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The upper Perron method for labelled complexes with applications to circle packings

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1. *Introduction*

The construction of geometric surfaces via labelled complexes was introduced by Thurston[**16**, chapter 13], and subsequent applications and developments have appeared in [**1**, **3**, **4**, **5**, **14**, **15**]. The basic idea of using labelled complexes to produce geometric structures is that the vertices of a simplicial triangulation of a surface can be labelled with positive real numbers that collectively determine a metric of constant curvature ± 1 or 0, with possible singularities at vertices, by using the label values to identify 2-simplices of the triangulation with geometric triangles. Beardon and Stephenson[**1**] developed a particularly simple method for producing non-singular surfaces via labelled complexes that is modelled after the classical Perron method for producing harmonic functions, and they applied their method in [**2**] to construct a fairly comprehensive theory of circle packings in general Riemann surfaces. This Perron method was developed more fully by Stephenson and the author in [**3**, **4**] and applied to the study of circle packing points in moduli space. At about the same time and independently of Beardon, Stephenson, and Bowers, Carter and Rodin [**5**] and Doyle[**8**] developed the method for flat surfaces and Minda and Rodin[**14**] developed the method for finite type surfaces. Minda and Rodin [**14**] applied their development to give partial solutions to the labelled complex version of the classical Schwarz–Picard problem that concerns the construction of singular hyperbolic metrics on surfaces with prescribed singularities. In this paper, we modify the aforementioned approaches and examine the upper Perron method for producing non-singular geometric surfaces. This upper method has several advantages over the Perron method as developed previously and provides a complete solution to the labelled complex version of the Schwarz–Picard problem.

Our main results are Theorem 3·3 and its generalization, Theorem 4·1. Theorem 3·3 concerns the existence of a label for a triangulation K of a compact surface that induces a non-singular hyperbolic metric on the realization $|K|$ with prescribed boundary values and prescribed cusp singularities. Theorem 3·3 provides a combinatorial invariant of K and of the number of desired cusp singularities that determines precisely when the upper Perron method produces a desired label. It shows further that this upper method produces a desired label when and only when such a label actually exists, and one exists except in seven exceptional cases. Moreover, a label with the desired properties, if it exists, is unique. Theorem 4·1 generalizes Theorem 3·3 and provides an invariant of K that is used to give necessary and sufficient conditions that determine whether a label exists for K that induces a

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singular hyperbolic metric with prescribed boundary values and prescribed singularities (angle sums) at the vertices.

The primary use of labelled complexes in the current literature is for the construction of circle packings on geometric surfaces, and much of the descriptive terminology involving labelled complexes reflects this association with circle packings. Those readers primarily interested in circle packings will be interested in Theorem 3·3 more so than its generalization, Theorem 4·1, and it is partly for this reason that we have chosen to generalize Theorem 4·1 from Theorem 3·3 rather than deduce the latter as a corollary of the former.

Section 2 reviews the terminology and notation for labelled complexes and presents a brief review of the Perron method as developed in [**4**]. Unlike previous developments, we do not restrict our considerations to orientable surfaces only, and we allow vertices to represent cusp points as in [**14**]. The Perron method produces a packing (non-singular) label as the supremum of subpacking labels, provided a subpacking label exists; thus, the method merely reduces the problem of finding a packing label to the equally difficult problem of finding a subpacking label.

The upper Perron method for finite complexes is introduced in Section 3 with the goal of producing a packing label as the infimum of superpacking labels. The immediate advantage that this method has over the Perron method is that the method is easy to start, i.e. automatically a suitable superpacking label exists. Theorem 3·3 is proved and several applications of the upper method are presented. One application gives a simple and elementary, yet elegant, proof of the Koebe Theorem on circle packings in the 2-sphere. Another analyses a practical algorithm for producing packing labels due in its original form to Thurston[**16**]. Yet another answers a question of Minda and Rodin[**14**] by observing as an immediate consequence of Theorem 3·3 that a hyperbolic packing label for any triangulation of the 2-sphere exists with three prescribed cusp singularities.

In Section 4, the success of the upper method is exploited to solve the compact case of the Boundary Value-Index problem [**14**, problem A], which in turn solves the labelled complex version of the Schwarz–Picard problem.

Section 5 presents an analysis of the seven exceptional cases from Section 3 when no hyperbolic packing label exists and shows that in each of these cases except for one, either a flat or a spherical packing label exists. Of particular interest are the cases where a flat label with cusps exists.

We have learned since the first version of this paper (1991) that at about the same time that we obtained our results, Brett Garret, working under the guidance of Bert Rodin, independently obtained a solution to the Schwarz–Picard problem for labelled complexes; see [**9**].

2. *Singular hyperbolic surfaces via labelled complexes*

Let K be a triangulation of a connected surface, compact or not, orientable or not, with or without boundary, and let $R:K^{(0)}\to(0,\infty]$ be a function that assigns a positive real number or ∞ to each vertex of K. We call $K(R)$ a labelled (simplicial) complex with label R. If R and R' are labels for K, then $R\leqslant R'$ means that $R(v)\leqslant R'(v)$ for every vertex v. If L is a subcomplex of K and R is a label for K, then slight abuse of notation allows us to write $L(R)$ for L labelled with the restriction of R

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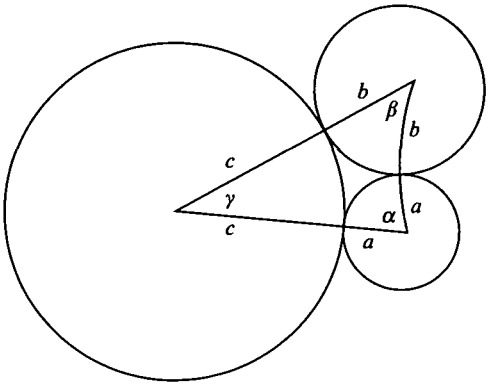


Fig. 1. The triangle $T(a,b,c)$.

to $L^{(0)}$. We use the label R to define a singular hyperbolic structure on the realization $|K|$ of K. First some terminology and elementary facts from hyperbolic geometry are recalled.

For $r=(a,b,c)\in(0,\infty]^3$, $T(r)=T(a,b,c)$ denotes the unique (up to hyperbolic isometry, possibly orientation-reversing) triangle in the hyperbolic plane $\mathbb{H}^2$ with side-lengths $a+b$, $b+c$, $a+c$ determined by circles of respective radii a,b,c. The vertices of $T(r)$ are centres of circles of respective radii a,b,c that are mutually tangent to one another (∞ corresponds to horocycles with ideal centres = ideal vertices). The hyperbolic area, denoted hyp-area, of a triangle with interior angles α, β, γ is $\pi-(\alpha+\beta+\gamma)$. Define partial orders $\leqslant$ and $<$ on 3-tuples via $r=(a,b,c)\leqslant(a',b',c')=r'$ provided that $a\leqslant a'$, $b\leqslant b'$ and $c\leqslant c'$, and $r<r'$ if $r\leqslant r'$ but $r\neq r'$. Henceforth, we shall mean by the term *triple* a 3-tuple in $(0,\infty]^3$. The following lemma describes monotonicity properties of hyperbolic triangles. Precise calculations using the hyperbolic cosine law verifying the lemma appear in [1].

LEMMA 2·1. (i) *If $r\leqslant r'$ for triples r and r', then* hyp-*area* $T(r)\leqslant$ hyp-*area* $T(r')$ *with strict inequality if $r<r'$.*

(ii) *Consider the configuration of Fig.* 1. *If the hyperbolic radii b and c are held constant, then the angles α, β and γ are continuous functions of a. Moreover $\alpha(a)$ is strictly decreasing with $\lim_{a\uparrow\infty}\alpha(a)=0$ and $\lim_{a\downarrow 0}\alpha(a)=\pi$, $\beta(a)$ and $\gamma(a)$ are increasing with strict monotonicity for β (respectively, γ) if $b<\infty$ (respectively, $c<\infty$) and $\lim_{a\downarrow 0}\beta(a)=\lim_{a\downarrow 0}\gamma(a)=0$.*

Let $K(R)$ be a labelled complex. We introduce a hyperbolic metric (Riemannian metric of constant curvature -1) on the geometric realization $|K|$ of K with singularities at the vertices by identifying, for each 2-simplex $\sigma=\langle v_0,v_1,v_2\rangle$ of K, the face σ with the hyperbolic triangle $T(R(v_0),R(v_1),R(v_2))$. One way to make this precise is first to introduce a convex distance function on each edge $\langle v_0,v_1\rangle$ of K that gives that edge length $R(v_0)+R(v_1)$. If one or two vertices have infinite labels, then this is interpreted to mean that the half-open or open edge is given a complete convex metric. Now for each 2-simplex $\sigma=\langle v_0,v_1,v_2\rangle$, let i_σ be an isometry of the boundary of $T(R(v_0),R(v_1),R(v_2))$ to $|\partial\sigma|$ that takes corresponding vertices to corresponding vertices, and extend i_σ to a homeomorphism of $T(R(v_0),R(v_1),R(v_2))$ onto $|\sigma|$ to make the identification. This is ambiguous and requires extra care only when R labels two

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adjacent vertices v_0 and v_1 of σ both with value ∞, in which case there is not a uniquely determined isometry i_σ as there is in all other cases. However, in a hyperbolic triangle with two ideal vertices and another vertex v, ideal or not, there is a unique geodesic containing v that intersects the side opposite v orthogonally. The extra requirement that the isometries i_σ and $i_{\sigma'}$ identify such points of orthogonal intersection whenever σ and σ' share an edge whose vertices both are labelled with ∞ is enough to specify uniquely up to isometry the metric on $|K|$. The resulting hyperbolic realization of the labelled complex $K(R)$ is denoted $|K(R)|$. Any interior vertex v with $R(v) = \infty$ is called a cusp point of $|K(R)|$. The singular metric on $|K(R)|$ determines a distance function ρ_R on the complement of the cusp points of $|K(R)|$: $\rho_R(x, y)$ is the infimum of the lengths of all paths in $|K(R)|$ from x to y. It is a rather subtle point that vertices, if labelled with ∞, are infinitely far from other points. This is true when there are two adjacent vertices both labelled with ∞ only because of the extra requirement that appropriate isometries i_σ and $i_{\sigma'}$ identify points of orthogonal intersection, as discussed above. The reader is referred to [**16**, pp. 3.16, 3.20–3.22] for an enlightening analysis of the consequences of ignoring this extra requirement. The label R is complete if its associated distance function ρ_R is complete. The geometric edge in $|K(R)|$ determined by the 1-simplex $\langle v_0, v_1 \rangle$ is the geodesic segment in $|K(R)|$ with endpoints v_0 and v_1 whose length is $R(v_0) + R(v_1)$. The geometric triangle $|\sigma(R)|$ in $|K(R)|$ determined by the 2-simplex σ is $|\sigma|$ with metric induced by that of $|K(R)|$. The geometric edges in $|K(R)|$ determine a geodesic triangulation of the singular hyperbolic surface $|K(R)|$.

The metric on $|K(R)|$ determines angles at each vertex of each 2-simplex of K; for any vertex v, summing up the angles at v of the geometric triangles to which it belongs determines an angle sum $\theta_R(v)$ at that vertex. Notice that $\theta_R(v) = 0$ for a vertex v if and only if $R(v) = \infty$, and always $\theta_R(v) < \pi \deg v$ where $\deg v$, the degree of the vertex v, is the number of vertices adjacent to v. Let $\mathbf{V}_{\text{cusps}}$ be a set, possibly empty, of interior vertices of K. The labelled complex $K(R)$ is a subpacking, packing, or superpacking with cusp set $\mathbf{V}_{\text{cusps}}$ if the angle sum at each interior vertex other than those in $\mathbf{V}_{\text{cusps}}$ is greater than or equal to 2π, equal to 2π, or less than or equal to 2π, respectively, and $R(v) = \infty$ for every v in $\mathbf{V}_{\text{cusps}}$. The angle sums at any boundary vertices are irrelevant. The terminology comes from applications of labelled complexes to circle packings. There is a circle packing on the hyperbolic surface $|K(R)|$ whenever $K(R)$ is a packing that is given as follows. Each finite label value $R(v)$ at an interior vertex v provides the radius for a hyperbolic circle in $|K(R)|$ centred at v. Each infinite label value $R(v) = \infty$ at an interior vertex corresponds to a circular horocyclic curve in $|K(R)|$ formed as the union of certain distinguished 'horocyclic sectors' in the geometric triangles $|\sigma(R)|$ that contain v. Recall that each such $|\sigma(R)|$ is isometric with a hyperbolic triangle in $\mathbb{H}^2$ determined by three mutually tangent circles, one of which is a horocycle whose 'centre' is the ideal vertex v. The distinguished 'horocyclic sector' is the intersection of that triangle with that horocycle. The interested reader is referred to [**1**, **3**, **4**, **14**, **15**] for more on the connections between labelled complexes and circle packings. The label R making $K(R)$ a subpacking, packing, or superpacking (with cusp set $\mathbf{V}_{\text{cusps}}$) is a subpacking, packing, or superpacking label (with cusp set $\mathbf{V}_{\text{cusps}}$), respectively. An interior vertex whose angle sum differs from 2π represents a singularity for the induced metric, and in particular represents a cusp singularity when the angle sum is equal to zero. If

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$K(R)$ is a packing with cusp set $\mathbf{V}_{\text{cusps}}$, then the metric extends to a non-singular hyperbolic metric on the complement in $|K|$ of the cusp points.

We review briefly without proofs the development of the Perron method presented in [**4**, section 3], augmented by the removal of the unnecessary restriction of orientability and by the observation of [**14**] that cusp singularities can be handled easily by infinite labels. Recall that K is a triangulation of a connected surface, possibly non-compact, non-orientable, or with non-empty boundary. If $\mathscr{R}$ is a collection of labels for K, let $\sup \mathscr{R}$ be the function $K^{(0)} \to [0, \infty]$ defined by

$$\sup \mathscr{R}(v) = \sup \{R(v) : R \in \mathscr{R}\},$$

for each vertex v of K (we use the convention $\sup \emptyset = 0$). Note that $\sup \mathscr{R}$ is a label whenever $\mathscr{R}$ is non-empty. The next proposition reduces the problem of finding a packing label for K to that of finding a subpacking label for K. The label $\sup \mathscr{R}$ described in this proposition, which is a slight generalization of [**4**, proposition 3·2], is termed the maximal label for K with cusp set $\mathbf{V}_{\text{cusps}}$.

PROPOSITION 2·2. *Let $\mathscr{R}$ be the collection of labels for K such that the angle sum $\theta_R(v)$ is greater than or equal to 2π whenever v is an interior vertex not in the cusp set $\mathbf{V}_{\text{cusps}}$. If $\mathscr{R}$ is non-empty, then* $\sup \mathscr{R}$ *is a packing label for K with cusp set $\mathbf{V}_{\text{cusps}}$ that assigns the value ∞ to each boundary vertex (and to each vertex in $\mathbf{V}_{\text{cusps}}$). Moreover, if, in addition, K has bounded degree, then* $\sup \mathscr{R}$ *is a complete label.*

Consider now the boundary value problem: given $f: \partial K^{(0)} \to (0, \infty]$ does there exist a packing label with boundary values determined by f? We say that K has finite boundary if $\partial K^{(0)}$ is finite.

PROPOSITION 2·3. *Let $\mathscr{R}$ be as in Proposition 2·2 and let $\mathscr{R}_f$ be the subset of $\mathscr{R}$ whose elements assign values to boundary vertices less than or equal to the corresponding ones assigned by $f: \partial K^{(0)} \to (0, \infty]$. If $\mathscr{R}_f$ is non-empty, then* $\sup \mathscr{R}_f$ *is a packing label for K with cusp set $\mathbf{V}_{\text{cusps}}$ whose boundary values are given by f, i.e.* $\sup \mathscr{R}_f | \partial K^{(0)} = f$. *Moreover, if K has finite boundary, then $\mathscr{R}_f$ is non-empty if and only if $\mathscr{R}$ is non-empty and if, in addition, K has bounded degree, then* $\sup \mathscr{R}_f$ *is a complete label.*

The proofs of propositions 2·2 and 2·3 are essentially those of propositions 3·2 and 3·3, respectively, of [**4**], with simple modifications to take care of the cusp singularities. Combining the previous proposition with [**2**, proposition 2 (*e*)] produces the following solution to the boundary value problem.

COROLLARY 2·4. *If K has bounded degree and finite, non-empty boundary, then for every $f: \partial K^{(0)} \to (0, \infty]$, the set $\mathscr{R}_f$ of Proposition 2·3 is non-empty and* $\sup \mathscr{R}_f$ *is a complete packing label for K with cusp set $\mathbf{V}_{\text{cusps}}$ whose boundary values are given by f.*

3. *The upper Perron method for superpackings with applications*

The Perron method as reviewed above produces a packing label with prescribed cusp singularities and boundary values by approaching that label from below through subpacking labels. The drawback of this method is that one must start by showing that a subpacking label exists ($\mathscr{R}$ or $\mathscr{R}_f$ is non-empty), which can be difficult. We present in this section an alternate method that does not have this deficiency; namely, form the infimum of a suitable collection $\mathscr{R}$ of superpacking labels. The

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immediate advantage that this has over the Perron method as developed thus far is that it is easy to start the method since a superpacking label with boundary values at least as large as the desired ones always exists: just label every vertex by ∞. The disadvantage is that the method is liable to produce a function $\inf \mathscr{R}$ that assigns value zero to all interior vertices that are not in the cusp set. This in fact happens, even for complexes for which packing labels exist, whenever K is infinite. We restrict ourselves to the case where K is finite and produce a combinatorial invariant of K that is a complete indicator of when this upper method produces a packing label (i.e. no zero labels) and when it does not. The result is that the upper method produces a packing label with prescribed cusp singularities and boundary values if and only if such a packing label exists. The finitely many cases where no such packing labels exist are explored in Section 5.

First recall some basic well-known facts about compact surfaces. All topological types of connected closed (= compact with empty boundary) surfaces are represented in the double sequence

$$\dots, S_{-2}, S_{-1}, S_0, S_1, S_2, \dots,$$

where $S_0 = \mathbb{S}^2$ is the 2-sphere, S_g for $g > 0$ is formed by attaching g handles to $\mathbb{S}^2$, and S_g for $g < 0$ is formed by attaching $|g|$ cross-caps to $\mathbb{S}^2$. The genus of S_g is $|g|$, S_g is orientable or non-orientable as $g \geqslant 0$ or $g < 0$, and the Euler characteristic of S_g is

$$\chi(S_g) = \begin{cases} 2-2g & g \geqslant 0 \\ 2+g & g < 0. \end{cases}$$

For integers g and n, when $n \geqslant 0$, let $S_{g,n}$ be the surface obtained by removing the interiors of n pairwise disjoint discs from S_g. Then the topological types of all compact connected surfaces are represented in the collection $\{S_{g,n} : g, n \in \mathbb{Z},\ n \geqslant 0\}$, and in addition $S_{g,n}$ is homeomorphic to $S_{g',n'}$ if and only if $(g,n) = (g',n')$. The genus of $S_{g,n}$ is defined to be $|g|$ and the Euler characteristics satisfies

$$\chi(S_{g,n}) = \chi(S_{g,0}) - n = \begin{cases} 2-2g-n & g \geqslant 0 \\ 2+g-n & g < 0. \end{cases}$$

Massey [13] is a standard reference for the elementary topology of surfaces and Thurston [16] and Casson [6] both contain a wealth of information on the geometry of surfaces.

Throughout the remainder of this section, K is a triangulation of a compact connected surface, possibly with non-empty boundary, that has F faces, E edges, and V vertices. The vertex set $K^{(0)}$ of K is decomposed into pairwise disjoint sets:

$$K^{(0)} = \mathbf{V}_{\text{int}} \cup \mathbf{V}_{\text{cusps}} \cup \mathbf{V}_{\partial},$$

where $\mathbf{V}_{\partial} = \partial K^{(0)}$, $\mathbf{V}_{\text{cusps}}$ is a fixed set, possibly empty, of interior vertices, and $\mathbf{V}_{\text{int}}$ is the complement in $K^{(0)}$ of $\mathbf{V}_{\text{cusps}} \cup \mathbf{V}_{\partial}$. The number of vertices in these three sets $\mathbf{V}_{\text{int}}$, $\mathbf{V}_{\text{cusps}}$, and $\mathbf{V}_{\partial}$ is denoted respectively by V_{int}, V_{cusps}, and V_{∂}. A function $f: \mathbf{V}_{\partial} \to (0, \infty]$ is assumed given, and our interest is in obtaining a packing label for K with cusp set $\mathbf{V}_{\text{cusps}}$ and boundary values given by f.

Let $\mathscr{R}$ be the set of all superpacking labels that assign infinite values to the vertices in $\mathbf{V}_{\text{cusps}}$ and assign values to the vertices in $\mathbf{V}_{\partial}$ greater than or equal to the corresponding ones that f assigns. Obviously $\mathscr{R}$ is non-empty since it contains the

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label that assigns value ∞ to every vertex of K. The function $\inf \mathscr{R}: K^{(0)} \to [0, \infty]$ is defined by

$$\inf \mathscr{R}(v) = \inf \{R(v) : R \in \mathscr{R}\},$$

for each vertex v of K. Notice that $\inf \mathscr{R}(v) = \infty$ if $v \in \mathbf{V}_{\text{cusps}}$, $\inf \mathscr{R}(v) \geqslant f(v)$ if $v \in \mathbf{V}_{\partial}$, and $\inf \mathscr{R}(v) < \infty$ if $v \in \mathbf{V}_{\text{int}}$. The latter statement follows from the monotonicity result Lemma 2·1(ii). We work now to determine when $\inf \mathscr{R}$ is a packing label for K with cusp set $\mathbf{V}_{\text{cusps}}$ and boundary values given by f.

Let R be a label for K, not necessarily in $\mathscr{R}$. If $\mathbf{V}$ is a collection of vertices of K, the angle sum of $\mathbf{V}$ determined by R is

$$\Theta_R(\mathbf{V}) = \sum_{v \in \mathbf{V}} \theta_R(v),$$

and the shortage of $\mathbf{V}$ determined by R is

$$s_R(\mathbf{V}) = \sum_{v \in \mathbf{V}} (2\pi - \theta_R(v)).$$

By convention, if $\mathbf{V}$ is empty, then $\Theta_R(\mathbf{V})$ and $s_R(\mathbf{V})$ are defined to be zero. Of particular importance are the total shortage s_R and the boundary angle sum Θ_R:

$$s_R = s_R(\mathbf{V}_{\text{int}}) \quad \text{and} \quad \Theta_R = \Theta_R(\mathbf{V}_{\partial}).$$

If L is a subcomplex of K, then $A_R(L)$ denotes the hyperbolic area of the subset $|L|$ of $|K(R)|$, i.e. $A_R(L)$ is the sum of the hyperbolic areas, determined by the label R, of the faces of L. We write $A_R = A_R(K)$.

LEMMA 3·1. *Let R and R' be labels for K. Then*

(i) *if $R \leqslant R'$ and $R = R'$ on $\mathbf{V}_{\partial}$, then $\Theta_R \leqslant \Theta_{R'}$;*

(ii) *the label $R'' = \min\{R, R'\}$ satisfies $\theta_{R''}(v) \leqslant \max\{\theta_R(v), \theta_{R'}(v)\}$ for each vertex v; in particular, $R'' \in \mathscr{R}$ whenever $R, R' \in \mathscr{R}$;*

(iii) *there exists a sequence $\{R_i\}$ in $\mathscr{R}$ such that $R_i \to \inf \mathscr{R}$ as $i \to \infty$.*

Proof. Statement (i) is a quick consequence of Lemma 2·1(ii). For statement (ii), let v be a vertex of K and without loss of generality assume that $R''(v) = \min\{R(v), R'(v)\} = R(v)$. The monotonicity result for angles in Lemma 2·1(ii) implies that $\theta_{R''}(v) \leqslant \theta_R(v)$ since $R''(v) = R(v)$ and $R''(w) \leqslant R(w)$ for any vertex w adjacent to v. For (iii), for each positive integer i and vertex v of K, let $R_{i,v} \in \mathscr{R}$ assign a value to v that lies within 2^{-i} of that assigned to v by $\inf \mathscr{R}$. The label $R_i = \min\{R_{i,v} : v \in K^{(0)}\}$ assigns values to the vertices of K that lie within 2^{-i} of the ones that $\inf \mathscr{R}$ assigns, and (ii) implies that $R_i \in \mathscr{R}$. ∎

Assume now that R is a label that assigns value ∞ to the vertices in $\mathbf{V}_{\text{cusps}}$ and, assuming an ordering on the faces of K, let $\alpha_i, \beta_i, \gamma_i$ for $i \in \{1, \ldots, F\}$ denote the angles that R determines on the ith face of K. Then

$$A_R = \sum_{i=1}^{F} [\pi - (\alpha_i + \beta_i + \gamma_i)] = F\pi - \Theta_R(K^{(0)}) = F\pi - (\Theta_R - s_R + 2\pi V_{\text{int}}),$$

$$A_R + \Theta_R - s_R = (F - 2V_{\text{int}})\pi. \tag{1}$$

Notice that the right hand side of (1), $(F - 2V_{\text{int}})\pi$, is a combinatorial invariant of K and V_{cusps}, i.e. it is independent of the label R and depends only on the complex K and the number of vertices in the cusp set. The number of angles in the complex K

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is $3F$ since each face has three angles. Counting the angles differently using the fact that there are two angles opposite every edge except for one opposite each boundary edge, and that the number of boundary vertices V_∂ equals the number of boundary edges, we get

$$3F = 2E - V_\partial.$$

Using $\chi = \chi(K) = F - E + V$, we get

$$F - 2V + V_\partial = -2\chi,$$

and using $V = V_{\text{int}} + V_{\text{cusps}} + V_\partial$, we get

$$F - 2V_{\text{int}} = -2\chi + 2V_{\text{cusps}} + V_\partial. \tag{2}$$

Let $|g|$ be the genus of $|K|$ with g positive or negative as K is orientable or not, and suppose that K has $n \geqslant 0$ boundary components. Then $|K|$ is homeomorphic to $S_{g,n}$ and (2) can be written

$$\left.\begin{aligned} F - 2V_{\text{int}} &= -2\chi(S_g) + 2n + 2V_{\text{cusps}} + V_\partial \\ &= \begin{cases} 4g - 4 + 2n + 2V_{\text{cusps}} + V_\partial & g \geqslant 0 \\ -2g - 4 + 2n + 2V_{\text{cusps}} + V_\partial & g < 0. \end{cases} \end{aligned}\right\} \tag{3}$$

Note that $2n + V_\partial$ is greater than or equal to 5 if K has non-empty boundary. Immediately, we have the following lemma.

LEMMA 3·2. *The invariant $F - 2V_{\text{int}}$ is positive except in the following seven exceptional cases: $n = 0$ and $(g, V_{\text{cusps}}) \in \{(1, 0), (0, 2), (0, 1), (0, 0), (-1, 1), (-1, 0), (-2, 0)\}$. The invariant is zero in the cases $(1, 0)$, $(0, 2)$, $(-1, 1)$, and $(-2, 0)$, which respectively correspond to the cases where K triangulates the torus, the 2-sphere with two cusps, the projective plane with one cusp, and the Klein bottle. The invariant is negative in the cases $(0, 1)$, $(0, 0)$, and $(-1, 0)$, which respectively correspond to the cases where K triangulates the 2-sphere with one cusp, the 2-sphere, and the projective plane.*

Define a collection $\mathbf{V}$ of vertices of K to be edge-path connected provided that every two vertices in $\mathbf{V}$ are endpoints of an edge-path in K, the vertices of which lie in $\mathbf{V}$. As usual, an edge-path component of $\mathbf{V}$ is a maximal edge-path connected subcollection of $\mathbf{V}$. The main result of this section is the next theorem, which determines when the function $\inf \mathscr{R}$ is a packing label for K with cusp set $\mathbf{V}_{\text{cusps}}$ and boundary values given by f.

THEOREM 3·3. *The following are equivalent:*

(i) *the invariant $F - 2V_{\text{int}}$ is positive, which occurs in all but the seven exceptional cases of Lemma 3·2;*

(ii) *the function $\inf \mathscr{R}$ is never zero;*

(iii) *the function $\inf \mathscr{R}$ is the unique packing label for K with cusp set $\mathbf{V}_{\text{cusps}}$ and boundary values given by f;*

(iv) *there exists a packing label for K with cusp set $\mathbf{V}_{\text{cusps}}$ and boundary values given by f.*

Proof. (i) $\Rightarrow$ (ii): Observe first that if $R \in \mathscr{R}$, then s_R is non-negative since R is a superpacking label. Assume that the invariant $F - 2V_{\text{int}}$ is positive. Our task is to show that the zero set $\mathbf{W} = (\inf \mathscr{R})^{-1}(0)$ of $\inf \mathscr{R}$ is empty. Let $\mathbf{V}$ be an edge-path component of $\mathbf{W}$ and observe that $\mathbf{W}$ and hence also $\mathbf{V}$ is contained in the set $\mathbf{V}_{\text{int}}$. Let L be the subcomplex of K that consists of all the simplices of K that meet $\mathbf{V}$. The subset $|L|$ of $|K|$ is the union of all faces of K that hit $\mathbf{V}$ and hence is a 2-manifold (with

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boundary) on the complement of a (possibly empty) set of vertices. There are three possibilities: either $L = \emptyset$, $L = K$, or $\emptyset \neq L \neq K$. We show that the latter two possibilities cannot occur, which forces L to be empty. This implies that $\mathbf{W}$ is empty also and hence (ii) holds.

If $L = K$, then every face in K contains a vertex in $\mathbf{V}$. Since $\inf \mathscr{R}$ is identically zero on $\mathbf{V}$, Lemma 3·1 (iii) implies that there exists a sequence $\{R_i\}$ in $\mathscr{R}$ such that $R_i(v) \to 0$ as $i \to \infty$ for every vertex v in $\mathbf{V}$. Since each face has a vertex in $\mathbf{V}$, the hyperbolic areas of each face determined by the labels $\{R_i\}$ converge to zero as $i \to \infty$, and this implies that $A_{R_i} \to 0$ as $i \to \infty$. Also, Lemma 2·1 (ii) implies that $\theta_{R_i}(v) \to 0$ as $i \to \infty$ for every boundary vertex v, since each face containing a boundary vertex contains also a vertex in $\mathbf{V}$ whose label values converge to zero while the label values on the boundary vertices are bounded away from zero. This implies that $\Theta_{R_i} \to 0$ as $i \to \infty$. Since s_R is non-negative for all $R \in \mathscr{R}$, equation (1) implies that

$$A_R + \Theta_R \geqslant (F - 2V_{\text{int}})\pi \tag{4}$$

for all $R \in \mathscr{R}$. We have argued though that $A_{R_i} + \Theta_{R_i} \to 0$ as $i \to \infty$, contradicting (4) since $F - 2V_{\text{int}}$ is positive. We conclude then that $L \neq K$.

We next assume that $\emptyset \neq L \neq K$, hence $|L|$ is a 2-manifold with non-empty boundary on the complement of a (possibly empty) set $\mathbf{U}$ of vertices. For simplicity, assume for the moment that $\mathbf{U}$ is empty so that $|L|$ is a compact bordered surface. Notice that $|L|$ is connected. We shall reach a contradiction by showing that this case reverts to that of the previous paragraph. Indeed, the vertex set $L^{(0)}$ of L partitions as $\mathbf{V}'_{\text{int}} \cup \mathbf{V}'_{\text{cusps}} \cup \mathbf{V}'_{\partial}$, where $\mathbf{V}'_{\partial}$ is the set of boundary vertices of L, $\mathbf{V}'_{\text{cusps}}$ consists of the interior vertices of L that lie in $\mathbf{V}_{\text{cusps}}$, and $\mathbf{V}'_{\text{int}}$ consists of the remaining vertices. Notice that the set $\mathbf{V}$ is contained in $\mathbf{V}'_{\text{int}}$, which in turn is contained in $\mathbf{V}_{\text{int}}$. For a boundary vertex v of L, $\inf \mathscr{R}(v)$ is not zero, for otherwise v must be a vertex in $\mathbf{W}$ that, by the definition of L, must be adjacent to a vertex in $\mathbf{V}$. Since $\mathbf{V}$ is an edge-path component of $\mathbf{W}$, this forces v to lie in $\mathbf{V}$, which in turn forces v to lie in the interior of $|L|$. Therefore $\inf \mathscr{R}(v)$ is not zero whenever $v \in \mathbf{V}'_{\partial}$ and we conclude that the minimum value μ of $\inf \mathscr{R}$ on the boundary vertices of L is positive.

Let $\mathscr{R}'$ be the set of superpacking labels for L with cusp set $\mathbf{V}'_{\text{cusps}}$ and boundary values all greater than or equal to the positive constant μ. For a label R in $\mathscr{R}$, let R' be the restriction of R to $L^{(0)}$. Then $R' \in \mathscr{R}'$ and this implies that $\inf \mathscr{R}'$ is identically zero on $\mathbf{V}$. Let F' be the number of faces of L and observe that since L has non-empty boundary, equation (3) implies that $F' - 2V'_{\text{int}}$ is positive. We now have reverted to the situation of the second paragraph of this proof: every face of L contains a vertex at which $\inf \mathscr{R}'$ is zero, and the invariant $F' - 2V'_{\text{int}}$ for L is positive. The argument of that paragraph produces a contradiction.

The remaining case to consider is when $\mathbf{U}$ is non-empty, in which case $|L|$ is a 2-manifold with non-empty boundary on the complement of the non-empty set $\mathbf{U}$ of vertices. The complex L then can be obtained as the quotient complex of a finite simplicial complex L' that triangulates a connected, compact bordered surface, where the quotient map merely identifies some of the boundary vertices of L'. As the reader may attest, the argument in the case that $\mathbf{U}$ is empty that is presented above, suitably 'lifted' to L', now works on L' to reach a contradiction. We conclude then that $K = L$ and $K \neq L \neq \emptyset$ both lead to contradictions and therefore L is empty, which finishes the proof of the implication (i) $\Rightarrow$ (ii).

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(ii) $\Rightarrow$ (iii): The proof is the 'upper' version of the proof of [1, theorem 3]. Assume $\inf \mathscr{R}$ is never zero. Continuity of angles with respect to radii (Lemma 2·1(ii)) guarantees that the angle sum determined by $\inf \mathscr{R}$ at each vertex in $\mathbf{V}_{\text{int}}$ is no greater than 2π. If there exist v in $\mathbf{V}_{\text{int}}$ such that $\theta_{\inf \mathscr{R}}(v) < 2\pi$, then Lemma 2·1(ii) guarantees that by reducing the value of $\inf \mathscr{R}$ only at the vertex v, we may obtain a label $R^{\dagger} \in \mathscr{R}$ with $R^{\dagger} = \inf \mathscr{R}$ on $K^{(0)} - \{v\}$ and $R^{\dagger}(v) < \inf \mathscr{R}(v)$, contradicting the definition of $\inf \mathscr{R}$. This follows because Lemma 2·1(ii) implies that reducing the value of the label R only at v causes a (continuous) increase in the angle sum at v, decreases in angle sums at vertices adjacent to v, and no change in the angle sums of remaining vertices. We conclude that $\inf \mathscr{R}$ is a packing label with cusp set $\mathbf{V}_{\text{cusps}}$. As for boundary values, let R be the label for K with $R = f$ on $\mathbf{V}_{\partial}$ and $R \equiv \infty$ on $\mathbf{V}_{\text{int}} \cup \mathbf{V}_{\text{cusps}}$. Then obviously $R \in \mathscr{R}$ and therefore $\inf \mathscr{R} = f$ on $\mathbf{V}_{\partial}$. Uniqueness follows from (1). Indeed, if R is any packing label with cusp set $\mathbf{V}_{\text{cusps}}$ and boundary values given by f, then the inequality $\inf \mathscr{R} \leqslant R$ and Lemma 2·1(i) imply that $A_{\inf \mathscr{R}} \leqslant A_R$ with strict inequality if $\inf \mathscr{R} \neq R$. In addition, since $\inf \mathscr{R} = R$ on $\mathbf{V}_{\partial}$, $\Theta_{\inf \mathscr{R}} \leqslant \Theta_R$ by Lemma 3·1(i). Since $\inf \mathscr{R}$ and R are both packing labels, the shortages $s_{\inf \mathscr{R}}$ and s_R are both zero and (1) then implies

$$A_{\inf \mathscr{R}} + \Theta_{\inf \mathscr{R}} = A_R + \Theta_R.$$

Since $\Theta_{\inf \mathscr{R}} \leqslant \Theta_R$, we conclude that $A_{\inf \mathscr{R}} \geqslant A_R$. Therefore, $A_{\inf \mathscr{R}} = A_R$ and this implies that $\inf \mathscr{R} = R$.

(iii) $\Rightarrow$ (iv): Immediate.

(iv) $\Rightarrow$ (i): Suppose R is a packing label with cusp set $\mathbf{V}_{cusps}$ and boundary values given by f. Then $s_R = 0$ and (1) becomes

$$(F - 2V_{\text{int}})\,\pi = A_R + \Theta_R.$$

Since $A_R > 0$ and $\Theta_R \geqslant 0$, we conclude that $F - 2V_{\text{int}}$ is positive. ∎

COROLLARY 3·4. *If the group G acts as a group of symmetries (i.e. simplicial isomorphisms) on K such that the cusp set $\mathbf{V}_{\text{cusps}}$ is G-invariant and the boundary function f is constant on G-orbits, then the packing label R for K with cusp set $\mathbf{V}_{\text{cusps}}$ and boundary values given by f is constant on G-orbits. Consequently, the action of G on the vertices of K extends to an isometric action on the hyperbolic realization $|K(R)|$ that preserves the geodesic triangulation determined by the geometric edges.*

COROLLARY 3·5. *The invariant $F - 2V_{\text{int}}$ is non-positive if and only if the function $\inf \mathscr{R}$ is identically zero on $\mathbf{V}_{\text{int}}$.*

Proof. The reverse implication is immediate from Theorem 3·3. For the forward implication, suppose that the invariant $F - 2V_{\text{int}}$ is non-positive so that in particular K has empty boundary. Then Theorem 3·3 implies that $\inf \mathscr{R}$ is zero at some vertex in $\mathbf{V}_{\text{int}}$ and our task is to show that $\inf \mathscr{R}$ is identically zero on $\mathbf{V}_{\text{int}}$. We claim first that every face of K contains a vertex in the zero set $\mathbf{W} = (\inf \mathscr{R})^{-1}(0)$. Indeed, suppose that there is a face $\sigma = \langle v_0, v_1, v_2 \rangle$ of K with none of its vertices v_0, v_1, v_2 in $\mathbf{W}$. Since K has empty boundary, the complex $K' = K - \{\sigma\}$ is a triangulation of the compact connected surface $|K| - \operatorname{int} \sigma$ with non-empty boundary $\partial K' = \partial \sigma$. Let $\mathbf{V}'_{\text{cusps}} = \mathbf{V}_{\text{cusps}} - \{v_0, v_1, v_2\}$, let $\mu = \min \{\inf \mathscr{R}(v_i) : i = 0, 1, 2\}$, and observe that μ is positive. Let $\mathscr{R}'$ be the set of all superpacking labels for K' with cusp set $\mathbf{V}'_{\text{cusps}}$ and boundary values at least as large as μ, and observe that $\mathscr{R}$ is contained in $\mathscr{R}'$. Since

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K' has non-empty boundary, equation (3) implies that the invariant of Theorem 3·3 for K' is positive, i.e. statement (i) of Theorem 3·3 holds for K'. Theorem 3·3 then implies that $\inf \mathscr{R}'$ is never zero and, since $\mathscr{R}$ is contained in $\mathscr{R}'$, $\inf \mathscr{R}$ is never zero. This contradicts the fact that $\inf \mathscr{R}$ is zero at some vertex of $\mathbf{V}_{\text{int}}$, and we conclude that every face of K contains a vertex v for which $\inf \mathscr{R}(v) = 0$.

Let v_0 be any vertex in $\mathbf{V}_{\text{int}}$ and assume that $\inf \mathscr{R}(v_0) \neq 0$. Let $\{R_i\}$ be a sequence in $\mathscr{R}$ for which $R_i \to \inf \mathscr{R}$ as $i \to \infty$ and for which the angle sum $\theta_{R_i}(v_0) = 2\pi$. Such a sequence exists by Lemma 3·1 (iii) since any label $R \in \mathscr{R}$ can be modified by reducing the value of R only at v_0 to obtain a label $R^\dagger \in \mathscr{R}$ with $R^\dagger = R$ on $K^{(0)} - \{v_0\}$ and $\theta_{R^\dagger}(v_0) = 2\pi$. To see this, recall from the proof of the implication (ii) $\Rightarrow$ (iii) of Theorem 3·3 that reducing the value of R only at v_0 causes either decreases or no changes in angle sums at all vertices of K with the exception of v_0, where the angle sum increases. By the continuity and monotonicity properties expressed in Lemma 2·1 (ii), since $\lim_{a \downarrow 0} \alpha(a) = \pi$, $\theta_R(v_0) \leqslant 2\pi$, and v_0 is an interior vertex, there is a unique positive value $R^\dagger(v_0) \leqslant R(v_0)$ at which the angle sum at v_0 is equal to 2π. Now let $\sigma = \langle v_0, v_1, v_2 \rangle$ be any face of K that contains v_0. Since $\inf \mathscr{R}(v_0) \neq 0$ we may assume without loss of generality that $\inf \mathscr{R}(v_1) = 0$. We have that $R_i(v_0) \geqslant \inf \mathscr{R}(v_0) > 0$ for all i and $R_i(v_1) \to 0$ as $i \to \infty$. This implies by Lemma 2·1 (ii) again, since $\lim_{a \downarrow 0} \beta(a) = 0$, that the angles in the face σ at the vertex v_0 determined by the labels R_i converge to zero as $i \to \infty$. Since σ is an arbitrary face that contains v_0, this forces $\theta_{R_i}(v_0)$ to converge to zero as $i \to \infty$ and this contradicts $\theta_{R_i}(v_0) = 2\pi$. We conclude that no such v_0 exists and $\inf \mathscr{R}$ is identically zero on $\mathbf{V}_{\text{int}}$. ∎

Our initial applications of Theorem 3·3 reproduce results concerning circle packings that have appeared in [**1**] and [**16**]. Let S be a surface equipped with intrinsic metric, meaning a complete metric of constant curvature ± 1 or 0. A disc in S is a closed metric ball (in the intrinsic metric) that is homeomorphic to a 2-cell and its boundary is a circle whose interior is the corresponding open ball. A circle packing in S is a collection $\mathscr{C}$ of circles whose interiors are pairwise disjoint and whose pairwise intersections are either empty or singleton sets. In the latter case, the two circles are tangent. Underlying each circle packing is a canonically defined 2-dimensional abstract simplicial complex $K(\mathscr{C})$, called the abstract carrier of $\mathscr{C}$, that encodes the combinatorics of the packing: the vertices of $K(\mathscr{C})$ are the circles of $\mathscr{C}$, two vertices determine an edge if and only if the two circles are tangent, and three vertices determine a face if and only if the three circles determine a unique triangular interstice that contains no circles from $\mathscr{C}$. See [**4**] for a more detailed review of terminology. Our first application of Theorem 3·3 is to circle packings in the 2-sphere $\mathbb{S}^2$.

Application 1. *Koebe Theorem.* Our purpose in this first application is to provide a self-contained elementary proof of a theorem of Koebe [**11**] that Thurston resurrected in [**16**, chapter 13]. Alternate elementary proofs that differ substantially from ours also have been given in [**1**, Appendix] and [**12**]. We should note that orientation-reversing (circle preserving) transformations are included under the term Möbius transformation.

KOEBE THEOREM. *If T is a triangulation of the 2-sphere $\mathbb{S}^2$, then there is a circle packing in $\mathbb{S}^2$ whose abstract carrier is isomorphic to T, and this circle packing is unique up to Möbius transformations of $\mathbb{S}^2$.*

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Proof. Let v_0 be a vertex in T and let K be the subcomplex of T obtained by deleting v_0 and simplices containing v_0. The complex K then triangulates a disc and the invariant $F-2V_{\text{int}}$ for K is positive by equation (3), where F is the number of faces of K and V_{int} is the number of interior vertices. Apply Theorem 3·3 with $\mathbf{V}_{\text{cusps}}$ empty and the boundary function f identically ∞ to obtain the packing label R for K with infinite boundary values. The surface $S(R)$ obtained by gluing V_∂ hyperbolic half-planes to $|K(R)|$ along the V_∂ boundary geodesics and deleting the boundary vertices is a complete simply connected hyperbolic surface with empty boundary and, as such, is isometric to $\mathbb{H}^2$. As above, the labelled complex $K(R)$ determines a circle packing $\mathscr{C}$ in $S(R)$ whose abstract carrier is isomorphic to K and for which the boundary vertices correspond to horocycles. Hence, by the usual identification of the lower hemisphere of $\mathbb{S}^2$ with $\mathbb{H}^2$, $\mathscr{C}$ is a circle packing in $\mathbb{S}^2$ contained in the lower hemisphere and the boundary circles (horocycles) are all tangent to the equator = unit circle $\partial\Delta$. The circle packing $\mathscr{C} \cup \{\partial\Delta\}$ has abstract carrier isomorphic to T. Uniqueness is just as straightforward. If $\mathscr{D}$ is any circle packing in $\mathbb{S}^2$ whose carrier is isomorphic to T, then by applying a Möbius transformation, we may assume that the circle corresponding to v_0 is $\partial\Delta$ and all other circles lie in the lower hemisphere, which is identified with $\mathbb{H}^2$. Labelling a vertex of K by the hyperbolic radius of its corresponding circle from $\mathscr{D}$ produces a packing label R' for K with infinite boundary values. By uniqueness in Theorem 3·3, $R = R'$, and this implies that there is a hyperbolic isometry carrying $\mathscr{D}-\{\partial\Delta\}$ to $\mathscr{C}$. Since, under our identifications, hyperbolic isometries are Möbius transformations of $\mathbb{S}^2$ that set-wise fix $\partial\Delta$ and preserve hemispheres, we are done. ∎

Application 2. *Beardon–Stephenson: circle packings on Riemann surfaces.* Beardon and Stephenson in the papers [**1**, **2**] generalized and extended Thurston's results on circle packings and constructed a fairly comprehensive theory of circle packings in general Riemann surfaces. In particular, they included both non-compact and bordered surfaces into the theory, and they solved the boundary value problem for triangulations of the disk. Though not contained explicitly in their papers, the solution of the boundary value problem in the setting of finite complexes is contained implicitly. Their methods are decidedly modelled after methods in the theory of Riemann surfaces that often rely upon passing to the universal covering surface. Contrasting this approach is the approach in [**3**, **4**], termed there the *in situ* approach, that uses the Perron method as reviewed in Section 2. Theorem 3·3 provides an alternate, fully self-contained *in situ* approach to the Beardon–Stephenson results in [**2**] that apply to compact hyperbolic surfaces, with the generalization that includes cusp singularities as in [**14**]. We see how Theorem 3·3 reproduces and extends the Beardon–Stephenson results that apply to the other compact surfaces in Section 5, where we analyse the seven exceptional cases. In the four cases where the invariant $F-2V_{\text{int}}$ is zero, there is a flat structure on $|K|$, and in two of the three cases where the invariant is negative, there is a spherical structure on $|K|$. In each of these cases, there is a circle packing whose combinatorics are encoded by the complex K. ∎

Application 3. *Minda–Rodin: prescribed singularities and the thrice-punctured sphere.* Minda and Rodin [**14**] used the Perron approach on finite complexes to produce labels for K with prescribed singularities and boundary values. Of particular interest there was the construction of labels with prescribed angle sums at various

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vertices. Minda and Rodin were unable to show the existence of a packing label on an arbitrary triangulation of the 2-sphere with three prescribed cusp singularities, and [**14**] closes with this question of existence as a stated problem. Theorem 3·3 with genus $g = 0$, $\mathbf{V}_{\text{cusps}}$ equal to any set of three distinct vertices of K, and $V_\partial = 0$ solves the problem of Minda and Rodin and produces a packing label for K with three prescribed cusp singularities. ∎

Application 4. *Analysis of algorithms*. Several practical algorithms have been used to find packing labels, primarily to produce computer graphics of circle packings and for computer experimentation. Thurston [**16**] presented one such algorithm whose convergence Colin de Verdière [**7**] confirmed. In the context of superpackings and the upper method, Thurston's algorithm takes the following form.

THURSTON'S ALGORITHM. *Let $R \in \mathscr{R}$, i.e. R is any superpacking label for K with cusp set $\mathbf{V}_{\text{cusps}}$ and boundary values at least as large as those given by f. Let R_0 be the label given by $R_0 = R$ on $\mathbf{V}_{\text{int}} \cup \mathbf{V}_{\text{cusps}}$ and $R_0 = f$ on $\mathbf{V}_\partial$. To obtain the $(k+1)$st approximation R_{k+1} from the kth approximation R_k, for each v in $\mathbf{V}_{\text{int}}$, let $R_{k+1}(v)$ be the unique positive real number for which the angle sum $\theta_{R^\dagger}(v) = 2\pi$, where $R^\dagger = R_k$ on $K^{(0)} - \{v\}$ and $R^\dagger(v) = R_{k+1}(v)$. This generates a sequence $\{R_k\}_{k=0}^\infty$ of labels for K.*

First note that if R_k exists, then R_{k+1} does also by the argument that appears in the second paragraph of the proof of Corollary 3·5. We make the following claims: (i) for each k, $R_{k+1} \leqslant R_k$; (ii) $\{R_k\}$ is contained in $\mathscr{R}$; (iii) $R_k \to \inf \mathscr{R}$ as $k \to \infty$. These claims follow from the dynamics of angle change with respect to label changes, as encoded in Lemma 2·1 (ii). If $R \in \mathscr{R}$, then since R is a superpacking label, the value of the label at a single vertex v in $\mathbf{V}_{\text{int}}$ must be reduced in order to gain an angle sum of 2π if originally the angle sum is not equal to, and therefore is less than 2π. This observation verifies (i) once (ii) is established. For (ii), $R_0 \in \mathscr{R}$ since decreasing only boundary values of $R \in \mathscr{R}$ decreases or leaves unchanged all angle sums at interior vertices. That $R_{k+1} \in \mathscr{R}$ whenever $R_k \in \mathscr{R}$ is seen most easily by decreasing the values of R_k to the corresponding ones of R_{k+1} first at a vertex v, then successively at those adjacent to v, all the while keeping track of the changes in angle sum at v using Lemma 2·1 (ii). Let $\hat{R} = \inf\{R_k\}$. If $\hat{R}$ is never zero, then continuity of angles implies that $\hat{R}$ is a superpacking label with cusp set $\mathbf{V}_{\text{cusps}}$ and boundary values given by f. Using continuity of angles and the construction of R_{k+1} from R_k, it is easy to reach a contradiction if $\theta_{\hat{R}}(v)$ is not equal to 2π for some v in $\mathbf{V}_{\text{int}}$. Thus $\hat{R}$ is a packing label and uniqueness in Theorem 3·3 forces $\hat{R} = \inf \mathscr{R}$. If $\hat{R}$ is zero at some vertex v, then $\inf \mathscr{R}$ is zero at that vertex and this implies by Corollary 3·5 that $\inf \mathscr{R}$ is identically zero on $\mathbf{V}_{\text{int}}$. A proof similar to that of Corollary 3·5 with $\inf \mathscr{R}$ replaced by $\hat{R}$ shows that $\hat{R}$ is identically zero on $\mathbf{V}_{\text{int}}$. Thus $\hat{R} = \inf \mathscr{R}$. The seed label R in Thurston's Algorithm may be taken to be the label that assigns an infinite value to every vertex of K. ∎

Application 5. *Totally geodesic boundaries*. Given a triangulation K of a surface with non-empty boundary, is there a packing label R with prescribed cusp singularities and all angle sums at boundary vertices equal to π? If so, the surface $|K(R)|$ is a complete hyperbolic surface with totally geodesic boundary and there is a naturally defined circle packing on the surface with boundary vertices corresponding to semi-circles that meet the boundary geodesics at right angles. One quick answer to this question

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is provided by forming the double DK of K constructed by gluing two disjoint copies of K together via the identity mapping on the boundaries. Actually, we must be more careful in our discussion. We can form the double DK and obtain a simplicial complex if and only if K has more than one face and every edge of K incident to two distinct boundary vertices is a boundary edge. This implies in particular that K has non-empty connected interior and every boundary vertex of K is adjacent to an interior vertex. For this application, we assume that K possesses these properties so that the double of K along ∂K exists and is a simplicial triangulation of a surface. Corollary 3·4 then implies that there is a packing label for K with V_{cusps} cusp singularities and angle sums of π at all boundary vertices if and only if there is a packing label for DK with $2V_{\text{cusps}}$ cusp singularities. Lemma 3·2 can be used to decide exactly when the latter occurs. Another approach to the answer is a direct one provided as follows: let $\mathscr{R}$ be the set of all superpacking labels for K with cusp set $\mathbf{V}_{\text{cusps}}$ and all angle sums at boundary vertices less than or equal to π. Then $\inf \mathscr{R}$ is the unique packing label for K with cusp set $\mathbf{V}_{\text{cusps}}$ and angle sums of π at all boundary vertices if and only if $\inf \mathscr{R}$ is never zero. The proof of this fact uses our assumption that every edge of K incident to two distinct boundary vertices is a boundary edge. If $R \in \mathscr{R}$, then Θ_R is no greater than πV_∂ and from equation (1),

$$A_R \geqslant (F - 2V_{\text{int}} - V_\partial)\pi.$$

A proof similar to the proof of Theorem 3·3 implies that $\inf \mathscr{R}$ is never zero if and only if $F - 2V_{\text{int}} - V_\partial$ is positive, and since n is non-zero in equation (3), this occurs except in the following four exceptional cases: $(g, V_{\text{cusps}}, n) \in \{(0,0,1), (0,1,1), (0,0,2), (-1, 0, 1)\}$. The invariant $F - 2V_{\text{int}} - V_\partial$ is zero in the cases $(0,1,1)$, $(0,0,2)$, and $(-1,0,1)$, which respectively correspond to the cases where K triangulates a disc with one cusp, an annulus, and a Möbius strip. The invariant is negative only in the case $(0,0,1)$ where K triangulates a disc. We shall see in Section 5 that in the three cases where $F - 2V_{\text{int}} - V_\partial$ is zero, there is a flat structure on $|K|$ with totally geodesic boundary, and in the remaining case there is a spherical structure with totally geodesic boundary. ∎

4. *The Schwarz–Picard problem and the Boundary Value-Index problem*

In this section, we generalize Theorem 3·3 and obtain a solution to the compact case of the Boundary Value-Index problem that was raised as problem A of [**14**]. Minda and Rodin [**14**] used the Perron approach on finite complexes to produce labels with prescribed singularities and boundary values. The parameter that they used to quantify how far an isolated singularity is from being removable is a conformal invariant of the metric in a neighbourhood of that singularity and is termed the index of the metric (at that singularity). It was introduced in [**10**] and given in [**14**] a geometric interpretation in terms of the angles of cone type singularities that becomes, in the setting of labelled complexes, nothing more than the shortage at the vertex of interest in the labelled complex $K(R)$. If $\nu_R(v)$ denotes the index determined by the label R at interior vertex v, then the relationship between index and angle sums or shortages is

$$2\pi\nu_R(v) = 2\pi - \theta_R(v) = s_R(v); \tag{5}$$

see the corollary to [**14**, theorem 5]. In particular, (5) gives a 1:1 correspondence

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between index values that range in $(-\infty, 1]$ and angle sums that range in $[0, \infty)$, with index one corresponding to angle sum zero or cusp singularities and index zero corresponding to angle sum 2π or removable singularities. In the terminology of labelled complexes as developed in this paper, [**14**, problem A] becomes the following problem.

BOUNDARY VALUE-INDEX PROBLEM. *Let K triangulate a connected surface. Let $f: \mathbf{V}_\partial \to (0, \infty]$ be a function defined on the boundary vertices $\mathbf{V}_\partial = \partial K^{(0)}$ of K and let $\theta: \mathbf{V}_{\text{Int}} \to [0, \infty)$ be a function defined on the interior vertices $\mathbf{V}_{\text{Int}} = K^{(0)} - \mathbf{V}_\partial$ of K. Give necessary and sufficient conditions for the existence of a label R with $R = f$ on $\mathbf{V}_\partial$ and $\theta_R = \theta$ on $\mathbf{V}_{\text{Int}}$.*

Theorem 3·3 solves the problem in the special case that K is finite and θ takes on values in $\{0, 2\pi\}$. Partial solutions to the problem in the special case that K is finite and $\mathbf{V}_\partial$ is empty, which is the labelled complex version of the classical Schwarz–Picard problem solved by Heins[**10**], were given in [**14**]. The main result of this section, Theorem 4·1, solves the problem in case K is finite, with no further restrictions on K and no restrictions on f and θ. In particular, Theorem 4·1 with $\mathbf{V}_\partial$ empty solves the Schwarz–Picard problem for labelled complexes.

As in the previous section and throughout the remainder of this section, K denotes a triangulation of a compact connected surface with F faces, E edges, and V vertices. The vertex set $K^{(0)}$ of K is decomposed into pairwise disjoint sets – the sets of interior vertices $\mathbf{V}_{\text{Int}}$ with V_{Int} elements and of boundary vertices $\mathbf{V}_\partial$ with V_∂ elements. Functions $f: \mathbf{V}_\partial \to (0, \infty]$ and $\theta: \mathbf{V}_{\text{Int}} \to [0, \infty)$ are assumed given, and our interest is in determining when a label R exists with boundary values given by f and interior angle sums given by θ. The set $\mathbf{V}_{\text{Int}}$ decomposes further as the disjoint union of the set $\mathbf{V}_{\text{cusps}} = \theta^{-1}(0)$ with V_{cusps} elements and its complement $\mathbf{V}_{\text{int}}$ with V_{int} elements. Let $\mathscr{R}$ be the set of all labels R with $\theta_R(v) \leqslant \theta(v)$ for every vertex v in $\mathbf{V}_{\text{Int}}$ and $R(v) \geqslant f(v)$ for every vertex v in $\mathbf{V}_\partial$. Again $\mathscr{R}$ is non-empty since it contains the label that assigns value ∞ to every vertex of K. Observe that $R(v) = \infty$ for every v in $\mathbf{V}_{\text{cusps}}$ and R in $\mathscr{R}$ and, as in Section 3, $\inf \mathscr{R}(v) = \infty$ if $v \in \mathbf{V}_{\text{cusps}}$, $\inf \mathscr{R}(v) \geqslant f(v)$ if $v \in \mathbf{V}_\partial$, and $\inf \mathscr{R}(v) < \infty$ if $v \in \mathbf{V}_{\text{int}}$.

Our efforts focus next on obtaining an analogue of the invariant $F - 2V_{\text{int}}$ of Theorem 3·3 – a combinatorial invariant of K and θ that is a complete indicator of when $\inf \mathscr{R}$ is never zero – and then on demonstrating that $\inf \mathscr{R}$ is never zero precisely when a desired label exists, one with boundary values given by f and interior angle sums given by θ. Suppose for the moment that a label R for K exists with the desired properties. Then

$$A_R + \Theta_R = F\pi - \Theta_R(\mathbf{V}_{\text{int}}) = F\pi - \sum_{v \in \mathbf{V}_{\text{int}}} \theta(v).$$

The right-hand side of this equation is an invariant of K and θ and must be positive since the left hand side is. Also, for every interior vertex v of K,

$$\theta(v) = \theta_R(v) < \pi \deg v.$$

These give two necessary conditions for a desired label to exist, but they are not sufficient; however, these two conditions are the extreme cases of the sufficient condition that we present in Theorem 4·1. If $\mathbf{V}$ is a collection of vertices of K, let $F_{\mathbf{V}}$

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denote the number of faces of K that contain a vertex of $\mathbf{V}$. Theorem 4·1 generalizes Theorem 3·3 and is our solution to the Boundary Value-Index problem for finite complexes.

THEOREM 4·1. *The following are equivalent*:

(i) *for every edge-path connected subset* $\mathbf{V}$ *of* $\mathbf{V}_{\text{int}}$, *the invariant*

$$F_{\mathbf{V}}\pi - \sum_{v\in\mathbf{V}} \theta(v)$$

is positive;

(ii) *the function* $\inf \mathscr{R}$ *is never zero*;

(iii) *the function* $\inf \mathscr{R}$ *is the unique label for* K *with boundary values given by* f *and interior angle sums given by* θ;

(iv) *there exists a label for* K *with boundary values given by* f *and interior angle sums given by* θ.

Proof. The proof in its essential aspects is the same as that of Theorem 3·3.

(i) $\Rightarrow$ (ii): Assume that the hypothesis expressed in (i) holds and that there exists a non-empty edge-path component $\mathbf{V}$ of the zero set $\mathbf{W} = (\inf \mathscr{R})^{-1}(0)$. As in the proof of Theorem 3·3, let L be the subcomplex of K that consists of the simplices of K that meet $\mathbf{V}$ and recall that L triangulates a compact connected surface with possible singularities (non-manifold points) at boundary vertices of L. For simplicity, assume that there are no singularities so that L does triangulate a surface. Let μ be the minimum value of $\inf \mathscr{R}$ on any boundary vertices and, exactly as in the proof of Theorem 3·3, observe that μ is positive if L has non-empty boundary. The vertex set of L partitions as the disjoint union $\mathbf{V}'_{\text{Int}} \cup \mathbf{V}'_{\partial}$ where $\mathbf{V}'_{\partial}$ is the set of boundary vertices of L. Denote by $\mathbf{V}'_{\text{int}}$ the complement of $\mathbf{V}'_{\text{cusps}} = \theta^{-1}(0) \cap \mathbf{V}'_{\text{Int}}$ in $\mathbf{V}'_{\text{Int}}$. Let $\mathscr{R}'$ be the set of labels R for L with $\theta_R(v) \leqslant \theta(v)$ for every vertex v in $\mathbf{V}'_{\text{Int}}$ and $R(v) \geqslant \mu$ for every vertex v in $\mathbf{V}'_{\partial}$. Each label R in $\mathscr{R}$ restricts to a label R' in $\mathscr{R}'$. This implies that since $\inf \mathscr{R}$ is identically zero on $\mathbf{V}$, $\inf \mathscr{R}'$ is identically zero on $\mathbf{V}$. Since every face of L contains a vertex in $\mathbf{V}$, $A_{R_i} \to 0$ as $i \to \infty$ if $\{R_i\}$ is a sequence in $\mathscr{R}'$ such that $R_i \to \inf \mathscr{R}'$ as $i \to \infty$. Also, since $\mu > 0$ whenever $\mathbf{V}'_{\partial}$ is non-empty, $\Theta_{R_i} \to 0$ as $i \to \infty$. For every R in $\mathscr{R}'$, since $\theta_R(v) \leqslant \theta(v)$, $F_{\mathbf{V}} = F_{\mathbf{V}'_{\text{int}}}$, and $\theta(v) = 0$ if v is in $\mathbf{V}'_{\text{cusps}}$, we have

$$A_R + \Theta_R = F_{\mathbf{V}}\pi - \Theta_R(\mathbf{V}'_{\text{Int}}) \geqslant F_{\mathbf{V}}\pi - \sum_{v\in\mathbf{V}'_{\text{int}}} \theta(v) = F_{\mathbf{V}'_{\text{int}}}\pi - \sum_{v\in\mathbf{V}'_{\text{int}}} \theta(v). \tag{6}$$

Since $\mathbf{V}'_{\text{int}}$ is edge-path connected (why?) and contained in $\mathbf{V}_{\text{int}}$, the hypothesis implies that the right-hand side of equation (6) is positive. This contradicts the fact that $A_{R_i} + \Theta_{R_i} \to 0$ as $i \to \infty$ for the sequence $\{R_i\}$ in $\mathscr{R}'$. We conclude then that $\mathbf{W}$ is empty and $\inf \mathscr{R}$ is never zero.

(ii) $\Rightarrow$ (iii): Minor modifications (replacing 2π by $\theta(v)$) in the proof of the implication (ii) $\Rightarrow$ (iii) of Theorem 3·3 shows that $\inf \mathscr{R}$ is a label with boundary values given by f and interior angle sums given by θ. It remains to prove uniqueness, which is accomplished also exactly as in the proof of (ii) $\Rightarrow$ (iii) of Theorem 3·3 except that the shortages $s_{\inf \mathscr{R}}$ and s_R no longer are necessarily zero, nor even non-negative. Nonetheless, equation (1) holds for any label R that assigns value ∞ to the elements of $\mathbf{V}_{\text{cusps}}$, though s_R may be negative. This implies that

$$A_{\inf \mathscr{R}} + \Theta_{\inf \mathscr{R}} - s_{\inf \mathscr{R}} = A_R + \Theta_R - s_R$$

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whenever R is another label with boundary values given by f and interior angle sums given by θ. Since $\theta_{\inf \mathscr{R}} = \theta = \theta_R$ on $\mathbf{V}_{\text{Int}}$, the shortages $s_{\inf \mathscr{R}}$ and s_R are equal and the previous equation reduces to

$$A_{\inf \mathscr{R}} + \Theta_{\inf \mathscr{R}} = A_R + \Theta_R.$$

The proof now concludes exactly as that of (ii) $\Rightarrow$ (iii) of Theorem 3·3.

(iii) $\Rightarrow$ (iv): Immediate.

(iv) $\Rightarrow$ (i): Let R be a label for K with boundary values given by f and interior angle sums given by θ and let $\mathbf{V}$ be any edge-path connected subset of $\mathbf{V}_{\text{int}}$. The label R restricts to a label R' of L with L as in the proof of (i) $\Rightarrow$ (ii), and R' is a label for L with positive (possibly infinite) boundary values and interior angle sums given by θ. Hence $A_{R'} + \Theta_{R'}$ is positive and, using the notation $\mathbf{V}'_{\text{Int}}, \mathbf{V}'_{\text{int}}$ from the proof of (i) $\Rightarrow$ (ii), we have

$$0 < A_{R'} + \Theta_{R'} = F_{\mathbf{V}}\pi - \Theta_{R'}(\mathbf{V}'_{\text{Int}}) = F_{\mathbf{V}}\pi - \Theta_R(\mathbf{V}'_{\text{Int}}) = F_{\mathbf{V}}\pi - \sum_{v \in \mathbf{V}'_{\text{Int}}} \theta(v).$$

Since $\mathbf{V}$ is contained in $\mathbf{V}'_{\text{Int}}$, the right-hand side of this equation is less than or equal to

$$F_{\mathbf{V}}\pi - \sum_{v \in \mathbf{V}} \theta(v),$$

which is therefore positive. We conclude that (i) holds. ∎

Theorem 3·3 is the special case of Theorem 4·1 that corresponds to restricting the function θ to have values in $\{0, 2\pi\}$, and we leave the reader the task of showing that statements (i) of the two theorems are equivalent in this setting. The analogue of Corollary 3·4 holds in this more general setting if each set of vertices of the form $\theta^{-1}(t)$, $0 \leqslant t < \infty$, is G-invariant, but there is no analogue of Corollary 3·5 that holds in this setting. The following corollary is a slight generalization of [**14**, Application 3].

COROLLARY 4·2. Minda and Rodin [**14**]. *Let K triangulate a compact surface and let $f: \mathbf{V}_\partial \to (0, \infty]$ be a function. If*

$$\theta(v) < \frac{\pi}{3} \deg v \tag{7}$$

for each interior vertex v of K, then a unique label for K with boundary values given by f and angle sums given by θ exists.

Proof. Let $\mathbf{V}$ be an edge-path connected subset of $\mathbf{V}_{\text{int}}$. Let α be the number of angles in K with vertices in $\mathbf{V}$ (an angle with vertex v is formed by two edges of a face of K that contain v). Obviously

$$\alpha = \sum_{v \in \mathbf{V}} \deg v. \tag{8}$$

On the other hand, each face of F that contains a vertex in $\mathbf{V}$ determines between one and three angles with vertices in $\mathbf{V}$; in particular,

$$\alpha \leqslant 3F_{\mathbf{V}}. \tag{9}$$

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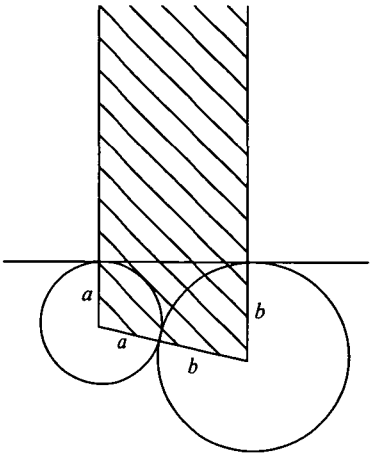


Fig. 2. The euclidean triangle $\Delta(a, b, \infty)$ with ideal vertex.

Inequality (7) along with (8) and (9) imply

$$F_V \pi - \sum_{v \in V} \theta(v) > F_V \pi - \frac{\pi}{3} \sum_{v \in V} \deg v = \frac{\pi}{3}(3F_V - \alpha) \geqslant 0.$$

Therefore, statement (i) of Theorem 4·1 holds. ∎

5. *The exceptional cases – flat and spherical metrics*

This section continues the dialogue of Section 3 by analysing the seven exceptional cases of Lemma 3·2 and the four exceptional cases of Application 5. All of these cases arise when either the invariant $F - 2V_{\text{int}}$ or the invariant $F - 2V_{\text{int}} - V_\partial$ is non-positive. The result of this section is that the complex K determines an essentially unique flat structure if the invariant is zero and a spherical structure if the invariant is negative. We continue to assume that K triangulates a compact connected surface that has F faces, E edges, and V vertices and we use freely the notation of Section 3. Before our analysis of the exceptional cases begins, we introduce the terminology and notation for singular flat surfaces that are associated with labelled complexes.

If (a, b, c) is a triple with a, b, and c finite, $\Delta(a, b, c)$ denotes the unique (up to euclidean isometry) triangle in the euclidean plane with side lengths $a+b$, $b+c$, $a+c$. The triangle $\Delta(a, b, c)$ can be formed by connecting the centres of three mutually tangent circles of respective radii a, b, c. If a and b are finite and $c = \infty$, we obtain the triangle $\Delta(a, b, \infty)$ with ideal vertex by considering the limiting behaviour of the triangles $\Delta(a, b, c)$ as $c \to \infty$. The result is that it is natural to define $\Delta(a, b, \infty)$ as the region of the plane bounded by the line segment connecting the centres of two tangent circles of respective radii a and b and by the two half-lines whose endpoints are these centres and that are perpendicular to a fixed line that is tangent to both circles (see Fig. 2).

The next lemma is the euclidean analogue of Lemma 2·1 (ii).

LEMMA 5·1. *Let α, β, and γ be the respective interior angles of the triangle $\Delta(a, b, c)$ that correspond to the circles of respective radii a, b, and c, where a and b are finite. If the radii b and c are held constant, then the angles α, β, and γ are continuous monotone functions*

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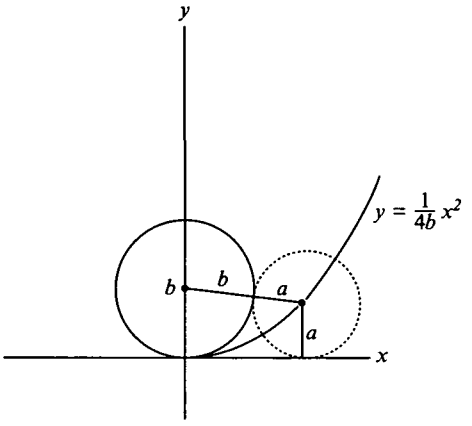


Fig. 3. The locus traced out by the centre.

of a with $\alpha(a)$ strictly decreasing and $\lim_{a\uparrow\infty}\alpha(a)=0$ and $\lim_{a\downarrow 0}\alpha(a)=\pi$, with $\beta(a)$ strictly increasing and $\lim_{a\downarrow 0}\beta(a)=0$, and with $\gamma(a)$ increasing, with strict monotonicity if $c<\infty$, and $\lim_{a\downarrow 0}\gamma(a)=0$.

Proof. The proof is straightforward using the law of cosines and even more easily seen geometrically. Note that our convention is that γ is zero if c is infinite. The only further elaboration that, if not necessary, then perhaps is helpful, is the establishment that $\lim_{a\uparrow\infty}\alpha(a)=0$ whenever $c=\infty$. This follows from the fact that the locus of points traced out by the centre of the circle of radius $a, 0\leqslant a<\infty$, is a parabola with axis perpendicular to the circle of infinite radius. In the coordinate system of Fig. 3, that locus has equation

$$y=\frac{1}{4b}x^2. \quad \blacksquare$$

Let $K(R)$ be a labelled complex such that no two vertices are adjacent if R assigns value ∞ to both. We introduce a flat metric on the geometric realization $|K|$ of K with singularities at the vertices by identifying, for each 2-simplex $\sigma=\langle v_0,v_1,v_2\rangle$ of K, the face σ with the euclidean triangle $\Delta(R(v_0),R(v_1),R(v_2))$. The resulting flat or euclidean realization of the labelled complex $K(R)$ is denoted $|K(R)|_{\text{flat}}$. Any interior vertex v with $R(v)=\infty$ is called a (flat or euclidean) cusp point of $|K(R)|_{\text{flat}}$. The metric on $|K(R)|_{\text{flat}}$ determines angles at each vertex of each 2-simplex of K and thus also determines a flat or euclidean angle sum $\theta_R^{\text{flat}}(v)$ at the vertex v (the angle at an ideal vertex is zero). The labelled complex $K(R)$ is a flat or euclidean packing and the label R is a flat or euclidean packing label if the angle sum $\theta_R^{\text{flat}}(v)$ is 2π at each interior vertex v for which $R(v)\neq\infty$. An interior vertex whose flat angle sum differs from 2π represents a singularity for the induced metric, and a cusp singularity if the flat angle sum is additionally zero. If the context makes clear the meaning, we shall at times drop the adjectives flat or euclidean and, conversely, for emphasis we shall at times include the adjective hyperbolic and use the symbols $|K(R)|_{\text{hyp}}$ and θ_R^{hyp} for $|K(R)|$ and θ_R, respectively.

We divide the total of eleven exceptional cases into four classes as follows, using the 3-tuple (g,V_{cusps},n) to specify the type of K:

Class I: $\{(1,0,0),\ (0,2,0),\ (-1,1,0),\ (-2,0,0)\}$

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Class II: $\{(0,1,0), (0,0,0), (-1,0,0)\}$
Class III: $\{(0,1,1), (0,0,2), (-1,0,1)\}$
Class IV: $\{(0,0,1)\}$.

Class I. In the four cases of Class I, K has empty boundary and at most two cusp points, and at first we assume that the two cusp points in the case $(0,2,0)$ are not adjacent. Later we address the case $(0,2,0)$ when the two cusp points are adjacent. Let $\mathscr{R}$ as in Section 3 be the set of superpacking labels for K with cusp set $\mathbf{V}_{\text{cusps}}$. Since the invariant $F-2V_{\text{int}}$ is zero, Corollary 3·5 implies that the function $\inf \mathscr{R}$ is identically zero on $\mathbf{V}_{\text{int}}$. Equation (1) (or (4)) implies that

$$A_R = s_R \tag{10}$$

for every R in $\mathscr{R}$ since $\Theta_R = 0 = F-2V_{\text{int}}$. Let $\{R_i\}$ be a sequence in $\mathscr{R}$ such that $R_i \to \inf \mathscr{R}$ as $i \to \infty$. Then $A_{R_i} \to 0$ as $i \to \infty$ and equation (10) implies that the angle sums $\theta^{\text{hyp}}_{R_i}(v) \to 2\pi$ as $i \to \infty$ for every vertex v in $\mathbf{V}_{\text{int}}$. Since euclidean geometry is the 'small scale' limit of hyperbolic geometry, this implies that $\theta^{\text{flat}}_{R_i}(v) \to 2\pi$ as $i \to \infty$ for every vertex v in $\mathbf{V}_{\text{int}}$. Thus the sequence $\{|K(R_i)|_{\text{flat}}\}$ is a sequence of singular flat surfaces whose singularities are removed in the limit. Of course there is no limiting flat surface for the sequence $\{|K(R_i)|_{\text{flat}}\}$ since $R_i \to \inf \mathscr{R}$ as $i \to \infty$ and $\inf \mathscr{R}$ is identically zero on $\mathbf{V}_{\text{int}}$; however, this can be remedied since rescalings (similarity transformations) exist in euclidean geometry. Indeed, by suitably rescaling the sequence $\{|K(R_i)|_{\text{flat}}\}$, a limiting non-singular flat surface can be obtained and it turns out that this surface essentially is independent of both the initial sequence $\{R_i\}$ and of the rescaling. This rather intuitive discussion is made precise in Theorem 5·5, which is presented after some additional discussion and preliminary lemmas.

Fix an arbitrary vertex $v^\dagger$ of $\mathbf{V}_{\text{int}}$ and for any label R of K, let $R^\dagger$ be the label R rescaled (i.e. multiplied) by the reciprocal of $R(v^\dagger)$:

$$R^\dagger(v) = \frac{R(v)}{R(v^\dagger)}$$

for each vertex v of K. Define a set $\mathscr{R}^\dagger$ of labels for K by

$$\mathscr{R}^\dagger = \{R^\dagger : R \in \mathscr{R} \text{ and } R(v^\dagger) \leqslant 1\}.$$

The next lemma is a slight strengthening of an observation by Peter Doyle that appears as lemma 2·3 of [**4**]. A geometric proof appears in [**3**, lemma 2·3] and an analytic proof appears in [**14**].

LEMMA 5·2. *Let $r = (a,b,c)$ be a triple with a finite and let $\alpha(\zeta)$ be the interior angle of $T(\zeta r)$ at the vertex corresponding to ζa, for $0 < \zeta \leqslant \infty$. Then α is a continuous strictly decreasing function of ζ.*

We apply Lemma 5·2 in the proof of Theorem 5·5 and to prove the next lemma.

LEMMA 5·3. *The set of labels $\mathscr{R}^\dagger$ is contained in the set $\mathscr{R}$.*

Proof. Obviously, $R^\dagger(v) = \infty$ if and only if $R(v) = \infty$ for a vertex v. Let $R \in \mathscr{R}$ with $R(v^\dagger) \leqslant 1$ so that

$$\zeta = \frac{1}{R(v^\dagger)} \geqslant 1.$$

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Lemma 5·2 implies that $\theta_{R^\dagger}(v) = \theta_{\zeta R}(v) \leqslant \theta_R(v)$ for every vertex v, and since R is a superpacking label, so is $R^\dagger$ and $R^\dagger$ lies in $\mathscr{R}$. ∎

LEMMA 5·4. *There exists a sequence $\{R_i'\}$ in $\mathscr{R}^\dagger$ such that $R_i' \to \inf \mathscr{R}^\dagger$ as $i \to \infty$.*

Proof. If $R_1^\dagger$ and $R_2^\dagger$ are elements of $\mathscr{R}^\dagger$, then lemmas 5·3 and 3·1(ii) imply that the label $R = \min\{R_1^\dagger, R_2^\dagger\}$ is an element of $\mathscr{R}$. Furthermore, $R(v^\dagger) = 1$, hence $R = R^\dagger \in \mathscr{R}^\dagger$. The result now follows exactly as (iii) follows from (ii) in Lemma 3·1. ∎

THEOREM 5·5. *In the cases of Class I, where $F - 2V_{\text{int}}$ is zero, assuming further that the cusp points in the case $(0, 2, 0)$ are not adjacent, there is a unique flat packing label with cusp set $\mathbf{V}_{\text{cusps}}$ that assigns value 1 to the distinguished vertex $v^\dagger$. Consequently, any two flat packing labels for K with cusp set $\mathbf{V}_{\text{cusps}}$ differ by scaling, i.e. one is a constant multiple of the other.*

Proof. Let $\{R_i\}$ be any sequence in $\mathscr{R}$ such that $R_i \to \inf \mathscr{R}$ as $i \to \infty$. Since $v^\dagger$ is in $\mathbf{V}_{\text{int}}$, Corollary 3·5 implies that $\inf \mathscr{R}(v^\dagger)$ is zero and thus, without loss of generality, we may assume that $R_i(v^\dagger) \leqslant 1$ for all i. This implies that the sequence $\{R_i^\dagger\}$ is in $\mathscr{R}^\dagger$. Let $R = \inf\{R_i^\dagger\}$. Our first task is to show that R is never zero, which follows if we show that $\inf \mathscr{R}^\dagger$ is never zero.

The argument used to reach a contradiction in the first paragraph of the proof of Corollary 3·5 with $\mathscr{R}$ there replaced by $\mathscr{R}^\dagger$, implies that $\inf \mathscr{R}^\dagger$ is never zero provided there exists a face σ of K none of whose vertices lie in the zero set $\mathbf{W} = (\inf \mathscr{R}^\dagger)^{-1}(0)$. Assume then that every face of K contains a vertex of $\mathbf{W}$ and let $\{R_i'\}$ be a sequence in $\mathscr{R}^\dagger$ as promised in Lemma 5·4 with $R_i' \to \inf \mathscr{R}^\dagger$ as $i \to \infty$. Since each face of K contains a vertex in $\mathbf{W}$, $A_{R_i'} \to 0$ as $i \to \infty$. Observe now that $\inf \mathscr{R}^\dagger(v^\dagger) = 1$. This implies by Lemma 2·1(ii), since each face containing $v^\dagger$ contains a vertex w such that $R_i'(w) \to 0$ as $i \to \infty$, that the angle sums $\theta_{R_i'}^{\text{hyp}}(v^\dagger) \to 0$ as $i \to \infty$, exactly as in the second paragraph of the proof of Corollary 3·5. By equation (10) and the definition of shortage,

$$A_{R_i'} \geqslant 2\pi - \theta_{R_i'}^{\text{hyp}}(v^\dagger), \tag{11}$$

which implies that the sequence $\{A_{R_i'}\}$ is bounded away from zero. This contradicts that $A_{R_i'} \to 0$ as $i \to \infty$. Thus there is a face of K that does not contain any vertex in $\mathbf{W}$ and we conclude that $\mathbf{W}$ is empty, i.e. $\inf \mathscr{R}^\dagger$ is never zero.

Since $R = \inf\{R_i^\dagger\}$ is never zero, R is a label for K and obviously $R(v^\dagger) = 1$, and $R(v) = \infty$ if and only if v is in $\mathbf{V}_{\text{cusps}}$. Since the sum of the interior angles of a euclidean triangle is π, even when the triangle has an ideal vertex, the sum of all the angles of the flat realization $|K(R)|_{\text{flat}}$ is

$$\sum_{v \in \mathbf{V}_{\text{int}}} \theta_R^{\text{flat}}(v) = \pi F = 2\pi V_{\text{int}}. \tag{12}$$

If it were known that the flat angle sum $\theta_R^{\text{flat}}(v)$ at each vertex v in $\mathbf{V}_{\text{int}}$ is no greater than 2π, then equation (12) would imply that each such angle sum is equal to 2π. We then could conclude that R is a flat packing label with cusp set $\mathbf{V}_{\text{cusps}}$ and $R(v^\dagger) = 1$ as desired. To confirm that $\theta_R^{\text{flat}}(v)$ is no greater than 2π for a vertex v in $\mathbf{V}_{\text{int}}$, by passing to a subsequence if necessary, assume that $R_i^\dagger(v) \to R(v)$. For each i, let R_i' be the label for K defined by $R_i' = R$ on $K^{(0)} - \{v\}$ and $R_i'(v) = R_i^\dagger(v)$. Then obviously $R_i' \to R$ as $i \to \infty$ and continuity of angles (Lemma 5·1) implies that

$$\theta_{R_i'}^{\text{flat}}(v) \to \theta_R^{\text{flat}}(v) \quad \text{as} \quad i \to \infty. \tag{13}$$

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On the other hand, since $R_i'(w) = R(w) \leqslant R_i^\dagger(w)$ for every vertex w adjacent to v, the monotonicity properties of Lemma 5·1 imply that

$$\theta_{R_i'}^{\text{flat}}(v) \leqslant \theta_{R_i^\dagger}^{\text{flat}}(v) = \theta_{R_i}^{\text{flat}}(v) \tag{14}$$

for all i, where the equality follows because euclidean scaling is conformal. As is well-known, euclidean geometry can be viewed as the small scale limit of hyperbolic geometry in the sense that the differences between the corresponding angles of the triangles $\Delta(a,b,c)$ and $T(a,b,c)$ converge to zero as $(a,b) \to (0,0)$ with c held fixed. This follows from considering the limiting behaviour of the hyperbolic and euclidean cosine rules as $(a,b) \to (0,0)$, which shows further that the previous statement is valid even when c is infinite. This implies that, since each face of K has at least two vertices in $\mathbf{V}_{\text{int}}$ and R_i converges to $\inf \mathscr{R}$, which is identically zero on $\mathbf{V}_{\text{int}}$, the difference $\theta_{R_i}^{\text{flat}}(v) - \theta_{R_i}^{\text{hyp}}(v)$ converges to zero as $i \to \infty$. Since $A_{R_i} \to 0$ as $i \to \infty$, inequality (11) (or equation (10)) implies that the angle sums $\theta_{R_i}^{\text{hyp}}(v) \to 2\pi$ as $i \to \infty$, hence the right-hand side of (14) converges to 2π as $i \to \infty$. (13) and (14) imply that

$$\theta_R^{\text{flat}}(v) \leqslant 2\pi, \tag{15}$$

as desired. Since v is an arbitrary vertex in $\mathbf{V}_{\text{int}}$, (12) implies that equality holds in (15) for every vertex in $\mathbf{V}_{\text{int}}$ and hence R is a flat packing label with cusp set $\mathbf{V}_{\text{cusps}}$ and $R(v^\dagger) = 1$.

It remains only to prove uniqueness. We use the fact that if R is a label for K, then

$$\theta_R^{\text{hyp}}(v) \leqslant \theta_R^{\text{flat}}(v), \tag{16}$$

with equality only if $R(v)$ is infinite, for each vertex v. This follows from the fact that the angles of $T(r)$ are strictly smaller than the corresponding angles of $\Delta(r)$ for any triple $r = (a,b,c)$ with a, b, and c finite. This fact in turn follows from Lemma 5·2, since the interior angles of $T(\zeta r)$ are decreasing functions of ζ on $(0,\infty)$ and the values of the interior angles of $T(\zeta r)$ converge to the corresponding ones of $\Delta(r)$ as $\zeta \downarrow 0$. Now let R and R' be flat packing labels for K with cusp set $\mathbf{V}_{\text{cusps}}$ and $R(v^\dagger) = 1 = R'(v^\dagger)$. Then for every $\epsilon > 0$, the labels, ϵR and $\epsilon R'$ are still flat packing labels for K with cusp set $\mathbf{V}_{\text{cusps}}$, but $\epsilon R(v^\dagger) = \epsilon = \epsilon R'(v^\dagger)$. By observation (16) and the definition of flat packing labels, ϵR and $\epsilon R'$ are elements of $\mathscr{R}$ for all $\epsilon > 0$. For each $\epsilon > 0$, let $R_\epsilon = \min\{\epsilon R, \epsilon R'\}$ and observe that R_ϵ is in $\mathscr{R}$, by Lemma 3·1 (ii), and $R_\epsilon \to \inf \mathscr{R}$ as $\epsilon \downarrow 0$, since $\inf \mathscr{R}$ is identically zero on $\mathbf{V}_{\text{int}}$ and infinite on $\mathbf{V}_{\text{cusps}}$. By the existence part of the proof of this theorem, $R_0 = \inf\{R_\epsilon^\dagger : 0 < \epsilon \leqslant 1\}$ is flat packing label for K with cusp set $\mathbf{V}_{\text{cusps}}$ and $R_0(v^\dagger) = 1$. Of course we worked with sequences above, but the same argument works for the 1-parameter family $\{R_\epsilon^\dagger : 0 < \epsilon \leqslant 1\}$. We have

$$R_0 \leqslant R_1^\dagger = \min\{R, R'\}.$$

Now Lemma 5·1 implies that if $R_0(v) = R(v)$ for some vertex v in $\mathbf{V}_{\text{int}}$, then $R_0(w) = R(w)$ for every vertex w adjacent to v. This follows, for otherwise $R_0(w) \leqslant R(w)$ for all such w with strict inequality for at least one such w, and the angle sum $\theta_{R_0}^{\text{flat}}(v)$ is strictly smaller than $\theta_R^{\text{flat}}(v)$, a contradiction since both angle sums are 2π. Since $R_0(v^\dagger) = 1 = R(v^\dagger)$ and $\mathbf{V}_{\text{int}}$ is edge-path connected, $R_0 = R$. Similarly, $R_0 = R'$ and this completes the proof. ∎

The cases $(1,0,0)$ and $(-2,0,0)$ correspond respectively to the torus and Klein

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bottle, and Theorem 5·5 says that any triangulation of either determines a corresponding flat structure via a label that is unique up to scaling (or homothety) that supports a circle packing whose combinatorics are encoded in the triangulation. The cases $(0,2,0)$ and $(-1,1,0)$ correspond respectively to the twice punctured sphere and the once punctured projective plane, and Theorem 5·5 says that any triangulation of either with any punctures corresponding to vertices (that are not adjacent in case of $(0,2,0)$) determines a complete flat structure on the complement of the punctures via a label that is unique up to scaling. The case $(0,2,0)$ thus corresponds to complete flat annuli and the two circles in the natural circle packing corresponding to the two (non-adjacent) cusps are geodesic circles that cut off two ends leaving a right circular cylinder. Likewise, the case $(-1,1,0)$ corresponds to complete flat Möbius strips and the circle in the natural circle packing corresponding to the one cusp is a geodesic circle that cuts off the one end leaving a compact bordered Möbius strip with totally geodesic boundary.

The only case remaining to consider is $(0,2,0)$ with the assumption that the two cusp points are adjacent. In this case, the first two paragraphs of the proof of Theorem 5·5 apply and show that $R = \inf\{R_i^\dagger\}$ is never zero whenever $\{R_i\}$ is a sequence in $\mathscr{R}$ that converges to $\inf \mathscr{R}$ as $i \to \infty$. Further, $R(v^\dagger) = 1$ and $R(v) = \infty$ if and only if v is one of the two adjacent cusp points. Let u_1 and u_2 be the two cusp points and let $\tau = \langle u, u_1, u_2\rangle$ and $\tau' = \langle u', u_1, u_2\rangle$ be the two faces of K that contain the edge $\langle u_1, u_2\rangle$. The label values for the vertices of τ and τ' are infinite at the two vertices u_1 and u_2 and the question arises of how this should be interpreted in attempting to build a flat realization for the labelled complex $K(R)$, i.e. how should we interpret a euclidean triangle with two ideal vertices? What turns out to work quite well is suggested by the proof of Lemma 5·1, where it is observed that the locus of points traced out by the centre of the circle of radius a, $0 \leqslant a < \infty$, is a parabola with axis perpendicular to the circle of infinite radius. This implies not only that $\alpha(a) \to 0$ as $a \uparrow \infty$ whenever $c = \infty$, but also that $\beta(a) \to \pi$ as $a \uparrow \infty$ whenever $c = \infty$. Thus the 'triangle' $\Delta(\infty, b, \infty)$ with b finite can be thought of as the limiting case of the triangle $\Delta(a, b, \infty)$ as $a \uparrow \infty$, which in some sense can be thought of as a straight line l in the euclidean plane with one vertex b, a point on l, with angle β interpreted as π, and two ideal vertices with angles zero. This degenerate triangle $\Delta(\infty, b, \infty)$ thus defined has empty interior and has angle sum equal to π. Essentially the same argument that appears in the third paragraph of the proof of Theorem 5·5, with minor modifications to handle the vertices u and u' in $\mathbf{V}_{\text{int}}$, implies that $\theta_R^{\text{flat}}(v) = 2\pi$ for each vertex v in $\mathbf{V}_{\text{int}}$. To define $|K(R)|_{\text{flat}}$, we must explain how to identify the labelled faces $\tau(R)$ and $\tau'(R)$ as euclidean triangles. This is accomplished by identifying both with a single euclidean line by removing the interiors of τ and τ' from $|K|$ and identifying the edge $\langle u_1, u_2\rangle$ with the edge-paths $\langle u_1, u\rangle \cup \langle u, u_2\rangle$ and $\langle u_1, u'\rangle \cup \langle u', u_2\rangle$ via isometries (recall the edges have complete convex metrics already) that in particular identify the vertices u and u'. The resulting flat angle sum at $u = u'$ determined by R is 2π since the flat angle sums at u and u' before the identification are 2π and the angle at the (non-ideal) vertex of a degenerate triangle is π. Since the flat angle sums at other vertices of $\mathbf{V}_{\text{int}}$ are all equal to 2π, $|K(R)|_{\text{flat}}$ is a complete flat surface, in fact a flat annulus that is 'triangulated' by the complex formed by removing τ, τ' and $\langle u, u_2\rangle$ from K and identifying $\langle u_1, u\rangle$ with $\langle u_1, u'\rangle$ and $\langle u, u_2\rangle$ with $\langle u', u_2\rangle$. This label R is called a flat packing label for K. Uniqueness

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follows exactly as in the proof of uniqueness in Theorem 5·5, with obvious interpretations to include degenerate triangles.

Class II. We do not have a reasonable interpretation of the case $(0,1,0)$. The remaining two cases, $(0,0,0)$ and $(-1,0,0)$, have no cusp points and have empty boundary. If $K(R)$ is a labelled complex with no cusp points and empty boundary, then both the hyperbolic and flat realizations of $K(R)$ exist since any three finite positive numbers determine unique hyperbolic and euclidean triangles. A definition of the spherical realization $|K(R)|_{\text{spher}}$ requires that the label R be restricted so that appropriate spherical triangles exist. The triple (a,b,c) determines a spherical triangle with side lengths $a+b$, $b+c$, $a+c$ whose sides are formed by connecting with geodesic segments the centres of three mutually tangent circles of $\mathbb{S}^2$ of respective radii a,b,c if and only if the sum $a+b+c$ is less than π. Such a triangle is unique up to isometry of $\mathbb{S}^2$. Notice that every circle in $\mathbb{S}^2$ can be thought of as the boundary of either of two complementary discs, and so each circle has two radii r_1 and r_2 depending on which disc we consider determining that circle. Obviously $r_1+r_2=\pi$, and we do allow a, b or c to be greater than $\pi/2$. Now if R is a label for K such that $R(v_0)+R(v_1)+R(v_2)<\pi$ whenever $\langle v_0,v_1,v_2\rangle$ is a face of K, then R is called acceptable and the spherical realization $|K(R)|_{\text{spher}}$ exists as a singular spherical surface with metric of constant curvature $+1$ with singularities at the vertices of K. Of course, R is a spherical packing label if all spherical angle sums determined by R are 2π, where spherical angle sum has the obvious meaning.

The case $(0,0,0)$ is covered by the Koebe Theorem, suitably interpreted. In this case, the Koebe Theorem guarantees that a circle packing $\mathscr{C}$ of $\mathbb{S}^2$ exists with abstract carrier isomorphic to K. Each such $\mathscr{C}$ provides an acceptable label R for K with all (spherical) angle sums equal to 2π; thus $|K(R)|_{\text{spher}}$ is isometric to $\mathbb{S}^2$ and the geodesic triangulation of $|K(R)|_{\text{spher}}$ corresponds to that of $\mathbb{S}^2$ determined by $\mathscr{C}$. Since $\mathscr{C}$ is determined up to Möbius transformations by K, a spherical packing label for K is not uniquely determined, but any two are related by a Möbius transformation.

For the case $(-1,0,0)$, K triangulates the projective plane $\mathbb{P}^2$ and is covered by a complex $\tilde{K}$ that triangulates $\mathbb{S}^2$. The Koebe Theorem provides a circle packing $\tilde{\mathscr{C}}$ of $\mathbb{S}^2$ whose abstract carrier is isomorphic to $\tilde{K}$ and, moreover, the uniqueness of the Koebe Theorem guarantees that there exists a (orientation-reversing) Möbius transformation μ of $\mathbb{S}^2$ that preserves $\tilde{\mathscr{C}}$ and realizes the antipodal action on $\tilde{K}$. If $\tilde{\mathscr{C}}$ is chosen so that the circles C and C' corresponding to the respective antipodal vertices v and v' of $\tilde{K}$ are parallel and have the same radius (i.e. C and C' lie on planes in $\mathbb{R}^3$ parallel to, say, the x–y plane, that lie the same distance above and below the x–y plane), then μ must be the antipodal map of $\mathbb{S}^2$. Then $\mathscr{C}=\tilde{\mathscr{C}}/\mu$ is a circle packing of $\mathbb{P}^2=\mathbb{S}^2/\mu$ whose abstract carrier is isomorphic to K, and as in the case $(0,0,0)$ this determines a spherical packing label R for K. The label R in this case is the unique spherical label for K.

Class III. Let K be as in Application 5, i.e. assume that every edge of K incident to two distinct boundary vertices is a boundary edge, and K contains more than one face. By doubling K along its boundary, we obtain a simplicial complex DK of one of the three types $(0,2,0)$, $(1,0,0)$, and $(-2,0,0)$ from Class I whenever K is of the respective types $(0,1,1)$, $(0,0,2)$, and $(-1,0,1)$. Theorem 5·5 implies that there is a flat packing label R for DK that is unique up to scaling. By symmetry of DK, using

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the uniqueness from Theorem 5·5, R labels the two copies of the same vertex from K with the same value. This implies that the complete flat surface $|K(R)|_{\text{flat}}$ has totally geodesic boundary since it forces each angle sum $\theta_R^{\text{flat}}(v)$ to equal π if v is a boundary vertex of K. Each such surface admits a circle packing where boundary vertices correspond to semi-circles that intersect the geodesic boundary orthogonally and whose abstract carrier is isomorphic to K. The case $(0, 1, 1)$ corresponds to a complete flat half-annulus with one totally geodesic boundary component, the case $(0, 0, 2)$ corresponds to a right circular cylinder, and the case $(-1, 0, 1)$ corresponds to a compact bordered Möbius strip with totally geodesic boundary.

Class IV. Again let K be as in Application 5. The double DK triangulates $\mathbb{S}^2$ and by the Koebe Theorem, there exists a circle packing $\mathscr{C}$ whose abstract carrier is isomorphic to DK. By symmetry of DK and uniqueness in the Koebe Theorem, we may assume that the circles of $\mathscr{C}$ that correspond to boundary vertices of K are centred on the equator of $\mathbb{S}^2$. This provides a spherical packing label R for K making $|K(R)|_{\text{spher}}$ isometric to a hemisphere of $\mathbb{S}^2$ with, of course, a great circle boundary curve. The corresponding circle packing on $|K(R)|_{\text{spher}}$ has semi-circular boundary circles that intersect the boundary orthogonally.

REFERENCES

[1] A. F. Beardon and K. Stephenson. The Schwartz–Pick Lemma for circle packings. *Ill. J. Math.* **141** (1991), 577–607.

[2] A. F. Beardon and K. Stephenson. The uniformization theorem for circle packings. *Indiana Univ. Math. J.* **39** (1990), 1383–1425.

[3] P. L. Bowers and K. Stephenson. The set of circle packing points in the Teichmüller space of a surface of finite conformal type is dense. *Math. Proc. Camb. Phil. Soc.* **111** (1992), 487–513.

[4] P. L. Bowers and K. Stephenson. Circle packings in surfaces of finite type: an *in situ* approach with applications to moduli. *Topology* **32** (1993), 157–183.

[5] I. Carter and B. Rodin. An inverse problem for circle packing and conformal mapping. *Trans. Amer. Math. Soc.* **334** (1992), 861–875.

[6] A. J. Casson. *Automorphisms of Surfaces after Neilson and Thurston*, notes by S. Bleiler. University of Texas, 1983.

[7] Y. Colin de Verdière. Empilements de cercles: convergence d'une methode de point fixe. *Forum Math.* **1** (1989), 395–402.

[8] P. Doyle. Oral communication to I. Carter and B. Rodin. See the introduction to Minda–Rodin [**22**].

[9] B. Garrett. Circle packings and polyhedral surfaces. *Discrete Comput. Geom.* **8** (1992), 429–440.

[10] M. Heins. On a class of conformal metrics. *Nagoya Math. J.* **21** (1962), 1–60.

[11] P. Koebe. Kontaktprobleme der Konformem Abbildung. *Math.-Psysiche* **88** (1936), 141–164.

[12] A. Marden and B. Rodin. On Thurston's formulation and proof of Andreev's Theorem. In *Computational Methods and Function Theory*. LNM 1435 (1990), 103–115.

[13] W. S. Massey. *Algebraic Topology: An Introduction* (Springer-Verlag, New York, 1967).

[14] D. Minda and B. Rodin. Circle packing and Riemann surfaces. *J. d'Analyse* **57** (1991), 221–249.

[15] K. Stephenson. Thurston's conjecture on circle packings in the nonhexagonal case, preprint.

[16] William Thurston. *The Geometry and Topology of 3-Manifolds*. Princeton University Notes.

https://doi.org/10.1017/S0305004100071619 Published online by Cambridge University Press