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Elliptic islands in strictly convex billiards

MÁRIO JORGE DIAS CARNEIRO, SYLVIE OLIFFSON KAMPHORST
and SÔNIA PINTO DE CARVALHO

Departamento de Matemática, ICEx, UFMG 30.123–970, Belo Horizonte, Brasil
(e-mail: {carneiro, syok, sonia}@mat.ufmg.br)

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Abstract. This paper addresses the question of genericity of existence of elliptic islands for the billiard map associated to strictly convex closed curves. More precisely, we study 2-periodic orbits of billiards associated to C^5 closed and strictly convex curves and show that the existence of elliptic islands is a dense property on the subset of those billiards having an elliptic 2-periodic point.

Our main tools are normal perturbations, the Birkhoff Normal Form for elliptic fixed points and Moser's Twist Theorem. In order to perform some of the long computations involved, Maple® software was employed.

1. *Introduction*

Let α be a planar curve, closed, regular, strictly convex and oriented counter-clockwise. Any curve with these properties may be parametrized by $\varphi \in [0, 2\pi]$, the angle between the tangent vector and a given direction. The billiard problem on α consists in the free motion of a point particle in the plane region enclosed by α, being reflected elastically at the impacts with the boundary. Since the particle moves with constant velocity inside the region, the motion is completely determined by the point of reflection at α and the direction of movement immediately after each reflection. So, the angle φ, which locates the point of reflection, and the angle θ between the direction of motion and the normal to the boundary at the reflection point, may be used to describe the system.

The billiard model defines a map T from the annulus $\mathcal{A} = [0, 2\pi) \times (-\pi/2, \pi/2)$ into itself. If α is C^k, T is a C^{k-1}-diffeomorphism [**10**] preserving the measure $d\mu = R(\varphi)\cos\theta\, d\theta\, d\varphi$ [**1**], where $R(\varphi)$ is the radius of curvature of α at φ. $(\mathcal{A}, \mu, T)$ defines a two-dimensional discrete dynamical system, whose orbits are given by $\{(\varphi_n, \theta_n) = T^n(\varphi_0, \theta_0), n \in \mathbb{Z}\}$.

A point $(\varphi_0, \theta_0) \in \mathcal{A}$ is n-periodic if n is the smallest positive integer such that $T^n(\varphi_0, \theta_0) = (\varphi_0, \theta_0)$. It is called elliptic if the eigenvalues of $DT^n_{(\varphi_0,\theta_0)}$ are complex, of norm one and with non-vanishing imaginary part. An elliptic island is a T^n-invariant subset, homeomorphic to a disk, surrounding a n-periodic elliptic point.

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We study 2-periodic orbits of billiards associated to C^5 closed and strictly convex curves and show that the existence of elliptic islands is a dense property on the subset of those billiards having an elliptic 2-periodic point.

It is well known that for strictly convex curves, sufficiently differentiable, the billiard map has invariant curves near the boundary of the phase space [**2, 7**], and then it is not ergodic. Here we study the dynamical properties of the region of the phase space between these curves, when they exist, or the whole phase space, otherwise. The existence of islands assures the non-ergodicity of this region.

The layout of the paper is as follows: we will deal with planar, closed, regular, strictly convex, oriented counter-clockwise curves. In §§2 and 3 we analyze the geometrical and dynamical properties of the 2-periodic orbits. In §4, we define normal perturbations and prove that the existence of elliptic or hyperbolic 2-periodic orbits is an open property. Sections 5 and 6 deal with the resonances and the existence of elliptic islands. In §7 we present a more topological description of the results obtained in the previous sections. We conclude this work with some comments and open questions. Our main tools are normal perturbations, the Birkhoff Normal Form for elliptic fixed points and Moser's Twist Theorem. The execution of these methods is not straightforward, as one needs to compute higher-order derivatives of the billiard map. The calculations involved are long and complicated. In order to achieve these goals we have employed the symbolic programming tool Maple®.

2. *The diameters of a convex curve*

Let a curve α be parametrized by $\varphi \in [0, 2\pi]$, the angle between the tangent vector and a given direction, hereafter denoted by $\mathbf{x}$. Let $\tau(\varphi)$ and $\eta(\varphi)$ be the unitary tangent and (inward) normal vectors. Then $\alpha'(\varphi) = R(\varphi)\tau(\varphi)$. Clearly, $\tau(\varphi + \pi) = -\tau(\varphi)$ and $\eta(\varphi + \pi) = -\eta(\varphi)$.

It is known that any closed convex curve has at least a width and a diameter, characterized by the global minimum and maximum of the distance between points with parallel tangents (or with double normal) (see, for instance, [**5**]). We broaden this concept defining the following.

Definition 2.1. φ_0 is a diameter of α if $\alpha(\varphi_0) - \alpha(\varphi_0 + \pi)$ is parallel to $\eta(\varphi_0)$.

PROPOSITION 2.2. *Let $l(\varphi) = |\alpha(\varphi) - \alpha(\varphi + \pi)|$. Then φ_0 is a diameter of α if and only if*

$$\frac{dl}{d\varphi}(\varphi_0) = 0.$$

Proof.

$$\begin{aligned} 2l\frac{dl}{d\varphi} &= 2\langle \alpha'(\varphi) - \alpha'(\varphi + \pi), \alpha(\varphi) - \alpha(\varphi + \pi)\rangle \\ &= 2\langle R(\varphi)\tau(\varphi) - R(\varphi + \pi)\tau(\varphi + \pi), \alpha(\varphi) - \alpha(\varphi + \pi)\rangle \\ &= 2\langle (R(\varphi) + R(\varphi + \pi))\tau(\varphi), \alpha(\varphi) - \alpha(\varphi + \pi)\rangle. \end{aligned}$$

So

$$\frac{dl}{d\varphi}(\varphi_0) = 0 \iff \langle \alpha(\varphi_0) - \alpha(\varphi_0 + \pi), \tau(\varphi_0)\rangle = 0. \qquad \square$$

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Let $L_0 = l(\varphi_0)$, $R_0 = R(\varphi_0)$ and $R_\pi = R(\varphi_0 + \pi)$.

PROPOSITION 2.3. *Let φ_0 be a diameter. If $L_0 - (R_0 + R_\pi) \neq 0$ then φ_0 is an isolated diameter.*

Proof.

$$\begin{aligned}\frac{d}{d\varphi}\left(2l\frac{dl}{d\varphi}\right) &= 2\left(\frac{dl}{d\varphi}\right)^2 + 2l\frac{d^2l}{d\varphi^2} \\ &= 2(\langle (R'(\varphi) + R'(\varphi+\pi))\tau(\varphi) + (R(\varphi) + R(\varphi+\pi))\eta(\varphi), \alpha(\varphi) - \alpha(\varphi+\pi)\rangle \\ &\quad + \langle (R(\varphi) + R(\varphi+\pi))\tau(\varphi), (R(\varphi) + R(\varphi+\pi))\tau(\varphi)\rangle).\end{aligned}$$

At the diameter φ_0,

$$2l(\varphi_0)\frac{d^2l}{d\varphi^2}(\varphi_0) = 2(-L_0(R_0 + R_\pi) + (R_0 + R_\pi)^2) = -2(R_0 + R_\pi)(L_0 - (R_0 + R_\pi))$$

and we have that if $L_0 - (R_0 + R_\pi) \neq 0$ then φ_0 is an isolated singular point of l. □

3. *Dynamical characterization of the diameters of a convex curve*

Let α be a C^k curve, $k \geq 2$, and T its associated billiard map.

If φ_0 is a diameter for α then $(\varphi_0, 0)$ is a 2-periodic orbit for T, and, as we have pointed out, any billiard on a strictly convex curve has at least two of those orbits. Its stability is given by the eigenvalues of

$$DT^2_{(\varphi_0,0)} = \frac{1}{R_0R_\pi}\begin{pmatrix} L_0 - R_\pi & -L_0 \\ -L_0 + (R_0 + R_\pi) & L_0 - R_0\end{pmatrix}\begin{pmatrix} L_0 - R_0 & -L_0 \\ -L_0 + (R_0 + R_\pi) & L_0 - R_\pi\end{pmatrix}$$

and it follows that:

1. $(\varphi_0, 0)$ is hyperbolic if
   - $L_0 - (R_0 + R_\pi) > 0$ or
   - $L_0 - (R_0 + R_\pi) < 0$ and $(L_0 - R_0)(L_0 - R_\pi) < 0$;
2. $(\varphi_0, 0)$ is elliptic if $L_0 - (R_0 + R_\pi) < 0$ and $(L_0 - R_0)(L_0 - R_\pi) > 0$;
3. $(\varphi_0, 0)$ is parabolic if
   - $L_0 - (R_0 + R_\pi) = 0$ or
   - $L_0 - (R_0 + R_\pi) < 0$ and $(L_0 - R_0)(L_0 - R_\pi) = 0$.

We will say that a diameter is hyperbolic, elliptic or parabolic if the associated 2-periodic orbit is, respectively, hyperbolic, elliptic or parabolic.

A nice consequence [**6**] of this characterization is that, for $k \geq 3$, the largest diameter of a strictly convex curve, if isolated, is hyperbolic due to the fact that

$$\frac{d^2l}{d\varphi^2}(\varphi_0) = -\frac{(R_0 + R_\pi)(L_0 - (R_0 + R_\pi))}{L_0} < 0.$$

The smallest one, however, can be of any type. If $R_0 = R_\pi$ (which happens, for instance, for the ellipse), it is elliptic or parabolic. Nevertheless, as we show in the last section, there are strictly convex curves such that all the diameters are hyperbolic.

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When φ_0 is elliptic, the eigenvalues of $DT^2_{(\varphi_0,0)}$ are given by

$$e^{\pm i\gamma} = \frac{\mathrm{tr}(DT^2)}{2} \pm i\sqrt{1 - \left(\frac{\mathrm{tr}(DT^2)}{2}\right)^2}$$

and so

$$\cos\gamma = 2\frac{(L_0 - R_0)(L_0 - R_\pi)}{R_0 R_\pi} - 1. \tag{1}$$

4. *Normal perturbations of a convex curve*

Let α be C^k, with $k \geq 4$, when parametrized by $\varphi \in [0, 2\pi]$, the angle between the tangent vector and $\mathbf{x}$. Using this direction and its perpendicular as a reference frame, the unitary tangent and (inward) normal vectors are written, respectively, as $\tau(\varphi) = (\cos\varphi, \sin\varphi)$ and $\eta(\varphi) = (-\sin\varphi, \cos\varphi)$. So, the normal bundle $(\alpha(\varphi), \eta(\varphi))$ is also C^k and $\eta(\varphi)$ is C^∞.

A normal perturbation of α is a curve β given by $\beta(\varphi) = \alpha(\varphi) + \lambda(\varphi)\eta(\varphi)$, where λ is a C^k, 2π-periodic function. Observe that in general φ does not represent the angle between the tangent vector of β and the fixed direction $\mathbf{x}$.

We will use the norm

$$|\lambda|_2 = \max_{\varphi\in[0,2\pi]} \{|\lambda(\varphi)|, |\lambda'(\varphi)|, |\lambda''(\varphi)|\}.$$

PROPOSITION 4.1. *If $|\lambda|_2$ is small enough then β is C^k, closed, regular and strictly convex.*

Proof. β is closed because λ is 2π-periodic. α is C^k, λ is C^k and η is C^∞. So, β is C^k. $\beta' = \alpha' + \lambda'\eta - \lambda\tau$. So, if α is regular, for ϵ small enough, β is regular. $\beta'' \times \beta' = (\alpha'' \times \alpha') + \lambda'(\alpha'' \times \eta) - \lambda(\alpha'' \times \tau) + (\lambda'' - \lambda)(\eta \times \alpha') + (\lambda(\lambda'' - \lambda) - 2(\lambda')^2)(\tau \times \eta)$. So, if α is strictly convex, for ϵ small enough, β is strictly convex. □

Let us denote by $\boldsymbol{t}(\varphi)$ and $\boldsymbol{n}(\varphi)$ the unitary tangent and (inward) normal vectors of β and $R_\alpha(\varphi)$ and by $R_\beta(\varphi)$ the radius of curvature of α and β at φ, respectively. Then

$$\begin{aligned}
\boldsymbol{t} &= \frac{(R_\alpha - \lambda)\tau + \lambda'\eta}{((R_\alpha - \lambda)^2 + (\lambda')^2)^{1/2}} \\
\boldsymbol{n} &= \frac{-\lambda'\tau + (R_\alpha - \lambda)\eta}{((R_\alpha - \lambda)^2 + (\lambda')^2)^{1/2}} \\
R_\beta &= \frac{((R_\alpha - \lambda)^2 + (\lambda')^2)^{3/2}}{(R_\alpha - \lambda)(R_\alpha - \lambda + \lambda'') - \lambda'(R'_\alpha - 2\lambda')}.
\end{aligned} \tag{2}$$

Let $h(\varphi)$ be the angle between $\boldsymbol{t}(\varphi)$ and the given direction that defines φ, as in Figure 1. It is easy to see that

$$h(\varphi) = \varphi + \arctan\frac{\lambda'}{R_\alpha - \lambda}.$$

According to Proposition 2.2, the diameters of β are such that the derivative of

$$l_\beta(\varphi) = |\beta(\varphi) - \beta(h^{-1}(h(\varphi) + \pi))|$$

vanishes.

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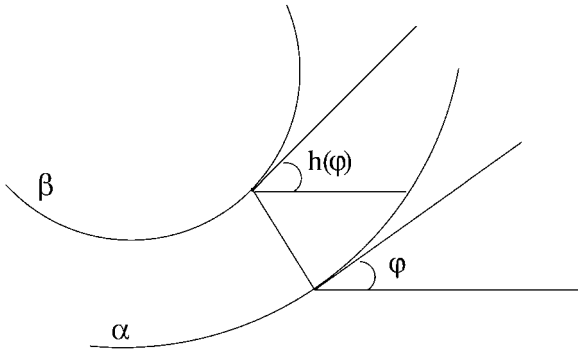


FIGURE 1. Definition of $h(\varphi)$.

LEMMA 4.2. *If $|\lambda|_2$ is sufficiently small then h is a C^2-diffeomorphism.*

Proof. As α is closed, $0 < c_1 \le R_\alpha$. Then, for $|\lambda|_2$ sufficiently small, $R_\alpha - \lambda > 0$ and h is well defined. With α and β being at least C^4 then λ is at least C^4, and so h is at least C^3.

Let s be the arclength parameter for β. Then

$$\frac{dh}{d\varphi} = \frac{dh}{ds}\frac{ds}{d\varphi} = k_\beta \left|\frac{d\beta}{d\varphi}\right| \ge c_2 > 0.$$

Then h is C^1 and invertible, and h^{-1} has continuous derivative.

$$\frac{d^2h}{d\varphi^2} = \frac{dk_\beta}{ds}\left|\frac{d\beta}{d\varphi}\right|^2 + k_\beta \frac{\left\langle \frac{d^2\beta}{d\varphi^2}, \frac{d\beta}{d\varphi}\right\rangle}{\left|\frac{d\beta}{d\varphi}\right|}$$

so h is C^2. Taking $y = h(\varphi)$ we have

$$\frac{d^2h^{-1}}{dy^2}(y) = -\frac{\frac{d^2h}{d\varphi^2}(\varphi)}{\left(\frac{dh}{d\varphi}(\varphi)\right)^3}$$

which shows that h is a C^2-diffeomorphism. □

LEMMA 4.3. *With the same hypothesis as Lemma 4.2, $f(\varphi) = h^{-1}(h(\varphi) + \pi)$ and $I_\pi(\varphi) = \varphi + \pi$ are C^2 close.*

Proof. Let $|\lambda|_2 < \epsilon$ and $1/c_1 < |R_\alpha - \lambda| < c_2$, then

$$|h(\varphi) - \varphi| = \left|\arctan\frac{\lambda'}{R_\alpha - \lambda}\right| \le \left|\frac{\lambda'}{R_\alpha - \lambda}\right| < c_1\epsilon.$$

Let $c_3 = \max |dh^{-1}/dy|$.

$$\begin{aligned} |f(\varphi) - I_\pi(\varphi)| &= |h^{-1}(h(\varphi) + \pi) - h^{-1}(h(\varphi + \pi))| \\ &\le c_3|h(\varphi) + \pi - h(\varphi + \pi)| \\ &= c_3|h(\varphi) - \varphi + \varphi + \pi - h(\varphi + \pi)| \\ &\le c_3(|h(\varphi) - \varphi| + |\varphi + \pi - h(\varphi + \pi)|) < 2c_1c_3\epsilon. \end{aligned}$$

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Let $0 < 1/c_4 < |R'_\alpha - \lambda'| < c_5$ and $\Delta(\varphi) = \arctan \lambda'/(R_\alpha - \lambda)$. Now, $dh/d\varphi = 1 + d\Delta/d\varphi$ and

$$\left|\frac{d\Delta}{d\varphi}\right| \le \left|\frac{d}{d\varphi}\frac{\lambda'}{R_\alpha - \lambda}\right| = \left|\frac{\lambda''(R_\alpha - \lambda) - \lambda'(R'_\alpha - \lambda')}{(R_\alpha - \lambda)^2}\right| < c_1^2 c_2 c_5 \epsilon. \tag{3}$$

Taking $y = h(\varphi)$,

$$\begin{aligned}
\left|\frac{df}{d\varphi} - 1\right| &= \left|\frac{dh^{-1}}{dy}(h(\varphi)+\pi)\frac{dh}{d\varphi}(\varphi) - 1\right| = \left|\frac{1 + \frac{d\Delta}{d\varphi}(\varphi)}{1 + \frac{d\Delta}{d\varphi}(h^{-1}(h(\varphi)+\pi))} - 1\right| \\
&= \left|\frac{1 + \frac{d\Delta}{d\varphi}(\varphi) + \frac{d\Delta}{d\varphi}(h^{-1}(h(\varphi)+\pi)) - \frac{d\Delta}{d\varphi}(h^{-1}(h(\varphi)+\pi))}{1 + \frac{d\Delta}{d\varphi}(h^{-1}(h(\varphi)+\pi))} - 1\right| \\
&= \left|\frac{\frac{d\Delta}{d\varphi}(\varphi) - \frac{d\Delta}{d\varphi}(h^{-1}(h(\varphi)+\pi))}{1 + \frac{d\Delta}{d\varphi}(h^{-1}(h(\varphi)+\pi))}\right| \\
&\le \left|\frac{dh^{-1}}{d\varphi}(h(\varphi)+\pi)\right| \left|\frac{d\Delta}{d\varphi}(\varphi)\right| + \left|\frac{d\Delta}{d\varphi}(h^{-1}(h(\varphi)+\pi))\right| < 2c_3 c_1^2 c_2 c_5 \epsilon.
\end{aligned}$$

Finally,

$$\begin{aligned}
\left|\frac{d^2 f}{d\varphi^2}(\varphi) - 0\right| &= \left|\frac{d}{d\varphi}\left(\frac{dh^{-1}}{dy}(h(\varphi)+\pi)\frac{dh}{d\varphi}(\varphi)\right)\right| \\
&\le \left|\frac{d^2 h^{-1}}{dy^2}(h(\varphi)+\pi)\right| \left|\frac{dh}{d\varphi}\right|^2 + \left|\frac{dh^{-1}}{dy}(h(\varphi)+\pi)\right| \left|\frac{d^2 h}{d\varphi^2}(\varphi)\right|.
\end{aligned}$$

We have

$$\left|\frac{dh}{d\varphi}\right| \quad \text{and} \quad \left|\frac{dh^{-1}}{dy}(h(\varphi)+\pi)\right|$$

are bounded from above and below because h is a C^2 diffeomorphism on $[0, 2\pi]$.

$$\begin{aligned}
\left|\frac{d^2 h}{d\varphi^2}(\varphi)\right| = \left|\frac{d^2\Delta}{d\varphi^2}(\varphi)\right| &\le \left|\frac{\lambda'''(R_\alpha - \lambda) - \lambda'(R''_\alpha - \lambda'')}{(R_\alpha - \lambda)^2 + (\lambda')^2}\right| \\
&\quad + 2\left|\frac{(\lambda''(R_\alpha - \lambda) - \lambda'(R'_\alpha - \lambda'))(R'_\alpha - \lambda')}{(R_\alpha - \lambda)((R_\alpha - \lambda)^2 + (\lambda')^2)}\right| + 2\left|\frac{\lambda'}{R_\alpha - \lambda}\right| \left|\frac{d\Delta}{d\varphi}(\varphi)\right|.
\end{aligned}$$

For ϵ small enough, $0 < 1/c_1 < |R_\alpha - \lambda| < c_2$, $|R'_\alpha - \lambda'| < c_5$, $|R''_\alpha - \lambda''| < c_6$ and $(R_\alpha - \lambda)^2 + (\lambda')^2 = |\beta'(\varphi)|^2 \ge c_7 > 0$, since β is regular. Together with equation (3), we have that

$$\left|\frac{d^2 h}{d\varphi^2}(\varphi)\right| < K\epsilon,$$

for a constant K.

$$\left|\frac{d^2 h^{-1}}{dy^2}(h(\varphi)+\pi)\right| = \left|\frac{\frac{d^2 h}{d\varphi^2}(h^{-1}(h(\varphi)+\pi))}{\left(\frac{dh}{d\varphi}(h^{-1}(h(\varphi)+\pi))\right)^3}\right| < K\epsilon \tag{4}$$

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for a constant K. So,

$$\left|\frac{d^2 f}{d\varphi^2}(\varphi)\right| < C\epsilon. \qquad \square$$

LEMMA 4.4. *Let $l_\alpha(\varphi) = |\alpha(\varphi) - \alpha(\varphi+\pi)|$ and $l_\beta(\varphi) = |\beta(\varphi) - \beta(h^{-1}(h(\varphi)+\pi))|$. If $|\lambda|_2$ is small enough then l_α and l_β are C^2 close.*

Proof. If $|\lambda|_2$ is sufficiently small then α and β are C^2 close. So $\alpha \circ I_\pi$ and $\beta \circ f$ are C^2 close and the result follows. $\square$

PROPOSITION 4.5. *Suppose that α has an isolated diameter at φ_0, with*

$$\frac{dl_\alpha}{d\varphi}(\varphi_0) = 0 \quad \textit{and} \quad \frac{d^2 l_\alpha}{d\varphi^2}(\varphi_0) > 0 \ (\textit{or } < 0).$$

Let δ be such that

$$\frac{d^2 l_\alpha}{d\varphi^2}(\varphi) > 0 \ (\textit{or } < 0) \quad \textit{for } \varphi \in [\varphi_0 - \delta, \varphi_0 + \delta].$$

Then there exists ϵ_0 such that if $|\lambda|_2 < \epsilon_0$ then any normal perturbation $\beta(\varphi) = \alpha(\varphi) + \lambda(\varphi)\eta(\varphi)$ has an isolated diameter $\varphi_1 \in (\varphi_0 - \delta, \varphi_0 + \delta)$, with

$$\frac{d^2 l_\beta}{d\varphi^2}(\varphi_1) > 0 \ (\textit{or } < 0).$$

Furthermore, $\epsilon_0 \to 0$ if $\delta \to 0$.

Proof. The proof follows immediately from the results above and from the transversality of the zero of $dl_\alpha/d\varphi$. $\square$

Observe that rotating the axis, if necessary, we can always suppose that the diameter of α is at $\varphi = 0$.

Let, as above, $L_0 = l_\alpha(0)$, $R_0 = R_\alpha(0)$, and $R_\pi = R_\alpha(\pi)$. An obvious consequence is the following.

COROLLARY 4.6. *If 0 is a hyperbolic diameter of α, with $L_0 - (R_0 + R_\pi) > 0$ (meaning that $\frac{d^2 l_\alpha}{d\varphi^2}(0) < 0$) then every normal perturbation β with $|\lambda|_2 < \epsilon_0$ has a hyperbolic diameter close to 0.*

PROPOSITION 4.7. *Let ϵ_0 and δ be as in Proposition 4.5 and suppose 0 is an elliptic (respectively hyperbolic) diameter for α, i.e. $L_0 - (R_0 + R_\pi) < 0$ and $(L_0 - R_0)(L_0 - R_\pi) > 0$ (respectively $(L_0 - R_0)(L_0 - R_\pi) < 0$). There exists $\epsilon_1 \le \epsilon_0$ such that if $|\lambda|_2 < \epsilon_1$ then any normal perturbation $\beta(\varphi) = \alpha(\varphi) + \lambda(\varphi)\eta(\varphi)$ has an elliptic (respectively hyperbolic) diameter at $\varphi_1 \in (\varphi_0 - \delta, \varphi_0 + \delta)$.*

Proof. If $|\lambda|_2 < \epsilon_0$ then, by Proposition 4.5, β has an isolated diameter at $\varphi_1 \in (\varphi_0 - \delta, \varphi_0 + \delta)$ with $l_\beta(\varphi_1) - (R_\beta(\varphi_1) + R_\beta(h^{-1}(h(\varphi_1) + \pi))) < 0$. Since α and β are C^2-close, R_α and R_β are C^0-close. This, together with Lemma 4.3 and $(L_0 - R_0)(L_0 - R_\pi) > 0$, gives that there exists $\epsilon_1 \le \epsilon_0$ such that if $|\lambda|_2 < \epsilon_1$ then $(l_\beta(\varphi_1) - R_\beta(\varphi_1))(l_\beta(\varphi_1) - R_\beta(h^{-1}(h(\varphi_1) + \pi))) > 0$.

The hyperbolic case $(L_0 - R_0)(L_0 - R_\pi) < 0$ is analogous. $\square$

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Remark. Propositions 4.5 and 4.7 and Corollary 4.6 only express, in the context of billiard maps, the well-known fact of openness of existence of elliptic or hyperbolic fixed points.

5. *Resonances*

We will assume that the curve α has an elliptic diameter with eigenvalues $e^{\pm i\gamma}, 0 < \gamma < \pi$. Without any loss of generality we can assume that this diameter is located at $\varphi = 0$.

Definition 5.1. An elliptic diameter is resonant if its eigenvalues satisfy $(e^{\pm i\gamma})^j = 1$, for $j = 2, 3, 4$, i.e. if $\gamma = \frac{1}{2}\pi, \frac{2}{3}\pi, \frac{4}{3}\pi, \frac{3}{2}\pi$. Otherwise, it is non-resonant.

PROPOSITION 5.2. *Given a C^k strictly convex curve α, $k \geq 4$, with a resonant elliptic diameter $\varphi = 0$, there is a C^k strictly convex curve β, C^2-close to α, such that β has a non-resonant elliptic diameter.*

Proof. For ϵ_1 as in Proposition 4.7 consider the normal perturbations $\beta(\varphi) = \alpha(\varphi) + \lambda(\varphi)\eta(\varphi)$, with $|\lambda|_2 < \epsilon_1$ and $\lambda'(0) = \lambda'(\pi) = 0$, which implies that $\varphi = 0$ is also an elliptic diameter for β.

Let $\lambda(0) = \lambda_0$, $\lambda(\pi) = \lambda_\pi$, $\lambda''(0) = \lambda''_0$, $\lambda''(\pi) = \lambda''_\pi$. We have that $l_0 = l_\beta(0) = |\beta(0) - \beta(\pi)| = L_0 - \lambda_0 - \lambda_\pi$ and, by equation (2),

$$r_0 = R_\beta(0) = \frac{(R_0 - \lambda_0)^2}{R_0 - \lambda_0 + \lambda''_0} = R_0 - \lambda_0 - \lambda''_0 \frac{R_0 - \lambda_0}{R_0 - \lambda_0 + \lambda''_0}$$

and

$$r_\pi = R_\beta(\pi) = \frac{(R_\pi - \lambda_\pi)^2}{R_\pi - \lambda_\pi + \lambda''_\pi} = R_\pi - \lambda_\pi - \lambda''_\pi \frac{R_\pi - \lambda_\pi}{R_\pi - \lambda_\pi + \lambda''_\pi}.$$

So, for $|\lambda|_2$ small enough, these normal perturbations have an elliptic diameter with eigenvalues $e^{\pm i\gamma(\beta)}$ satisfying, by equation (1),

$$\begin{aligned}\cos\gamma(\beta) &= 2\frac{(l_0 - r_0)(l_0 - r_\pi)}{r_0 r_\pi} - 1 \\ &= 2\frac{(L_0 - R_0 - \lambda_\pi + \lambda''_0\Delta_0)(L_0 - R_\pi - \lambda_0 + \lambda''_\pi\Delta_\pi)}{(R_0 - \lambda_0 - \lambda''_0\Delta_0)(R_\pi - \lambda_\pi - \lambda''_\pi\Delta_\pi)} - 1 \\ &= F(\lambda_0, \lambda_\pi, \lambda''_0, \lambda''_\pi),\end{aligned}$$

where

$$\Delta_a = \frac{R_a - \lambda_a}{R_a - \lambda_a + \lambda''_a}.$$

Clearly,

$$F(0, 0, 0, 0) = 2\frac{(L_0 - R_0)(L_0 - R_\pi)}{R_0 R_\pi} - 1$$

gives the eigenvalues of the elliptic diameter of α.

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Moreover

$$\frac{\partial F}{\partial \lambda_0}(0,0,0,0) = 2\frac{(L_0 - R_0)(L_0 - R_0 - R_\pi)}{R_0^2 R_\pi} \neq 0$$
$$\frac{\partial F}{\partial \lambda_\pi}(0,0,0,0) = 2\frac{(L_0 - R_\pi)(L_0 - R_0 - R_\pi)}{R_\pi^2 R_0} \neq 0$$
$$\frac{\partial F}{\partial \lambda_0''}(0,0,0,0) = 2\frac{(L_0 - R_\pi)L_0}{R_0^2 R_\pi} \neq 0$$
$$\frac{\partial F}{\partial \lambda_\pi''}(0,0,0,0) = 2\frac{(L_0 - R_0)L_0}{R_\pi^2 R_0} \neq 0$$

which shows that F is a non-constant rational function and the resonance conditions of Definition 5.1 are fulfilled only on a union of codimension 1 submanifolds. Hence it is enough to choose a normal perturbation with λ small, $\lambda'(0) = \lambda'(\pi) = 0$ and outside the above union of manifolds, to obtain a curve β, close to α, with a non-resonant elliptic diameter. □

PROPOSITION 5.3. *If* 0 *is a non-resonant elliptic diameter for a strictly convex* C^k *curve* α, $k \geq 3$, *then there exists* ϵ_2 *such that every normal perturbation* β, *with* $|\lambda|_2 < \epsilon_2$, *has a non-resonant elliptic diameter.*

Proof. As 0 is a non-resonant elliptic diameter for α then

$$\cos\gamma(\alpha) = 2\frac{(l_\alpha(0) - R_\alpha(0))(l_\alpha(0) - R_\alpha(\pi))}{R_\alpha(0)R_\alpha(\pi)} - 1$$
$$= 2\frac{(L_0 - R_0)(L_0 - R_\pi)}{R_0 R_\pi} - 1 \neq 0, 1, -1, \cos\frac{2}{3}\pi, \cos\frac{4}{3}\pi.$$

For ϵ_1 as in Proposition 4.7, let $\beta(\varphi) = \alpha(\varphi) + \lambda(\varphi)\eta(\varphi)$, with $|\lambda|_2 < \epsilon_1$. Then β has an elliptic diameter at $\varphi_1 \in (-\delta, \delta)$ with eigenvalues determined by

$$\cos\gamma(\beta) = 2\frac{(l_\beta(\varphi_1) - R_\beta(\varphi_1))(l_\beta(\varphi_1) - R_\beta(h^{-1}(h(\varphi_1) + \pi)))}{R_\beta(\varphi_1)R_\beta(h^{-1}(h(\varphi_1) + \pi))} - 1.$$

For $|\lambda|_2$ sufficiently small, R_α and R_β are C^0 close. This, together with Lemmas 4.3 and 4.4, gives the desired result. □

6. *Elliptic islands*

In order to seek out islands, we will change coordinates from (φ, θ) to (s, p), where s is the arclength parameter of the curve α and $p = \sin\theta$. The advantage of these coordinates is that the billiard map T preserves the area $ds \wedge dp$.

Now suppose that, for $k \geq 5$, the C^k curve α has a non-resonant elliptic diameter at $s = 0$, so that $(0, 0)$ is a non-resonant elliptic fixed point of T^2.

This map then has the Birkhoff Normal Form $e^{i(\gamma + \tau_1(\alpha)|\zeta|^2)}\zeta + h(\zeta, \overline{\zeta})$, where $h(\zeta, \overline{\zeta}) = \mathcal{O}(|\zeta|^4)$. Moreover, if the first Birkhoff coefficient $\tau_1(\alpha) \neq 0$ then, by Moser's Twist Theorem [**9**], $\zeta = 0$ is a stable fixed point, i.e. $(0, 0)$ has T^2-invariant curves surrounding it, which means that there is an elliptic island of positive measure.

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Hence, by assuming $\tau_1(\alpha) = 0$, the goal is to show that small C^2-normal perturbations of α may have non-zero Birkhoff coefficients.

The canonical way of calculating τ_1 (see, for instance [**8**] or [**4**]) is as follows. Let the three-jet of T^2 at $(0, 0)$ be written as

$$J_3T^2_{(0,0)}(s, p) = (a_{10}s + a_{01}p + a_{20}s^2 + a_{11}sp + \cdots \\ \cdots + a_{03}p^3, b_{10}s + b_{01}p + b_{20}s^2 + \cdots + b_{03}p^3).$$

After complexification and diagonalization of the linear part, it takes the form

$$Z = e^{i\gamma}(z + c_{20}z^2 + c_{11}z\overline{z} + c_{02}\overline{z}^2 + c_{30}z^3 + c_{21}z^2\overline{z} + c_{12}z\overline{z}^2 + c_{03}\overline{z}^3).$$

By means of a polynomial change of coordinates the map assumes its Birkhoff Normal Form, and the first Birkhoff coefficient is given by

$$\tau_1 = \frac{1}{i}\left(c_{21} + 2|c_{20}|^2\left(\frac{2e^{i\gamma}+1}{e^{i\gamma}-1}\right) + 2|c_{02}|^2\left(\frac{1}{e^{3i\gamma}-1}\right)\right).$$

Clearly, c_{20} and c_{02} depend only on the first and second derivatives of T^2 at $(0, 0)$. The crucial term, for the kind of perturbation we are going to take, will be c_{21}, which is given by

$$c_{21} = \frac{i}{8}a_{10}\left(-a_{21} + 3\frac{b_{10}a_{03}}{a_{01}} - 3\frac{a_{01}b_{30}}{b_{10}} + b_{12}\right) \\ - \frac{i}{8}b_{10}\left(-3\frac{a_{01}a_{30}}{b_{10}} + a_{12} - \frac{a_{01}b_{21}}{b_{10}} + 3b_{03}\right).$$

To achieve the calculation of c_{21} let $(S(s, p), P(s, p)) = T(s, p)$, so $T^2(s, p) = S(S(s, p), P(s, p)), P(S(s, p), P(s, p))$. Then, by the chain rule, the coefficients of $J_3T^2_{(0,0)}$ will depend algebraically on the first and second derivatives and linearly on the third derivatives of S and P at $(0, 0)$ and at $(S(0, 0), P(0, 0)) = (s_1, 0)$.

The first derivatives of the billiard map can be written in a closed form as

$$\frac{\partial S}{\partial s} = \frac{l(s, p) - R(s)\cos\theta(p)}{R(s)\cos\theta(P(s, p))} \\ \frac{\partial S}{\partial p} = -\frac{l(s, p)}{\cos\theta(p)\cos\theta(P(s, p))} \\ \frac{\partial P}{\partial s} = -\frac{l(s, p) - R(s)\cos\theta(p) - R(S(s, p))\cos\theta(P(s, p))}{R(s)R(S(s, p))} \\ \frac{\partial P}{\partial p} = \frac{l(s, p) - R(S(s, p))\cos\theta(P(s, p))}{R(S(s, p))\cos\theta(p)},$$

where $l(s, p) = |\alpha(s) - \alpha(S(s, p))|$, $R(s)$ is the radius of curvature of α and $\theta(p) = \arcsin p$.

The first derivatives of S and P at $(0, 0)$ and $(s_1, 0)$ depend on $l(0, 0) = l(s_1, 0) = L_0$, $R(0) = R_0$, $R(s_1) = R_\pi$ and $\theta(0) = 0$. The second derivatives also depend on the first derivatives of R, θ and l. Clearly, $\frac{d\theta}{dp}(0) = 1$ and since $s = 0$ is an isolated elliptic diameter, all the first derivatives of l vanish. So, the second derivatives of S and P at $(0, 0)$ and at

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$(s_1, 0)$ depend only on L_0, R_0, R_π, $\frac{dR}{ds}(0)$ and $\frac{dR}{ds}(s_1)$. The third derivatives will depend on all the previous parameters, on $\frac{d^2R}{ds^2}(0)$ and $\frac{d^2R}{ds^2}(s_1)$, on $\frac{d^2\theta}{dp^2}(0) = 0$ and on the second derivatives of l at $(0, 0)$ and $(s_1, 0)$, which are functions only of L_0, R_0, R_π, $\frac{dR}{ds}(0)$ and $\frac{dR}{ds}(s_1)$. The dependence of the third derivatives on $\frac{d^2R}{ds^2}(0)$ and $\frac{d^2R}{ds^2}(s_1)$ is clearly linear. Then, the coefficient c_{21} depends linearly on $\frac{d^2R}{ds^2}(0)$ and $\frac{d^2R}{ds^2}(s_1)$.

Because of its recurrent structure, the calculation of the coefficients of the three-jet is suitable to be implemented as a computer program.

We have employed the symbolic software Maple® to evaluate the a_{kj} and b_{kj} and we obtained†

$$c_{21} = -\frac{i}{16}\left[\frac{L_0(L_0 - R_\pi)}{(L_0 - R_0)(L_0 - R_0 - R_\pi)}\right]\frac{d^2R}{ds^2}(0) + if(L_0, R_0, R_\pi)\frac{d^2R}{ds^2}(s_1) + g\left(L_0, R_0, R_\pi, \frac{dR}{ds}(0), \frac{dR}{ds}(s_1)\right).$$

Using these results we can prove the following.

PROPOSITION 6.1. *Let α be a C^k strictly convex curve, $k \geq 5$, with a non-resonant elliptic diameter at $s = 0$, and such that $\tau_1(\alpha) = 0$. Then there is a C^k strictly convex curve β, C^2-close to α, such that β has a non-resonant elliptic diameter, with $\tau_1(\beta) \neq 0$.*

Proof. Let $\beta(\varphi) = \alpha(\varphi) + \lambda(\varphi)\eta(\varphi)$ be a normal perturbation of α, with a third-order contact with α at $\varphi(0) = 0$ and $\varphi(s_1) = \pi$. This means that $\lambda(0) = \lambda(\pi) = \lambda'(0) = \lambda'(\pi) = \lambda''(0) = \lambda''(\pi) = \lambda'''(0) = \lambda'''(\pi) = 0$ and then β has a non-resonant elliptic diameter at $\varphi = 0$.

Let σ be the arclength parameter for β and $(\sigma_1, 0) = T_\beta^2(0, 0)$. The third-order contact implies that the elliptic diameters coincide, with the same length L_0 and

$$R_\beta(0) = R_\alpha(0) = R_0, \quad R_\beta(\sigma_1) = R_\alpha(s_1) = R_\pi,$$
$$\frac{dR_\beta}{d\sigma}(0) = \frac{dR_\alpha}{ds}(0), \quad \frac{dR_\beta}{d\sigma}(\sigma_1) = \frac{dR_\alpha}{ds}(s_1),$$
$$\frac{d^2R_\beta}{d\sigma^2}(0) = \frac{d^2R_\alpha}{ds^2}(0) - \frac{1}{R_0^2}\frac{d^4\lambda}{d\varphi^4}(0), \quad \frac{d^2R_\beta}{d\sigma^2}(\sigma_1) = \frac{d^2R_\alpha}{ds^2}(s_1) - \frac{1}{R_0^2}\frac{d^4\lambda}{d\varphi^4}(\pi).$$

Then $\gamma(\beta) = \gamma(\alpha)$, $c_{20}(\beta) = c_{20}(\alpha)$, $c_{02}(\beta) = c_{02}(\alpha)$ and

$$c_{21}(\beta) = \frac{i}{16R_0^2}\left[\frac{L_0(L_0 - R_\pi)}{(L_0 - R_0)(L_0 - R_0 - R_\pi)}\right]\frac{d^4\lambda}{d\varphi^4}(0) + if(L_0, R_0, R_\pi)\frac{d^4\lambda}{d\varphi^4}(\pi) + c_{21}(\alpha),$$

where f is a C^{k-4} function.

Hence

$$\tau_1(\beta) = \frac{1}{16R_0^2}\left[\frac{L_0(L_0 - R_\pi)}{(L_0 - R_0)(L_0 - R_0 - R_\pi)}\right]\frac{d^4\lambda}{d\varphi^4}(0) + f(L_0, R_0, R_\pi)\frac{d^4\lambda}{d\varphi^4}(\pi) + \tau_1(\alpha).$$

† The program is available at `http://www.mat.ufmg.br/~syok/papers`.

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As $\sigma = 0$ is a non-resonant elliptic diameter for α and β, then $L_0 \neq 0$, $L_0 - R_0 \neq 0$, $L_0 - R_\pi \neq 0$ and $L_0 - R_0 - R_\pi \neq 0$. Therefore, by choosing

$$\frac{d^4\lambda}{d\varphi^4}(0) \neq 0 \quad \text{and} \quad \frac{d^4\lambda}{d\varphi^4}(\pi) = 0$$

we obtain a perturbation λ such that $\tau_1(\beta) \neq 0$. □

7. *Density of elliptic islands*

Let α be a closed, regular, strictly convex planar curve and let T_α be its associated billiard map. Planar rigid motions do not change the geometrical features of α and do not change the dynamical characteristics of T_α. This defines an equivalence relation on the set of curves.

Let $\mathcal{C}$ be the set of equivalence classes, denoted by $[\alpha]$, of closed, regular, strictly convex planar curves α, that are C^5 when parametrized by the angle φ.

Given a representative α of an equivalence class $[\alpha] \in \mathcal{C}$, the normal bundle $(\alpha(\varphi), \eta(\varphi))$ is then C^5. Given $\epsilon > 0$, let us consider a tubular neighbourhood

$$N_\epsilon(\alpha) = \{\alpha(\varphi) + \lambda\eta(\varphi), 0 \leq \varphi < 2\pi, -\epsilon < \lambda < \epsilon\} \simeq S^1 \times (-\epsilon, \epsilon)$$

and the canonical projection

$$\Pi : S^1 \times (-\epsilon, \epsilon) \to S^1$$

that projects $N_\epsilon(\alpha)$ onto the image of α.

Definition 7.1. $[\beta] \in \mathcal{C}$ is ϵ-close to $[\alpha] \in \mathcal{C}$ if:

(1) there exist representatives α_1 in $[\alpha]$ and β_1 in $[\beta]$ such that the image of β_1, $\beta_1([0, 2\pi))$, is in $N_\epsilon(\alpha_1)$;

(2) the restriction of Π to the image of β_1 is a diffeomorphism.

As a consequence, β can be written as $\beta(\varphi) = \alpha(\varphi) + \lambda(\varphi)\eta(\varphi)$, with λ periodic.

Definition 7.2. $[\beta] \in \mathcal{C}$ is ϵ-C^2-close to $[\alpha] \in \mathcal{C}$ if $[\beta]$ is ϵ-close to $[\alpha]$ and $|\lambda|_2 < \epsilon$.

In this context, Propositions 4.7 and 5.3 can be rewritten as follows.

PROPOSITION 7.3. *Let $\mathcal{E} \subset \mathcal{C}$ be the subset of equivalence classes of curves with an elliptic diameter. Then $\mathcal{E}$ is open in $\mathcal{C}$.*

PROPOSITION 7.4. *Let $\mathcal{R} \subset \mathcal{E}$ be the subset of equivalence classes of curves with a non-resonant diameter. Then $\mathcal{R}$ is open in $\mathcal{C}$.*

Propositions 5.2 and 6.1 lead to the following.

THEOREM 1. *Any curve in an equivalence class of $\mathcal{E}$ can be approximated by curves such that the associated billiard map has an elliptic island of positive measure.*

Proof. Let α be a representative of $[\alpha] \in \mathcal{E}$. So, α has an elliptic diameter at, we can suppose, $\varphi = 0$. There are three possible cases:

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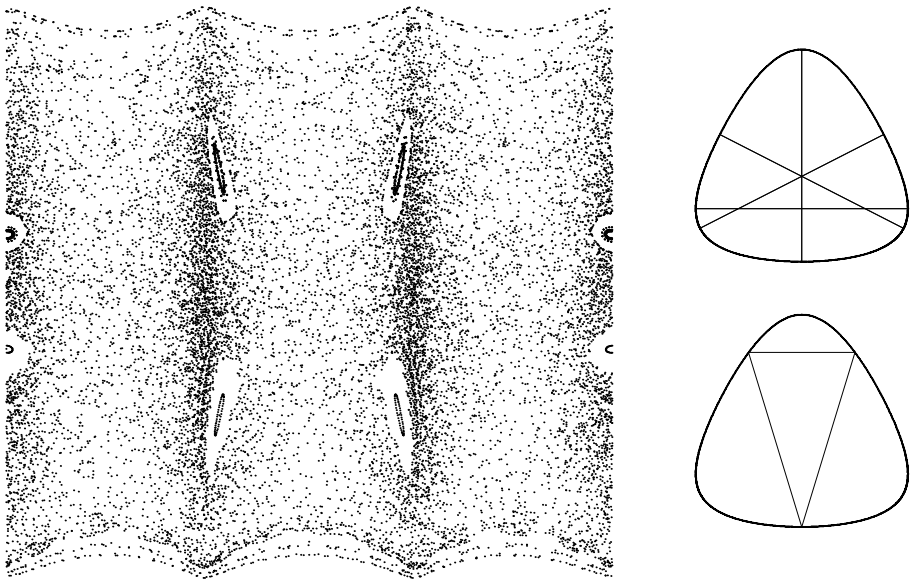


FIGURE 2. The curve $x(t) = \cos t$, $y(t) = 3/(2 - \sin t)$ with its four diameters and a period 3 trajectory and the phase space of the associated billiard map.

(1) 0 is a non-resonant diameter with $\tau_1(\alpha) \neq 0$;
(2) 0 is a non-resonant diameter with $\tau_1(\alpha) = 0$;
(3) 0 is a resonant diameter.

In the first case, there is nothing to prove.

If 0 is a non-resonant diameter with $\tau_1(\alpha) = 0$ then, by Proposition 6.1, α can be approximated by a sequence β_n such that for each n, β_n has a non-resonant diameter with $\tau_1(\beta_n) \neq 0$.

If 0 is a resonant diameter, by Proposition 5.2, α can be approximated by a sequence β_n such that for each n, β_n has a non-resonant diameter. The same reasoning used on the second case leads to the result. □

8. *Conclusions and comments*

Although we can prove the density of elliptic islands surrounding orbits associated to diameters, many questions remain open.

The existence of islands is not equivalent to a non-vanishing first Birkhoff coefficient. For instance, any Birkhoff coefficient different from zero assures, for a sufficiently differentiable curve and avoiding higher-order resonances, the existence of such islands. However, one may ask if there is a strictly convex C^∞ curve with an elliptic diameter such that all the Birkhoff coefficients vanish. If so, how does this fact translate into the dynamics of the associated billiard map?

It may also be that all the isolated diameters of a given curve are hyperbolic or parabolic. This is the case, for instance, of an ellipse with half-axes 1 and $\sqrt{2}$, which has only two diameters: the largest axis, which is hyperbolic, and the smallest one, which is parabolic.

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However, due to the integrability of the system, the parabolic orbit has an island. In [**6**] Kozlov constructs a strictly convex curve, as differentiable as desired, that has exactly three diameters, all of them hyperbolic.

Even when all the diameters are hyperbolic, the examples suggest that a higher-period trajectory is elliptic, if the curve is strictly convex. This is the case for the billiard associated to $x(t) = \cos t$, $y(t) = 3/(2 - \sin t)$. It has exactly four diameters, all of them hyperbolic. The numerical investigation of the phase-space of its associated billiard map shows the existence of a pair of 3-periodic orbits, corresponding to a same triangular trajectory followed in both senses, and its islands (see Figure 2). One may wonder if there is always an elliptic periodic orbit for any sufficiently differentiable and strictly convex curve. A partial answer was given by Dvorin and Lazutkin [**3**] who proved that generically, billiards on C^k convex curves, $k \geq 8$, have a periodic elliptic trajectory with sufficiently large period.

If there are elliptic periodic orbits, can they have any period? In the example above only elliptic orbits of period 3 or multiples of 3, belonging to the same island, are visible in the region of the phase space between the invariant curves predicted by Lazutkin and Douady.

Anyway, if there are elliptic orbits of higher period, we believe that our method of normal perturbations with a third-order contact at the points of reflection of the periodic trajectory with the boundary will give the results we obtained for the 2-periodic orbit.

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