r-1}^\mu))^2 \leqslant (d(E_0^\mu))^2$.

Let $E^\mu = \prod_{r=0}^{\infty} E_r^\mu$.

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LEMMA 4. $\Lambda^2(E^\mu) \leqslant \frac{1}{4}\pi(d(E_0^\mu))^2$.

Proof. E^μ is closed and hence measurable. Given any $\rho > 0$ choose an r such that $d(e_r^\mu) \leqslant \rho$ for all e_r^μ in E_r^μ. Then

$$\begin{aligned}\Lambda_\rho^2 E^\mu &\leqslant \sum_{E_r^\mu} \tfrac{1}{4}\pi(d(e_r^\mu))^2 \\ &\leqslant \tfrac{1}{4}\pi\,(d(E_0^\mu))^2, \quad \text{by property } E\,3.\end{aligned}$$

This being true for any $\rho > 0$ the lemma follows.

LEMMA 5. *There exists a positive constant C such that for all z in Z (see property $E\,2$),* $\Lambda E^\mu(z) \geqslant d(E_0^\mu(z)) - C\epsilon$.

Proof. The details of this proof will be omitted since they are both long and tedious. For a completely rigorous treatment of an exactly analogous situation the reader is again referred to (3). Briefly, the point is that the construction of the E-shapes in non-overlapping H-shapes ensures that for each z in Z the distance between the centres of two adjacent $e_r^\mu(z)$'s is at least equal to the sum of their diameters. It can be shown that this implies that $E_r^\mu(z)$ is an efficient ρ-covering (for suitable r) of $E^\mu(z)$. We can then use property $E\,2$ to yield the lemma. We can now prove Theorem 3.

$$\begin{aligned}\int \Lambda E^\mu(z)\,dz &\geqslant \int_Z \Lambda E^\mu(z)\,dz \\ &\geqslant \int_Z d(E_0^\mu(z))\,dz - c\epsilon, \quad \text{by Lemma 5.}\end{aligned}$$

Hence, since $d(E_0^\mu(z)) \leqslant 1$ and $\Lambda Z \geqslant z_0'' - z_0' - \epsilon$

$$\begin{aligned}\int \Lambda E^\mu(z)\,dz &\geqslant \int d(E_0^\mu(z))\,dz - (C+1)\,\epsilon \\ &= \int d(E_0(z))\,dz - (C+1)\,\epsilon - \tfrac{1}{2}\mu, \quad \text{by (6).} \qquad (9)\end{aligned}$$

Hence, by Theorem 1 and the definition of E_0,

$$\begin{aligned}\int \Lambda E^\mu(z)\,dz &\geqslant \phi^{-1}\tfrac{1}{4}\pi(d(E_0))^2 - (C+1)\,\epsilon - \tfrac{1}{2}\mu \\ &\geqslant \phi^{-1}\Lambda^2 E^\mu - (C+1)\,\epsilon - \tfrac{1}{2}\mu\end{aligned}$$

by Lemma 4, and since $d(E_0) = d(E_0^\mu)$. Hence

$$\Lambda^2 E^\mu \leqslant \phi\left(\int \Lambda E^\mu(z)\,dz + (C+1)\,\epsilon + \tfrac{1}{2}\mu\right). \qquad (10)$$

ϕ is less than one and since $\int d(E_0(z))\,dz = (\frac{1}{4}\pi + \frac{1}{2})/\sqrt{2} > \frac{7}{8}$, we have (equation (9) above) that $\int \Lambda E^\mu(z)\,dz > \frac{3}{4}$ for all sufficiently small ϵ and μ. Hence by (10) we can make ϵ so small that

$$\Lambda^2 E^\mu < (\phi + \mu)\int \Lambda E^\mu(z)\,dz.$$

Remark. There seems no reason why this greatest lower bound for $\Lambda^2 E / \int \Lambda E(z)\,dz$ should not be attained for some sets, though how such a set can be constructed is not

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clear. There is no obvious reason, for example, why a sequence of sets E^{μ_i} $(\mu_i \to 0)$ should in any sense 'converge' to a set of finite positive measure satisfying

$$\Lambda^2 E = \phi \int \Lambda E(z)\, dz.$$

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