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Fixed point theorems for metric spaces with a conical geodesic bicombing

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Abstract. We derive two fixed point theorems for a class of metric spaces that includes all Banach spaces and all complete Busemann spaces. We obtain our results by the use of a 1-Lipschitz barycenter construction and an existence result for invariant Radon probability measures. Furthermore, we construct a bounded complete Busemann space that admits an isometry without fixed points.

1. *Introduction*

Let (X, d) be a metric space. It is well known that if (X, d) is a complete CAT(0) space, then every subgroup of the isometry group of X with bounded orbits has a non-empty fixed point set (see [**BH99**, Corollary II.2.8]). Analogous results are known for a wide variety of metric spaces. For example, the above statement holds if the metric space (X, d) is an L-embedded Banach space or an injective metric space (cf. [**BGM12**, Theorem A] and [**Lan13**, Proposition 1.2]). Further results can be found in [**Leu10**, **Ede64**].

A metric space (X, d) is said to be a *Busemann space* if (X, d) is a geodesic metric space and if the function $t \mapsto d(\sigma(t), \tau(t))$ is convex on [0, 1] for all pairs of constant speed geodesics $\sigma, \tau : [0, 1] \to X$. It turns out that the initial statement does not hold if (X, d) is a complete Busemann space instead of a complete CAT(0) space. A counterexample is discussed in §2.

The first main result of this article, Theorem 1.1, is a fixed point theorem for the class of metric spaces that admit a conical geodesic bicombing. We follow Descombes and Lang and say that a map $\sigma : X \times X \times [0, 1] \to X$ is a *geodesic bicombing* if, for all points x, y in X, the map $\sigma_{xy}(\cdot) := \sigma(x, y, \cdot)$ is a constant speed geodesic from x to y: that is, if $\sigma_{xy}(0) = x$, $\sigma_{xy}(1) = y$ and $d(\sigma_{xy}(s), \sigma_{xy}(t)) = |s - t| d(x, y)$ for all s, t in [0, 1] (cf. [**DL15**]). Furthermore, a *conical geodesic bicombing* is a geodesic bicombing $\sigma : X \times X \times [0, 1] \to X$ such that the inequality

$$d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \leq (1 - t)d(x, x') + td(y, y')$$

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holds for all points x, x', y, y' in X and all t in $[0, 1]$. The term bicombing was coined by Epstein and Thurston in the context of combinatorial group theory (see [**EPC+92**, p. 84]). To the author's knowledge, conical geodesic bicombings have first been considered by Itoh under the name W-convexity mappings that satisfy condition (III) (see [**Ito79**]). Examples of metric spaces that admit a conical geodesic bicombing include all convex subsets of normed vector spaces and all Busemann spaces. Furthermore, every 1-Lipschitz retract of a normed vector space admits a conical geodesic bicombing (see [**DL15**, p. 370]). For further examples and a thorough discussion of conical geodesic bicombings we refer the reader to [**DL15**].

Let $A \subset X$ be a subset of X and let $\sigma : X \times X \times [0, 1] \to X$ be a conical geodesic bicombing. The *σ-convex hull of A* is the set $\operatorname{conv}_\sigma(A) := \bigcup_{k\geq 1} A_k$, where the sequence $(A_k)_{k\geq 1}$ of subsets of X is given by the recursive rule

$$A_1 := A \quad \text{and} \quad A_{k+1} := \{\sigma_{xy}(t) : x, y \in A_k, t \in [0, 1]\} \quad \text{for all } k \geq 1.$$

We use $\overline{\operatorname{conv}_\sigma}(A)$ to denote the closure of the σ-convex hull of A. Let $\varphi : X \to X$ be an isometry of (X, d) and let $\sigma : X \times X \times [0, 1] \to X$ be a conical geodesic bicombing. We say that φ is *σ-equivariant* if $\varphi \circ \sigma_{xy} = \sigma_{\varphi(x)\varphi(y)}$ for all points x, y in X. Moreover, a subsemigroup Σ of the isometry group of X is called σ-equivariant if every isometry of Σ is σ-equivariant. We say that a conical geodesic bicombing $\sigma : X \times X \times [0, 1] \to X$ has the *midpoint property* if $\sigma_{xy}(1/2) = \sigma_{yx}(1/2)$ for all points x, y in X.

Now, we have everything at hand to state our first result.

THEOREM 1.1. *Let (X, d) denote a complete metric space and let $\sigma : X \times X \times [0, 1] \to X$ be a conical geodesic bicombing that has the midpoint property. Suppose that Σ is a σ-equivariant subsemigroup of the isometry group of X. If there is a non-empty compact subset $K \subset X$ such that $s(K) = K$ for all s in Σ, then there is a point $x_\star$ in the closed σ-convex hull $\overline{\operatorname{conv}_\sigma}(K)$ such that $s(x_\star) = x_\star$ for all s in Σ.*

In [**Nav13**, p. 620], Navas introduced a simple geometric argument that implies Theorem 1.1 if one requires, additionally, that the closed σ-convex hull of K is compact. Unfortunately, Navas's method seems not to work without this additional assumption.

Note that the assumption in Theorem 1.1 of the metric space (X, d) having a conical geodesic bicombing is necessary as, for instance, the unit circle $S^1 \subset \mathbb{R}^2$ clearly admits isometries without fixed points.

The proof strategy of Theorem 1.1 may be roughly described as follows. We use Ryll-Nardzewski's fixed point theorem to construct an invariant Radon probability measure first, and then we use the equivariant contracting barycenter map from Theorem 3.5 to obtain a fixed point.

How restrictive is the assumption in Theorem 1.1 that the subsemigroup Σ is σ-equivariant? Fortunately, metric spaces (X, d) that admit a conical geodesic bicombing $\sigma : X \times X \times [0, 1] \to X$ such that every isometry of X is σ-equivariant provide a broad spectrum of examples. For instance, every isometry of a Busemann space is σ-equivariant with respect to the conical geodesic bicombing σ given by the unique geodesics. Moreover, it is a consequence of a generalized version of the Mazur–Ulam Theorem that every isometry of an open convex subset of a normed vector space is σ-equivariant with respect

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to the conical geodesic bicombing σ given by the linear geodesics (see [**Man72**, p. 368]). Furthermore, [**Lan13**, Proposition 3.8] asserts that every injective metric space (X, d) admits a conical geodesic bicombing σ such that every isometry of X is σ-equivariant.

Our second main result is a strengthened version of Theorem 1.1 if the subsemigroup Σ is generated by a single isometry.

THEOREM 1.2. *Let (X, d) denote a complete metric space and let $\sigma : X \times X \times [0, 1] \to X$ be a conical geodesic bicombing that has the midpoint property. Suppose that $\varphi : X \to X$ is a σ-equivariant isometry of (X, d). If there is a point x_0 in X and a compact subset $K \subset X$ such that the strict inequality*

$$\limsup_{k\to+\infty}\left(\sup_{l\geq 0}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}_K(\varphi^{i+l}(x_0))\right) > 0 \tag{1.1}$$

holds, then there is a point $x_\star$ in $\overline{conv}_\sigma(\{\varphi^k(x_0) : k \geq 0\})$ such that $\varphi(x_\star) = x_\star$.

The function $\mathbb{1}_K : X \to \{0, 1\}$ in Theorem 1.2 denotes the indicator function of the subset $K \subset X$.

Note that the left-hand side of (1.1) is equal to the *upper Banach density* (see [**Fur14**, Definition 3.7]), of the set $D := \{k \geq 0 : \mathbb{1}_{K_0}(\varphi^k(x_0)) = 1\}$. This fact allows us to invoke a basic result from combinatorial number theory in order to show that the orbits of the isometry φ are bounded (see Lemma 5.2).

One key ingredient in the proof of Theorem 1.2 is a generalization of a classical existence result for invariant Radon measures (see Theorem 4.5).

The paper is structured as follows. In §2, we construct a bounded complete Busemann space that admits an isometry without fixed points. In §3, we transfer a 1-Lipschitz barycenter construction introduced by Es-Sahib and Heinich (cf. [**ESH99**] and Navas [**Nav13**]), into the setting of metric spaces that admit a conical geodesic bicombing. The primary result of §3 is Theorem 3.5. The purpose of §4 is to derive Theorem 4.5, which is a generalization of a classical result from topological dynamics that is due to Oxtoby and Ulam (see [**OU39**, Theorem 1]). The results from §4 may be of independent interest. Finally, in §5, we use Theorems 3.5 and 4.5 to prove our main results.

2. *Counterexample*

In this section, we construct a bounded complete Busemann space that admits an isometry without fixed points. As usual, let $\ell^1(\mathbb{Z}) \subset \mathbb{R}^{\mathbb{Z}}$ denote the linear subspace of $\mathbb{R}^{\mathbb{Z}}$ that consists of all sequences $x := (x_k)_{k\in\mathbb{Z}}$ such that

$$\|x\|_1 := \sum_{k\in\mathbb{Z}}|x_k| < +\infty.$$

Now, we use a standard technique (cf. [**JL01**, p. 786]), to renorm the Banach space $(\ell^1(\mathbb{Z}), \|\cdot\|_1)$ into a strictly convex Banach space. We define the map $\|\cdot\|_\star : \ell^1(\mathbb{Z}) \to \mathbb{R}$ through the assignment

$$(x_k)_{k\in\mathbb{Z}} \mapsto \sqrt{\left(\sum_{k\in\mathbb{Z}}|x_k|\right)^2 + \sum_{k\in\mathbb{Z}}|x_k|^2}.$$

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It is straightforward to show that the map $\|\cdot\|_\star$ defines a norm on $\ell^1(\mathbb{Z})$. Elementary estimates show that

$$\frac{1}{\sqrt{2}}\|\cdot\|_\star \leq \|\cdot\|_1 \leq \|\cdot\|_\star;$$

hence, the norms $\|\cdot\|_1$ and $\|\cdot\|_\star$ are equivalent. It follows that $(\ell^1(\mathbb{Z}), \|\cdot\|_\star)$ is a Banach space. Recall that a normed vector space $(V, \|\cdot\|_V)$ is said to be *strictly convex* if, for all distinct points x, y in V with $\|x\|_V = \|y\|_V = 1$ and for all λ in $(0, 1)$, the strict inequality $\|(1-\lambda)x + \lambda y\|_V < 1$ holds.

LEMMA 2.1. *The Banach space $(\ell^1(\mathbb{Z}), \|\cdot\|_\star)$ is strictly convex.*

Proof. Let x and y denote two distinct points of $\ell^1(\mathbb{Z})$ that satisfy $\|x\|_\star = \|y\|_\star = 1$ and let λ in $(0, 1)$ be a real number. Since x and y are distinct, there is an integer k_0 such that $x_{k_0} \neq y_{k_0}$. It follows that

$$((1-\lambda)x_{k_0} + \lambda y_{k_0})^2 < (1-\lambda)x_{k_0}^2 + \lambda y_{k_0}^2, \tag{2.1}$$

as the real valued function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is strictly convex. Now, elementary estimates and the strict inequality in (2.1) imply that $\|(1-\lambda)x + \lambda y\|_\star^2 < (1-\lambda)\|x\|_\star^2 + \lambda\|y\|_\star^2 = 1$; hence, the Banach space $(\ell^1(\mathbb{Z}), \|\cdot\|_\star)$ is strictly convex, as was to be shown. □

The shift map $T : \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ given by the assignment

$$(x_k)_{k\in\mathbb{Z}} \mapsto (x_{k-1})_{k\in\mathbb{Z}}$$

is a linear map and an isometry of $(X, \|\cdot\|_\star)$. Note that the zero sequence is the only fixed point of T. Let $x_0 \in \ell^1(\mathbb{Z})$ be the sequence that is equal to one if $k = 0$ and equal to zero if $k \neq 0$. We define the set $A := \{T^k(x_0) : k \in \mathbb{Z}\}$. Let $\mathrm{conv}(A)$ denote the convex hull of A.

LEMMA 2.2. *If x is an element of $\mathrm{conv}(A)$, then $1 \leq \|x\|_\star \leq \sqrt{2}$.*

Proof. Let x be an element of $\mathrm{conv}(A)$. By the definition of $\mathrm{conv}(A)$, there is an integer $n \geq 0$, an element $(\alpha_0, \ldots, \alpha_n)$ in the n-dimensional standard simplex $\Delta^n \subset \mathbb{R}^{n+1}$ and $n+1$ distinct integers $l_0, \ldots, l_n$ such that $x = \alpha_0 T^{l_0}(x_0) + \cdots + \alpha_n T^{l_n}(x_0)$. For every integer $0 \leq i \leq n$, the sequence $T^{l_i}(x_0) \in \ell^1(\mathbb{Z})$ is equal to one if $k = l_i$ and equal to zero if $k \neq l_i$. Therefore, we compute

$$\left(\sum_{k\in\mathbb{Z}} |x_k|\right)^2 = \left(\sum_{i=0}^{n} |\alpha_i|\right)^2 = 1$$

and

$$\sum_{k\in\mathbb{Z}} |x_k|^2 = \sum_{i=0}^{n} \alpha_i^2.$$

As a result, we obtain that

$$1 \leq \|x\|_\star = \sqrt{1 + \sum_{i=0}^{n} \alpha_i^2} \leq \sqrt{2},$$

since $\alpha_i^2 \leq \alpha_i$ for all integers $0 \leq i \leq n$. □

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Let $\overline{\mathrm{conv}}(A)$ denote the closure of $\mathrm{conv}(A)$. By the use of Lemma 2.2, we obtain that $1 \leq \|x\|_\star \leq \sqrt{2}$ for all points x in $\overline{\mathrm{conv}}(A)$. Thus, in particular, the zero sequence is not an element of $\overline{\mathrm{conv}}(A)$. A straightforward calculation shows that $T(\overline{\mathrm{conv}}(A)) = \overline{\mathrm{conv}}(A)$; hence, the map T is an isometry of the bounded metric space $(\overline{\mathrm{conv}}(A), \|\cdot\|_\star)$ without fixed points. We claim that $(\overline{\mathrm{conv}}(A), \|\cdot\|_\star)$ is a complete Busemann space. It is well known that every convex subset of a strictly convex normed vector space is a Busemann space (see [**Pap14**, Propositions 8.1.6 and 8.1.5]). Hence, it follows that $(\overline{\mathrm{conv}}(A), \|\cdot\|_\star)$ is a Busemann space as $\overline{\mathrm{conv}}(A)$ is a convex subset of $\ell^1(\mathbb{Z})$. Note that $(\overline{\mathrm{conv}}(A), \|\cdot\|_\star)$ is complete. Thus, we have constructed a complete bounded Busemann space that admits an isometry without fixed points.

3. *Existence of contracting barycenter maps*

This section is divided into two parts. In the first part, we build up prerequisite material from optimal transportation theory and, in the second part, we transfer a 1-Lipschitz barycenter construction that traces back to Es-Sahib and Heinich (cf. [**ESH99**] and Navas [**Nav13**]), into the setting of metric spaces that admit a conical geodesic bicombing. The primary result of this section is Theorem 3.5.

3.1. *Prerequisite material from optimal transportation theory.*

In this subsection, we collect some facts from the theory of optimal transportation. Let $(\mathfrak{X}, \mathcal{T}_\mathfrak{X})$ be a Hausdorff topological space. We denote by $\mathcal{B}_\mathfrak{X}$ the Borel σ-algebra of $(\mathfrak{X}, \mathcal{T}_\mathfrak{X})$ and by $\mathcal{K}_\mathfrak{X}$ the set that consists of all compact subsets of $(\mathfrak{X}, \mathcal{T}_\mathfrak{X})$. A non-negative Borel measure $\mu : \mathcal{B}_\mathfrak{X} \to [0, +\infty]$ is called a *Radon measure* if $\mu(K) < +\infty$ for all compact subsets K of $(\mathfrak{X}, \mathcal{T}_\mathfrak{X})$ and

$$\mu(B) = \sup\{\mu(K) : K \subset B,\ K \in \mathcal{K}_\mathfrak{X}\}$$

for all Borel measurable sets B of $(\mathfrak{X}, \mathcal{T}_\mathfrak{X})$. A signed finite Borel measure $\mu : \mathcal{B}_\mathfrak{X} \to \mathbb{R}$ is called a signed finite Radon measure if the total variation $|\mu| : \mathcal{B}_\mathfrak{X} \to [0, +\infty)$ is a Radon measure. Let $P(\mathfrak{X})$ denote the set that consists of all non-negative Borel measures on $(\mathfrak{X}, \mathcal{B}_\mathfrak{X})$ that are Radon probability measures. Suppose that $f : \mathfrak{X} \to \mathfrak{X}$ is a continuous map. We define the map

$$f_* : P(\mathfrak{X}) \to P(\mathfrak{X})$$
$$\mu \mapsto \begin{cases} f_*\mu : \mathcal{B}_\mathfrak{X} \to [0, 1] \\ B \mapsto \mu(f^{-1}(B)). \end{cases}$$

The map f_* is well defined and, for every μ in $P(\mathfrak{X})$, the measure $f_*\mu$ is called the *pushforward of* μ *by* f. A measure μ in $P(\mathfrak{X})$ is called f*-invariant* if $f_*\mu = \mu$.

For the rest of this subsection let (X, d) be a metric space. Suppose that the Borel measure $\mu : \mathcal{B}_X \to [0, 1]$ is a Radon probability measure. The subset $\mathrm{spt}(\mu)$ of all points x in X such that $\mu(U) > 0$ for all open neighborhoods of x is called the *support* of μ. We say that μ has a *finite first moment* if there is a point x_0 in X such that

$$\int_X d(x, x_0)\, \mu(dx) < +\infty.$$

We let $P_1(X)$ be the set that consists of all measures of $(X, \mathcal{B}_X)$ that are Radon probability measures with a finite first moment. We denote by $W^1 : P_1(X) \times P_1(X) \to \mathbb{R}$ the first

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Wasserstein distance on $P_1(X)$. Due to the Kantorovich–Rubinstein duality theorem, the first Wasserstein distance W^1 is given by

$$W^1(\mu, \nu) = \sup\left\{\int_X f\, d\mu - \int_X f\, d\nu : f : X \to \mathbb{R} \text{ is 1-Lipschitz}\right\}$$

and thus defines a metric on $P_1(X)$ (see [**Edw11**]). We define

$$P_{\mathbb{Q}}(X) := \left\{\sum_{k=1}^{n} \alpha_k \delta_{x_k} : n \geq 1, \sum_{k=1}^{n} \alpha_k = 1, \alpha_k \in \mathbb{Q}_{\geq 0}, x_k \in X\right\}.$$

On $P_{\mathbb{Q}}(X)$, there is an explicit formula for the first Wasserstein distance.

PROPOSITION 3.1. *Let (X, d) denote a metric space. If $n \geq 1$ is an integer and $x_1, y_1, \ldots, x_n, y_n$ are points in X, then*

$$W^1\left(\frac{1}{n}(\delta_{x_1} + \cdots + \delta_{x_n}), \frac{1}{n}(\delta_{y_1} + \cdots + \delta_{y_n})\right) = \frac{1}{n} \min_{\tau \in S_n} \sum_{k=1}^{n} d(x_k, y_{\tau(k)}).$$

Proof. See [**Vil03**, p. 5]. □

It turns out that the set $P_{\mathbb{Q}}(X)$ is W^1-dense in $P_1(X)$. This is the content of the following proposition.

PROPOSITION 3.2. *Let (X, d) be a metric space and let $\varepsilon > 0$ be a positive real number. If the Borel measure $\mu : \mathcal{B}_X \to [0, 1]$ is a Radon probability measure contained in $P_1(X)$, then there exists a measure ν_ε contained in $P_{\mathbb{Q}}(X)$ with $spt(\nu_\varepsilon) \subset spt(\mu)$ such that $W^1(\mu, \nu_\varepsilon) < \varepsilon$.*

Proof. See [**Edw11**, Theorem 6.1] and [**Vil09**, Theorem 6.18]. □

Suppose that the map $\varphi : X \to X$ is 1-Lipschitz. By the use of the Kantorovich–Rubinstein duality theorem, it is readily verified that the map $\varphi_* : P_1(X) \to P_1(X)$ is well defined and 1-Lipschitz as well.

3.2. *A 1-Lipschitz barycenter construction.* The following definition is due to Sturm (see [**Stu03**, Remark 6.4]).

Definition 3.3. (Contracting barycenter map) Let (X, d) denote a metric space. A contracting barycenter map is a 1-Lipschitz map $\beta : (P_1(X), W^1) \to (X, d)$ such that $\beta(\delta_x) = x$ for all points x in X.

If a metric space (X, d) admits a contracting barycenter map $\beta : P_1(X) \to X$, then (X, d) admits a conical geodesic bicombing. Indeed, it is immediately verified that the map $\sigma : X \times X \times [0, 1] \to X$ given by the assignment $(x, y, t) \mapsto \beta((1 - t)\delta_x + t\delta_y)$ defines a conical geodesic bicombing on (X, d).

In this subsection, we are interested in the reversed situation. Does every metric space that admits a conical geodesic bicombing admit a contracting barycenter map? It turns out that every complete metric space that admits a conical geodesic bicombing that has the midpoint property admits a contracting barycenter map (see Theorem 3.5).

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In [**ESH99**], Es-Sahib and Heinich developed a barycenter construction for non-empty finite subsets of separable complete Busemann spaces. Es-Sahib and Heinich's barycenter construction translates with no effort to complete metric spaces that admit a conical geodesic bicombing. This is the content of the subsequent proposition.

PROPOSITION 3.4. *Let (X, d) be a complete metric space. If X admits a conical geodesic bicombing $\sigma : X \times X \times [0, 1] \to X$, then there exists a collection $\{b_n : X^n \to X\}_{n\geq 1}$ of maps that satisfy the following four conditions.*

(1) *(Locality) For all integers $n \geq 1$ and all points $\mathbf{x} := (x_1, \ldots, x_n)$ in X^n, the point $b_n(\mathbf{x})$ is contained in $\overline{\mathrm{conv}_\sigma}(\{x_1, \ldots, x_n\})$.*

(2) *(Recursion) For all integers $n \geq 3$ and all points $\mathbf{x} := (x_1, \ldots, x_n)$ in X^n,*

$$b_n(\mathbf{x}) = b_n(b_{n-1}(\mathbf{x}_1), \ldots, b_{n-1}(\mathbf{x}_n)),$$

where $\mathbf{x}_k := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ for all integers $1 \leq k \leq n$.

(3) *(Non-expansiveness) For all integers $n \geq 1$ and all points $\mathbf{x} := (x_1, \ldots, x_n)$ and $\mathbf{y} := (y_1, \ldots, y_n)$ in X^n,*

$$d(b_n(\mathbf{x}), b_n(\mathbf{y})) \leq \frac{1}{n}\sum_{k=1}^{n} d(x_k, y_k).$$

(4) *(W^1-Non-expansiveness) If σ has the midpoint property, then*

$$d(b_n(\mathbf{x}), b_n(\mathbf{y})) \leq \frac{1}{n}\min_{\tau\in S_n}\sum_{k=1}^{n} d(x_k, y_{\tau(k)})$$

for all integers $n \geq 1$ and all points $\mathbf{x} := (x_1, \ldots, x_n)$ and $\mathbf{y} := (y_1, \ldots, y_n)$ in X^n.

Proof. Let b_1 denote the identity map of X and define the map $b_2 : X^2 \to X$ through the assignment $(x, y) \mapsto \sigma_{xy}(1/2)$. It is straightforward to show that the map b_2 satisfies all four conditions. Now, we proceed by induction. Let $n \geq 3$ be an integer and suppose that b_{n-1} is defined and satisfies all four conditions. Let $\mathbf{x} := (x_1, \ldots, x_n)$ be a point in X^n. We define the sequence $(\mathbf{x}^{(k)})_{k\geq 0} \subset X^n$ via the recursive rule

$$\mathbf{x}^{(0)} := \mathbf{x}, \quad \mathbf{x}^{(k+1)} := (b_{n-1}(\mathbf{x}_1^{(k)}), \ldots, b_{n-1}(\mathbf{x}_n^{(k)})),$$

where, for each integer $k \geq 0$ and each integer $1 \leq l \leq n$, the $(n-1)$-tuple $\mathbf{x}_l^{(k)}$ is obtained from the n-tuple $\mathbf{x}^{(k)}$ by deleting the lth entry. From now on, let $x_l^{(k)}$ denote the lth entry of the n-tuple $\mathbf{x}^{(k)}$. For every integer $k \geq 0$, we set $A_k := \overline{\mathrm{conv}_\sigma}(\{x_1^{(k)}, \ldots, x_n^{(k)}\})$ and $D_k := \mathrm{diam}(A_k)$. The definition of the closed σ-convex hull $\overline{\mathrm{conv}_\sigma}$ is given in the introduction. Note that the sequence $(D_k)_{k\geq 0}$ is non-increasing. We claim that $D_{2k} \leq (1/(n-1)^k)D_0$ for all integers $k \geq 1$. Let $k \geq 1$ be an integer and suppose that $1 \leq l < l' \leq n$. We compute

$$\begin{aligned} d(x_l^{(2k)}, x_{l'}^{(2k)}) &= d(b_{n-1}(\mathbf{x}_l^{(2k-1)}), b_{n-1}(\mathbf{x}_{l'}^{(2k-1)})) \\ &\leq \frac{1}{n-1}\sum_{i=l}^{l'-1} d(b_{n-1}(\mathbf{x}_i^{(2k-2)}), b_{n-1}(\mathbf{x}_{i+1}^{(2k-2)})) \\ &\leq \frac{1}{n-1} D_{2(k-1)}. \end{aligned}$$

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Since taking the σ-convex hull of a subset does not increase the diameter, we have shown that $D_{2k} \leq (1/(n-1))D_{2(k-1)}$. Hence, it follows that $D_{2k} \leq (1/(n-1)^k)D_0$ for all integers $k \geq 1$. As a result, we obtain that the intersection $\bigcap_{k\geq 0} A_k$ consists precisely of one point which we call x^∞. For later use, observe that, for each integer $1 \leq l \leq n$, $x_l^{(k)} \to x^\infty$ as $k \to +\infty$. We define $b_n(x) := x^\infty$. It is readily verified that the map b_n satisfies the first two conditions. Next, we show that b_n satisfies the non-expansiveness condition. Suppose that $\mathbf{x} := (x_1, \ldots, x_n)$ and $\mathbf{y} := (y_1, \ldots, y_n)$ are points in X^n. For each integer $k \geq 1$, we compute

$$\begin{aligned} d(x_l^{(k)}, y_l^{(k)}) &= d(b_{n-1}(\mathbf{x}_l^{(k-1)}), b_{n-1}(\mathbf{y}_l^{(k-1)})) \\ &\leq \frac{1}{n-1} \sum_{i=1, i\neq l}^{n} d(x_i^{(k-1)}, y_i^{(k-1)}) \end{aligned} \tag{3.1}$$

for all integers $1 \leq l \leq n$. By the use of (3.1),

$$\sum_{l=1}^{n} d(x_l^{(k)}, y_l^{(k)}) \leq \sum_{l=1}^{n} d(x_l^{(k-1)}, y_l^{(k-1)})$$

for all integers $k \geq 1$. Hence, by passing to the limit $k \to +\infty$, we conclude that

$$n \cdot d(x^\infty, y^\infty) \leq \sum_{l=1}^{n} d(x_l, y_l),$$

as desired. Now, we are left to show that the map b_n satisfies the W^1-non-expansiveness condition. Suppose that σ has the midpoint property. Let $\tau \in S_{n-1}$ be a permutation. Due to the W^1-non-expansiveness of b_{n-1},

$$d(b_{n-1}(x_1, \ldots, x_{n-1}), b_{n-1}(x_{\tau(1)}, \ldots, x_{\tau(n-1)})) \leq \frac{1}{n-1} \min_{\rho\in S_{n-1}} \sum_{k=1}^{n-1} d(x_k, x_{\tau(\rho(k))}) = 0$$

for all points $x_1, \ldots, x_n$ in X. Consequently, we obtain that b_{n-1} is invariant under permutations: that is,

$$b_{n-1}(x_1, \ldots, x_{n-1}) = b_{n-1}(x_{\tau(1)}, \ldots, x_{\tau(n-1)})$$

for all permutations τ in S_{n-1}. Hence, it follows from the second condition that b_n is invariant under permutations as well. Thus, the fourth condition is a consequence of the third condition and the permutation invariance of b_n. The proposition follows. □

If the complete metric space (X, d) in Proposition 3.4 is a Banach space and the conical geodesic bicombing $\sigma : X \times X \times [0, 1] \to X$ is given by $(x, y, t) \mapsto (1-t)x + ty$, then it is readily verified that the collection $\{\mathrm{bar}_n : X^n \to X\}_{n\geq 1}$, where, for each integer $n \geq 1$, the map bar_n is given by the assignment

$$(x_1, \ldots, x_n) \mapsto \frac{1}{n}(x_1 + \cdots + x_n),$$

satisfies all four conditions of Proposition 3.4.

Let (X, d) denote a complete metric space. Suppose that (X, d) admits a conical geodesic bicombing $\sigma : X \times X \times [0, 1] \to X$ that has the midpoint property. Let $\{b_n :$

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$X^n \to X\}_{n\geq 1}$ denote the collection of maps that we have constructed in Proposition 3.4, let $n \geq 1$ be an integer and let $\mathbf{x}$ be a point in X^n. For every integer $k \geq 1$, we denote by $Q^k(\mathbf{x})$ the element in X^{kn} that is equal to $(\mathbf{x}, \ldots, \mathbf{x})$. It is tempting to assume that

$$b_{nk_1}(Q^{k_1}(\mathbf{x})) = b_{nk_2}(Q^{k_2}(\mathbf{x})) \quad \text{for all } k_1, k_2 \geq 1. \tag{3.2}$$

However, this is not necessarily true. A counterexample can be found in [**Nav13**, p. 614]. Since the equality in (3.2) does not hold in general, one might ask: does at least the limit

$$\lim_{k\to+\infty} b_{nk}(Q^k(\mathbf{x})) \tag{3.3}$$

exist? Navas showed that the limit (3.3) exists for all integers $n \geq 1$ and all points $\mathbf{x}$ in X^n if X is a complete separable Busemann space (see [**Nav13**, Proposition 1.2]). As Navas's proof relies solely on the fact that the collection $\{b_n : X^n \to X\}_{n\geq 1}$ satisfies the recursion and the W^1-non-expansiveness condition, Navas's proof translates verbatim to collections $\{b_n : X^n \to X\}_{n\geq 1}$ that arose from complete metric spaces that admit a conical geodesic bicombing with the midpoint property.

A streamlined version of Navas's proof can be found in [**Des16**]. If X satisfies a weak local compactness assumption, then it is possible to draw the conclusion that the limit in (3.3) exists via a martingale convergence theorem (cf. [**ESH99**, Theorem 2]). Navas used the existence of the limit (3.3) to construct a contracting barycenter map for every complete separable Busemann space (see [**Nav13**]). Essentially, the same construction yields a contracting barycenter map for every complete metric space that admits a conical geodesic bicombing that has the midpoint property.

THEOREM 3.5. *Let (X, d) be a complete metric space and let $\sigma : X \times X \times [0, 1] \to X$ denote a conical geodesic bicombing. If σ has the midpoint property, then the map $\beta_\sigma : P_{\mathbb{Q}}(X) \to X$ given by the assignment*

$$\mu = \frac{1}{n}(\delta_{x_1} + \cdots + \delta_{x_n}) \mapsto \lim_{k\to+\infty} b_{nk}(Q^k((x_1, \ldots, x_n)) \tag{3.4}$$

is well defined and extends uniquely to a contracting barycenter map $\beta_\sigma : P_1(X) \to X$ that has the following properties.

(1) *(Locality) For all measures μ in $P_1(X)$, the point $\beta_\sigma(\mu)$ is contained in $\overline{conv_\sigma}(spt(\mu))$.*
(2) *(Equivariance) If $\varphi : X \to X$ is a 1-Lipschitz and σ-equivariant self-map of (X, d), then β_σ is φ-equivariant: that is, $\varphi \circ \beta_\sigma = \beta_\sigma \circ \varphi_*$.*

Proof. It is readily verified that the map $\beta_\sigma : P_{\mathbb{Q}}(X) \to X$ is well defined: that is, the assignment (3.4) does not depend on the representation of μ. Let μ and ν denote two elements of $P_{\mathbb{Q}}(X)$. Note that there is an integer $n \geq 1$ and there are points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in X^n such that $\mu = (1/n)(\delta_{x_1} + \cdots + \delta_{x_n})$ and $\nu = (1/n)(\delta_{y_1} + \cdots + \delta_{y_n})$. Due to Equation (3.4), the W^1-non-expansiveness condition and Proposition 3.1,

$$d(\beta_\sigma(\mu), \beta_\sigma(\nu)) \leq \frac{1}{n} \min_{\tau\in S_n} \sum_{k=1}^{n} d(x_k, y_{\tau(k)}) = W^1(\mu, \nu);$$

hence, the map $\beta_\sigma : P_{\mathbb{Q}}(X) \to X$ is 1-Lipschitz. Proposition 3.2 tells us that $P_{\mathbb{Q}}(X)$ is dense in $(P_1(X), W^1)$; thus, as X is complete, the map $\beta_\sigma : P_{\mathbb{Q}}(X) \to X$ extends uniquely

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to the whole space $P_1(X)$. We denote this map again by β_σ. Note that the extended map β_σ is 1-Lipschitz, by construction, and $\beta_\sigma(\delta_x) = x$ for all points x in X; hence, the map β_σ is a contracting barycenter map on (X, d).

The fact that the point $\beta_\sigma(\mu)$ is contained in $\overline{\mathrm{conv}_\sigma}(\mathrm{spt}(\mu))$ for all measures μ in $P_1(X)$ is a direct consequence of Proposition 3.2.

To conclude the proof, we show that if $\varphi : X \to X$ is a 1-Lipschitz and σ-equivariant self-map of (X, d), then β_σ is φ-equivariant. As the map φ is σ-equivariant, $\varphi(b_2(x, y)) = b_2(\varphi(x), \varphi(y))$ for all points x, y in X. A straightforward induction shows that, for all integers $n \geq 2$ and all points $\mathbf{x}$ in X^n,

$$\varphi(b_n(\mathbf{x})) = b_n(\boldsymbol{\varphi}(\mathbf{x})), \tag{3.5}$$

where the map $\boldsymbol{\varphi} : X^n \to X^n$ is given by the assignment $(x_1, \ldots, x_n) \mapsto (\varphi(x_1), \ldots, \varphi(x_n))$. Suppose that μ is a measure in $P_{\mathbb{Q}}(X)$. There is an integer $n \geq 1$ and a point $(x_1, \ldots, x_n)$ in X^n such that $\mu = (1/n)(\delta_{x_1} + \cdots + \delta_{x_n})$. Note that $\varphi_*\mu = (1/n)(\delta_{\varphi(x_1)} + \cdots + \delta_{\varphi(x_n)})$. We compute

$$\varphi(\beta_\sigma(\mu)) = \lim_{k\to+\infty} \varphi(b_{nk}(Q^k(\mathbf{x}))) \overset{(3.5)}{=} \lim_{k\to+\infty} b_{nk}(Q^k(\boldsymbol{\varphi}(\mathbf{x}))) = \beta_\sigma(\varphi_*\mu).$$

Since the two 1-Lipschitz maps $\varphi \circ \beta_\sigma$ and $\beta_\sigma \circ \varphi_*$ agree on the W^1-dense subset $P_{\mathbb{Q}}(X) \subset P_1(X)$, we obtain that they coincide on the whole space $P_1(X)$. The theorem follows. □

We call the map β_σ from Theorem 3.5 the *contracting barycenter map associated to* σ. The rest of this section is devoted to contracting barycenter maps on Banach spaces. In the subsequent proposition we show that there is precisely one contracting barycenter map on a Banach space.

PROPOSITION 3.6. *Let $(E, \|\cdot\|)$ be a Banach space, let λ be the conical geodesic bicombing on E that consists of the linear geodesics and let $\beta_\lambda : P_1(E) \to E$ denote the contracting barycenter map associated to λ. It holds that the map $\beta_\lambda : P_1(E) \to E$ is given through the assignment*

$$\mu \mapsto \int_E x \, d\mu(x) \tag{3.6}$$

and that the map β_λ is the only contracting barycenter map on $(E, \|\cdot\|)$.

Proof. Suppose that $\beta : P_1(E) \to E$ is a contracting barycenter map on $(E, \|\cdot\|)$. Let μ be a measure contained in $P_1(E)$. The point $\beta(\mu)$ satisfies

$$\|\beta(\mu) - y\| \leq W^1(\mu, \delta_y) = \int_E \|x - y\| \, d\mu(x) \quad \text{for all } y \in E. \tag{3.7}$$

It is well known that $\mathrm{spt}(\mu)$ is separable and that $\mu(E \setminus \mathrm{spt}(\mu)) = 0$; hence, the identity map $\mathrm{id} : (E, \mathcal{B}_E) \to (E, \mathcal{B}_E)$ is μ-essentially separably valued. Now, the Pettis measurability theorem tells us that the identity map id is μ-measurable. Hence, we can use the definition of $P_1(E)$ and Bochner's criterion for integrability to deduce that the identity map id is Bochner integrable with respect to the measure μ. Thus, as the point

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$\beta(\mu)$ satisfies the inequality (3.7), [**Mol06**, Theorem 3.6], which is a direct consequence of the strong law of large numbers, tells us that

$$\beta(\mu) = \int_E \mathrm{id}(x)\, d\mu(x). \tag{3.8}$$

Since the map β_λ is a contracting barycenter map, we have shown that β_λ is given through the assignment (3.8). Furthermore, as the contracting barycenter map β was arbitrary, we have also shown that β_λ is the unique contracting barycenter map on $(E, \|\cdot\|)$. □

Having Theorem 3.5 and Proposition 3.6 on hand we can deduce the following corollary.

COROLLARY 3.7. *Let $(E, \|\cdot\|)$ be a Banach space. If μ is a measure in $P_1(E)$, then the Bochner integral $\int_E x\, d\mu(x)$ is contained in the closure of the convex hull of spt(μ).*

Proof. This is a consequence of Theorem 3.5 and Proposition 3.6. □

4. *Existence of invariant measures*

The primary result of this section, Theorem 4.5, is proved in §4.2. In §4.1, we introduce some results that we will need in order to derive Theorem 4.5.

4.1. *Maximal values of generalized limits.* Let Σ denote a countable semigroup. A sequence $(\Sigma_k)_{k\geq 1}$ of non-empty finite subsets of Σ is a *Følner sequence* if

$$\lim_{k\to+\infty} \frac{|s\,\Sigma_k \,\Delta\, \Sigma_k|}{|\Sigma_k|} = 0$$

for all s in Σ. Here the symbol Δ denotes the symmetric difference of two sets. Recall that the sequence $(\{0, \ldots, k-1\})_{k\geq 1}$ is a Følner sequence of the semigroup of the non-negative integers.

Definition 4.1. (Generalized limit) Let Σ denote a countable semigroup. A generalized limit is a positive linear functional $\Theta : \ell^\infty(\Sigma) \to \mathbb{R}$ such that $\Theta((1)_{g\in G}) = 1$ and $\Theta((x_s)_{s\in\Sigma}) = \Theta((x_{s_0 s})_{s\in G})$ for all s_0 in Σ and x in $\ell^\infty(\Sigma)$.

For convenience, we use the notation $\mathbf{1} := (1)_{g\in G}$ and $s_0.x := (x_{s_0 s})_{s\in\Sigma}$ for all s_0 in Σ and for all x in $\ell^\infty(\Sigma)$. The subsequent lemma is an extension of [**Suc64**, Theorem 1].

LEMMA 4.2. *Let Σ be a countable semigroup and let $(\Sigma_k)_{k\geq 1}$ denote a Følner sequence of Σ. Suppose that $\Theta : \ell^\infty(\Sigma) \to \mathbb{R}$ is a linear functional. Then the following statements are equivalent.*

(1) *The linear functional $\Theta : \ell^\infty(\Sigma) \to \mathbb{R}$ is positive and a generalized limit.*

(2) *For all points x in $\ell^\infty(\Sigma)$,*

$$\Theta(x) \leq \liminf_{k\to+\infty}\left(\sup_{s\in\Sigma} \frac{1}{|\Sigma_k|} \sum_{h\in\Sigma_k} x_{hs}\right).$$

(3) *For all points x in $\ell^\infty(\Sigma)$,*

$$\Theta(x) \leq \limsup_{k\to+\infty}\left(\sup_{s\in\Sigma} \frac{1}{|\Sigma_k|} \sum_{h\in\Sigma_k} x_{hs}\right).$$

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Proof. We show that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1).
(1) $\Longrightarrow$ (2). Let x in $\ell^\infty(\Sigma)$ be a point and let $k \geq 1$ be an integer. For every h in Σ_k, $\Theta(h.x) = \Theta(x)$; hence, it follows that

$$\Theta(x) = \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} \Theta(h.x) \leq \left(\sup_{s \in \Sigma} \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} x_{hs} \right) \Theta(\mathbf{1}).$$

Since $\Theta(\mathbf{1}) = 1$, we have shown the desired inequality.
(2) $\Longrightarrow$ (3). This is trivial.
(3) $\Longrightarrow$ (1). To begin, we show that Θ is positive. Suppose that x in $\ell^\infty(\Sigma)$ is a point with $x \geq 0$. We have

$$\Theta(-x) \leq \limsup_{k \to +\infty} \left(\sup_{s \in \Sigma} \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} -x_{hs} \right) \leq 0,$$

and hence it follows that $\Theta(x) \geq 0$. Next, we show that $\Theta(\mathbf{1}) = 1$. Since $\Theta(\mathbf{1}) \leq 1$ and $\Theta(-\mathbf{1}) \leq -1$, we obtain that $\Theta(\mathbf{1}) = 1$, as desired. To conclude, we show that Θ is left Σ-invariant. Let x in $\ell^\infty(\Sigma)$ be a point and let s_0 be an element of Σ. We define the point $y := x - s_0.x$. Note that y is contained in $\ell^\infty(\Sigma)$. We claim that $\Theta(y) = 0$. We have that

$$\begin{aligned} |\Theta(y)| &\leq \limsup_{k \to +\infty} \left(\sup_{s \in \Sigma} \left| \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} (x_{hs} - x_{s_0 hs}) \right| \right) \\ &\leq \|x\|_\infty \limsup_{k \to +\infty} \left(\sup_{s \in \Sigma} \frac{|(s_0 \Sigma_k s) \Delta \Sigma_k s|}{|\Sigma_k|} \right). \end{aligned} \tag{4.1}$$

Let s be an element of Σ. Observe that, since $(s_0\Sigma_k \cup \Sigma_k)s = s_0\Sigma_k s \cup \Sigma_k s$ and $(s_0\Sigma_k \cap \Sigma_k)s \subset s_0\Sigma_k s \cap \Sigma_k s$, it follows that $s_0\Sigma_k s \Delta \Sigma_k s \subset (s_0\Sigma_k \Delta \Sigma_k)s$. As $|(s_0\Sigma_k \Delta \Sigma_k)s| \leq |s_0\Sigma_k \Delta \Sigma_k|$, we obtain $|(s_0\Sigma_k s)\Delta\Sigma_k s| \leq |s_0\Sigma_k \Delta \Sigma_k|$. Now, inequality (4.1) implies that $\Theta(y) = 0$, since $(\Sigma_k)_{k \geq 1}$ is a Følner sequence. Thus, we have shown that $\Theta(s_0.x) = \Theta(x)$, as desired. □

We proceed with two immediate corollaries of Lemma 4.2.

COROLLARY 4.3. *Let Σ denote a countable semigroup. If Σ admits a Følner sequence $\boldsymbol{\Sigma} := (\Sigma_k)_{k \geq 1}$, then, for every point $x^\star$ in $\ell^\infty(\Sigma)$, there is a generalized limit $\Theta_\star^{\boldsymbol{\Sigma}} : \ell^\infty(\Sigma) \to \mathbb{R}$ such that*

$$\Theta_\star^{\boldsymbol{\Sigma}}(x^\star) = \limsup_{k \to +\infty} \left(\sup_{s \in \Sigma} \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} x^\star_{hs} \right).$$

Proof. Fix a vector $x^\star$ in $\ell^\infty(\Sigma)$. Let $U \subset \ell^\infty(\Sigma)$ denote the linear span of the vector $x^\star$ and let $f : U \to \mathbb{R}$ denote the unique linear functional such that

$$f(x^\star) = \limsup_{k \to +\infty} \left(\sup_{s \in \Sigma} \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} x^\star_{hs} \right).$$

The Hahn–Banach theorem tells us that there is a linear map $\Theta_\star^{\boldsymbol{\Sigma}} : \ell^\infty(\Sigma) \to \mathbb{R}$ such that $\Theta_\star^{\boldsymbol{\Sigma}}|_U = f$ and such that

$$\Theta_\star^{\boldsymbol{\Sigma}}(x) \leq \limsup_{k \to +\infty} \left(\sup_{s \in \Sigma} \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} x_{hs} \right)$$

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for all points $x \in \ell^\infty(\Sigma)$. Due to Lemma 4.2, the map $\Theta_\star^{\mathbf{\Sigma}}$ is a generalized limit and hence the corollary follows. □

COROLLARY 4.4. *Let Σ denote a countable semigroup. If $\mathbf{\Sigma} := (\Sigma_k)_{k\geq 1}$ is a Følner sequence of Σ, then the map $\mathcal{L}_{\mathbf{\Sigma}} : \ell^\infty(\Sigma) \to \mathbb{R}$ given by the assignment*

$$x \mapsto \lim_{k\to+\infty}\left(\sup_{s\in\Sigma}\frac{1}{|\Sigma_k|}\sum_{h\in\Sigma_k} x_{hs}\right)$$

is well defined. Moreover, the equality $\mathcal{L}_{\mathbf{\Sigma}} = \mathcal{L}_{\mathbf{\Sigma}'}$ holds for all Følner sequences $\mathbf{\Sigma} := (\Sigma_k)_{k\geq 1}$ and $\mathbf{\Sigma}' := (\Sigma'_k)_{k\geq 1}$.

Proof. This is a direct consequence of Lemma 4.2 and Corollary 4.3. □

4.2. *A generalization of a theorem of Oxtoby and Ulam.* Let (X, d) be a complete separable metric space and let $T : X \to X$ be a homeomorphism of X. In [**OU39**], Oxtoby and Ulam showed that if there is a point x_0 in X and a compact subset $K_0 \subset X$ such that

$$\limsup_{k\to+\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\mathbb{1}_{K_0}(T^i(x_0))\right) > 0,$$

then there exists a T-invariant Radon probability measure $\mu : \mathcal{B}_X \to [0, 1]$ such that $\mu(K_0) > 0$. In Oxtoby and Ulam's proof, the measure μ is obtained by the use of Carathéodory's extension theorem from a T-invariant metric outer measure, which is constructed by means of generalized limits. In [**Ada89**], Adamski used the well-known construction of Radon measures via inner approximation due to Kisyński and Topsøe to generalize the result of Oxtoby and Ulam to Hausdorff topological spaces. In the following, we use Adamski's approach to prove a further generalization of Oxtoby and Ulam's result.

THEOREM 4.5. *Let $(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}})$ be a Hausdorff topological space and let Σ be a countable subsemigroup of the semigroup of continuous self-maps of $(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}})$. Suppose that Σ admits a Følner sequence $(\Sigma_k)_{k\geq 1}$. If there is a point x_0 in $\mathfrak{X}$ and a compact subset $K_0 \subset \mathfrak{X}$ such that*

$$\limsup_{k\to+\infty}\left(\sup_{s\in\Sigma}\frac{1}{|\Sigma_k|}\sum_{h\in\Sigma_k}\mathbb{1}_{K_0}(h\circ s(x_0))\right) > 0,$$

then there exists a Σ-invariant Radon probability measure $\mu : \mathcal{B}_{\mathfrak{X}} \to [0, 1]$ such that $\mu(K_0) > 0$.

Proof. We define the sequence $x_0 := (\mathbb{1}_{K_0}(s(x_0)))_{s\in\Sigma}$. By the virtue of Corollary 4.3 there exists a generalized limit $\Theta : \ell^\infty(\Sigma) \to \mathbb{R}$ such that

$$\Theta(x_0) = \limsup_{k\to+\infty}\left(\sup_{s\in\Sigma}\frac{1}{|\Sigma_k|}\sum_{h\in\Sigma_k}\mathbb{1}_{K_0}(h\circ s(x_0))\right).$$

The set function $\beta : \mathcal{T}_{\mathfrak{X}} \to [0, 1]$ given by the assignment

$$U \mapsto \Theta((\mathbb{1}_U(s(x_0)))_{s\in\Sigma})$$

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satisfies $\beta(\varnothing) = 0$ and $\beta(U \cap V) + \beta(U \cap V) = \beta(U) + \beta(V)$ for all U, V in $\mathcal{T}_{\mathfrak{X}}$. Moreover, for all U, V in $\mathcal{T}_{\mathfrak{X}}$, $\beta(U) \leq \beta(V)$, whenever $U \subset V$. Thus [**Top70**, Theorem 2] asserts that the map $\mu : \mathcal{B}_{\mathfrak{X}} \to [0, 1]$ given by the assignment

$$B \mapsto \sup_{\substack{K \subset B,\\ K \in \mathcal{K}_{\mathfrak{X}}}} \inf_{\substack{K \subset U,\\ U \in \mathcal{T}_{\mathfrak{X}}}} \beta(U)$$

is a Radon measure. Note that $\mu(U) \leq \beta(U)$ for all U in $\mathcal{T}_{\mathfrak{X}}$. Let s be an element of Σ. We claim that $s_*\mu = \mu$. Note that $\beta(s^{-1}(U)) = \beta(U)$ for all U in $\mathcal{T}_{\mathfrak{X}}$. Let K be a compact subset of $(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}})$. We compute

$$\mu(s^{-1}(K)) \leq \inf_{\substack{K \subset U,\\ U \in \mathcal{T}_{\mathfrak{X}}}} \mu(s^{-1}(U)) \leq \inf_{\substack{K \subset U,\\ U \in \mathcal{T}_{\mathfrak{X}}}} \beta(s^{-1}(U)) = \inf_{\substack{K \subset U,\\ U \in \mathcal{T}_{\mathfrak{X}}}} \beta(U) = \mu(K).$$

As a result, we obtain that $s_*\mu \leq \mu$, as μ is a Radon measure. Also

$$\mu(\mathfrak{X}) = s_*\mu(\mathfrak{X}) = s_*\mu(B) + s_*\mu(\mathfrak{X} \setminus B) \leq \mu(B) + \mu(\mathfrak{X} \setminus B) = \mu(\mathfrak{X})$$

for all B in $\mathcal{B}_{\mathfrak{X}}$. Hence, it follows that $s_*\mu = \mu$, as claimed. By construction, $\mu(K_0) > 0$. Thus by rescaling μ, if necessary, we obtain a Σ-invariant Radon probability measure on $(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}})$ such that $\mu(K_0) > 0$, as desired. □

5. *Proofs of the main results*

We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. Throughout the following proof, we employ the notation from §3. Fix a measure μ_0 in $P(K)$. Let $M^0(K)$ denote the vector space of all signed finite Radon measures $\mu : \mathcal{B}_K \to \mathbb{R}$ such that $\mu(K) = 0$. The map $\|\cdot\|_0 : M^0(K) \to \mathbb{R}$ given by the assignment

$$\mu \mapsto \sup\left\{ \int_K f \, d\mu : f : X \to \mathbb{R} \text{ is 1-Lipschitz} \right\}$$

defines a norm on $M^0(K)$ (cf. [**Edw11**, Theorem 4.4]). Due to [**Edw11**, Theorem 4.1], $W^1(\mu, \nu) = \|\mu - \nu\|_0$ for all measures μ and ν in $P(K)$; hence, the map $\Phi : P(K) \to M^0(K)$ given by $\mu \mapsto \mu - \mu_0$ is an isometric embedding. It is well known that the metric space $(P(K), W^1)$ is compact (see [**Vil09**, Remark 6.19]). As a result, the set $\Phi(P(K))$ is a non-empty compact convex subset of $M^0(K)$. Note that the restriction map $s|_K : K \to K$ is an isometry of K. For each s in Σ we define the map $\overline{s} : \Phi(P(K)) \to \Phi(P(K))$ through the assignment $\mu - \mu_0 \mapsto (s|_K)_*\mu - \mu_0$. Observe that $\overline{s}$ is an affine isometry of $\Phi(K)$. Ryll-Nardzewski's fixed-point theorem (see [**RN67**]), asserts that there is a point $\mu_\star - \mu_0$ in $\Phi(P(K))$ such that $\overline{s}(\mu_\star - \mu_0) = \mu_\star - \mu_0$ for all s in Σ. Hence, the probability measure $\mu_\star : \mathcal{B}_K \to [0, 1]$ is $s|_K$-invariant for all s in Σ. Let $i : K \hookrightarrow X$ denote the inclusion map. It is readily verified that the probability measure $i_*\mu_\star : \mathcal{B}_X \to [0, 1]$ is contained in $P_1(X)$. Furthermore, the measure $i_*\mu_\star$ is Σ-invariant. Let $\beta_\sigma : P_1(X) \to X$ denote the contracting barycenter map associated to σ. We define the point $x_\star := \beta_\sigma(i_*\mu_\star)$. Clearly, as $\mathrm{spt}(i_*\mu_\star)$ is a subset of K, Theorem 3.5 tells us that the point $x_\star$ is contained in $\overline{\mathrm{conv}_\sigma}(K)$. Furthermore, we compute $s(x_\star) = \beta_\sigma(s_*i_*\mu_\star) = \beta_\sigma(i_*\mu_\star) = x_\star$ for all s in Σ, since Σ is σ-equivariant and $i_*\mu_\star$ is Σ-invariant. The theorem follows. □

In order to derive Theorem 1.2 we establish two results, Theorem 5.1 and Lemma 5.2, whose combination will directly imply Theorem 1.2.

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THEOREM 5.1. *Let (X, d) denote a complete metric space and let $\sigma : X \times X \times [0, 1] \to X$ be a conical geodesic bicombing that has the midpoint property. Suppose that Σ is a countable σ-equivariant subsemigroup of the semigroup of 1-Lipschitz self-maps of (X, d). Suppose that Σ admits a Følner sequence $(\Sigma_k)_{k\geq 1}$. If there is a point x_0 in X and a compact subset $K_0 \subset X$ such that the set $A := \{s(x_0) : s \in \Sigma\}$ is bounded and the inequality*

$$\limsup_{k\to+\infty}\left(\sup_{s\in\Sigma}\frac{1}{|\Sigma_k|}\sum_{h\in\Sigma_k}\mathbb{1}_{K_0}(h\circ s(x_0))\right) > 0 \tag{5.1}$$

holds, then there is a point $x_\star$ in $\overline{conv_\sigma}(A)$ such that $s(x_\star) = x_\star$ for all s in Σ.

Proof. The intersection $\overline{A} \cap K_0$ is a compact subset of X. Theorem 4.5 tells us that there is a Σ-invariant Radon probability measure $\mu : \mathcal{B}_X \to [0, 1]$ such that $\mu(\overline{A} \cap K_0) > 0$. It is readily verified that the Borel measure

$$\begin{aligned}\mu_\star : \mathcal{B}_X &\to [0, 1]\\ B &\mapsto \frac{1}{\mu(\overline{A})}\mu(\overline{A} \cap B)\end{aligned}$$

is a Radon probability measure. Note that $\overline{A} \subset s^{-1}(\overline{A})$ for all $s \in \Sigma$. Since μ is Σ-invariant, it follows that $\mu(s^{-1}(\overline{A}) \cap \overline{A}^c) = 0$ for all s in Σ. Now, it is straightforward to show that $\mu_\star$ is Σ-invariant. By construction, the support $\mathrm{spt}(\mu_\star)$ is a subset of $\overline{A}$. Since the subset $\overline{A}$ is bounded, we obtain that the measure $\mu_\star$ has a finite first moment and is thus contained in $P_1(X)$. Let $\beta_\sigma : P_1(X) \to X$ denote the contracting barycenter map associated to σ. We define the point $x_\star := \beta_\sigma(\mu_\star)$. Clearly, as $\mathrm{spt}(\mu_\star)$ is a subset of $\overline{A}$, Theorem 3.5 tells us that the point $x_\star$ is contained in $\overline{\mathrm{conv}_\sigma}(\overline{A}) = \overline{\mathrm{conv}_\sigma}(A)$. Furthermore, we compute $s(x_\star) = \beta_\sigma(s_*\mu_\star) = \beta_\sigma(\mu_\star) = x_\star$ for all s in Σ, since Σ is σ-equivariant and $\mu_\star$ is Σ-invariant. The theorem follows. □

Note that Corollary 4.4 asserts that the limit (5.1) does not depend on the Følner sequence $(\Sigma_k)_{k\geq 1}$.

LEMMA 5.2. *Let (X, d) be a metric space and let $\varphi : X \to X$ be an isometry of X. If there is a point x_0 in X and a bounded subset $B \subset X$ such that*

$$\limsup_{k\to+\infty}\left(\sup_{l\geq 0}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}_B(\varphi^{i+l}(x_0))\right) > 0,$$

then φ has bounded orbits.

Proof. We define the set $A := \{\varphi^k(x_0) : k \geq 0\}$. Note that it suffices to show that $\mathrm{diam}(A) < +\infty$. We define the set $D := \{k \geq 0 : \mathbb{1}_{K_0}(\varphi^k(x_0)) = 1\}$. Theorem 3.19(a) in [**Fur14**] asserts that there is an integer $k_0 \geq 1$ such that, for every integer $k \geq 0$, at least one of the integers $k, k + 1, \ldots, k + k_0$ is contained in the set $D - D := \{d - d' : d, d' \in D, d \geq d'\}$. We define the real number $C := \max\{d(x_0, \varphi^k(x_0)) : 0 \leq k \leq k_0\}$. We claim that $\mathrm{diam}(A) \leq \mathrm{diam}(B) + C$. Let $k \geq 0$ be an integer. By the above, there is an integer $0 \leq l \leq k_0$ such that the integer $k + l$ is contained in $D - D$. We compute

$$d(x_0, \varphi^k(x_0)) \leq d(x_0, \varphi^{k+l}(x_0)) + d(\varphi^{k+l}(x_0), \varphi^k(x_0)) \leq \mathrm{diam}(B) + C.$$

This concludes the proof, since $\mathrm{diam}(A) \leq \sup\{d(x_0, \varphi^k(x_0)) : k \geq 0\}$. □

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Proof of Theorem 1.2. Since the sequence $(\{0, \ldots, k-1\})_{k\geq 1}$ is a Følner sequence of the semigroup of the non-negative integers, it follows that Theorem 1.2 is a direct consequence of Theorem 5.1 and Lemma 5.2. □

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