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PAPERS

On the dynamics of a diffusive SEIRS epidemic model

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Abstract

This work investigates the dynamics of positive classical solutions to a diffusive susceptible-exposed-infected-recovered-susceptible epidemic model with a mass-action incidence mechanism in spatially heterogeneous environments. Under minimal assumptions on the initial data, the global existence of classical solutions is established. Moreover, the eventual boundedness of these solutions is proved when either the spatial domain has dimension five or lower or the susceptible and exposed subpopulations share the same diffusion rate. Next, we define the basic reproduction number, $\mathcal{R}_0$, and demonstrate that the disease-free equilibrium is globally stable when $\mathcal{R}_0$ is sufficiently small. However, due to the complex interaction between population movement and spatial variation in transmission rates, we find that the disease may persist even when $\mathcal{R}_0$ is slightly less than one. In such cases, we show that the system admits at least two endemic equilibrium (EE) solutions, an outcome not observed under the frequency-dependent incidence mechanism. These results highlight the significant influence of the transmission mechanism on disease dynamics. Furthermore, we examine the spatial profiles of the EE solutions when diffusion rates are small. Our analysis suggests that limiting the movement of the susceptible population can significantly reduce disease prevalence, provided that the total population remains below a specific threshold. In contrast, restricting the movement of the infected, exposed, or recovered populations alone may not eradicate the disease. Overall, our findings provide important insights into the spatial dynamics of infectious diseases and may offer guidance for developing and implementing effective containment strategies.

1. Introduction

Over recent decades, the study of mathematical models of epidemics has become indispensable for accurately simulating real-world infectious disease dynamics. The insights gained from these studies and their biological interpretations have significantly contributed to the development of effective disease control and containment strategies by public health authorities. In particular, the important role played by population movements and the spatial heterogeneity of the transmission rates on the emergence and outbreak of epidemics have been well documented in the current literature [3, 6, 7, 9, 11, 13, 22, 23, 36].

Indeed, in the seminal work [3], Allen et al. introduced the susceptible-infected-susceptible (SIS) epidemic model

$$\begin{cases} S_t = d_S \Delta S + \gamma(x) I - \beta(x) \frac{SI}{S+I} & x \in \Omega,\ t > 0, \\ I_t = d_I \Delta I - \gamma(x) I + \beta(x) \frac{SI}{S+I} & x \in \Omega,\ t > 0, \\ 0 = \partial_n S = \partial_n I & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

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where S and I are the densities of the susceptible and infected individuals, respectively, Ω is bounded open domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega$. In (1.1), β and γ are positive Hölder continuous functions on $\bar{\Omega}$ and represent the disease transmission and recovery rates, respectively. Moreover, the positive constant numbers d_S and d_I are the diffusion rates of the susceptible and infected individuals, respectively. Through a rigorous mathematical analysis, Allen [3] introduced the concept of the basic reproduction number (BRN) for system (1.1), demonstrating that the disease persists if and only if the BRN exceeds one. Furthermore, in the case where this condition is satisfied, the study establishes the existence of a unique endemic equilibrium (EE) solution for the system. Notably, when the environment includes a low-risk region, defined by the condition $\gamma(x) > \beta(x)$, the asymptotic analysis of EE solutions in [3] reveals that the infected population tends toward extinction as the diffusion rate of the susceptible population, d_S, approaches zero. The same conclusion is reached in [22] if the moderate-risk region, i.e. the set where $\gamma(x) = \beta(x)$, has a positive area. Biologically, this indicates that the model (1.1) predicts successful disease eradication via reduced mobility of the susceptible population, provided the habitat includes a low-risk region, or a moderate-risk region of positive area. This conclusion has been further substantiated by subsequent studies [23, 33, 36], which confirm its validity in the case of the degenerate ODE-PDE epidemic system (1.1) with $d_S = 0$ and $d_I > 0$.

However, when the diffusion rate of the infected population is reduced, the findings of [28] regarding the spatial profile of EE solutions of (1.1) suggest that the disease may still persist. Similarly, the study in [22], which analyzes disease prevalence based on the epidemic model (1.1), reveals (under the assumption that the population eventually stabilizes at the EE) that the total number of infected individuals increases when the diffusion rate of the susceptible population is low and that of the infected population is even lower. The work in [29] also explores how epidemic risk and population movement affect disease persistence. For further recent developments in this area, readers may refer to [12, 26, 27, 40, 41] and the references therein. In particular, studies such as [1, 2, 23, 24, 34, 36, 37] examine the impact of population movement and spatial heterogeneity on the dynamics of multi-strain infectious diseases, while the studies [8, 18, 19, 21, 32, 38, 39] considered cross-diffusive and diffusive-advection-reaction epidemic models.

An important hypothesis in the modeling of the diffusive epidemic model (1.1) is that people after contracting the disease turn right away to infectious individual. However, for several epidemic diseases such as West Nile virus, HIV/AIDS, Covid19, etc, infected individuals can experience incubation before showing symptoms. To account for this fact, Song et al. [37] modified the epidemic model (1.1) by incorporating the exposed and recovered populations. Specifically, the diffusive susceptible-exposed-infected-recovered-susceptible (SEIRS) epidemic model,

$$\begin{cases} \partial_t S = d_S \Delta S + \alpha(x) R - \beta(x) \frac{IS}{S+E+I+R} & x \in \Omega,\ t > 0, \\ \partial_t E = d_E \Delta E + \beta(x) \frac{IS}{S+E+I+R} - \sigma(x) E & x \in \Omega,\ t > 0, \\ \partial_t I = d_I \Delta I + \sigma(x) E - \gamma(x) I & x \in \Omega,\ t > 0, \\ \partial_t R = d_R \Delta R + \gamma(x) I - \alpha(x) R & x \in \Omega,\ t > 0, \\ 0 = \partial_{\boldsymbol{n}} S = \partial_{\boldsymbol{n}} E = \partial_{\boldsymbol{n}} I = \partial_{\boldsymbol{n}} R & x \in \partial\Omega,\ t > 0, \\ N = \int_\Omega (S + E + I + R), \end{cases} \tag{1.2}$$

was proposed and investigated in [37]. In (1.2), E and R represent the exposed and recovered populations, respectively. The functions σ and α are positive Hölder continuous functions on $\bar{\Omega}$: the quantity $\frac{1}{\sigma(x)}$ is the local latent period for exposed individual to become infectious, while $\alpha(x)$ is the local immunity loss rate of people who have recovered from the disease. The positive numbers d_E and d_R are, respectively, the diffusion rates of the exposed and recovered populations. In system (1.2), Ω, S, I, β, γ, d_I, and d_S have the same meanings as in the diffusive SIS epidemic model (1.1).

Assuming that σ and α are spatially homogeneous, [37] gives a comprehensive analytical study of the diffusive epidemic model (1.2). Indeed, under this assumption and following the approach of the next-generation operators theory, the authors of [37] obtained the BRN for system (1.2). Moreover, similar to

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the dynamics of the solutions to (1.1), it is shown in [37] that system (1.2) predicts disease persistence if and only if its BRN exceeds one. However, unlike system (1.1), the BRN of (1.2) is not generally monotonic with respect to the diffusion rate of the infected population, highlighting the influence of the exposed population on disease dynamics. In fact, [37] shows that the BRN of system (1.2) depends on all model parameters except for d_S, d_R, α, and N. Moreover, the results in [37] on the spatial profiles of the EE solutions of (1.2) suggest that restricting the diffusion rate of the susceptible population can significantly reduce disease prevalence, under suitable assumptions on parameters d_R, α, β, and γ. In particular, if d_R is small and $\gamma(x) > \beta(x)$ at some location $x \in \Omega$, then [37, Theorem 1.4] shows that the total sizes of the exposed, infected, and recovered populations are approximately proportional to the diffusion rate d_S, provided that d_S is sufficiently small.

While the work [37] on the SEIRS diffusive epidemic model (1.2) highlights some important effects of the exposed and recovered population movements on the dynamics of some infectious diseases, it is important to note the transmission mechanism used in (1.2) is through the term $\frac{SI}{S+E+I+R}$, which is referred to as the standard transmission mechanism in the literature. Such a disease transmission mechanism, which assumes a random mixing of the population in the sense that the probability that each susceptible individual S contacts the infection depends on the proportion $I/(S+E+I+R)$ of encounters involving infected individuals, goes back the work of De Jong [16]. It is often used when the transmission of the disease does not significantly change with population size and density, or for diseases with constant interaction rates (e.g., sexually transmitted diseases or diseases spread by vectors). However, not all infectious diseases follows the standard transmission mechanism. Moreover, it is well known that a change in the transmission mechanism in an epidemic model may lead to significant difference on predictions on the disease dynamics. It is the aim of the current study to investigate the dynamics of the diffusive epidemic model (1.2) with the mass-action transmission mechanism. Our study falls in the same direction as the investigations in [6, 7, 11, 40, 41], which studied the epidemic system (1.1) by replacing its mode of transmission mechanism with that of the mass-action incidence mechanism. In particular, the work [7] shows that system (1.1) with the mass-action incidence mechanism may exhibit very colorful asymptotic dynamics.

The diffusive epidemic model. Motivated by the previously mentioned results on the epidemic model (1.2), and recognizing that the standard incidence mechanism may not adequately capture the dynamics of certain viral infectious diseases, we propose and analyze the behavior of solutions to the diffusive SEIRS epidemic model

$$\begin{cases} \partial_t S = d_S \Delta S + \alpha(x) R - \beta(x) IS & x \in \Omega,\ t > 0, \\ \partial_t E = d_E \Delta E + \beta(x) IS - \sigma(x) E & x \in \Omega,\ t > 0, \\ \partial_t I = d_I \Delta I + \sigma(x) E - \gamma(x) I & x \in \Omega,\ t > 0, \\ \partial_t R = d_R \Delta R + \gamma(x) I - \alpha(x) R & x \in \Omega,\ t > 0, \\ 0 = \partial_n S = \partial_n E = \partial_n I = \partial_n R & x \in \partial\Omega,\ t > 0, \\ N = \int_\Omega (S + E + I + R), \end{cases} \tag{1.3}$$

where the parameters have the same meanings as those of system (1.2). The diffusive epidemic SEIRS model (1.3) uses the mass-action transmission mechanism, which is represented by the bilinear term SI. Such a transmission mechanism can be traced back to the work of Kermack and McKendrick [17]. This classic mechanism is based on the homogeneous-mixing assumption, that is transmission occurs via direct contact between susceptible and infective hosts that mix completely with each other and move randomly in a fixed domain. The mass-action incidence is more suited for environments where interactions between individuals are frequent and not constrained by resources or space. In the current work, we investigate the dynamics of positive solutions to system (1.3). In particular, Proposition 2.1 asserts that nonnegative solutions of (1.3) are always defined for all time, while the boundedness of these solutions when either the spatial dimension n is less than or equal to 5, or $d_E = d_S$ is obtained in Theorem 2.2.

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Moreover, system (1.3) always admits a unique disease-free equilibrium (DFE), given by the constant state $(\frac{N}{|\Omega|}, 0, 0, 0)$. Its BRN, denoted by $\mathcal{R}_0$, is defined in (2.11). In Theorem 2.3, we show that the DFE is locally stable when $\mathcal{R}_0 < 1$ and becomes unstable when $\mathcal{R}_0 > 1$. Under the latter condition, Theorem 2.5 guarantees the existence of at least one EE solution for system (1.3). However, Theorem 2.4 further reveals that the DFE is globally stable when $\mathcal{R}_0$ is sufficiently small. This contrast naturally leads to the following question: Can system (1.3) admit an EE solution even when $\mathcal{R}_0 < 1$?

Interestingly, Theorem 2.7 offers an affirmative answer to this question. Specifically, we demonstrate that system (1.3) can admit at least two EE solutions for some parameters range with $\mathcal{R}_0 < 1$. This is one of the main novel findings of the present work. In particular, it shows that replacing the standard incidence mechanism by the mass-action incidence mechanism can fundamentally alter the structure of the EE set of the diffusive SEIRS model. Such multiplicity does not occur in the corresponding kinetic ODE system, nor in the diffusive epidemic model (1.2). Therefore, our finding highlights the crucial role played by spatial heterogeneity and diffusion rates in shaping the qualitative behavior of system (1.3). Moreover, it reveals a fundamental distinction between the dynamics of (1.2) and (1.3).

Theorems 2.11, 2.13, 2.14, and 2.15 further explore the structure of EE solutions in the regime of small diffusion rates. Notably, we identify a sharp critical population size, N_* (as defined in (2.20)), such that if $N \leq N_*$, the densities of the exposed, infected, and recovered individuals at the EE tend to zero as the susceptible diffusion rate d_S approaches zero. From a biological perspective, this suggests that limiting the movement of the susceptible population can substantially mitigate the spread of the disease, provided that the total population remains below a certain threshold.

The rest of our manuscript is organized as follows. Section 2 contains the main results of the current work. Some preliminary results are collected in Section 3. The EE solution problem is investigated in Section 5, where an equivalent problem for the EE solution is formulated. The proofs of our main results are given in Sections 4, 6, and 7.

2. Main results

We present our main results on the dynamics of positive solutions to system (1.3). As a first step, we investigate the global boundedness of solutions to the initial-boundary value problem in Section 2.1. Next, we define the BRN, and investigate the stability of the DFE, the existence and multiplicity of the EE solutions in Section 2.2. The spatial profiles of the EE with respect to small population diffusion rates are investigated in Section 2.3. Finally, a discussion of our results is provided in Section 2.4.

2.1 Eventual boundedness of solutions

Let $(S_0, E_0, I_0, R_0) \in [C^+(\bar{\Omega})]^4$ be given. Since the nonlinearity in (1.3) is locally Lipschitz in $(S, E, I, R)^T$ uniformly in $x \in \bar{\Omega}$, it follows from [25] and the regularity theory for parabolic equations that there is $t_{\max} \in (0, \infty]$, depending on initial data, such that system (1.3) has a unique classical solution $(S(t,\cdot), E(t,\cdot), I(t,\cdot), R(t,\cdot))$ on $[0, t_{\max})$ satisfying $(S(0,\cdot), E(0,\cdot), I(0,\cdot), R(0,\cdot)) = (S_0(\cdot), E_0(\cdot), I_0(\cdot), R_0(\cdot))$. Moreover, if $T_{\max} < \infty$, then

$$\limsup_{t \to T_{\max}} (\|S(t,\cdot)\|_\infty + \|E(t,\cdot)\|_\infty + \|I(t,\cdot)\|_\infty + \|R(t,\cdot)\|_\infty) = \infty. \tag{2.1}$$

By the maximum principle for parabolic equations, $(S(t,\cdot), E(t,\cdot), I(t,\cdot), R(t,\cdot)) \in [C^+(\bar{\Omega})]^4$ for all $t \in [0, t_{\max})$. Moreover, it holds that

$$\frac{d}{dt} \int_\Omega (S(t) + E(t) + I(t) + R(t)) = 0 \quad 0 < t < T_{\max}.$$

Therefore, the total population size is constant over time, that is

$$N := \int_\Omega (S(t) + E(t) + I(t) + R(t)) = \int_\Omega (S_0 + E_0 + I_0 + R_0) \quad \forall\, 0 < t < T_{\max}. \tag{2.2}$$

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Observe that, in addition if $I_0 = E_0 \equiv 0$, then $E(t,\cdot) = I(t,\cdot) \equiv 0$ for all $t \in [0, T_{\max})$. In this case, it follows from the comparison principle for parabolic equations that $R(t,\cdot) \le e^{-t\alpha_{\min}} \|R_0\|_\infty$ and $S(t,\cdot) \le \|S_0\|_\infty + \frac{\|\alpha\|_\infty \|R_0\|_\infty}{\alpha_{\min}}(1 - e^{-t\alpha_{\min}})$ for all $0 < t < T_{\max}$. (Here we used the notation $\alpha_{\min} := \min_{x\in\bar\Omega} \alpha(x)$). Hence, $T_{\max} = \infty$, and it can be easily established that $\|R(t,\cdot)\|_\infty + \|S(t,\cdot) - \frac{\int_\Omega (S_0+R_0)}{|\Omega|}\|_\infty \to 0$ as $t \to \infty$. This shows that the dynamics of (1.3) is simple when $(S_0, R_0) \in [C^+(\bar\Omega)]^2$ and $E_0 = I_0 \equiv 0$. Therefore, throughout the work, we shall always assume that $(S(t,x), E(t,x), I(t,x), R(t,x))$ has a positive initial data in the sense that $(S_0, E_0, I_0, R_0) \in [C^+(\bar\Omega)]^4$ and $(E_0, I_0) \not\equiv (0,0)$. Note that if $(S(t,x), E(t,x), I(t,x), R(t,x))$ has a positive initial data, then $(S(t,\cdot), E(t,\cdot), I(t,\cdot), R(t,\cdot)) \in [C^{++}(\Omega)]^4$ for all $t \in (0, T_{\max})$.

In the particular case of $d := d_S = d_E = d_I = d_R$, then $Z := S + E + I + R$ satisfies

$$\begin{cases} \partial_t Z = d\Delta Z & 0 < t < T_{\max},\ x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}} Z & 0 < t < T_{\max},\ x \in \partial\Omega. \end{cases} \tag{2.3}$$

Thus, by the maximum principle for the heat equation, we have that $\|Z(t,\cdot)\|_\infty \le \|Z(0,\cdot)\|_\infty$ for all $0 \le t < T_{\max}$, and hence $T_{\max} = \infty$. Furthermore, $Z(t,\cdot) \to \frac{1}{|\Omega|}\int_\Omega Z(0,\cdot)$. This shows that

$$S(t) + E(t) + I(t) + R(t) \to \frac{N}{|\Omega|} \quad \text{as } t \to \infty \text{ uniformly on } \bar\Omega. \tag{2.4}$$

However, when the diffusion rates are not all equal, the proof of the global existence of solutions turns out to be challenging. Nonetheless, system (1.3) is quasi-positive and the total mass is preserved by (2.2). Moreover, if we set

$$\begin{aligned} &f_1(x,\mathbf{r}) = \alpha(x) r_4 - \beta(x) r_1 r_3, \qquad f_2(x,\mathbf{r}) = \beta(x) r_1 r_3 - \sigma(x) r_2, \quad x \in \bar\Omega,\ \mathbf{r} \in \mathbb{R}^4, \\ &f_3(x,\mathbf{r}) = \sigma(x) r_2 - \gamma(x) r_3, \quad \text{and} \quad f_4(x,\mathbf{r}) = \gamma(x) r_3 - \alpha(x) r_4 \quad x \in \bar\Omega,\ \mathbf{r} \in \mathbb{R}^4, \end{aligned}$$

then system (1.3) can be rewritten in the compact form

$$\begin{cases} \partial_t \mathbf{r} = D\Delta \mathbf{r} + f(x,\mathbf{r}) & x \in \Omega,\ t > 0, \\ 0 = \partial_{\boldsymbol{n}} \mathbf{r} & x \in \partial\Omega,\ t > 0, \\ \mathbf{r}(0,\cdot) = \mathbf{r}^0(\cdot) \in [C^+(\bar\Omega)]^4, \end{cases} \tag{2.5}$$

where $D = \operatorname{diag}(d_S, d_E, d_I, d_R)$ is the diagonal matrix. For convenience, we have substituted $(S, E, I, R)^T$ with $\mathbf{r} \in \mathbb{R}^4$. Setting $f = (f_1, f_2, f_3, f_4)^T$. Introducing the lower triangular matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

then

$$Pf(x,\mathbf{r}) = \begin{pmatrix} f_1(x,\mathbf{r}) \\ f_1(x,\mathbf{r}) + f_2(x,\mathbf{r}) \\ f_1(x,\mathbf{r}) + f_2(x,\mathbf{r}) + f_3(x,\mathbf{r}) \\ f_1(x,\mathbf{r}) + f_2(x,\mathbf{r}) + f_3(x,\mathbf{r}) + f_4(x,\mathbf{r}) \end{pmatrix} \le \|\alpha\|_\infty r_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \forall\, x \in \bar\Omega,\ \mathbf{r} \in [0,\infty)^4.$$

This shows that [30, Inequality (3.17)] is satisfied. Therefore, since the solution operator of (2.5) leaves invariant positive cone $[C^+(\bar\Omega)]^4$, we can apply [30, Theorem 3.5] to deduce that for any given initial data $\mathbf{r}^0 \in [C^+(\bar\Omega)]^4$, system (2.5) has a unique global classical solution $\mathbf{r}(t,x)$ defined for all $x \in \bar\Omega$ and $t \ge 0$. As a result, we conclude the following result on the existence of global solutions of system (1.3).

Proposition 2.1. *For any given nonnegative initial data $(S_0, E_0, I_0, R_0) \in [C^+(\bar\Omega)]^4$, the system of parabolic equations (1.3) has a unique nonnegative global classical solution $(S, E, I, R)(t,x)$ defined for all $t \ge 0$ and $x \in \bar\Omega$.*

Proposition 2.1 guarantees the existence of a unique global classical solution of system (1.3) irrespective of the choice of the population diffusion rates and the spatial dimension, but does not assert the

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boundedness of the solution. It is clear from (2.3) and (2.4) that the solutions are uniformly bounded if $d_S = d_E = d_I = d_R$. In the following, we relax the latter hypothesis and prove the global existence and boundedness of solutions if either $d_S = d_E$ or $n \leq 5$. Specifically, the following result holds.

Theorem 2.2. *Assume that either* (i) $n \in \{1, \cdots, 5\}$ *or* (ii) $d_S = d_E$. *Let* $(S_0, E_0, I_0, R_0) \in [C^+(\overline{\Omega})]^4$ *with* $(E_0, I_0) \not\equiv (0, 0)$ *be an initial data satisfying* $\int_\Omega (S_0 + E_0 + I_0 + R_0) = N$. *Then, there exists a unique nonnegative classical and globally bounded solution* $(S(t,x), E(t,x), I(t,x), R(t,x))$ *to (1.3) defined for all* $t > 0$. *Moreover, there is a positive constant* $M > 0$, *independent of initial data and N, such that*

$$\limsup_{t\to\infty} \left(\|S(t,\cdot)\|_\infty + \|E(t,\cdot)\|_\infty + \|I(t,\cdot)\|_\infty + \|R(t,\cdot)\|_\infty\right) \leq MN(1+N^2). \tag{2.6}$$

2.2 Stability of the DFE and multiplicity of EE solutions

To study the global dynamics of solutions of (1.3), it is necessary to understand the stability of its equilibrium solutions. An equilibrium solution of (1.3) is a classical solution of the system of elliptic equations

$$\begin{cases} 0 = d_S \Delta S + \alpha(x)R - \beta(x)IS & x \in \Omega, \\ 0 = d_E \Delta E + \beta(x)IS - \sigma(x)E & x \in \Omega, \\ 0 = d_I \Delta I + \sigma(x)E - \gamma(x)I & x \in \Omega, \\ 0 = d_R \Delta R + \gamma(x)I - \alpha(x)R & x \in \Omega, \\ 0 = \partial_n S = \partial_n E = \partial_n I = \partial_n R & x \in \partial\Omega, \\ N = \int_\Omega (S + E + I + R). \end{cases} \tag{2.7}$$

A DFE is an equilibrium solution satisfying either $I \equiv 0$ or $E \equiv 0$. Note that in either case, by the strong maximum principle for elliptic equations, $I \equiv E \equiv R \equiv 0$. Therefore, any DFE must be of the form $(S, 0, 0, 0)$. An EE solution of (1.3) is a nonnegative classical solution (S, E, I, R) of (2.7) for which either $I \geq, \not\equiv 0$ or $E \geq, \not\equiv 0$. Note that if $I \geq, \not\equiv 0$, then $IS \geq, \not\equiv 0$. Otherwise, the strong maximum principle for elliptic equations apply to the second equation of (2.7) would give $E \equiv 0$. This, again by the strong maximum principle for elliptic equations apply to the third equation of (2.7) would give $I \equiv 0$, which is contrary to our assumption that $I \geq, \not\equiv 0$. Hence, we must have that $IS \geq, \not\equiv 0$. Using this fact, we can appeal to the strong maximum principle for elliptic equations to the second, third, and fourth equations of (2.7), respectively, to obtain that $E > 0$, $I > 0$, and $R > 0$ on $\bar{\Omega}$. The latter in turn together with the strong maximum principle for elliptic equations apply to the first equation of (2.7) yields $S > 0$ on $\bar{\Omega}$. Similarly, if $E \geq, \not\equiv 0$, by the strong maximum principle for elliptic equations, we have $I > 0$, $R > 0$, $S > 0$ and $E > 0$ on $\bar{\Omega}$. Hence, if (S, E, I, R) is an EE of (1.3), then $S > 0$, $E > 0$, $I > 0$, and $R > 0$ on $\bar{\Omega}$.

It is easy to see that $(\frac{N}{|\Omega|}, 0, 0, 0)$ is the unique DFE of (1.3). Linearizing (1.3) at the DFE gives rise to the system of linear parabolic equations

$$\begin{cases} \partial_t S = d_S \Delta S + \alpha R - \frac{N}{|\Omega|}\beta I & x > 0,\ t > 0, \\ \partial_t E = d_E \Delta E + \frac{N}{|\Omega|}\beta I - \sigma E & x > 0,\ t > 0, \\ \partial_t I = d_I \Delta I + \sigma E - \gamma I & x > 0,\ t > 0, \\ \partial_t R = d_R \Delta R + \gamma I - \alpha R & x > 0,\ t > 0, \\ 0 = \partial_n S = \partial_n E = \partial_n I = \partial_n R & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.8}$$

Following the next generation operators method, we shall consider the weighted eigenvalue problem

$$\begin{cases} 0 = d_E \Delta \Psi - \sigma \Psi + \mu\beta\Phi & x \in \Omega, \\ 0 = d_I \Delta \Phi + \sigma \Psi - \gamma \Phi & x \in \Omega, \\ 0 = \partial_n \Psi = \partial_n \Phi & x \in \partial\Omega, \end{cases} \tag{2.9}$$

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where $\mu \in \mathbb{R}$. A scalar μ is called an eigenvalue of (2.9), if the system has a nonzero solution. Note that (2.9) is a strongly cooperative system for every $\mu > 0$. Thanks to [37, Lemm 2.2], system (2.9) has a unique principal eigenvalue $\mu^* > 0$, so that there is a positive eigenfunction $\Psi^* > 0$ and $\Phi^* > 0$, that is (μ^*, Ψ^*, Φ^*) satisfies

$$\begin{cases} 0 = d_E \Delta \Psi^* - \sigma \Psi^* + \mu^* \beta \Phi^* & x \in \Omega, \\ 0 = d_I \Delta \Phi^* + \sigma \Psi^* - \gamma \Phi^* & x \in \Omega, \\ 0 = \partial_n \Psi^* = \partial_n \Phi^* & x \in \partial\Omega, \\ \Psi^* > 0 \quad \text{and} \quad \Phi^* > 0 & x \in \overline{\Omega}. \end{cases} \tag{2.10}$$

Moreover, μ^* is a simple eigenvalue. For convenience, we normalize (Ψ^*, Φ^*) such that $\|\Psi^* + \Phi^*\|_\infty = 1$. The positive number

$$\mathcal{R}_0 = \frac{N}{|\Omega|\mu^*} \tag{2.11}$$

is the BRN of system (1.3). It is easily seen from (2.11) that $\mathcal{R}_0$ increases with respect to the total population size while it decreases with respect to the living habitat size. Note also that $\mathcal{R}_0$ is independent of the diffusion rates d_S and d_R of the susceptible and recovered groups, respectively. Our next result confirms the uniform persistence of the disease when $\mathcal{R}_0 > 1$.

Theorem 2.3.

(i) *If $\mathcal{R}_0 < 1$, then the DFE is locally asymptotically stable.*

(ii) *If $\mathcal{R}_0 > 1$, then the DFE is unstable. In addition, if either $n \in \{1, \cdots, 5\}$ or $d_S = d_E$, there is $m_* > 0$ (independent of initial data) such that*

$$\liminf_{t\to\infty} \min_{x\in\overline{\Omega}} \min\{(I(t,x), E(t,x))\} \geq m_*, \tag{2.12}$$

for every classical solution $(S(t,\cdot), E(t,\cdot), I(t,\cdot), R(t,\cdot))$ of (1.3) with a positive initial data. Furthermore, system (1.3) has at least one EE solution.

The global stability of the DFE can be shown if the BRN is sufficiently small. Precisely, we have the following result.

Theorem 2.4 (Global stability of the DFE). *Suppose that either* (i) $n \in \{1, \cdots, 5\}$, *or* (ii) $d_S = d_E$. *Then, there is $N_0 > 0$, such that for any $0 < N < N_0$, every solution $(S, E, I, R)(t, x)$ of (1.3) with a positive initial data satisfies*

$$\lim_{t\to\infty} \left(\left\| S(t) - \frac{N}{|\Omega|} \right\|_\infty + \|E(t)\|_\infty + \|I(t)\|_\infty + \|R(t)\|_\infty \right) = 0. \tag{2.13}$$

If $d_S = d_E = d_I = d_R$, we can take $N_0 = |\Omega|\mu^*$, where $\mu^* > 0$ is the principal eigenvalue of (2.10).

Under the hypotheses of the global boundedness of solutions of (1.3) as in Theorem 2.2, Theorem 2.4 establishes the global stability of the DFE if the BRN is very small. Furthermore, in the special case of equal diffusion rates of all the subgroups of the population, that is $d_S = d_E = d_I = d_R$, it follows from Theorem 2.4 that the DFE is globally stable when $\mathcal{R}_0 < 1$. In this case, thanks to Theorem 2.3, the BRN serves as a sharp threshold number for the persistence of the disease.

Next, we study the existence and nonexistence of EE solution of system (1.3) without any restriction on the model's parameters as in Theorem 2.4. In this direction, we establish the following result.

Theorem 2.5.

(i) *If $\mathcal{R}_0 > 1$, then (1.3) has at least one EE solution.*

(ii) *If $\mathcal{R}_0 \leq \frac{1}{1+\left(\frac{d_E}{d_S}-1\right)_+ + \left(\frac{d_I}{d_S}-1\right)_+ + \left(\frac{d_R}{d_S}-1\right)_+}$, then (1.3) has no EE solution.*

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Remark 2.6. *Theorem 2.5*-(i) *shows that system (1.3) has at least one EE solution whenever* $\mathcal{R}_0 > 1$ *with no restriction on the space dimension or the diffusion rates. However, system (1.3) has no such solution if* $\mathcal{R}_0 \leq \frac{1}{1+\left(\frac{d_E}{d_S}-1\right)_+ + \left(\frac{d_I}{d_S}-1\right)_+ + \left(\frac{d_R}{d_S}-1\right)_+}$ *as in Theorem 2.5*-(ii). *When the latter inequality holds, it remains open whether the DFE is globally stable.*

Theorem 2.5 establishes the existence of EE solution of (1.3) when $\mathcal{R}_0 > 1$. However, it is unclear whether system (1.3) has an EE solution for a range of parameter satisfying $\mathcal{R}_0 < 1$. Our next result complements Theorems 2.4 and 2.5 and shows that the DFE is not in general globally stable when $\mathcal{R}_0 < 1$. To clearly state the next result, some notations are in order.

Let λ_{d_E,d_I,d_R} denote the principal eigenvalue of the cooperative system

$$\begin{cases} \lambda\varphi_E = d_E\Delta\varphi_E - \sigma\varphi_E + \alpha\varphi_R & x \in \Omega, \\ \lambda\varphi_I = d_I\Delta\varphi_I - \gamma\varphi_I + \sigma\varphi_E & x \in \Omega, \\ \lambda\varphi_R = d_R\Delta\varphi_R - \alpha\varphi_R + \gamma\varphi_I & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}\varphi_E = \partial_{\boldsymbol{n}}\varphi_I = \partial_{\boldsymbol{n}}\varphi_R & x \in \partial\Omega, \end{cases} \tag{2.14}$$

and $(\varphi_E, \varphi_I, \varphi_R)$ be the corresponding eigenvector satisfying $\varphi_E > 0$, $\varphi_I > 0$, $\varphi_R > 0$ on $\overline{\Omega}$ and $\|\varphi_E + \varphi_I + \varphi_R\|_\infty = 1$. The existence of λ_{d_E,d_I,d_R} follows from the Krein–Rutman theorem. Moreover, λ_{d_E,d_I,d_R} is a simple eigenvalue, hence the quantity $\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}$ is independent of the chosen eigenvector. In fact, as shall be shown in Proposition 3.1, we have that $\lambda_{d_E,d_I,d_R} = 0$.

Theorem 2.7. *Fix* $d_I > 0$, $d_E > 0$, *and* $d_R > 0$. *Let* $(\varphi_E, \varphi_I, \varphi_R)$ *be the unique positive eigenvector of (2.14) associated with* λ_{d_E,d_I,d_R} *satisfying* $\max_{x\in\bar{\Omega}}(\varphi_E(x) + \varphi_I(x) + \varphi_R(x)) = 1$. *Assume that*

$$\mathcal{R}_1^* := \frac{\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}}{\mu^*|\Omega|} < 1, \tag{2.15}$$

where $\mu^* > 0$ *is the principal eigenvalue of (2.10). Then, for every choice of N such that* $\mathcal{R}_1^* < \mathcal{R}_0 < 1$, *there is* $d_S^* > 0$ *such that system (1.3) has at least two EE solutions for every* $0 < d_S < d_S^*$.

Remark 2.8. *The limit of the quantity* $\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}$ *with respect to small diffusion rates is established in Proposition 2.12. In particular, it follows from Proposition 2.12*-(i) *that* $\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I} \to \int_\Omega \gamma/\beta$ *as* $d_R \to 0$. *Note also by Proposition 2.9*-(v) *below that* $1/\mu^* \to \int_\Omega \beta / \int_\Omega \gamma$ *as* $\min\{d_E, d_I\} \to \infty$. *Thus, if* $\int_\Omega(\gamma/\beta) < |\Omega|(\int_\Omega \gamma / \int_\Omega \beta)$, *(2.15) holds for small values of* d_R *and large values of* $\min\{d_E, d_I\}$.

Given that $\mathcal{R}_0$ is an important threshold quantity for disease persistence as confirmed by Theorem 2.3, it is important to examine its dependence with respect to the model's parameters. To this end, we first introduce the following notation. Given positive Hölder continuous functions h and f on $\overline{\Omega}$ and $d > 0$, let

$$\mathcal{R}(d, h, f) := \sup_{\varphi\in W^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega f\varphi^2}{\int_\Omega \left[d|\nabla\varphi|^2 + h\varphi^2\right]}. \tag{2.16}$$

It is well known that the supremum in (2.16) is achieved at some positive function $\varphi \in C^2(\overline{\Omega})$. Moreover,

$$\begin{cases} 0 = d\Delta\varphi - h\varphi + \frac{1}{\mathcal{R}(d,h,f)} f\varphi & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}\varphi & x \in \partial\Omega. \end{cases} \tag{2.17}$$

With regards to the limits of $\mathcal{R}_0$ with respect to d_E and d_I, the following result follows from [37, Theorem 3.1].

Proposition 2.9.

(i) *Fix* $d_I > 0$. *Then,* $\mathcal{R}_0 \to \frac{N}{|\Omega|}\frac{\int_\Omega \beta(\gamma - d_I\Delta)^{-1}(\sigma)dx}{\int_\Omega \sigma}$ *as* $d_E \to \infty$, *and* $\mathcal{R}_0 \to \frac{N}{|\Omega|}\mathcal{R}(d_I, \gamma, \beta)$ *as* $d_E \to 0$.

(ii) *Fix* $d_E > 0$. *Then,* $\mathcal{R}_0 \to \frac{N}{|\Omega|}\mathcal{R}(d_E, \sigma, \frac{\sigma\beta}{\gamma})$ *as* $d_I \to 0$, *and* $\mathcal{R}_0 \to \frac{N}{|\Omega|}\frac{\int_\Omega \sigma(\sigma - d_E\Delta)^{-1}(\beta)}{\int_\Omega \gamma}$ *as* $d_I \to \infty$.

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(iii) $\mathcal{R}_0 \to \frac{N}{|\Omega|}\left\|\frac{\beta}{\gamma}\right\|_\infty$ *as* $\max\{d_E, d_I\} \to 0$.

(iv) $\mathcal{R}_0 \to \frac{N}{|\Omega|}\frac{\int_\Omega \frac{\sigma\beta}{\gamma}}{\int_\Omega \sigma}$ *as* $d_E \to \infty$ *and* $d_I \to 0$.

(v) $\mathcal{R}_0 \to \frac{N}{|\Omega|}\frac{\int_\Omega \beta}{\int_\Omega \gamma}$ *as* $\min\{d_E, d_I\} \to \infty$.

We note that similar results (i)-(iv) of Proposition 2.9 are obtained in [37] for the case where the function σ is spatially homogeneous. The arguments used for that special case can be extended to the case of a non-constant function σ. The result (v) of Proposition 2.9 can also be established using similar arguments in [37, Theorem 3.1]. So, to avoid duplicating the ideas developed in [37], the proof of Proposition 2.9 will be omitted. Our next result discusses the effects of large values of β or σ on $\mathcal{R}_0$.

Proposition 2.10.

(i) *Assume that* $\beta = n\beta_1$, $n > 0$, *for some positive and Hölder continuous function* β_1 *on* $\bar{\Omega}$. *Denote by* $\mathcal{R}_0^{(n)}$ *the BRN of (1.3) for every* $n > 0$. *Then,* $\mathcal{R}_0^{(n)} = n\mathcal{R}_0^{(1)}$ *for all* $n > 0$. *Hence,* $\mathcal{R}_0^{(n)} < 1$ *if* $n < \frac{1}{\mathcal{R}_0^{(1)}}$, $\mathcal{R}_0^{(n)} = 1$ *if* $n = \frac{1}{\mathcal{R}_0^{(1)}}$, *and* $\mathcal{R}_0^{(n)} > 1$ *if* $n > \frac{1}{\mathcal{R}_0^{(1)}}$.

(ii) *Assume that* $\sigma = n\sigma_1$, $n > 0$, *for some positive and Hölder continuous function* σ_1 *on* $\bar{\Omega}$. *Denote by* $\tilde{\mathcal{R}}_0^{(n)}$ *the BRN of (1.3) for every* $n > 0$. *Then*

$$\lim_{n\to\infty} \tilde{\mathcal{R}}_0^{(n)} = \frac{N}{|\Omega|}\mathcal{R}(d_I, \gamma, \beta) \tag{2.18}$$

and

$$\lim_{n\to 0^+} \tilde{\mathcal{R}}_0^{(n)} = \frac{N}{|\Omega|}\frac{\int_\Omega \beta(\gamma - d_I\Delta)^{-1}(\sigma_1)dx}{\int_\Omega \sigma_1}. \tag{2.19}$$

Proposition 2.10-(i) confirms that an increase in the infection rate consistently leads to a higher BRN value, which aligns with real-world observations. Recall that $\frac{1}{\sigma}$ represents the latent period, i.e., the time it takes for exposed individuals to become infectious. Therefore, a large value of σ implies that the exposed population transitions to the infectious state more rapidly. In such cases, our result in (2.18) shows that the BRN of the diffusive epidemic model with an exposed class system (1.3) closely approximates that of a SIS diffusive epidemic model without an exposed class, as studied in [6, 7, 11, 40, 41].

2.3 Asymptotic profiles of EE solutions

To further understand the spatial distribution of the EE solutions of system (1.3), we shall study their asymptotic profiles with respect to the diffusion rates.

First, we study the limit of EE solution as d_S approaches zero. As shown in our next result, the quantity $\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}$ serves as a sharp critical number for the total population size for the asymptotic limits of EE of (1.3) as d_S approaches zero.

Theorem 2.11. *Fix* $d_E > 0$, $d_I > 0$ *and* $d_R > 0$, *and define*

$$N_* := \int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}, \tag{2.20}$$

where $(\varphi_E, \varphi_I, \varphi_R)$ *is the positive eigenvector associated with* λ_{d_E,d_I,d_R} *satisfying* $\|\varphi_E + \varphi_I + \varphi_R\|_\infty = 1$.

(i) *Fix* $N > 0$ *such that* $\mathcal{R}_0 \neq 1$ *and assume that there is* $d_0 > 0$ *such that system (1.3) has at least one EE solution for every* $0 < d_S \le d_0$. *The following conclusions hold.*

(i-1) *If* $N < N_*$, *then there is a positive constant* $C > 0$ *such that every EE solution* (S, E, I, R) *of (1.3) for* $0 < d_S \le d_0$ *satisfies*

$$\max\{E, I, R\} \le Cd_S \quad \forall\, 0 < d_S \le d_0. \tag{2.21}$$

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Furthermore, up to a subsequence, as d_S tends to zero, it holds that

$$\left\| S - l^*(1 - d_E\tilde{E}^0 - d_I\tilde{I}^0 - d_R\tilde{R}^0) \right\|_{C^1(\bar{\Omega})} \to 0, \tag{2.22}$$

and

$$\left\| \frac{E}{d_S} - l^*\tilde{E}^0 \right\|_{C^1(\bar{\Omega})} + \left\| \frac{I}{d_S} - l^*\tilde{I}^0 \right\|_{C^1(\bar{\Omega})} + \left\| \frac{R}{d_S} - l^*\tilde{R}^0 \right\|_{C^1(\bar{\Omega})} \to 0, \tag{2.23}$$

where $(\tilde{E}^0, \tilde{I}^0, \tilde{R}^0)$ is a positive classical solution of

$$\begin{cases} 0 = d_E\Delta\tilde{E}^0 + N\frac{(1-d_E\tilde{E}^0-d_I\tilde{I}^0-d_R\tilde{R}^0)}{\int_\Omega(1-d_E\tilde{E}^0-d_I\tilde{I}^0-d_R\tilde{R}^0)}\beta\tilde{I}^0 - \sigma\tilde{E}^0 & x\in\Omega, \\ 0 = d_I\Delta\tilde{I}^0 + \sigma\tilde{E}^0 - \gamma\tilde{I}^0 & x\in\Omega, \\ 0 = d_R\Delta\tilde{R}^0 + \gamma\tilde{I}^0 - \alpha\tilde{R}^0 & x\in\Omega, \\ 0 = \partial_n\tilde{E}^0 = \partial_n\tilde{I}^0 = \partial_n\tilde{R}^0 & x\in\partial\Omega, \\ \tilde{E}^0 > 0, \quad \tilde{I}^0 > 0, \quad \tilde{R}^0 > 0, \quad \& \quad d_E\tilde{E}^0 + d_I\tilde{I}^0 + d_R\tilde{R}^0 < 1 & x\in\overline{\Omega}, \end{cases} \tag{2.24}$$

and $l^* = N/\int_\Omega(1 - d_E\tilde{E}^0 - d_I\tilde{I}^0 - d_R\tilde{R}^0)$.

(i-2) *If $N = N_*$, then every EE solution (S, E, I, R) of (1.3) for $0 < d_S \le d_0$ satisfies $\int_\Omega S \to N$ and $\|E\|_\infty + \|I\|_\infty + \|R\|_\infty \to 0$ as $d_S \to 0$.*

(ii) *Fix $N > N_*$. Then, there is a sequence $\{d_S^l\}_{l>0}$ of diffusion rates of the susceptible population, converging to zero as $l \to \infty$, and a sequence of EE solutions $\{(S^l, E^l, I^l, R^l)\}_{l>0}$ of (1.3) such that as $l \to \infty$,*

$$S^l \to \frac{\alpha}{\beta}\frac{\tilde{R}^*}{\tilde{I}^*} \quad \text{in } C(\overline{\Omega}), \tag{2.25}$$

$$E^l \to \left(\frac{N - \int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}}{\int_\Omega(\tilde{E}^* + \tilde{I}^* + \tilde{R}^*)} \right)\tilde{E}^* \quad \text{in} \quad C^1(\overline{\Omega}), \tag{2.26}$$

$$I^l \to \left(\frac{N - \int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}}{\int_\Omega(\tilde{E}^* + \tilde{I}^* + \tilde{R}^*)} \right)\tilde{I}^* \quad \text{in} \quad C^1(\overline{\Omega}), \tag{2.27}$$

and

$$R^l \to \left(\frac{N - \int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}}{\int_\Omega(\tilde{E}^* + \tilde{I}^* + \tilde{R}^*)} \right)\tilde{R}^* \quad \text{in} \quad C^1(\overline{\Omega}), \tag{2.28}$$

where $(\tilde{E}^*, \tilde{I}^*, \tilde{R}^*)$ is the unique positive classical solution of

$$\begin{cases} 0 = d_E\Delta\tilde{E}^* + \alpha\tilde{R}^* - \sigma\tilde{E}^* & x\in\Omega, \\ 0 = d_I\Delta\tilde{I}^* + \sigma\tilde{E}^* - \gamma\tilde{I}^* & x\in\Omega, \\ 0 = d_R\Delta\tilde{R}^* + \gamma\tilde{I}^* - \alpha\tilde{R}^* & x\in\Omega, \\ 0 = \partial_n\tilde{E}^* = \partial_n\tilde{I}^* = \partial_n\tilde{R}^* & x\in\partial\Omega, \end{cases} \tag{2.29}$$

satisfying

$$1 = d_E\tilde{E}^* + d_I\tilde{I}^* + d_R\tilde{R}^*. \tag{2.30}$$

In addition, if $N < \mu^*|\Omega|$, where $\mu^* > 0$ is the principal eigenvalue of (2.10), then there is d_S^* such that for every $0 < d_S < d_S^*$, system (1.3) has an EE solution (S_2, E_2, I_2, R_2) satisfying

$$\|E_2\|_\infty + \|I_2\|_\infty + \|R_2\|_\infty \le d_S C_1 \quad 0 < d_S < d_S^*, \tag{2.31}$$

where C_1 is independent of d_S. Furthermore, up to a subsequence, S_2 has the same asymptotic profile as in (2.22) as $d_S \to 0$.

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Thanks to Theorem 2.11–(i), reducing the diffusion rate of the susceptible population can significantly lower disease prevalence at EE solutions when the total population size N is less than or equal to the critical threshold N_* defined in (2.20). However, according to Theorem 2.11–(ii), if $N > N_*$, there always exists a branch of EE solutions for small diffusion rates of susceptible individuals, where the disease persists even as $d_S \to 0$. Interestingly, when $N_* < N < \mu^*|\Omega|$, another branch of EE solutions may arise in which the infected population goes extinct as d_S decreases. This highlights how population mobility can complicate disease dynamics, as control strategies that reduce the movement of susceptible individuals may not always lead to disease eradication. The limit of N_* as d_R, d_E, or d_I tends to zero is established in the next result.

Proposition 2.12. *For every $d_R > 0$, $d_I > 0$ and $d_E > 0$, let N_* be defined by (2.20). The following conclusions hold.*

(i) $N_* \to \int_\Omega \gamma/\beta$ *as* $d_R \to 0$.

(ii) $N_* \to \int_\Omega \frac{\gamma}{\beta}\frac{d_E\alpha\tilde{\varphi}_R^*}{\sigma(1-d_R\tilde{\varphi}_R^*)}$ *as* $d_I \to 0$, *where* $0 < \tilde{\varphi}_R^* < 1/d_R$ *is the unique positive solution of*

$$\begin{cases} 0 = d_R\Delta\tilde{\varphi}_R^* - \alpha\tilde{\varphi}_R^* + \frac{\sigma}{d_E}\left(1 - d_R\tilde{\varphi}_R^*\right) & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}\tilde{\varphi}_R^* & x \in \partial\Omega. \end{cases} \tag{2.32}$$

In particular, if $\frac{\alpha}{\sigma}$ is a constant function, then $N_* \to \int_\Omega \gamma/\beta$ as $d_I \to 0$.

(iii) $N_* \to \int_\Omega \frac{\alpha}{\beta}\frac{d_I\tilde{\varphi}_R^{**}}{(1-d_R\tilde{\varphi}_R^{**})}$ *as* $d_E \to 0$, *where* $0 < \tilde{\varphi}_R^{**} < \frac{1}{d_R}$ *is the unique positive solution of*

$$\begin{cases} 0 = d_R\Delta\tilde{\varphi}_R^{**} - \alpha\tilde{\varphi}_R^{**} + \frac{\gamma}{d_I}\left(1 - d_R\tilde{\varphi}_R^{**}\right) & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}\tilde{\varphi}_R^{**} & x \in \partial\Omega. \end{cases} \tag{2.33}$$

In particular, if $\frac{\alpha}{\gamma}$ is a constant function, then $N_* \to \int_\Omega \gamma/\beta$ as $d_E \to 0$.

Next, we discuss the limit of EE as d_I approaches zero.

Theorem 2.13. *Fix $d_E > 0$, $d_R > 0$ and $d_S > 0$, and suppose that*

$$\frac{N}{|\Omega|}\mathcal{R}\left(d_E, \sigma, \frac{\sigma\beta}{\gamma}\right) \neq 1. \tag{2.34}$$

Assume also that there is $d_* > 0$ such that system (1.3) has at least one EE solution (S, E, I, R) for every $0 < d_I \leq d_*$. Then, up to a subsequence, as $d_I \to 0$, it holds that

$$\lim_{d_I \to 0}\left(\|S - S^*\|_\infty + \|E - E^*\|_\infty + \|I - I^*\|_\infty + \|R - R^*\|_\infty\right) = 0, \tag{2.35}$$

with

$$S^* = \frac{N\left(1 - d_E\tilde{E}^* - d_R\tilde{R}^*\right)}{\int_\Omega\left(1 - d_E\tilde{E}^* - d_R\tilde{R}^*\right) + d_S\int_\Omega\left(\tilde{R}^* + \tilde{E}^* + \frac{\sigma}{\gamma}\tilde{E}^*\right)} > 0, \tag{2.36}$$

$$I^* = \frac{d_S N(\sigma/\gamma)\tilde{E}^*}{\int_\Omega\left(1 - d_E\tilde{E}^* - d_R\tilde{R}^*\right) + d_S\int_\Omega\left(\tilde{R}^* + \tilde{E}^* + \frac{\sigma}{\gamma}\tilde{E}^*\right)}, \tag{2.37}$$

$$E^* = \frac{d_S N\tilde{E}^*}{\int_\Omega\left(1 - d_E\tilde{E}^* - d_R\tilde{R}^*\right) + d_S\int_\Omega\left(\tilde{R}^* + \tilde{E}^* + \frac{\sigma}{\gamma}\tilde{E}^*\right)}, \tag{2.38}$$

and

$$R^* = \frac{d_S N\tilde{R}^*}{\int_\Omega\left(1 - d_E\tilde{E}^* - d_R\tilde{R}^*\right) + d_S\int_\Omega\left(\tilde{R}^* + \tilde{E}^* + \frac{\sigma}{\gamma}\tilde{E}^*\right)}, \tag{2.39}$$

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where $(\tilde{E}^*, \tilde{R}^*)$ *is a positive classical solution of the system of elliptic equations*

$$\begin{cases} 0 = d_E \Delta \tilde{E}^* + \frac{N(\sigma\beta/\gamma)}{\int_\Omega (1 - d_E \tilde{E}^* - d_R \tilde{R}^*) + d_S \int_\Omega (\tilde{R}^* + \tilde{E}^* + \frac{\sigma}{\gamma} \tilde{E}^*)} \big(1 - d_E \tilde{E}^* - d_R \tilde{R}^*\big) \tilde{E}^* - \sigma \tilde{E}^* & x \in \Omega, \\ 0 = d_R \Delta \tilde{R}^* + \sigma \tilde{E}^* - \alpha \tilde{R}^* & x \in \Omega, \\ 0 = \partial_n \tilde{E}^* = \partial_n \tilde{R}^* & x \in \partial\Omega. \end{cases} \tag{2.40}$$

Our third result on the asymptotic profiles of EEs concerns the case of small values of d_E and read as follows.

Theorem 2.14. *Fix* $d_S > 0$, $d_I > 0$, $d_R > 0$, *and suppose that*

$$\frac{N}{|\Omega|} \mathcal{R}(d_I, \gamma, \beta) \neq 1. \tag{2.41}$$

Assume also that there is $d^* > 0$ *such that system (1.3) has at least one EE solution for every* $0 < d_E \leq d^*$. *Then, as* $d_E \to 0$, *up to a subsequence, EE solutions* (S, E, I, R) *of (1.3) for* $0 < d_E \leq d^*$ *satisfy*

$$\lim_{d_E \to 0} \left(\|S - l^* \big(1 - d_I \tilde{I}^* - d_R \tilde{R}^*\big)\|_\infty + \|E - \frac{d_S (l^*)^2 \beta}{\sigma} \big(1 - d_I \tilde{I}^* - d_R \tilde{R}^*\big) \tilde{I}^* \|_\infty \right) = 0, \tag{2.42}$$

and

$$\lim_{d_E \to 0} \big(\|I - d_S l^* \tilde{I}^* \|_\infty + \|R - d_S l^* \tilde{R}^* \|_\infty \big) = 0, \tag{2.43}$$

where $l^* > 0$ *is a real number and* $(\tilde{I}^*, \tilde{R}^*)$ *is a positive classical solution of*

$$\begin{cases} 0 = d_I \Delta \tilde{I}^* - \gamma \tilde{I}^* + l^* \beta \big(1 - d_I \tilde{I}^* - d_R \tilde{R}^*\big) \tilde{I}^* & x \in \Omega, \\ 0 = d_R \Delta \tilde{R}^* - \alpha \tilde{R}^* + \gamma \tilde{I}^* & x \in \Omega, \\ 0 = \partial_n \tilde{R}^* = \partial_n \tilde{I}^* & x \in \partial\Omega, \\ N = l^* \big(\int_\Omega \big(1 - d_I \tilde{I}^* - d_R \tilde{R}^*\big) + d_S \int_\Omega \big(\tilde{R}^* + \tilde{I}^* + l^* \frac{\beta}{\sigma} \big(1 - d_I \tilde{I}^* - d_R \tilde{R}^*\big) \tilde{I}^* \big) \big) \end{cases} \tag{2.44}$$

and satisfies $1 - d_I \tilde{I}^* - d_R \tilde{R}^* > 0$ *on* $\bar{\Omega}$.

Finally, we consider the scenario of lowering the movement of the recovered population by letting d_R tend to zero. In this case, we have the following result.

Theorem 2.15. *Fix* $d_S > 0$, $d_I > 0$, $d_E > 0$ *and assume that* $\mathcal{R}_0 \neq 1$. *Assume also that there is* $d^{**} > 0$ *such that system (1.3) has at least one EE solution for every* $0 < d_R \leq d^{**}$. *Then, as* $d_R \to 0^+$, *up to a subsequence, EE solutions* (S, E, I, R) *of (1.3) for* $0 < d_R \leq d^{**}$ *satisfy*

$$\lim_{d_R \to 0^+} \left(\|S - l^{**} \big(1 - d_E \tilde{E}^{**} - d_I \tilde{I}^{**}\big)\|_\infty + \|E - d_S l^{**} \tilde{E}^{**}\|_\infty + \|I - d_S l^{**} \tilde{I}^{**}\|_\infty + \left\| R - d_S \frac{\gamma}{\alpha} l^{**} \tilde{I}^{**} \right\|_\infty \right) = 0 \tag{2.45}$$

where $(\tilde{E}^{**}, \tilde{I}^{**}) \in [C^{++}(\bar{\Omega})]^2$ *and* l^{**} *solves*

$$\begin{cases} 0 = d_E \Delta \tilde{E}^{**} + l^{**} \beta \big(1 - d_E \tilde{E}^{**} - d_I \tilde{I}^{**}\big) \tilde{I}^{**} - \sigma \tilde{E}^{**} & x \in \Omega, \\ 0 = d_I \Delta \tilde{I}^{**} + \sigma \tilde{E}^{**} - \gamma \tilde{I}^{**} & x \in \Omega, \\ 0 = \partial_n \tilde{E}^{**} = \partial_n \tilde{I}^{**} & x \in \partial\Omega, \\ N = l^{**} \big(\int_\Omega \big(1 - d_E \tilde{E}^{**} - d_I \tilde{I}^{**}\big) + d_S \int_\Omega \big(\tilde{E}^{**} + \tilde{I}^{**} + \frac{\gamma}{\alpha} \tilde{I}^{**} \big) \big), \end{cases} \tag{2.46}$$

and $d_E \tilde{E}^{**} + d_I \tilde{I}^{**} < 1$ *on* $\bar{\Omega}$.

Theorems 2.13–2.15 examined the effect of small diffusion rates of the infected, exposed, and recovered populations on the spatial profiles of the EE solutions of the diffusive SEIRS epidemic model (1.3). In theses three results, the infected population is not wiped out as opposed to the conclusion of Theorem 2.11-(i) when the susceptible population dispersal rate is lowered and the total population size does not exceed the threshold number N_*.

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2.4 Discussion

This work examined the dynamics of solutions to the diffusive SEIRS epidemic model (1.3). In the first part of the paper, we investigate the global boundedness of classical solutions. The analysis of the boundedness of theses solutions turn out to present significant mathematical chanllenges as opposed to that of the corresponding model with the standard incidence mechanism (1.2) studied in [37]. Despite these complexities inherent from the mass-action transmission mechanism in system (1.3), Theorem 2.2 shows that solutions are eventually bounded if either $d_S = d_E$ or the dimension of spatial domain is less than or equal to 5. It is important to note from Proposition 2.1, that no positive solution to the epidemic model (1.3) blows up in finite time. Hence, we conjectured that the additional hypotheses imposed in Theorem 2.2 are only related to our mathematical approach. The proof of Theorem 2.2 is mainly based on the semigroup theory, in particular, the variation of constant formula and the $L^p - L^q$ estimates for the heat semigroup.

In the second part of our paper, we analyze the stability of the DFE and investigate the existence and multiplicity of the EE solutions for system (1.3). This leads us to define the BRN, $\mathcal{R}_0$ (see (2.11)), which is inversely proportional to the weighted principal eigenvalue of the associated cooperative system (2.10). Theorem 2.4 demonstrates that the DFE is globally stable when $\mathcal{R}_0$ is sufficiently small, while Theorem 2.5 establishes the existence of at least one EE solution when $\mathcal{R}_0 > 1$.

Interestingly, unlike the system studied in [37], which admits no EE solutions when its BRN is less than one, Theorem 2.7 reveals that the dynamics of the diffusive SEIRS epidemic model (1.3) are more complex. Specifically, under appropriate conditions on the model parameters, Theorem 2.7 shows that multiple EE solutions may exist even when $\mathcal{R}_0$ is slightly less than one. This multiplicity highlights a key challenge in studying the dynamics of system (1.3). Notably, our approach to establishing the existence of EE solutions differs significantly from the methods used in [37]. A central idea is to recast the EE problem as a coexistence problem for the three-equation elliptic system (5.1), as shown in Lemma 5.8. The resulting system (5.1) is strongly coupled and highly nonlinear. To study it, we combine tools from spectral theory, parabolic and elliptic PDE theory, and bifurcation theory. Building on this framework, Theorem 5.7 proves the existence of a continuum of positive coexistence solutions for (5.1). The argument also draws on powerful tools from the global bifurcation theory developed by Shi and Wang [35] and López-Gómez [20].

In the third part of our work, we investigate the spatial profile of the EE solutions to the epidemic model (1.3) with respect to small diffusion rates. Theorem 2.11 highlights the critical role played by the total population size N in determining the effectiveness of reducing susceptible movement to contain the spread of an infectious disease. Specifically, when $N \leq N_*$, where N_* is defined in (2.20), limiting the diffusion rate of the susceptible population may lead to disease eradication. However, if $N > N_*$, strategies that aim to restrict the movement of susceptibles may not always yield the desired outcome, as the disease may either persist or die out. The proof of Theorem 2.11 relies on the transformed system (5.1) and the analytic results established for that system.

These biological implications differ notably from those regarding the spatial profile of EE solutions to the system (1.2), as established in [37]. In that work, the analytical results do not identify any link between the total population size and the existence or profile of EE solutions for small values of d_S. Furthermore, Theorems 2.13–2.15 in our study suggest that restricting the movement rates of the infected, exposed or recovered populations individually does not lead to disease eradication, implying persistence of the disease in these scenarios. In particular, our results for system (1.3) as $d_I \to 0$ align, to some extent, with those for the SIS diffusive epidemic model. However, for system (1.3), the disease persists throughout the entire domain as $d_I \to 0$, whereas in the SIS diffusive epidemic model, the disease may be eradicated in low-risk regions in the same limit (see [6, Theorem 2.5]). Notably, the asymptotic behavior of EEs for the system (1.2) for small d_I, d_E, or d_R was not addressed in [37], and thus remains an open question.

In summary, our results on the epidemic model (1.3) reveal significant differences from those based on the standard incidence mechanism studied in [37]. Moreover, the biological interpretations of our

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analytical findings underscore the importance of spatial heterogeneity in transmission rates and population movement, offering valuable insights for the design and implementation of effective disease control strategies.

3. Preliminary results and proofs of Propositions 2.10 and 2.12

This section contains some preliminary results necessary for the clarity of this work. We start with the following result which shows that the principal eigenvalue of system (2.14) is zero.

Proposition 3.1. *Let λ_{d_E,d_I,d_R} denote the principal eigenvalue of (2.14). Then, the following conclusions hold.*

(i) $\lambda_{d_E,d_I,d_R} = 0$.

(ii) *There is a unique positive solution $(\tilde{E}^*, \tilde{I}^*, \tilde{R}^*)$ of (2.29) satisfying (2.30).*

Proof. Let $\lambda = \lambda_{d_E,d_I,d_R}$. Integrating (2.14) both sides, we obtain

$$\begin{cases} 0 = \int_\Omega (-\sigma\varphi_E + \alpha\varphi_R - \lambda\varphi_E) \\ 0 = \int_\Omega (-\gamma\varphi_I + \sigma\varphi_E - \lambda\varphi_I) \\ 0 = \int_\Omega (-\alpha\varphi_R + \gamma\varphi_I - \lambda\varphi_R). \end{cases} \tag{3.1}$$

Add up the equations in (3.1) to get

$$\lambda \int_\Omega (\varphi_E + \varphi_I + \varphi_R) = 0.$$

Since $\varphi_E, \varphi_I, \varphi_R$ are all positive, then $\int_\Omega (\varphi_E + \varphi_I + \varphi_R)$ is positive and so $\lambda = 0$. So (i) holds.

Observe that since $\lambda_{d_E,d_I,d_R} = 0$, then (E, I, R) solves (2.29) if and only if it is an eigenvector associated with λ_{d_E,d_I,d_R}. Therefore, recalling that λ_{d_E,d_I,d_R} is a simple eigenvalue, then any solution of (2.29) is of the form $(E, I, R) = \kappa(\varphi_E, \varphi_I, \varphi_R)$ for some $\kappa \in \mathbb{R}$. But, adding up the first three equations of (2.14) yields that

$$\begin{cases} 0 = \Delta(d_E\varphi_E + d_I\varphi_I + d_R\varphi_R) & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}(d_E\varphi_E + d_I\varphi_I + d_R\varphi_R) & x \in \partial\Omega. \end{cases}$$

We deduce that there is a positive constant $c_* > 0$ such that $d_E\varphi_E + d_I\varphi_I + d_R\varphi_R = c_*$. Therefore,

$$(\tilde{E}^*, \tilde{I}^*, \tilde{R}^*) := \frac{1}{c_*}(\varphi_E, \varphi_I, \varphi_R) \tag{3.2}$$

is the unique positive solution of (2.29) satisfying (2.30). Hence, (ii) holds. □

Proof of Proposition 2.10. (i) Taking $\beta = n\beta_1$ in (2.10), and let μ_n^* be the principal eigenvalue of the weighted eigenvalue problem (2.10). By the uniqueness of μ^*, we have that $\mu_1^* = n\mu_n^*$. Hence, by (2.11), it holds that $\mathcal{R}_0^{(n)} = n\mathcal{R}_0^{(1)}$ for all $n > 0$.

(ii) Assume that $\sigma = n\sigma_1$ for $n > 0$. Let $\mu^{*,n}$ denote the principal eigenvalue of (2.10). Let (Φ^n, Ψ^n) be a positive eigenfunction associated with $\mu^{*,n}$. Then, setting $(\tilde{\Phi}^n, \tilde{\Psi}^n) = (\Phi^n, \frac{\Psi^n}{n})$, it holds that

$$\begin{cases} 0 = \frac{d_E}{n}\Delta\tilde{\Phi}^n - \sigma_1\tilde{\Phi}^n + \mu^{*,n}\beta\tilde{\Psi}^n & x \in \Omega, \\ 0 = d_I\Delta\tilde{\Psi}^n + \sigma_1\tilde{\Phi}^n - \gamma\tilde{\Psi}^n & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}\tilde{\Phi}^n = \partial_{\boldsymbol{n}}\tilde{\Psi}_n & x \in \partial\Omega, \\ \tilde{\Phi}^n > 0, \quad \tilde{\Psi}^n > 0 & x \in \bar{\Omega}. \end{cases}$$

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Then, $\mu^{*,n}$ is the principal eigenvalue of the weighted eigenvalue problem (2.10) with σ replaced with σ_1 and d_E replaced with $\frac{d_E}{n}$, respectively. Hence, by (2.11) and Proposition 2.9-(i), we have that

$$\lim_{n\to\infty} \tilde{\mathcal{R}}_0^{(n)} = \lim_{n\to\infty} \frac{N}{|\Omega|\mu^{*,n}} = \frac{N}{|\Omega|}\mathcal{R}(d_I, \gamma, \beta),$$

and

$$\lim_{n\to 0^+} \tilde{\mathcal{R}}_0^{(n)} = \lim_{n\to 0^+} \frac{N}{|\Omega|\mu^{*,n}} = \frac{N}{|\Omega|}\frac{\int_\Omega \beta(\gamma - d_I\Delta)^{-1}(\sigma_1)dx}{\int_\Omega \sigma_1 dx}.$$

□

Proof of Proposition 2.12. Let $(\varphi_E, \varphi_I, \varphi_R)$ be the positive eigenfunction associated with λ_{d_E,d_I,d_R} satisfying $\max_{x\in\bar{\Omega}} (\varphi_E(x) + \varphi_I(x) + \varphi_R(x)) = 1$. Define

$$\xi := d_E\varphi_E + d_I\varphi_I + d_R\varphi_R,$$

then

$$0 = \Delta\xi, \quad x \in \Omega; \quad 0 = \partial_{\boldsymbol{n}}\xi, \quad x \in \partial\Omega.$$

This shows that ξ is constant. So, setting $\tilde{\varphi}_E = \frac{\varphi_E}{\xi}$, $\tilde{\varphi}_I = \frac{\varphi_I}{\xi}$, and $\tilde{\varphi}_R = \frac{\varphi_R}{\xi}$, it holds that

$$\tilde{\varphi}_E = \frac{1}{d_E}\big(1 - d_I\tilde{\varphi}_I - d_R\tilde{\varphi}_R\big) > 0 \tag{3.3}$$

$$\begin{cases} 0 = d_E(d_I\Delta\tilde{\varphi}_I - \gamma\tilde{\varphi}_I) + \sigma(1 - d_I\tilde{\varphi}_I - d_R\tilde{\varphi}_R) & x \in \Omega, \\ 0 = d_R\Delta\tilde{\varphi}_R - \alpha\tilde{\varphi}_R + \gamma\tilde{\varphi}_I & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}\tilde{\varphi}_I = \partial_{\boldsymbol{n}}\tilde{\varphi}_R & x \in \partial\Omega, \end{cases} \tag{3.4}$$

and

$$N_* := \int_\Omega \frac{\alpha}{\beta}\frac{\tilde{\varphi}_R}{\tilde{\varphi}_I},$$

where N_* is defined by (2.20).

(i) We first find the profiles of $(\tilde{\varphi}_I, \tilde{\varphi}_R)$ as $d_R \to 0$. Since by (3.3), $\|\sigma(1 - d_I\tilde{\varphi}_I - d_R\tilde{\varphi}_R)\|_\infty \le \|\sigma\|_\infty$, then by the regularity theory for elliptic equations, we have that $\sup_{d_R>0} \|\tilde{\varphi}_I\|_{C^{1+\nu}(\bar{\Omega})} < \infty$. So, the set $\{\tilde{\varphi}_I : d_R > 0\}$ is relatively compact in $C^1(\bar{\Omega})$. Let $\tilde{\varphi}_I^0$ be a limit point of $\{\tilde{\varphi}_I : d_R > 0\}$ in $C^1(\bar{\Omega})$ as $d_R \to 0$. After passing to a subsequence, we may suppose that $\|\tilde{\varphi}_I - \tilde{\varphi}_I^0\|_{C^1(\bar{\Omega})} \to 0$ as $d_R \to 0$. Hence, apply the singular perturbation theory for elliptic equations, it follows from the second equation of (3.4) that $\|\tilde{\varphi}_R - \frac{\gamma}{\alpha}\tilde{\varphi}_I^0\|_\infty \to 0$ as $d_R \to 0$. This, in turn, implies that $d_R\|\tilde{\varphi}_R\|_\infty \to 0$ as $d_R \to 0$. This along with the regularity theory for elliptic equations apply to the first equation of (3.4) implies that $\tilde{\varphi}_I^0 \in C^2(\bar{\Omega})$ and is a classical solution of

$$\begin{cases} 0 = d_E\big(d_I\Delta\tilde{\varphi}_I^0 - \gamma\tilde{\varphi}_I^0\big) + \sigma\big(1 - d_I\tilde{\varphi}_I^0\big) & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}\tilde{\varphi}_I^0 & x \in \partial\Omega. \end{cases} \tag{3.5}$$

However, it is clear that (3.5) has a unique classical solution. Therefore, $\tilde{\varphi}_I^0$ is independent of the chosen subsequence. Consequently, $N_* \to \int_\Omega \frac{\alpha}{\beta}\frac{(\gamma/\alpha)\tilde{\varphi}_I^0}{\tilde{\varphi}_I^0} = \int_\Omega \frac{\gamma}{\beta}$ as $d_R \to 0$.

(ii) By the proper modifications of the arguments employed in the proof of (i), we can show that $\|\tilde{\varphi}_R - \tilde{\varphi}_R^*\|_{C^1(\bar{\Omega})} \to 0$ and $\|\tilde{\varphi}_I - \frac{\sigma(1-d_R\tilde{\varphi}_R^*)}{\gamma d_E}\|_\infty \to 0$ as $d_I \to 0$ where $\tilde{\varphi}_R^*$ is the unique positive solution of (2.32). Hence, $N_* \to \int_\Omega \frac{\gamma}{\beta}\frac{d_E\alpha\tilde{\varphi}_R^*}{\sigma(1-d_R\tilde{\varphi}_R^*)}$ as $d_I \to 0$. If $\frac{\alpha}{\sigma}$ is a constant function, then $\tilde{\varphi}_R^*$ is also a constant function and $\frac{d_E\alpha\tilde{\varphi}_R^*}{\sigma(1-d_R\tilde{\varphi}_R^*)} = 1$. Therefore, $N_* \to \int_\Omega \gamma/\beta$ as $d_I \to 0$.

(iii) The result can also be obtained by a proper modification of the arguments of the proof of (i). Hence, we omit the details. □

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Given $h \in C(\overline{\Omega})$ and $d > 0$, let $\lambda(d, h)$ denote the principal eigenvalue of the eigenvalue problem

$$\begin{cases} \lambda\varphi = d\Delta\varphi + h\varphi & x \in \Omega, \\ 0 = \partial_n \varphi & x \in \partial\Omega. \end{cases} \tag{3.6}$$

It is well known that $\lambda(d, h)$ is given by the variational formula

$$\lambda(d, h) = \max_{\varphi \in W^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[h\varphi^2 - d|\nabla\varphi|^2\right]}{\int_\Omega \varphi^2}. \tag{3.7}$$

Note also that all eigenvalues of (3.6) are real. The next lemma collects a few information on $\lambda(d, h)$.

Lemma 3.2. *If h is constant, then $\lambda(d, h) = h$ for any $d > 0$. However, if h is not constant, then $\lambda(d, h)$ is strictly decreasing in d and*

$$\lim_{d\to 0^+} \lambda(d, h) = h_{\max} \quad \text{and} \quad \lim_{d\to\infty} \lambda(d, h) = \frac{1}{|\Omega|}\int_\Omega h. \tag{3.8}$$

For a proof of Lemma 3.2, we refer to [5].

4. Boundedness of classical solutions: Proof of Theorem 2.2

We prove a few lemmas. First, we recall the following $L^p - L^q$ estimates for the heat semigroup. Let $\{e^{t\Delta}\}_{t\geq 0}$ denote the heat semigroup subject to the homogeneous Neumann boundary conditions on $\partial\Omega$. Then, for every $1 \leq q \leq p \leq \infty$, there is $C > 0$ such that

$$\|e^{t\Delta}u\|_{L^p(\Omega)} \leq C\left(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)\|u\|_{L^q(\Omega)} \quad \forall\, u \in L^q(\Omega). \tag{4.1}$$

See [31, Proposition 48.4*] as a reference for (4.1). Throughout this section, we fix $0 < a_0 \ll \frac{1}{\max\{d_I, d_S, d_E, d_R\}} \min\{\sigma_{\min}, \alpha_{\min}, \gamma_{\min}\}$. Whenever $1 \leq q \leq p \leq \infty$, we suppose that $C > 0$ is fixed for which (4.1) holds. We use the conventional notation $\frac{a}{b_+} = \infty$ if $b \leq 0$ and $a > 0$.

Lemma 4.1. *Fix $1 \leq p < \frac{n}{(n-2)_+}$. Then, for every solution $(S, E, I, R)(t, x)$ of (1.3), there is a positive constant $M_{1,p}$, independent of initial data and N, such that*

$$\max\left\{\|I(t)\|_{L^p(\Omega)}, \|S(t)\|_{L^p(\Omega)}\right\} \leq NM_{1,p}\left(\left(1 + (\min\{d_I, d_S\}t)^{-\frac{n}{2}(1-\frac{1}{p})}\right)e^{-\min\{d_S, d_I\}a_0 t} + 1\right) \quad \forall\, t > 0. \tag{4.2}$$

Proof. Fix $1 \leq p < \frac{n}{(n-2)_+}$. Then, $\frac{n}{2}(1 - \frac{1}{p}) < 1$. Next, thanks to the variation of constant formula,

$$I(t) = e^{d_I t(\Delta - a_0)}I_0 + \int_0^t e^{d_I(t-s)(\Delta-a_0)}(\sigma E(s) - (\gamma - a_0 d_I)I(s))ds, \quad \forall\, t > 0. \tag{4.3}$$

Hence, thanks to the positivity of the C_0-semigroup $\{e^{t\Delta}\}_{t\geq 0}$, the fact that $\gamma \geq d_I a_0$, and the positivity of E and I, we get

$$0 \leq I(t) \leq e^{d_I t(\Delta-a_0)}I_0 + \|\sigma\|_\infty \int_0^t e^{d_I(t-s)(\Delta-a_0)}E(s)ds, \quad \forall\, t > 0.$$

This, along with (4.1), the triangle inequality, and the Minkowski's inequality, yields

$$\begin{aligned} \|I(t)\|_{L^p(\Omega)} &\leq \left\| e^{d_I t(\Delta-a_0)}I_0 + \|\sigma\|_\infty \int_0^t e^{d_I(t-s)(\Delta-a_0)}E(s)ds \right\|_{L^p(\Omega)} \\ &\leq \left\| e^{d_I t(\Delta-a_0)}I_0 \right\|_{L^p(\Omega)} + \|\sigma\|_\infty \int_0^t \left\| e^{d_I(t-s)(\Delta-a_0)}E(s) \right\|_{L^p(\Omega)} ds \\ &\leq \frac{C\left(1 + (d_I)^{-\frac{n}{2}(1-\frac{1}{p})}\right)}{e^{a_0 d_I t}}\|I_0\|_{L^1(\Omega)} + C\|\sigma\|_\infty \int_0^t \frac{\left(1 + (d_I(t-s))^{-\frac{n}{2}(1-\frac{1}{p})}\right)}{e^{a_0 d_I(t-s)}}\|E(s)\|_{L^1(\Omega)}ds \end{aligned}$$

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$$\begin{aligned}
&\leq CN\big(1+(d_I t)^{-\frac{n}{2}(1-\frac{1}{p})}\big)e^{-a_0 d_I t} + CN\|\sigma\|_\infty \int_0^t \big(1+(d_I(t-s))^{-\frac{n}{2}(1-\frac{1}{p})}\big)e^{-a_0 d_I (t-s)}ds\\
&\leq CN\big(1+(d_I t)^{-\frac{n}{2}(1-\frac{1}{p})}\big)e^{-a_0 d_I t} + CN\|\sigma\|_\infty \int_0^\infty \big(1+(d_I s)^{-\frac{n}{2}(1-\frac{1}{p})}\big)e^{-d_I a_0 s}ds\\
&= \frac{CN\big(1+(d_I t)^{-\frac{n}{2}(1-\frac{1}{p})}\big)}{e^{a_0 d_I t}} + \frac{CN\|\sigma\|_\infty}{d_I a_0}\left(1+a_0^{\frac{n}{2}(1-\frac{1}{p})}\Gamma\left(1-\frac{n}{2}\left(1-\frac{1}{p}\right)\right)\right) \quad t>0, \qquad (4.4)
\end{aligned}$$

where Γ is the Gamma function. Similarly, noting that

$$\begin{aligned}
0\leq S(t) =& e^{d_S t(\Delta-a_0)}S_0 + \int_0^t e^{d_S(t-s)(\Delta-a_0)}(\alpha R(s) + a_0 d_S S(s) - \beta SI(s))ds\\
\leq& e^{d_S t(\Delta-a_0)}S_0 + \int_0^t e^{d_S(t-s)(\Delta-a_0)}(\alpha R(s) + a_0 d_S S(s))ds, \qquad (4.5)
\end{aligned}$$

and $\|\alpha R(t) + a_0 d_S S(t)\|_{L^1(\Omega)} \leq 2\|\alpha\|_\infty N$ for all $t\geq 0$, we can proceed as above to obtain

$$\|S(t)\|_{L^p(\Omega)} \leq CN\big(1+(d_S t)^{-\frac{n}{2}(1-\frac{1}{p})}\big)e^{-d_S a_0 t} + \frac{2CN\|\alpha\|_\infty}{d_S a_0}\left(1+(d_S a_0)^{\frac{n}{2}(1-\frac{1}{p})}\Gamma\left(1-\frac{n}{2}\left(1-\frac{1}{p}\right)\right)\right) \quad t>0,$$

which completes the proof of the result. □

Lemma 4.2. *Fix $1\leq p_1 < \frac{n}{(n-2)_+}$ and $p_1\leq p_2 < \frac{np_1}{(n-2p_1)_+}$. Then, for every solution $(S,E,I,R)(t,x)$ of (1.3), there is a positive constant M_{2,p_1,p_2}, independent of N and initial data, such that*

$$\max\left\{\|R(t)\|_{L^{p_2}(\Omega)}, \|S(t)\|_{L^{p_2}(\Omega)}\right\} \leq NM_{2,p_1,p_2}\left(\frac{(1+(\min\{d_R,d_S\}t)^{-\frac{n}{2}(1-\frac{1}{p_2})})}{e^{a_0\min\{d_S,d_R\}t}}+1\right) \quad \forall\, t>1. \qquad (4.6)$$

Proof. Fix $1\leq p_1 < \frac{n}{(n-2)_+}$ and $p_1\leq p_2 < \frac{np_1}{(n-2p_1)_+}$. By (4.2), there is $\tilde{M}_{1,p_1}$, independent of initial data and N such that

$$\max\left\{\|I(t)\|_{L^{p_1}(\Omega)}, \|S(t)\|_{L^{p_1}(\Omega)}\right\} \leq N\tilde{M}_{1,p_1} \quad \forall\, t>1. \qquad (4.7)$$

By the variation of constant formula,

$$\begin{aligned}
R(t+1) =& e^{d_R t(\Delta-a_0)}R(1) + \int_0^t e^{d_R(t-s)(\Delta-a_0)}(\gamma I(1+s))ds\\
&- \int_0^t e^{d_R(t-s)(\Delta-a_0)}((\alpha-a_0 d_R)R(1+s))ds\\
\leq& e^{d_R t(\Delta-a_0)}R(1) + \int_0^t e^{d_R(t-s)(\Delta-a_0)}(\gamma I(1+s))ds \quad t>0. \qquad (4.8)
\end{aligned}$$

Hence, thanks to (4.7), (4.1), and Minkowski's inequality,

$$\begin{aligned}
&\|R(1+t)\|_{L^{p_2}(\Omega)}\\
\leq& \left\|e^{d_R t(\Delta-a_0)}R(1)\right\|_{L^{p_2}(\Omega)} + \int_0^t \left\|e^{d_R(t-s)(\Delta-a_0)}(\gamma I(1+s))\right\|_{L^{p_2}(\Omega)}ds\\
\leq& C\big(1+(d_R t)^{-\frac{n}{2}(1-\frac{1}{p_2})}\big)e^{-a_0 d_R t}\|R(1)\|_{L^1(\Omega)}\\
&+ C\|\gamma\|_\infty \int_0^t \big(1+(d_R(t-s))^{-\frac{n}{2}(\frac{1}{p_1}-\frac{1}{p_2})}\big)e^{-a_0 d_R(t-s)}\|I(1+s)\|_{L^{p_1}(\Omega)}ds\\
\leq& CN\big(1+(d_R t)^{-\frac{n}{2}(1-\frac{1}{p_2})}\big)e^{-a_0 d_R t} + CN\tilde{M}_{1,p_1}\|\gamma\|_\infty \int_0^t \frac{\big(1+(d_R(t-s))^{-\frac{n}{2}(\frac{1}{p_1}-\frac{1}{p_2})}\big)}{e^{a_0 d_R(t-s)}}ds,\\
\leq& \frac{CN\big(1+(d_R t)^{-\frac{n}{2}(1-\frac{1}{p_2})}\big)}{e^{a_0 d_R t}} + \frac{CN\tilde{M}_{1,p_1}\|\gamma\|_\infty}{d_R a_0}\left(1+a_0^{\frac{n}{2}(\frac{1}{p_1}-\frac{1}{p_2})}\Gamma\left(1-\frac{n}{2}\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\right)\right) \quad t>0.
\end{aligned}$$

Next, since (4.5) and (4.7) hold, we can proceed as above to derive similar estimates for $\|S(t)\|_{L^{p_2}(\Omega)}$. □

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Lemma 4.3. *Suppose that $n \in \{1, \cdots, 5\}$. Then, there is a positive constant M_∞, independent of N and initial data, such that every solution $(S, E, I, R)(t, x)$ of (1.3) with a positive initial data satisfies*

$$\|S(t)\|_\infty \le NM_\infty \quad \forall\, t \ge 3. \tag{4.9}$$

Proof. Since $n \in \{1, \cdots, 5\}$, then $\max\{\frac{n}{4}, 1\} < \frac{n}{(n-2)_+}$. So, we can first choose

$$\max\left\{\frac{n}{4}, 1\right\} < p_1 < \frac{n}{(n-2)_+}.$$

Next, we select p_2 as follows:

Case 1. $\frac{n}{2} \le p_1$, that is $(n - 2p_1)_+ = 0$. In this case we fix any $p_2 > \max\{n, p_1\}$.

Case 2. $\frac{n}{2} > p_1$. In this case, we have

$$\frac{np_1}{n-2p_1} - \frac{n}{2} = \frac{n(4p_1 - n)}{2(n-2p_1)} > 0.$$

So, we have $p_1 < \frac{n}{2} < \frac{np_1}{n-2p_1}$. We can then choose $\frac{n}{2} < p_2 < \frac{np_1}{n-2p_1}$.

In both cases, we have $\max\{\frac{n}{2}, p_1\} < p_2 < \frac{np_1}{(n-2p_1)_+}$. So, by (4.6), there is $M_2 > 0$, independent of N and initial data, such that

$$\max\left\{\|R(t)\|_{L^{p_2}(\Omega)}, \|S(t)\|_{L^{p_2}(\Omega)}\right\} \le NM_2 \quad \forall\ t \ge 2.$$

Therefore, by (4.1) and (4.5), we have that

$$\begin{aligned}
S(t+2) \le& \|e^{d_S t(\Delta - a_0)} S(2)\|_\infty + \int_0^t \|e^{d_S(t-s)(\Delta - a_0)}(\alpha R(2+s) + a_0 d_S S(2+s))\|_\infty ds \\
\le& C\big(1 + (d_S t)^{-\frac{n}{2p_2}}\big) e^{-a_0 d_S t} \|S(2)\|_{L^{p_2}(\Omega)} \\
&+ C\int_0^t \big(1 + (d_S(t-s))^{-\frac{n}{2p_2}}\big) e^{-a_0 d_S(t-s)} \|\alpha R(2+s) + a_0 d_S S(2+s)\|_{L^{p_2}(\Omega)} ds \\
\le& CNM_2\Big(\big(1 + (d_S t)^{-\frac{n}{2p_2}}\big) e^{-a_0 d_S t} + 2\|\alpha\|_\infty \int_0^t \big(1 + (d_S(t-s))^{-\frac{n}{2p_2}}\big) e^{-a_0 d_S(t-s)} ds\Big) \\
\le& CNM_2 \left(\big(1 + (d_S t)^{-\frac{n}{2p_2}}\big) e^{-a_0 d_S t} + \frac{2\|\alpha\|_\infty}{d_S a_0}\left(1 + a_0^{\frac{n}{2p_2}} \Gamma\left(1 - \frac{n}{2p_2}\right)\right)\right) \quad \forall\, t > 0,
\end{aligned}$$

which yields the desired result. □

Lemma 4.4. *Suppose that $n \in \{1, \cdots, 5\}$. Then, there is a positive constant $\tilde{M}_\infty$ such that every solution $(S, E, I, R)(t, x)$ of (1.3) with a positive initial data satisfies*

$$\limsup_{t\to\infty} \max\left\{\|E(t)\|_\infty, \|I(t)\|_\infty, \|R(t)\|_\infty\right\} \le N^3 \tilde{M}_\infty. \tag{4.10}$$

Proof. Let M_∞ as in (4.9). We proceed in five steps.

Step 1. Fix $1 \le p < \frac{n}{(n-2)_+}$ so that $\frac{n}{2}(1 - \frac{1}{p}) < 1$. We claim that there is $\tilde{M}_{1,p} > 0$, independent of initial data and N such that

$$\|E(t)\|_{L^p(\Omega)} \le N^2 \tilde{M}_{1,p} \quad \forall\, t \ge 4. \tag{4.11}$$

Indeed, thanks to the variation of constant formula, and the positivity of $\{e^{t(\Delta - a_0)}\}_{t>0}$, we have

$$\begin{aligned}
E(t+3) =& e^{d_E t(\Delta - a_0)} E(3) + \int_0^t e^{d_E(t-s)(\Delta - a_0)}(\beta S(s+3) I(s+3) - (\sigma - a_0 d_E) E(s+3)) ds \\
\le& e^{d_E t(\Delta - a_0)} E(3) + NM_\infty \|\beta\|_\infty \int_0^t e^{d_E(t-s)(\Delta - a_0)} I(s+3) ds \quad t > 0.
\end{aligned} \tag{4.12}$$

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Therefore, by the similar arguments leading to (4.4), we get

$$\|E(t+3)\|_{L^p(\Omega)} \le \frac{CN\left(1+(d_E t)^{-\frac{n}{2}(1-\frac{1}{p})}\right)}{e^{a_0 d_E t}} + \frac{CN^2 M_\infty \|\beta\|_\infty}{d_E a_0}\left(1 + a_0^{\frac{n}{2}(1-\frac{1}{p})}\Gamma\left(1-\frac{n}{2}\left(1-\frac{1}{p}\right)\right)\right) \quad t>0.$$

Therefore, there is a positive number $\tilde{M}_{1,p}$, independent of N and initial data such that (4.11) holds.

Step 2. Fix $1 \le p_1 < \frac{n}{(n-2)_+}$ and $p_1 < p_2 < \frac{np_1}{(n-2p_1)_+}$. Since (4.11) holds, we can proceed as in (4.2) to establish that there is $\tilde{M}_{2,p_1,p_2} > 0$ independent of N and initial data such that

$$\|I(t)\|_{L^{p_2}(\Omega)} \le N^2 \tilde{M}_{1,p_1,p_2} \quad \forall\, t \ge 5. \tag{4.13}$$

Step 3. Here, we fix p_1 and p_2 as in the proof of Lemma 4.3. Then, thanks to (4.9), (4.11), and (4.12), we can proceed as in the proof of (4.9) to show the existence of $\tilde{M}_2 > 0$, independent of N and initial data, such that

$$\|E(t)\|_\infty \le N^3 \tilde{M}_2 \quad \forall\, t \ge 6. \tag{4.14}$$

Step 4. Here we derive the estimates on $\|I(t)\|_\infty$. To this end, thanks to (4.14) and the third equation of (1.3), we have

$$\begin{cases} \partial_t I \le d_I \Delta I + \|\sigma\|_\infty N^3 \tilde{M}_2 - \gamma_{\min} I, & x \in \Omega,\ t > 6, \\ 0 = \partial_n I, & x \in \partial\Omega,\ t > 6. \end{cases}$$

We can now invoke the comparison principle for parabolic equations to conclude that

$$\|I(t)\|_\infty \le \frac{\|\sigma\|_\infty N^3 \tilde{M}_2}{\gamma_{\min}} + \left(\|I(6,\cdot)\|_\infty - \frac{\|\sigma\|_\infty N^3 \tilde{M}_2}{\gamma_{\min}}\right) e^{-\gamma_{\min}(t-6)}, \quad \forall\, t > 6.$$

Consequently,

$$\limsup_{t\to\infty} \|I(t)\|_\infty \le \frac{\|\sigma\|_\infty N^3 \tilde{M}_2}{\gamma_{\min}}. \tag{4.15}$$

Step 5. Here we derive the estimate on $\|R(t)\|_\infty$. From (4.15) and the fourth equation of (1.3), for every $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that

$$\begin{cases} \partial_t R \le d_R \Delta R + (1+\varepsilon)\frac{\|\gamma\|_\infty \|\sigma\|_\infty N^3 \tilde{M}_2}{\gamma_{\min}} - \alpha_{\min} R, & x \in \Omega,\ t > t_\varepsilon, \\ 0 = \partial_n R, & x \in \partial\Omega,\ t > t_\varepsilon. \end{cases}$$

We can now invoke the comparison principle for parabolic equations, and proceed by the similar arguments leading to (4.15) to conclude that

$$\limsup_{t\to\infty} \|R(t)\|_\infty \le (1+\varepsilon)\frac{\|\gamma\|_\infty \|\sigma\|_\infty N^3 \tilde{M}_2}{\gamma_{\min}\alpha_{\min}} \quad \forall\ \varepsilon > 0.$$

Letting $\varepsilon \to 0$, we get

$$\limsup_{t\to\infty} \|R(t)\|_\infty \le \frac{\|\gamma\|_\infty \|\sigma\|_\infty N^3 \tilde{M}_2}{\gamma_{\min}\alpha_{\min}}.$$

This, along with (4.14) and (4.15), yields (4.10). □

Lemma 4.5. *Suppose that $d_S = d_E$. Let (S_0, E_0, I_0, R_0) be as above. Then, there is $M_* > 0$, independent of initial data and N, such that*

$$\limsup_{t\to\infty} \left(\|S(t)\|_\infty + \|E(t)\|_\infty + \|I(t)\|_\infty + \|R(t)\|_\infty\right) \le M_* N. \tag{4.16}$$

Proof. Suppose that $d_E = d_S$. Fix $0 < \tau_1 < \frac{1}{2}\frac{\min\{1, d_{\min}\}}{(\|\sigma\|_\infty + \|\gamma\|_\infty + \|\alpha\|_\infty)}$ where $d_{\min} = \min\{d_S, d_R, d_I\}$. Since $\tau_1 > 0$, by the Harnack's inequality for parabolic equations [15, Theorem 2.5], there is $c_{\tau_1} > 0$ such that

$$\|e^{t\Delta} u_0\|_\infty \le c_{\tau_1} \min_{x\in\bar{\Omega}} (e^{t\Delta} u_0)(x) \quad \forall\, t \ge \tau_1, \quad u_0 \in C^+(\bar{\Omega}),$$

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where $\{e^{t\Delta}\}_{t\geq 0}$ denotes the heat C_0-semigroup subject to the homogeneous Neumann boundary conditions on $\partial\Omega$. From this point, we proceed in two steps.

Step 1. In this step, we prove global boundedness. To this end, let $(S(t,x), E(t,x), I(t,x), R(t,x))$ be a classical solution of (1.3) with positive initial data. For every $T \geq T_1 := \frac{\tau_1}{d_{\min}}$, define

$$M_T = \max_{0\leq t\leq T} (\|S(t)\|_\infty + \|E(t)\|_\infty + \|I(t)\|_\infty + \|R(t)\|_\infty).$$

We claim that

$$M_T \leq \max\left\{M_{T_1}, \frac{\frac{(\|S_0+E_0\|_\infty+\|I_0\|_\infty+\|R_0\|_\infty)}{e^{\tau_1 a_0}} + \frac{c_{\tau_1}N}{|\Omega|a_0 d_{\min}}(\|\sigma\|_\infty + \|\gamma\|_\infty + \|\alpha\|_\infty)}{1 - \frac{\tau_1}{d_{\min}}(\|\sigma\|_\infty + \|\gamma\|_\infty + \|\alpha\|_\infty)}\right\} \quad \forall\ T \geq T_1. \tag{4.17}$$

Thanks to the first equations in (4.5) and (4.12),

$$\begin{aligned}
S(t) + E(t) &= e^{td_S(\Delta - a_0)}(E_0 + S_0) + \int_0^t e^{(t-s)d_S(\Delta-a_0)}(\alpha R(s) + a_0 d_S S(s) - (\sigma - a_0 d_S)E(s))\,\mathrm{d}s \\
&\leq e^{-td_S a_0}\|E_0 + S_0\|_\infty + \|\alpha\|_\infty \int_0^t e^{(t-s)d_S(\Delta-a_0)}(R(s) + S(s))\,\mathrm{d}s \\
&= e^{-td_S a_0}\|E_0 + S_0\|_\infty + \|\alpha\|_\infty \int_0^{t-\frac{\tau_1}{d_S}} e^{(t-s)d_S(\Delta-a_0)}(R(s) + S(s))\,\mathrm{d}s \\
&+ \|\alpha\|_\infty \int_{t-\frac{\tau_1}{d_S}}^t e^{(t-s)d_S(\Delta-a_0)}(R(s) + S(s))\,\mathrm{d}s \\
&\leq e^{-td_S a_0}\|E_0 + S_0\|_\infty + c_{\tau_1}\|\alpha\|_\infty \int_0^{t-\frac{\tau_1}{d_S}} e^{-(t-s)d_S a_0}(e^{(t-s)d_S\Delta}(R(s) + S(s)))_{\min}\,\mathrm{d}s \\
&+ \|\alpha\|_\infty \int_{t-\frac{\tau_1}{d_S}}^t e^{-(t-s)d_S a_0}\|e^{(t-s)d_S\Delta}(R(s) + S(s))\|_\infty\,\mathrm{d}s \\
&\leq e^{-td_S a_0}\|E_0 + S_0\|_\infty + \frac{c_{\tau_1}\|\alpha\|_\infty}{|\Omega|} \int_0^{t-\frac{\tau_1}{d_S}} e^{-(t-s)d_S a_0}\|e^{(t-s)d_S\Delta}(R(s) + S(s))\|_{L^1(\Omega)}\,\mathrm{d}s \\
&+ \|\alpha\|_\infty M_T \int_{t-\frac{\tau_1}{d_S}}^t e^{-(t-s)d_S a_0}\,\mathrm{d}s \\
&\leq e^{-ta_0 d_S}\|E_0 + S_0\|_\infty + \frac{c_{\tau_1}N\|\alpha\|_\infty}{|\Omega|} \int_0^{t-\frac{\tau_1}{d_S}} e^{-(t-s)a_0 d_S}\,\mathrm{d}s + \frac{\tau_1}{d_S}\|\alpha\|_\infty M_T \\
&\leq e^{-ta_0 d_S}\|E_0 + S_0\|_\infty + \frac{c_{\tau_1}N\|\alpha\|_\infty}{|\Omega|a_0 d_S} + \frac{\tau_1}{d_S}\|\alpha\|_\infty M_T \quad \frac{\tau_1}{d_S} < t < T.
\end{aligned} \tag{4.18}$$

Similarly, thanks to the variation of constant formula, (4.3) and (4.8), we can proceed as above to establish that

$$I(t) \leq e^{-td_I a_0}\|I_0\|_\infty + \frac{c_{\tau_1}N\|\sigma\|_\infty}{|\Omega|a_0 d_I} + \frac{\tau_1}{d_I}\|\sigma\|_\infty M_T \quad \frac{\tau_1}{d_I} < t < T \tag{4.19}$$

and

$$R(t) \leq e^{-td_R a_0}\|R_0\|_\infty + \frac{c_{\tau_1}N\|\gamma\|_\infty}{|\Omega|d_R a_0} + \frac{\tau_1}{d_R}\|\gamma\|_\infty M_T \quad \frac{\tau_1}{d_R} < t < T. \tag{4.20}$$

Adding up side by side inequalities (4.18), (4.19) and (4.20), we get

$$(S+E+I+R)(t) \leq \frac{(\|S_0 + E_0\|_\infty + \|I_0\|_\infty + \|R_0\|_\infty)}{e^{ta_0 d_{\min}}} + \left(\frac{c_{\tau_1}N}{|\Omega|a_0 d_{\min}} + \frac{\tau_1 M_T}{d_{\min}}\right)(\|\sigma\|_\infty + \|\gamma\|_\infty + \|\alpha\|_\infty) \tag{4.21}$$

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for all $T_1 < t < T$. Hence

$$M_T \le \max\left\{M_{T_1}, \frac{(\|S_0+E_0\|_\infty + \|I_0\|_\infty + \|R_0\|_\infty)}{e^{\tau_1 a_0}} + \left(\frac{c_{\tau_1}N}{|\Omega|a_0 d_{\min}} + \frac{\tau_1 M_T}{d_{\min}}\right)(\|\sigma\|_\infty + \|\gamma\|_\infty + \|\alpha\|_\infty)\right\},$$

from which (4.17) follows.

Step 2. We complete the proof of (4.16) here. Indeed, set

$$A = \limsup_{t\to\infty} \|S(t)+E(t)+I(t)+R(t)\|_\infty.$$

By Step 1, we know that $A < \infty$. Next, by the definition of the lim sup, for every $\varepsilon > 0$ there is $t_\varepsilon \gg \tau_1$ such that

$$\|S(t)+E(t)+I(t)+R(t)\|_\infty \le A+\varepsilon \quad \forall\, t > t_\varepsilon.$$

This, along with (4.21), yields for every $t > \frac{\tau_1}{d_{\min}}$,

$$\|S(t+t_\varepsilon)+E(t+t_\varepsilon)+I(t+t_\varepsilon)+R(t+t_\varepsilon)\|_\infty \le \frac{(\|S(t_\varepsilon)+E(t_\varepsilon)\|_\infty + \|I(t_\varepsilon)\|_\infty + \|R(t_\varepsilon)\|_\infty)}{e^{ta_0 d_{\min}}} + \left(\frac{c_{\tau_1}N}{|\Omega|a_0 d_{\min}} + \frac{\tau_1(A+\varepsilon)}{d_{\min}}\right)(\|\sigma\|_\infty + \|\gamma\|_\infty + \|\alpha\|_\infty).$$

Taking lim sup as $t \to \infty$, we obtain that

$$A \le \left(\frac{c_{\tau_1}N}{|\Omega|a_0 d_{\min}} + \frac{\tau_1}{d_{\min}}(A+\varepsilon)\right)(\|\sigma\|_\infty + \|\gamma\|_\infty + \|\alpha\|_\infty).$$

Finally, letting $\varepsilon \to 0^+$ in the last inequality, we get

$$A \le \frac{c_{\tau_1}(\|\sigma\|_\infty + \|\gamma\|_\infty + \|\alpha\|_\infty)}{|\Omega|a_0(d_{\min} - \tau_1(\|\sigma\|_\infty + \|\gamma\|_\infty + \|\alpha\|_\infty))}N.$$

Taking $M_* = \frac{c_{\tau_1}(\|\sigma\|_\infty+\|\gamma\|_\infty+\|\alpha\|_\infty)}{|\Omega|a_0(d_{\min}-\tau_1(\|\sigma\|_\infty+\|\gamma\|_\infty+\|\alpha\|_\infty))}$, the result follows. □

Now, we can give a proof of Theorem 2.2.

Proof of Theorem 2.2. Collecting Lemmas 4.3–4.5, the result follows. □

5. The EE solution problem

Here we present an equivalent problem for the EE solution of system (1.3). To this end, for every positive real number $l > 0$, consider the one parameter family of system of elliptic equations

$$\begin{cases} 0 = d_E\Delta\tilde{E} + l\beta(1 - d_E\tilde{E} - d_I\tilde{I} - d_R\tilde{R})\tilde{I} - \sigma\tilde{E} & x\in\Omega,\\ 0 = d_I\Delta\tilde{I} + \sigma\tilde{E} - \gamma\tilde{I} & x\in\Omega,\\ 0 = d_R\Delta\tilde{R} + \gamma\tilde{I} - \alpha\tilde{R} & x\in\Omega,\\ 0 = \partial_n\tilde{E} = \partial_n\tilde{I} = \partial_n\tilde{R} & x\in\partial\Omega. \end{cases} \tag{5.1}$$

Clearly, the trivial solution $\mathbf{0} := (0,0,0)^T$ solves (5.1). Linearizing (5.1) at the trivial solution $\mathbf{0}$, we obtain the following eigenvalue problem

$$\begin{cases} \lambda\psi_E = d_E\Delta\psi_E - \sigma\psi_E + l\beta\psi_I & x\in\Omega,\\ \lambda\psi_I = d_I\Delta\psi_I - \gamma\psi_I + \sigma\psi_E & x\in\Omega,\\ \lambda\psi_R = d_R\Delta\psi_R - \alpha\psi_R + \gamma\psi_I & x\in\Omega,\\ 0 = \partial_n\psi_E = \partial_n\psi_I = \partial_n\psi_R & x\in\partial\Omega. \end{cases} \tag{5.2}$$

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It is easy to see that (5.2) is cooperative and that the subsytem

$$\begin{cases} \lambda\psi_E = d_E\Delta\psi_E - \sigma\psi_E + l\beta\psi_I & x\in\Omega,\\ \lambda\psi_I = d_I\Delta\psi_I - \gamma\psi_I + \sigma\psi_E & x\in\Omega,\\ 0 = \partial_{\boldsymbol{n}}\psi_E = \partial_{\boldsymbol{n}}\psi_I & x\in\partial\Omega. \end{cases} \tag{5.3}$$

decouples from (5.2). By the Krein–Rutman theorem, (5.3) has a principal eigenvalue, which we denote by λ^l. We shall denote by (ψ_E^l, ψ_I^l) the positive eigenfunction associated with λ^l satisfying $\|\psi_E^l + \psi_I^l\|_\infty = 1$.

Let $\lambda^l_{\max}$ denote the maximal eigenvalue of the matrix

$$A_{\max} = \begin{pmatrix} -\sigma_{\min} & l\beta_{\max}\\ \sigma_{\max} & -\gamma_{\min} \end{pmatrix},$$

and $\lambda^l_{\min}$ denote the maximal eigenvalue of the matrix

$$A_{\min} = \begin{pmatrix} -\sigma_{\max} & l\beta_{\min}\\ \sigma_{\min} & -\gamma_{\max} \end{pmatrix}.$$

It follows from the comparison principle for principal eigenvalues that

$$\lambda^l_{\min} \le \lambda^l \le \lambda^l_{\max}. \tag{5.4}$$

By computations,

$$\lambda^l_{\min} = \frac{1}{2}\left[\sqrt{4(l\beta_{\min}\sigma_{\min} - \sigma_{\max}\gamma_{\max}) + (\sigma_{\max}+\gamma_{\max})^2} - (\sigma_{\max}+\gamma_{\max})\right]$$

and

$$\lambda^l_{\max} = \frac{1}{2}\left[\sqrt{4(l\beta_{\max}\sigma_{\max} - \sigma_{\min}\gamma_{\min}) + (\sigma_{\min}+\gamma_{\min})^2} - (\sigma_{\min}+\gamma_{\min})\right].$$

It then follows from (5.4) that

$$\sqrt{\beta_{\min}\sigma_{\min}} \le \liminf_{l\to\infty}\frac{\lambda^l}{\sqrt{l}} \le \limsup_{l\to\infty}\frac{\lambda^l}{\sqrt{l}} \le \sqrt{\beta_{\max}\sigma_{\max}}. \tag{5.5}$$

This shows that $\lambda^l\to\infty$ as $l\to\infty$. Thanks to (5.5), we see that $\frac{\lambda^l}{\sqrt{l}}$, up to a subsequence, converges to a positive number as $l\to\infty$. In fact as we shall see in the next lemma, $\frac{\lambda^l}{\sqrt{l}}$ converges as $l\to\infty$.

Lemma 5.1. *For every $l>0$, let λ^l denote the principal eigenvalue of (5.3). The function $l\mapsto\lambda^l$ is smooth, strictly increasing,*

$$\lambda^0 := \lim_{l\to 0^+}\lambda^l < 0 \quad \textit{and} \quad \lim_{l\to\infty}\frac{\lambda^l}{\sqrt{l}} = \max_{x\in\overline{\Omega}}\sqrt{\beta(x)\sigma(x)}. \tag{5.6}$$

Moreover, letting μ^* be the principal eigenvalue of (2.10), it holds that $\lambda^l<0$ if $l<\mu^*$, $\lambda^l=0$ if $l=\mu^*$, and $\lambda^l>0$ if $l>\mu^*$.

Proof. Since the principle eigenvalue λ^l is single, it follows from standard regularity theory of principal eigenvalue that λ^l is continuously differentiable $l>0$, and the mapping $l\mapsto(\psi_E^l,\psi_I^l)\in[C^2(\overline{\Omega})]^2$ is also continuously differentiable (see [14, Lemma 15.1, Chapter 2]). Denoting by $\sigma^l := \frac{d\lambda^l}{dl}$, $\psi_E^{'} := \partial_l\psi_E^l$ and $\psi_I^{'} := \partial_l\psi_I^l$, it follows from (5.3) that

$$\begin{cases} \sigma^l\psi_E^l + \lambda^l\psi_E^{'} = d_E\Delta\psi_E^{'} - \sigma\psi_E^{'} + l\beta\psi_I^{'} + \beta\psi_I^l & x\in\Omega,\\ \sigma^l\psi_I^l + \lambda^l\psi_I^{'} = d_I\Delta\psi_I^{'} - \gamma\psi_I^{'} + \sigma\psi_E^{'} & x\in\Omega,\\ 0 = \partial_{\boldsymbol{n}}\psi_E^{'} = \partial_{\boldsymbol{n}}\psi_I^{'} & x\in\partial\Omega. \end{cases} \tag{5.7}$$

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Now, let (ψ_E^*, ψ_I^*) be the positive eigenfunction satisfying $\|\psi_E^* + \psi_I^*\|_\infty = 1$ of the adjoint problem of (5.3), that is (ψ_E^*, ψ_I^*) solves

$$\begin{cases} \lambda^l \psi_E^* = d_E \Delta \psi_E^* - \sigma \psi_E^* + \sigma \psi_I^* & x \in \Omega, \\ \lambda^l \psi_I^* = d_I \Delta \psi_I^* - \gamma \psi_I^* + l\beta \psi_E^* & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}} \psi_E^* = \partial_{\boldsymbol{n}} \psi_I^* & x \in \partial\Omega. \end{cases} \tag{5.8}$$

Multiply the first equations of (5.7) and (5.8) by ψ_E^* and $\psi_E^{'}$, respectively, then integrate by parts the resulting equations, then take the difference to obtain

$$\sigma^l \int_\Omega \psi_E^* \psi_E^l = l \int_\Omega \beta \psi_I^{'} \psi_E^* + \int_\Omega \beta \psi_E^* \psi_I^l - \int_\Omega \sigma \psi_I^* \psi_E^{'}. \tag{5.9}$$

Similarly, multiplying the second equations of (5.7) and (5.8) by ψ_I^* and $\psi_I^{'}$, respectively, integrate by parts the resulting equations, and then take the difference to obtain

$$\sigma^l \int_\Omega \psi_I^l \psi_I^* = \int_\Omega \sigma \psi_E^{'} \psi_I^* - l \int_\Omega \beta \psi_E^* \psi_I^{'}. \tag{5.10}$$

Now, taking the sum of (5.9) and (5.10) yields

$$\frac{d\lambda^l}{dl} = \sigma^l = \frac{\int_\Omega \beta \psi_E^* \psi_I^l}{\int_\Omega \left(\psi_E^l \psi_E^* + \psi_I^l \psi_I^*\right)} > 0$$

since $\beta \geq, \not\equiv 0$ and $\psi_I^l, \psi_I^*, \psi_E^l, \psi_E^* > 0$. This shows that λ^l is strictly increasing. Next, clearly, λ^l is continuous on $[0, \infty)$ where λ^0 is the maximal eigenvalue of (5.3) when $l = 0$. Observe that $\lambda^0 < 0$. Indeed, let (ψ_E^0, ψ_I^0) be a nonnegative eigenfunction associated with λ^0. If $\psi_E^0 \equiv 0$, then λ^0 is an eigenvalue of the single species elliptic equation

$$\begin{cases} \lambda^0 \varphi = d_I \Delta \varphi - \gamma \varphi & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}} \varphi & x \in \Omega. \end{cases} \tag{5.11}$$

Hence, it follows from Lemma 3.2 that $\lambda^0 \leq \lambda(d_I, -\gamma) \leq -\gamma_{\min} < 0$. On the other hand, if $\|\varphi_E^0\|_\infty > 0$, then λ^0 is an eigenvalue of (5.11) with d_I and γ replaced with d_E and σ. Hence, by Lemma 3.2 again, we have $\lambda^0 \leq \lambda(d_E, -\sigma) \leq -\sigma_{\min} < 0$. Therefore, we always have $\lim_{l \to 0^+} \lambda^l = \lambda^0 < 0$.

Note that we can apply [4, Proposition 3] to conclude that $\lambda^l / \sqrt{l} \to \max_{x \in \bar{\Omega}} \sqrt{\beta(x)\sigma(x)}$ as $l \to \infty$.

Finally, noting that $\lambda^{\mu^*} = 0$, where μ^* is the principal eigenvalue of (2.10), the conclusion follows from the strict monotonicity of λ^l in $l > 0$. □

For our purpose, we will be interested in nonnegative solutions of (5.1). We first make the following simple fact.

Lemma 5.2. *Fix $l > 0$. If $(\tilde{E}^l, \tilde{I}^l, \tilde{R}^l)$ is a classical solution of (5.1) with $\tilde{E}^l \geq, \not\equiv 0$, $\tilde{R}^l > 0$, and $\tilde{I}^l > 0$, then $\tilde{E}^l > 0$, and $d_E \tilde{E}^l + d_I \tilde{I}^l + d_R \tilde{R}^l < 1$ on $\overline{\Omega}$.*

Proof. Suppose that $(\tilde{E}^l, \tilde{I}^l, \tilde{R}^l)$ is a classical solution of (5.1) with $\tilde{E}^l \geq, \not\equiv 0$. It follows from the second equation of (5.1), the fact that $\gamma_{\min} > 0$, and the strong maximum principle principle for elliptic equations that $\tilde{I}^l > 0$ on $\overline{\Omega}$. This, in turn, thanks the third equation of (5.1), the fact that $\alpha_{\min} > 0$, and the strong maximum principle principle that $\tilde{R}^l > 0$ on $\overline{\Omega}$. Next, set $S^l = l(1 - d_E \tilde{E}^l - d_I \tilde{I}^l - d_R \tilde{R}^l)$. Then, S^l satisfies

$$\begin{cases} 0 = \frac{1}{l} \Delta S^l + \alpha \tilde{R}^l - \beta \tilde{I}^l S^l & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}} S^l & x \in \partial\Omega. \end{cases} \tag{5.12}$$

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As a result, since $(\alpha\tilde{R}^l)_{\min} > 0$ and $(\beta\tilde{I}^l)_{\min} > 0$, we employ the strong maximum principle principle for elliptic equations to deduce that $S^l_{\min} > 0$. Therefore, observing that $\tilde{E}^l$ is a classical solution of

$$\begin{cases} 0 = d_E \Delta \tilde{E}^l + \beta S^l \tilde{I}^l - \sigma \tilde{E}^l & x \in \Omega, \\ 0 = \partial_n \tilde{E}^l & x \in \partial\Omega, \end{cases}$$

with $\sigma_{\min} > 0$ and $(\beta S^l \tilde{I}^l)_{\min} > 0$, we employ again the strong maximum principle principle for elliptic equations to obtain that $\tilde{E}^l > 0$ on $\overline{\Omega}$, which completes the proof of the lemma. □

We complement the previous lemma with a result on the asymptotic profiles of positive solutions of (5.1) for large values of l.

Lemma 5.3. *Let $(\tilde{E}^*, \tilde{I}^*, \tilde{R}^*)$ be the unique positive classical solution of (2.29) satisfying (2.30) as given by Proposition 3.1. For every $l > \mu^*$, let $(\tilde{E}^l, \tilde{I}^l, \tilde{R}^l)$ be a positive solution of (5.1) and set $S^l = l(1 - d_E\tilde{E}^l - d_I\tilde{I}^l - d_R\tilde{R}^l)$. Then, $(\tilde{E}^l, \tilde{I}^l, \tilde{R}^l) \to (\tilde{E}^*, \tilde{I}^*, \tilde{R}^*)$ as $l \to \infty$ in $C^1(\overline{\Omega})$ and $S^l \to \frac{\alpha}{\beta}\frac{\tilde{R}^*}{\tilde{I}^*}$ as $l \to \infty$ in $C(\overline{\Omega})$.*

Proof. Let $(\tilde{E}^l, \tilde{I}^l, \tilde{R}^l)$ be a sequence of positive solution of (5.1) for $l > \mu^*$. Observe that

$$\|\tilde{E}^l\|_\infty \le \frac{1}{d_E}, \quad \|\tilde{I}^l\|_\infty \le \frac{1}{d_I} \quad \text{and} \quad \|\tilde{R}^l\|_\infty \le \frac{1}{d_R} \quad \forall\, l > \mu^*, \tag{5.13}$$

since $1 > d_E\tilde{E}^l + d_I\tilde{I}^l + d_R\tilde{R}^l$. Therefore,

$$\|\Delta\tilde{I}^l\|_\infty \le \frac{1}{d_I}\|\gamma\tilde{I}^l - \sigma\tilde{E}^l\|_\infty \le \frac{1}{d_I}\left(\frac{\|\gamma\|_\infty}{d_I} + \frac{\|\sigma\|_\infty}{d_E}\right) \quad \forall\, l > \mu^*$$

and

$$\|\Delta\tilde{R}^l\|_\infty \le \frac{1}{d_R}\|\gamma\tilde{I}^l - \alpha\tilde{R}^l\|_\infty \le \frac{1}{d_R}\left(\frac{\|\gamma\|_\infty}{d_I} + \frac{\|\alpha\|_\infty}{d_R}\right) \quad \forall\, l > \mu^*.$$

Hence, after passing to a subsequence, we employ the Sobolev compact embedding of $W^{2,p}(\Omega)$ in $C^1(\Omega)$ for $p \gg n$, to deduce that there exist nonnegative functions $I^*, R^* \in C^1(\overline{\Omega})$ such that

$$\lim_{l\to\infty} \|\tilde{I}^l - I^*\|_{C^1(\overline{\Omega})} = 0 \quad \text{and} \quad \lim_{l\to\infty} \|\tilde{R}^l - R^*\|_{C^1(\overline{\Omega})} = 0. \tag{5.14}$$

Furthermore, thanks to the third equation of (5.1) and the regularity theory for elliptic equations, we have that $R^* \in C^2(\overline{\Omega})$ and solves (in the classical sense)

$$\begin{cases} 0 = d_R \Delta R^* - \alpha R^* + \gamma I^* & x \in \Omega, \\ 0 = \partial_n R^* & x \in \partial\Omega. \end{cases} \tag{5.15}$$

Moreover, since $\sup_{l>\mu^*} \|\tilde{E}^l\|_\infty < \infty$ by (5.13), it follows from the Banach-Alaoglu Theorem that, after passing to a furtehr subsequence, there is $E^* \in L^\infty(\Omega)$ such that $\tilde{E}^l \to E^*$ as $l \to \infty$ in the weak-star topology of $L^\infty(\Omega)$. Therefore, it follows from the second equation of (5.1) and the L^p- regularity theory for elliptic equations that $I^* \in W^{2,p}(\Omega)$ for every $p > 1$ and is a weak solution of

$$\begin{cases} 0 = d_I \Delta I^* - \gamma I^* + \sigma E^* & x \in \Omega \\ 0 = \partial_n I^* & x \in \partial\Omega. \end{cases} \tag{5.16}$$

Note that $E^* \ge 0$ almost everywhere since $\tilde{E}^l > 0$ for all $l > \mu^*$.

Next, observe from (5.1) that

$$\begin{cases} 0 \ge d_E \Delta \tilde{E}^l + l\beta_{\min}\big(1 - \|d_I\tilde{I}^l + d_R\tilde{R}^l\|_\infty\big)\tilde{I}^l - \big(\sigma_{\max} + l\beta_{\max} d_E\|\tilde{I}^l\|_\infty\big)\tilde{E}^l & x \in \Omega, \\ 0 \ge d_I \Delta \tilde{I}^l + \sigma_{\min}\tilde{E}^l - \gamma_{\max}\tilde{I}^l & x \in \Omega, \\ 0 = \partial_n \tilde{E}^l = \partial_n \tilde{I}^l & x \in \partial\Omega. \end{cases}$$

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Hence, it follows as in (5.4) that

$$0 \geq \frac{1}{2}\Bigg[\sqrt{4\big(l\beta_{\min}\big(1-\|d_I\tilde{I}^l + d_R\tilde{R}^l\|_\infty\big)\sigma_{\min} - \big(\sigma_{\max} + l\beta_{\max}d_E\|\tilde{I}^l\|_\infty\big)\gamma_{\max}\big) + \big(\big(\sigma_{\max}+l\beta_{\max}d_E\|\tilde{I}^l\|_\infty\big) + \gamma_{\max}\big)^2}$$
$$- \big(\big(\sigma_{\max} + l\beta_{\max}d_E\|\tilde{I}^l\|_\infty\big) + \gamma_{\max}\big)\Bigg].$$

Equivalently,

$$0 \geq l\beta_{\min}\big(1 - \|d_I\tilde{I}^l + d_R\tilde{R}^l\|_\infty\big)\sigma_{\min} - \big(\sigma_{\max} + l\beta_{\max}d_E\|\tilde{I}^l\|_\infty\big)\gamma_{\max},$$

which gives

$$\frac{\beta_{\max}}{\sigma_{\min}\beta_{\min}}d_E\|\tilde{I}^l\|_\infty\gamma_{\max} + \frac{\sigma_{\max}\gamma_{\max}}{l\sigma_{\min}\beta_{\min}} + \|d_I\tilde{I} + d_R\tilde{R}^l\|_\infty \geq 1.$$

Letting $l \to \infty$ in the last inequality yields

$$1 \leq \frac{d_E\beta_{\max}\gamma_{\max}}{\sigma_{\min}\beta_{\min}}\|I^*\|_\infty + \|d_II^* + d_RR^*\|_\infty \leq \left(\frac{d_E\beta_{\max}\gamma_{\max}}{\sigma_{\min}\beta_{\min}} + d_E\right)\|I^*\|_\infty + d_R\|R^*\|_\infty. \tag{5.17}$$

However, observe from (5.15) and the maximum principle for elliptic equations that

$$\|R^*\|_\infty \leq \left\|\frac{\gamma}{\alpha}\right\|_\infty \|I^*\|_\infty.$$

Hence, it follows from (5.17) that $\|I^*\|_\infty > 0$. It then follows from the strong maximum principle principle and (5.15) and (5.16) that

$$I^* > 0 \quad \text{and} \quad R^* > 0 \quad \text{on } \overline{\Omega}. \tag{5.18}$$

Now, observing that

$$\begin{cases} 0 = \frac{1}{l}\Delta S^l + \alpha\tilde{R}^l - \beta\tilde{I}^lS^l & x \in \Omega, \\ 0 = \partial_n S^l & x \in \partial\Omega, \end{cases}$$

we infer to the singular perturbation arguments to conclude that

$$S^l \to \frac{\alpha}{\beta}\frac{R^*}{I^*} \quad \text{as } l \to \infty \quad \text{uniformly on } \overline{\Omega}. \tag{5.19}$$

Therefore, rewriting (5.1) as

$$\begin{cases} 0 = d_E\Delta\tilde{E} + \beta S^l\tilde{I} - \sigma\tilde{E} & x \in \Omega, \\ 0 = d_I\Delta\tilde{I} + \sigma\tilde{E} - \gamma\tilde{I} & x \in \Omega, \\ 0 = d_R\Delta\tilde{R} + \gamma\tilde{I} - \alpha\tilde{R} & x \in \Omega, \\ 0 = \partial_n\tilde{E} = \partial_n\tilde{I} = \partial_n\tilde{R} & x \in \partial\Omega. \end{cases} \tag{5.20}$$

Then, by regularity theory for elliptic equations, we can invoke (5.1), (5.15), (5.16), and (5.19) to conclude that (E^*, I^*, R^*) is a positive classical solution of (2.29) and $(\tilde{E}^l, \tilde{I}^l, \tilde{R}^l) \to (E^*, I^*, R^*)$ as $l \to \infty$ in $C^1(\overline{\Omega})$. Finally, recalling that

$$1 = \frac{1}{l}S^l + d_E\tilde{E}^l + d_I\tilde{I}^l + d_R\tilde{R},$$

then as $l \to \infty$, since (5.19) holds, we conclude that (E^*, I^*, R^*) also satisfies (2.30). Hence, by Proposition 3.1, $(E^*, I^*, R^*) = (\tilde{E}^*, \tilde{I}^*, \tilde{R}^*)$. □

Lemma 5.4. *Fix $l^* > \mu^*$. Then, there is a positive number $\eta_* > 0$ such that*

$$\min_{x\in\overline{\Omega}} \min\{\tilde{E}^l(x), \tilde{I}^l(x), \tilde{R}^l(x)\} \geq \eta_*, \tag{5.21}$$

whenever $l \geq l^*$ and $(\tilde{E}^l, \tilde{I}^l, \tilde{R}^l)$ is a positive classical solution of (5.1).

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Proof. Thanks to Lemma 5.3, it is enough to show that for every $k > 0$, there is $\eta_k > 0$ such that

$$\min_{x\in\overline{\Omega}} \min\{\tilde{E}^l(x), \tilde{I}^l(x), \tilde{R}^l(x)\} \geq \eta_k, \tag{5.22}$$

whenever $l \in [l^*, l^* + k]$ and $(\tilde{E}^l, \tilde{I}^l, \tilde{R}^l)$ is a positive classical solution of (5.1). We proceed by contradiction to establish that (5.22) holds. So, suppose that there is some $k > 0$ such that there exist a sequence $\{l_m\}_{m\geq 1} \subset [l^*, l^* + k]$ and positive classical solutions $(\tilde{E}^{l_m}, \tilde{I}^{l_m}, \tilde{R}^{l_m})$ of (5.1) for each $l = l_m$ satisfying

$$\lim_{m\to\infty} \min_{x\in\overline{\Omega}} \min\{\tilde{E}^{l_m}(x), \tilde{I}^{l_m}(x), \tilde{R}^{l_m}(x)\} = 0. \tag{5.23}$$

Since $(\tilde{E}^{l_m}, \tilde{I}^{l_m}, \tilde{R}^{l_m})$ is uniformly bounded, then after passing to a subsequence, by the regularity theory for elliptic equations, we may suppose that there exist $l^\infty \in [l^*, l^* + k]$ and a nonnegative classical solution $(\tilde{E}^\infty, \tilde{I}^\infty, \tilde{R}^\infty)$ of (5.1) with $l = l^\infty$ such that $(\tilde{E}^{l_m}, \tilde{I}^{l_m}, \tilde{R}^{l_m}) \to (\tilde{E}^\infty, \tilde{I}^\infty, \tilde{R}^\infty)$ as $m \to \infty$ in $C^1(\overline{\Omega})$. Note from (5.23) that $\min_{x\in\overline{\Omega}} \min\{\tilde{E}^\infty(x), \tilde{I}^\infty(x), \tilde{R}^\infty(x)\} = 0$. Hence, by Lemma 5.2, we must have that $\tilde{E}^\infty = \tilde{I}^\infty = \tilde{R}^\infty \equiv 0$. This implies that

$$\lim_{m\to\infty} \left(\|\tilde{E}^{l_m}\|_\infty + \|\tilde{I}^{l_m}\|_\infty + \|\tilde{R}^{l_m}\|_\infty\right) = 0. \tag{5.24}$$

Now, set $\hat{E}^{l_m} := \frac{\tilde{E}^{l_m}}{\|\tilde{E}^{l_m}\|_\infty}$, $\hat{I}^{l_m} := \frac{\tilde{I}^{l_m}}{\|\tilde{E}^{l_m}\|_\infty}$, and $\hat{R}^{l_m} := \frac{\tilde{R}^{l_m}}{\|\tilde{E}^{l_m}\|_\infty}$ for all $m \geq 1$. Observe that $(\hat{E}^{l_m}, \hat{I}^{l_m}, \hat{R}^{l_m})$ is a classical solution of

$$\begin{cases} 0 = d_E\Delta\hat{E}^{l_m} + l_m\beta\left(1 - d_E\tilde{E}^{l_m} - d_I\tilde{I}l_m - d_R\tilde{R}^{l_m}\right)\hat{I}^{l_m} - \sigma\hat{E}^{l_m} & x \in \Omega, \\ 0 = d_I\Delta\hat{I}^{l_m} + \sigma\hat{E}^{l_m} - \gamma\hat{I}^{l_m} & x \in \Omega, \\ 0 = d_R\Delta\hat{R}^{l_m} + \gamma\hat{I}^{l_m} - \alpha\hat{R}^{l_m} & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}\tilde{E}^{l_m} = \partial_{\boldsymbol{n}}\tilde{I}^{l_m} = \partial_{\boldsymbol{n}}\tilde{R}^{l_m} & x \in \partial\Omega, \end{cases}$$

and satisfies

$$\|\hat{E}^{l_m}\|_\infty = 1, \quad \|\hat{I}^{l_m}\|_\infty \leq \left\|\frac{\sigma}{\gamma}\hat{E}^{l_m}\right\|_\infty \leq \left\|\frac{\sigma}{\gamma}\right\|_\infty, \quad \text{and} \quad \|\hat{R}^{l_m}\|_\infty \leq \left\|\frac{\gamma}{\alpha}\hat{I}^{l_m}\right\|_\infty \leq \left\|\frac{\gamma}{\alpha}\right\|_\infty \quad \forall\, m \geq 1.$$

Hence, since (5.24) holds, we can employ the regularity theory for elliptic equations, to conclude that, if possible after passing to a further subsequence, there exist a nonnegative function $(\hat{E}^\infty, \hat{I}^\infty, \hat{R}^\infty)$ satisfying $\|\hat{E}^\infty\|_\infty = 1$ and

$$\begin{cases} 0 = d_E\Delta\hat{E}^\infty + l^\infty\beta\hat{I}^\infty - \sigma\hat{E}^\infty & x \in \Omega, \\ 0 = d_I\Delta\hat{I}^\infty + \sigma\hat{E}^\infty - \gamma\hat{I}^\infty & x \in \Omega, \\ 0 = d_R\Delta\hat{R}^\infty + \gamma\hat{I}^\infty - \alpha\hat{R}^\infty & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}\hat{E}^\infty = \partial_{\boldsymbol{n}}\hat{I}^\infty = \partial_{\boldsymbol{n}}\hat{R}^\infty & x \in \partial\Omega, \end{cases}$$

such that $(\hat{E}^{l_m}, \hat{I}^{l_m}, \hat{R}^{l_m}) \to (\hat{E}^\infty, \hat{I}^\infty, \hat{R}^\infty)$ as $m \to \infty$ in $C^1(\overline{\Omega})$. Since $\hat{E}^\infty \geq, \not\equiv 0$, it then follows from the similar arguments as in the proof of Lemma 5.2 that $\hat{E}^\infty > 0$, $\hat{I}^\infty > 0$, and $\hat{R}^\infty > 0$ on $\overline{\Omega}$. This shows that $\lambda^{l^\infty} = 0$, which is not possible in view of Lemma 5.1 since $l^\infty > \mu^*$. Therefore, (5.22) must hold, which complete the proof of the lemma. □

The next lemma shows that (5.1) has no positive solution for $0 \leq l \leq \mu^*$.

Lemma 5.5. *System (5.1) has no positive solution for every* $0 \leq l \leq \mu^*$.

Proof. If $l = 0$, it follows from the first equation of (5.1) that $\tilde{E} \equiv 0$ for any solution $(\tilde{E}, \tilde{I}, \tilde{R})$ of (5.1), which gives the desired result. Next, fix $0 < l \leq \mu^*$. Note if $(\tilde{E}, \tilde{I}, \tilde{R})$ is a positive solution of (5.1), then by the monotonicity of the principal eigenvalue with repect to parameters, we must have that the principal eigenvalue λ^l of (5.3) is positive, which contradicts with Lemma 5.1. Hence, the result follows. □

Lemma 5.6. *Assume that* $\{l^n\}_{n\geq 1}$ *is a sequence of elements of* (μ^*, ∞) *converging to* μ^* *such that system (5.1) has a positive solution* $(\tilde{E}^n, \tilde{I}^n, \tilde{R}^n)$ *when* $l = l^n$ *for each* $n \geq 1$. *Then,* $\|\tilde{E}^n\|_\infty + \|\tilde{I}^n\|_\infty + \|\tilde{R}^n\|_\infty \to 0$ *as* $n \to \infty$.

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Proof. By Lemma 5.2, we have that $\|\tilde{E}^n\|_\infty < \frac{1}{d_E}$, $\|\tilde{I}^n\|_\infty \le \frac{1}{d_I}$, and $\|\tilde{R}^n\|_\infty \le \frac{1}{d_R}$ for all $n \ge 1$. Hence, possibly after passing to a subsequence, we can apply the regularity theory for elliptic equations to conclude that there is $(\tilde{E}^*, \tilde{I}^*, \tilde{R}^*) \in [C^2(\Omega)]^3 \cap [C^+(\bar{\Omega})]^3$ such that $\|\tilde{E}^n - \tilde{E}^*\|_\infty + \|\tilde{I}^n - \tilde{I}^*\|_\infty + \|\tilde{R}^n - \tilde{R}^*\|_\infty \to 0$ as $n \to \infty$. Moreover, $(\tilde{E}^*, \tilde{I}^*, \tilde{R}^*)$ is a classical solution of (5.1) for $l = \mu^*$. Hence, by Lemma 5.5, we must have that $(\tilde{E}^*, \tilde{I}^*, \tilde{R}^*) = (0, 0, 0)$. □

Thanks to Lemmas 5.1–5.6, we can now employ the bifurcation theory to assess the existence of positive solutions of (5.1).

Theorem 5.7. *The system of elliptic equations (5.1) has positive solution if and only if $l > \mu^*$. Moreover, taking $l > 0$ as the moving parameter, there is a connected component Γ^+ of nonnegative solutions of (5.1) such that $\mathbf{u} \in [C^{++}(\bar{\Omega})]^3$ whenever $l > \mu^*$ and $(l, \mathbf{u}) \in \Gamma^+$,*

$$\{l \in \mathbb{R}^+ : (l, \mathbf{u}) \in \Gamma^+ \text{ for some } \mathbf{u} \in [C^+(\bar{\Omega})]^3\} = [\mu^*, \infty), \tag{5.25}$$

and

$$\lim_{l \to \mu^*, (l,\mathbf{u}) \in \Gamma^+} \left[\|u_1\|_\infty + \|u_2\|_\infty + \|u_3\|_\infty\right] = 0. \tag{5.26}$$

The proof of Theorem 5.7 uses the bifurcation theory and is given in the appendix section.

The next result shows the relationship between solutions of system (5.1) and EE solutions of system (1.3).

Lemma 5.8.

(i) *Let (S, E, I, R) be an EE of system (1.3) and set*

$$\kappa := d_S S + d_E E + d_I I + d_R R \tag{5.27}$$

and

$$\tilde{S} := \frac{S}{\kappa}, \ \tilde{E} := \frac{E}{\kappa}, \ \tilde{I} := \frac{I}{\kappa}, \quad \text{and} \quad \tilde{R} := \frac{R}{\kappa}. \tag{5.28}$$

Then, κ is a positive constant,

$$1 = d_S \tilde{S} + d_E \tilde{E} + d_I \tilde{I} + d_R \tilde{R}, \tag{5.29}$$

and $(\tilde{E}, \tilde{I}, \tilde{R})$ solves the system (5.1) with $l = \frac{\kappa}{d_S}$.

(ii) *If $l > 0$ and $(\tilde{E}^l, \tilde{I}^l, \tilde{R}^l)$ solves the system (5.1) such that*

$$N = l \int_\Omega \left[\left(1 - d_E \tilde{E}^l - d_I \tilde{I}^l - d_R \tilde{R}^l\right) + d_S(\tilde{E}^l + \tilde{I}^l + \tilde{R}^l)\right], \tag{5.30}$$

then (S^l, E^l, I^l, R^l) is an EE of system (1.3) where

$$S^l = l\left(1 - d_E \tilde{E}^l - d_I \tilde{I}^l - d_R \tilde{R}^l\right), \quad E^l = d_S l \tilde{E}^l, \quad I^l = d_S l \tilde{I}^l, \quad \text{and} \quad R^l = d_S l \tilde{R}^l. \tag{5.31}$$

Proof. (i) It is apparent that κ satisfies

$$\begin{cases} 0 = \Delta \kappa, & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}} \kappa, & x \in \partial\Omega. \end{cases}$$

Therefore, κ is a constant function. By dividing (5.27) by κ, we obtain (5.29). Lastly, using $S = \kappa \tilde{S} = \frac{\kappa}{d_S}(1 - d_E \tilde{E} - d_I \tilde{I} - d_R \tilde{R})$, a direct computation shows that $(\tilde{E}, \tilde{I}, \tilde{R})$ solves (5.1) with $l = \frac{\kappa}{d_S}$.

(ii) By Lemma 5.2, we have that $S^l = l(1 - d_E \tilde{E} - d_I \tilde{I} - d_R \tilde{R}) > 0$. It now follows by inspection that (S^l, E^l, I^l, R^l) is an EE solution of (1.3). □

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We complete this section with the following lemma.

Lemma 5.9. *Let $l > 0$ and $(\tilde{E}^l, \tilde{I}^l, \tilde{R}^l)$ be a positive classical solution of system (5.1). If (5.30) holds for some $N > 0$, then*

$$l \leq \frac{2N}{|\Omega|} \max\left\{1, \frac{d_E}{d_S}, \frac{d_R}{d_S}, \frac{d_I}{d_S}\right\}.$$

Proof. We distinguish two cases.

Case 1. Here we assume that $d_E \int_\Omega \tilde{E}^l + d_I \int_\Omega \tilde{I}^l + d_R \int_\Omega \tilde{R}^l \leq \frac{|\Omega|}{2}$. In this case, we have

$$N = l\left(|\Omega| - d_E \int_\Omega \tilde{E}^l - d_I \int_\Omega \tilde{I}^l - d_R \int_\Omega \tilde{R}^l\right) + d_S l \int_\Omega (\tilde{E}^l + \tilde{I}^l + \tilde{R}^l) \geq l\frac{|\Omega|}{2},$$

from which we get $l \leq \frac{2N}{|\Omega|}$.

Case 2. Here we assume that $d_E \int_\Omega \tilde{E}^l + d_I \int_\Omega \tilde{I}^l + d_R \int_\Omega \tilde{R}^l \geq \frac{|\Omega|}{2}$. Then

$$d_S \int_\Omega (\tilde{E}^l + \tilde{I}^l + \tilde{R}^l) \geq \min\left\{\frac{d_S}{d_E}, \frac{d_S}{d_I}, \frac{d_S}{d_R}\right\}\left(d_E \int_\Omega \tilde{E}^l + d_I \int_\Omega \tilde{I}^l + d_R \int_\Omega \tilde{R}^l\right) \geq \frac{|\Omega|}{2} \min\left\{\frac{d_S}{d_E}, \frac{d_S}{d_I}, \frac{d_S}{d_R}\right\}.$$

As a result, we get

$$N = l\left(|\Omega| - d_E \int_\Omega \tilde{E}^l - d_I \int_\Omega \tilde{I}^l - d_R \int_\Omega \tilde{R}^l\right) + d_S l \int_\Omega (\tilde{E}^l + \tilde{I}^l + \tilde{R}^l) \geq l\frac{|\Omega|}{2} \min\left\{\frac{d_S}{d_E}, \frac{d_S}{d_I}, \frac{d_S}{d_R}\right\},$$

from which we get

$$l \leq \frac{2N}{|\Omega| \min\left\{\frac{d_S}{d_E}, \frac{d_S}{d_I}, \frac{d_S}{d_R}\right\}} = \frac{2N}{|\Omega|} \max\left\{\frac{d_E}{d_S}, \frac{d_I}{d_S}, \frac{d_R}{d_S}\right\}.$$

Combining both cases, we obtain $l \leq \frac{2N}{|\Omega|} \max\{1, \frac{d_E}{d_S}, \frac{d_R}{d_S}, \frac{d_I}{d_S}\}$. This completes the proof of the lemma. □

6. Proofs of Theorems 2.3–2.5, 2.7

Proof of Theorem 2.3. The eigenvalue problem associated with the linearization of (1.3) at the DFE, $(\frac{N}{|\Omega|}, 0, 0, 0)$, is given by

$$\begin{cases} \lambda\psi_1 = d_S\Delta\psi_1 + \alpha\psi_4 - \frac{N}{|\Omega|}\beta\psi_3 & x \in \Omega, \\ \lambda\psi_2 = d_E\Delta\psi_2 + \frac{N}{|\Omega|}\beta\psi_3 - \sigma\psi_2 & x \in \Omega, \\ \lambda\psi_3 = d_I\Delta\psi_3 + \sigma\psi_2 - \gamma\psi_3 & x \in \Omega, \\ \lambda\psi_4 = d_R\Delta\psi_4 + \gamma\psi_3 - \alpha\psi_4 & x \in \Omega, \\ 0 = \partial_n\psi_1 = \partial_n\psi_2 = \partial_n\psi_3 = \partial_n\psi_4 & x \in \partial\Omega, \\ 0 = \int_\Omega (\psi_1 + \psi_2 + \psi_3 + \psi_4). \end{cases} \tag{6.1}$$

We first take $\mathcal{R}_0 < 1$. Suppose that $(\psi_1, \psi_2, \psi_3, \psi_4)$ solves (6.1) with at least one of $\psi_1, \psi_2, \psi_3, \psi_4$ not identically 0. We show that $\Re e(\lambda) < 0$. We proceed in two cases.

Case 1. Here, we suppose that $\psi_2 = \psi_3 \equiv 0$. Then, ψ_1 and ψ_4 are not both 0 and (6.1) reduces to

$$\begin{cases} \lambda\psi_1 = d_S\Delta\psi_1 + \alpha\psi_4 & x \in \Omega, \\ \lambda\psi_4 = d_R\Delta\psi_4 - \alpha\psi_4 & x \in \Omega, \\ 0 = \partial_n\psi_1 = \partial_n\psi_4 & x \in \partial\Omega, \\ 0 = \int_\Omega (\psi_1 + \psi_4). \end{cases} \tag{6.2}$$

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Next, we distinguish two subcases: either $\psi_4 \equiv 0$ or $\psi_4 \not\equiv 0$. If $\psi_4 \equiv 0$, then (6.2) reduces to

$$\begin{cases} \lambda\psi_1 = d_S\Delta\psi_1 & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}\psi_1 & x \in \partial\Omega, \\ 0 = \int_\Omega \psi_1 \quad \text{and} \quad \int_\Omega |\psi_1| > 0. \end{cases} \tag{6.3}$$

Hence, λ is an eigenvalue of (3.6) with $d = d_S$ and $h \equiv 0$. Thus λ is real. Multiplying (6.3) by ψ_1 and integrating by parts yields

$$\lambda = -\frac{d_S \int_\Omega |\nabla\psi_1|^2}{\int_\Omega \psi_1^2}.$$

Moreover, since $\int_\Omega \psi_1 = 0$ while $\int_\Omega \psi_1^2 > 0$, then ψ_1 is not constant almost everywhere, and hence $\lambda < 0$. Similarly, if $\psi_4 \not\equiv 0$, then by the second equation of (6.2), λ is an eigenvalue of (3.6) with $d = d_R$ and $h = -\alpha$. Thus λ is real and by Lemma 3.2, we have that $\lambda = \lambda(d_R, -\alpha) \le -\alpha_{\min} < 0$. Hence, in this case, we have $\Re e(\lambda) = \lambda < 0$.

Case 2. In this case, we suppose that $\|\psi_2 + \psi_3\|_\infty > 0$. Hence, the second and third equations of (6.1) decouple to give

$$\begin{cases} \lambda\psi_2 = d_E\Delta\psi_2 + \frac{N}{|\Omega|}\beta\psi_3 - \sigma\psi_2 & x \in \Omega, \\ \lambda\psi_3 = d_I\Delta\psi_3 + \sigma\psi_2 - \gamma\psi_3 & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}}\psi_2 = \partial_{\boldsymbol{n}}\psi_3 & x \in \partial\Omega, \\ 0 = \int_\Omega (\psi_2 + \psi_3), \end{cases}$$

which yield that λ is an eigenvalue of (5.3) with $l = l_0 = \frac{N}{|\Omega|}$. Therefore, we deduce from the Krein-Rutman theorem that

$$\Re e(\lambda) \le \lambda^{l_0}.$$

However, since $\mathcal{R}_0 = \frac{N}{|\Omega|\mu^*} < 1$, then $\mu^* > \frac{N}{|\Omega|} = l_0$, which implies from Lemma 5.1 that $\lambda^{l_0} < \lambda^{\mu^*} = 0$. Therefore, in this case as well, we get $\Re e(\lambda) < 0$, which yields the desired result. This completes the proof of statement (i).

Next, we suppose that $\mathcal{R}_0 > 1$. We show that the DFE is linearly unstable. First, note that $l_0 := \frac{N}{|\Omega|} > \mu^*$ since $\mathcal{R}_0 > 1$. Hence, by Lemma 5.1, $\lambda^{l_0} > 0$, where λ^{l_0} is the principal eigenvalue of (5.3) with $l = l_0$. Let (ψ_2^0, ψ_3^0) be a positive eigenvector associated with λ^{l_0}. Since $\lambda^{l_0} > 0$, then the densely defined linear operator $\mathrm{Dom}_\infty(\Delta) \ni u \mapsto (\lambda^{l_0} + \alpha)u - d_R\Delta u \in C(\bar{\Omega})$, where $\mathrm{Dom}_\infty(\Delta) := \{u \in \cap_{p>1} W^{2,p}(\Omega) : \partial_{\boldsymbol{n}} u = 0 \text{ on } \partial\Omega\}$, is invertible. Therefore, there is a unique $\psi_4^0 \in \mathrm{Dom}_\infty(\Delta)$ solving the fourth equation of (6.1) with $\lambda = \lambda^{l_0}$ and $\psi_3 = \psi_3^0$. By the regularity theory for elliptic equations, we have that $\psi_4^0 \in C^2(\bar{\Omega})$. Again, since $\lambda^{l_0} > 0$, then the densely defined linear operator $\mathrm{Dom}_\infty(\Delta) \ni u \mapsto \lambda^{l_0}u - d_S\Delta u \in C(\bar{\Omega})$ is invertible. Hence, there is a unique $\psi_1^0 \in \mathrm{Dom}_\infty(\Delta)$ solving the first equation of (6.1) for $\lambda = \lambda^{l_0}$ and $(\psi_4, \psi_3) = (\psi_4^0, \psi_3^0)$. Therefore, $(\psi_1^0, \psi_2^0, \psi_3^0, \psi_4^0) \neq (0, 0, 0, 0)$ solves the first five equations of (6.1) for $\lambda = \lambda^{l_0} > 0$. Now, adding up all the first four equations of (6.1) and integrate by parts on Ω yields $\lambda^{l_0} \int_\Omega (\psi_1^0 + \psi_2^0 + \psi_3^0 + \psi_4^0) = \int_\Omega (d_S\Delta\psi_1^0 + d_E\Delta\psi_2^0 + d_I\Delta\psi_3^0 + d_R\Delta\psi_4^0) = 0$, from which we derive that $\int_\Omega (\psi_1^0 + \psi_2^0 + \psi_3^0 + \psi_4^0) = 0$. This shows that $(\frac{N}{|\Omega|}, 0, 0, 0)$ is linearly unstable.

Finally, assume that $\mathcal{R}_0 > 1$ and that either of the hypotheses of Theorem 2.2 holds, so that the system (1.3) is eventually bounded. Hence, $(\frac{N}{|\Omega|}, 0, 0, 0)$ is linearly unstable. As a result, the proof of the persistence of the disease and the existence of at least one EE solution when $\mathcal{R}_0 > 1$ can be also established by appealing to the persistence theory in population dynamics as in the proofs of [37, Theorem 1.1-(ii)] or [9, Theorem 1.1-(b)]. Since the arguments here are standard, then it will be omitted, which completes the proof of statement (ii). □

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Proof of Theorem 2.4. Let $M > 0$ be as in (2.6) and μ^* be the principal eigenvalue of the weighted eigenvalue problem (2.10). Let $N_0 > 0$ be the unique positive number satisfying

$$2MN_0\big(1+N_0^2\big) := \mu^*. \tag{6.4}$$

Now, we fix $0 < N < N_0$ and let $(S, E, I, R)(t, x)$ be a classical solution of (1.3). Thanks to (2.6), there is $T_1 > 0$ such that

$$\|S(t,\cdot)+E(t,\cdot)+I(t,\cdot)+R(t,\cdot)\|_\infty \le 2MN(1+N^2) \quad \forall\, t \ge T_1. \tag{6.5}$$

Hence, from the second and third equations of (1.3), we have that

$$\begin{cases} \partial_t E \le d_E \Delta E - \sigma(x)E + 2MN(1+N^2)\beta(x)I & x \in \Omega,\ t > T_1, \\ \partial_t I = d_I \Delta I + \sigma(x)E - \gamma(x)I & x \in \Omega,\ t > T_1, \\ 0 = \partial_n E = \partial_n I & x \in \partial\Omega,\ t > T_1. \end{cases} \tag{6.6}$$

Since $0 < N < N_0$, it follows from (6.4) that

$$2MN(1+N^2) < \mu^*. \tag{6.7}$$

Hence, since μ^* is the principal eigenvalue of the weighted eigenvalue problem (2.10), then by the monotonicity of the principal eigenvalue with respect to parameters, we have that the principal eigenvalue, say λ_N, of the system of cooperative equations

$$\begin{cases} \lambda\varphi = d_E\varphi - \sigma(x)\varphi + 2MN(1+N^2)\beta(x)\psi & x \in \Omega, \\ \lambda\psi = d_I\Delta\psi + \sigma(x)\varphi - \gamma(x)\psi & x \in \Omega, \\ 0 = \partial_n\varphi = \partial_n\psi & x \in \partial\Omega, \end{cases} \tag{6.8}$$

is less than zero. Let (φ_N, ψ_N) be the principal eigenvector associated with λ_N satisfying

$$\min_{x\in\bar\Omega} \min\{\varphi_N(x), \psi_N(x)\} = 1. \tag{6.9}$$

Then, since (φ_N, ψ_N) solves (6.8), we have that

$$(\bar E(t,x), \bar I(t,x)) := 2MN(1+N^2)e^{\lambda_N(t-T_1)}(\varphi_N(x), \psi_N(x)) \quad t \ge T_1,\ x \in \bar\Omega,$$

is a super-solution of (6.6). Note from (6.4) and (6.9) that $(\bar E(T_1,\cdot), \bar I(T_1,\cdot)) \ge (E(T_1,\cdot), I(T_1,\cdot))$. Hence, by the comparison principle for cooperative systems, we have that

$$(E(t,\cdot), I(t,\cdot)) \le (\bar E(t,\cdot), \bar I(t,\cdot)) \quad \forall\, t > T_1,$$

where the inequality holds componentwise. Recalling that $\lambda_N < 0$, then

$$\|E(t,\cdot)\|_\infty + \|I(t,\cdot)\|_\infty \le 2MN(1+N^2)e^{\lambda_N(t-T_1)}(\|\varphi_N\|_\infty + \|\psi_N\|_\infty) \to 0 \quad \text{as} \quad t \to \infty. \tag{6.10}$$

This along with the fourth equation of (1.3) implies that

$$\|R(t,\cdot)\|_\infty \to 0 \quad \text{as } t \to \infty. \tag{6.11}$$

By (6.10) and (6.11),

$$\int_\Omega S = N - \int_\Omega (E(t)+I(t)+R(t)) \to N \quad \text{as} \quad t \to \infty. \tag{6.12}$$

Finally, thanks to (6.10)–(6.12), we can employ standard perturbation arguments to the first equation of (1.3) to deduce that $\|S(t,\cdot) - \frac{N}{|\Omega|}\|_\infty \to 0$ as $t \to \infty$.

Finally, assume that $d_S = d_E = d_I = d_R$. Then, by (2.4), for all $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that

$$\|S(t,\cdot)+E(t,\cdot)+I(t,\cdot)+R(t,\cdot)\|_\infty < \frac{(1+\varepsilon)}{|\Omega|}N. \tag{6.13}$$

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So, if $0 < N < N_0 = |\Omega|\mu^*$, and we fix $0 < \varepsilon \ll \frac{N_0 - N}{N}$, we have

$$\frac{(1+\varepsilon)}{|\Omega|} N < \mu^*. \tag{6.14}$$

Note that (6.13) is similar to (6.5), while (6.14) is similar to (6.7). Thus, we can proceed as in the above to obtain that $\|S(t,\cdot) - \frac{N}{|\Omega|}\|_\infty \to 0$ and $\|E(t,\cdot) + I(t,\cdot) + R(t,\cdot)\|_\infty \to 0$ as $t \to \infty$. □

Next, we give a proof of Theorem 2.5.

Proof of Theorem 2.5. (i) Suppose that $\mathcal{R}_0 > 1$, that is $N > \mu^*|\Omega|$. Let $\Gamma^+ \subset \mathbb{R}^+ \times [C^1(\overline{\Omega})]^3$ be the unbounded connected branch of nonnegative solutions of (5.1) satisfying (5.25) and (5.26) given by Theorem 5.7. Note by Theorem 5.7, $\mathbf{u} \in [C^{++}(\bar{\Omega})]^3$ whenever $l > \mu^*$ and $(l, \mathbf{u}) \in \Gamma^+$. Define the continuous map $\mathcal{N} : \Gamma^+ \to [0, \infty)$ by

$$\mathcal{N}(l, \mathbf{u}) = l \int_\Omega [(1 - d_E u_1 - d_I u_2 - d_R u_3) + d_S(u_1 + u_2 + u_3)] \quad \forall\, (l, \mathbf{u}) \in \Gamma^+. \tag{6.15}$$

Clearly, $\mathcal{N}$ is a continuous function and $\mathcal{N}(\mu^*, \mathbf{0}) = |\Omega|\mu^*$. Given any sequence $\{(l^n, \mathbf{u}^n)\}_{n\geq 1}$ of elements of Γ^+ with $l^n \to \infty$ as $n \to \infty$, it follows from Lemma 5.3 that $(u_1^n, u_2^n, u_3^n) \to (\tilde{E}^*, \tilde{I}^*, \tilde{R}^*)$ as $n \to \infty$ in $C^1(\overline{\Omega})$ and $l(1 - d_E u_1^n - d_I u_2^n - d_R u_3^n) \to \frac{\alpha}{\beta}\frac{\tilde{R}^*}{\tilde{I}^*}$ as $n \to \infty$ in $C(\overline{\Omega})$, where $(\tilde{E}^*, \tilde{I}^*, \tilde{R}^*)$ is the unique positive classical solution of (2.29) satisfying (2.30) as given by Proposition 3.1. Hence, $\mathcal{N}(l^n, \mathbf{u}^n) \to \infty$ as $n \to \infty$. This shows that $\mathcal{N}$ is not bounded above. Recalling that $\mathcal{N}(\Gamma^+)$ is also an interval as the image of a connected set under a continuous map, we get that $[\mu^*, \infty) \subset \mathcal{N}(\Gamma^+)$. Hence, given $N > \mu^*|\Omega|$, or equivalently $\mathcal{R}_0 > 1$, there is $(l, \mathbf{u}^l) \in \Gamma^+$ with $l > \mu^*$ such that $\mathcal{N}(l, \mathbf{u}^l) = N$. This along with Lemma 5.8 shows that system (1.3) has at least one EE solution whenever $\mathcal{R}_0 > 1$.

(ii) For the nonexistence, note that from Lemmas 5.5 and 5.8-(i) that if (S, E, I, R) is an EE solution, then

$$\frac{\kappa}{d_S} > \mu^* = \frac{N}{|\Omega|\mathcal{R}_0}.$$

On the other hand, by (5.27),

$$\begin{aligned}
\frac{\kappa}{d_S}|\Omega| &= \int_\Omega S + \frac{d_E}{d_S}\int_\Omega E + \frac{d_I}{d_S}\int_\Omega I + \frac{d_R}{d_S}\int_\Omega R \\
&= N - \int_\Omega (E + I + R) + \frac{d_E}{d_S}\int_\Omega E + \frac{d_I}{d_S}\int_\Omega I + \frac{d_R}{d_S}\int_\Omega R \\
&= N + \left(\frac{d_E}{d_S} - 1\right)\int_\Omega E + \left(\frac{d_I}{d_S} - 1\right)\int_\Omega I + \left(\frac{d_R}{d_S} - 1\right)\int_\Omega R \\
&\leq \left(1 + \left(\frac{d_E}{d_S} - 1\right)_+ + \left(\frac{d_I}{d_S} - 1\right)_+ + \left(\frac{d_R}{d_S} - 1\right)_+\right) N.
\end{aligned}$$

Therefore,

$$\frac{N}{|\Omega|\mathcal{R}_0} < \frac{N}{|\Omega|}\left(1 + \left(\frac{d_E}{d_S} - 1\right)_+ + \left(\frac{d_I}{d_S} - 1\right)_+ + \left(\frac{d_R}{d_S} - 1\right)_+\right),$$

which implies that $\mathcal{R}_0 > \frac{1}{1+\left(\frac{d_E}{d_S}-1\right)_+ + \left(\frac{d_I}{d_S}-1\right)_+ + \left(\frac{d_R}{d_S}-1\right)_+}$. Hence, system (1.3) has no EE solution if $\mathcal{R}_0 \leq \frac{1}{1+\left(\frac{d_E}{d_S}-1\right)_+ + \left(\frac{d_I}{d_S}-1\right)_+ + \left(\frac{d_R}{d_S}-1\right)_+}$. □

We end this subsection with a proof of Theorem 2.7.

Proof of Theorem 2.7. Fix $d_I > 0, d_E > 0$, and $d_R > 0$. Let $(\varphi_E, \varphi_I, \varphi_R)$ be the unique positive eigenvector of (2.14) associated with λ_{d_E,d_I,d_R} satisfying $\max_{x\in\bar{\Omega}}(\varphi_E(x) + \varphi_I(x) + \varphi_R(x)) = 1$. Assume (2.15) holds.

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Fix $\mathcal{R}_0 \in (\mathcal{R}_1^*, 1)$. Since $\mathcal{R}_0 = \frac{N}{|\Omega|\mu^*}$, we must have that

$$\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I} < N < \mu^*|\Omega|. \tag{6.16}$$

Let Γ^+ be as in Theorem 5.7. Then, by Lemma 5.3, it holds that

$$\lim_{l\to\infty,(l,\mathbf{u})\in\Gamma^+} \|l(1 - d_E u_1 - d_I u_2 - d_R u_2) - \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}\|_\infty = 0. \tag{6.17}$$

Hence, since (6.16) holds, there exist $l_0 \gg \mu^*$ and $0 < \varepsilon_0 \ll 1$ such that

$$M_0 := \sup_{l\geq l_0,(l,\mathbf{u})\in\Gamma^+} \int_\Omega l(1 - d_E u_1 - d_I u_2 - d_R u_2) < (1 - \varepsilon_0)N < N. \tag{6.18}$$

Define

$$d_S^* := \frac{((1 - \varepsilon_0)N - M_0)}{l_0\left(\frac{1}{d_E} + \frac{1}{d_I} + \frac{1}{d_R}\right)|\Omega|}. \tag{6.19}$$

Fix $0 < d_S < d_S^*$ and let $\mathcal{N}$ be the function introduced in (6.15). For every $\mathbf{u} \in [C^{++}(\bar{\Omega})]^3$ satisfying $(l_0, \mathbf{u}) \in \Gamma^+$, we have

$$\begin{aligned}\mathcal{N}(l_0, \mathbf{u}) =& \int_\Omega l_0(1 - d_E u_1 - d_I u_2 - d_R u_3) + l_0 d_S \int_\Omega (u_1 + u_2 + u_3)\\ \leq& M_0 + d_S l_0|\Omega|\left(\frac{1}{d_E} + \frac{1}{d_I} + \frac{1}{d_R}\right)\\ <& M_0 + d_S^* l_0|\Omega|\left(\frac{1}{d_E} + \frac{1}{d_I} + \frac{1}{d_R}\right) = (1 - \varepsilon_0)N.\end{aligned}$$

This shows that

$$N_0 := \sup_{(l_0,\mathbf{u})\in\Gamma^+} \mathcal{N}(l_0, \mathbf{u}) \leq (1 - \varepsilon_0)N < N. \tag{6.20}$$

Since the set $\Gamma_1^+ := \{(l, \mathbf{u}) \in \Gamma^+ : l \geq l_0\}$ is connected, then $\mathcal{N}(\Gamma_1^+)$ is an interval. Recalling that $\mathcal{N}(l^n, \mathbf{u}^n) \to \infty$ as $n \to \infty$ for any sequence $(l^n, \mathbf{u}^n) \in \Gamma^+$ with $l^n \to \infty$ as $n \to \infty$ (see Lemma 5.3), we conclude from (6.20) that

$$N \in (N_0, \infty) \subset \mathcal{N}(\Gamma_1^+).$$

Thus there is $l_1 > l_0$ and $\mathbf{u}^{l_1} > 0$ such that $(l_1, \mathbf{u}^{l_1}) \in \Gamma_1^+$ and $\mathcal{N}(l_1, \mathbf{u}^{l_1}) = N$. Therefore, by Lemma 5.8-(ii),

$$(S_1, E_1, I_1, R_1) := \left(l_1\left(1 - d_E u_1^{l_1} - d_I u_2^{l_1} - d_R u_3^{l_1}\right), d_S l_1 u_1^{l_1}, d_S l_1 u_2^{l_1}, d_S l_1 u_3^{l_1}\right)$$

is an EE solution of (1.3).

Note also that the set $\Gamma_2^+ := \{(l, \mathbf{u}) \in \Gamma^+ : \mu^* \leq l \leq l_0\}$ is a connected set, and hence $\mathcal{N}(\Gamma_2^+)$ is an interval. Since $\mathcal{N}(\mu^*, \mathbf{0}) = \mu^*|\Omega|$ and (6.20) holds, we have that $N \in (N_0, \mu^*|\Omega|) \subset \mathcal{N}(\Gamma_2^+)$. Thus, there is $l_2 \in (\mu^*, l_0)$ and $\mathbf{u}^{l_2} > 0$ such that $(l_2, \mathbf{u}^{l_2}) \in \Gamma_2^+$ satisfying $\mathcal{N}(l_2, \mathbf{u}^{l_2}) = N$. Therefore, by Lemma 5.8-(ii),

$$(S_2, E_2, I_2, R_2) := \left(l_2\left(1 - d_E u_1^{l_2} - d_I u_2^{l_2} - d_R u_3^{l_2}\right), d_S l_2 u_1^{l_2}, d_S l_2 u_2^{l_2}, d_S l_2 u_3^{l_2}\right)$$

is an EE solution of (1.3). Finally, we claim that

$$(S_1, E_1, I_1, R_1) \neq (S_2, E_2, I_2, R_2). \tag{6.21}$$

If this was false, then we must have $(d_S l_1 u_1^{l_1}, d_S l_1 u_2^{l_1}, d_S l_1 u_3^{l_1}) = (d_S l_2 u_1^{l_2}, d_S l_2 u_2^{l_2}, d_S l_2 u_3^{l_2})$ which in turn yields

$$S_1 - S_2 = l_1 - l_2 > 0,$$

which is a contradiction. Therefore, (6.21) holds, which completes the proof of the theorem. □

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7. Proof of Theorems 2.11, 2.13–2.15

Proof of Theorem 2.11. (i) Fix $d_I > 0$, $d_R > 0$, and $d_E > 0$. Suppose that $\mathcal{R}_0 \neq 1$ and assume that for some $d_0 > 0$, system (1.3) has at least one EE solution (S, E, I, R) for every $0 < d_S \leq d_0$. Set $\kappa = d_S S + d_E E + d_I I + d_R R$ as in (5.27), and let $(\tilde{E}, \tilde{I}, \tilde{R})$ be defined by (5.28). Then, by Lemma 5.8, $(\tilde{E}, \tilde{I}, \tilde{R})$ satisfies (5.29) and (5.1) with $l = \frac{\kappa}{d_S}$.

(i-1) Assume that $N < \int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}$. We claim that

$$\limsup_{d_S \to 0} \frac{\kappa}{d_S} < \infty. \tag{7.1}$$

Suppose to the contrary that (7.1) is false. Without loss of generality, after passing to a subsequence, we may suppose that

$$\lim_{d_S \to 0} \frac{\kappa}{d_S} = \infty. \tag{7.2}$$

Then, by Lemma 5.3, if possible after passing to a subsequence, $(\tilde{E}, \tilde{I}, \tilde{R}) \to (\tilde{E}^*, \tilde{I}^*, \tilde{R}^*)$ as $d_S \to 0$ uniformly in $C^1(\overline{\Omega})$ where $(\tilde{E}^*, \tilde{I}^*, \tilde{R}^*)$ is the unique positive classical solution of (2.29) satisfying (2.30) and

$$S \to \frac{\alpha}{\beta}\frac{\tilde{R}^*}{\tilde{I}^*} \quad \text{as} \quad d_S \to 0 \quad \text{uniformly on } \overline{\Omega}. \tag{7.3}$$

However, since

$$N = \int_\Omega S + \int_\Omega (E + I + R) > \int_\Omega S,$$

then letting $d_S \to 0$ in this inequality yields $N \geq \int_\Omega \frac{\alpha}{\beta}\frac{\tilde{R}^*}{\tilde{I}^*}$. But, by (3.2), there is a positive constant $c_* > 0$ such that $(\tilde{E}^*, \tilde{I}^*, \tilde{R}^*) = \frac{1}{c_*}(\varphi_E, \varphi_I, \varphi_R)$, which implies that $\int_\Omega \frac{\alpha}{\beta}\frac{\tilde{R}^*}{\tilde{I}^*} = \int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}$. So, we have established that $N \geq \int_\Omega \frac{\alpha}{\beta}\frac{\tilde{R}^*}{\tilde{I}^*} = \int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}$, which contradicts our initial assumption on N. Hence, (7.1) holds. This implies that there is some positive constant $C = C(d_E, d_R, d_I)$ such that (2.21) holds.

Next, by (7.1), possibly after passing to a subsequence, we may suppose that there is $l^* \in [\mu^*, \infty)$ such that $l = \frac{\kappa}{d_S} \to l^*$ as $d_S \to 0^+$. Note that we have used the fact that $l > \mu^*$ (Lemma 5.5) to conclude that $l^* \geq \mu^*$. We first show that $l^* > \mu^*$. If this was false, we get from Lemma 5.6 that $\|\tilde{E}\|_\infty + \|\tilde{I}\|_\infty + \|\tilde{R}\|_\infty \to 0$ as $d_S \to 0^+$. Hence

$$N = \lim_{d_S \to 0^+} \int_\Omega \left(l\left(1 - d_E\tilde{E} - d_I\tilde{I} - d_R\tilde{R}\right) + l d_S(\tilde{E} + \tilde{I} + \tilde{R})\right) = \mu^*|\Omega|,$$

which gives $\mathcal{R}_0 = 1$. This contradicts our initial assumption that $\mathcal{R}_0 \neq 1$. Therefore, $l^* > \mu^*$. Now, since $l^* > \mu^*$, we can employ the regularity theory for elliptic equations, coupled with Lemma 5.4, to conclude that (possibly after passing to a further subsequence) there is $(\tilde{E}^0, \tilde{I}^0, \tilde{R}^0) \in [C^{++}(\bar{\Omega})]^3 \cap [C^2(\bar{\Omega})]^3$ such that $\|\tilde{E} - \tilde{E}^0\|_{C^1(\bar{\Omega})} + \|\tilde{I} - \tilde{I}^0\|_{C^1(\bar{\Omega})} + \|\tilde{R} - \tilde{R}^0\|_{C^1(\bar{\Omega})} \to 0$ as $d_S \to 0$. Moreover, $(\tilde{E}^0, \tilde{I}^0, \tilde{R}^0)$ is a positive solution of (5.1) with $l = l^*$. Recalling that $S = l(1 - d_E\tilde{E} - d_I\tilde{I} - d_R\tilde{R})$ satisfies (5.12), then we can employ the regularity theory for elliptic equations to conclude that $\|S - S^0\|_{C^1(\bar{\Omega})} \to 0$ as $d_S \to 0$, where $S^0 \in C^+(\bar{\Omega})$ is the unique solution of the system of elliptic equations

$$\begin{cases} 0 = \frac{1}{l^*}\Delta S^0 - \beta\tilde{I}^0 S^0 + \alpha\tilde{R}^0 & x \in \Omega, \\ 0 = \partial_n S^0 & x \in \partial\Omega. \end{cases}$$

By the strong maximum principle principle for elliptic equations, we have that $S^0 > 0$ on $\bar{\Omega}$. Noting also that $S = l(1 - d_E\tilde{E} - d_I\tilde{I} - d_R\tilde{R}) \to l^*(1 - d_E\tilde{E}^0 - d_I\tilde{I}^0 - d_R\tilde{R}^0)$ as $d_S \to 0^+$ in $C^1(\bar{\Omega})$, we conclude that $l^*(1 - d_E\tilde{E}^0 - d_I\tilde{I}^0 - d_R\tilde{R}^0) = S^0 > 0$ on $\bar{\Omega}$. Finally, noting that

$$l^* = \lim_{d_S \to 0^+} \frac{N}{\int_\Omega (1 - d_E\tilde{E} - d_I\tilde{I} - d_R\tilde{R}) + d_S \int_\Omega (\tilde{E} + \tilde{I} + \tilde{R})} = \frac{N}{\int_\Omega (1 - d_E\tilde{E}^0 - d_I\tilde{I}^0 - d_R\tilde{R}^0)},$$

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and

$$\left(\frac{E}{d_S},\frac{I}{d_S},\frac{R}{d_S}\right)=(l\tilde{E},l\tilde{I},l\tilde{R})\to(l^*\tilde{E}^0,l^*\tilde{I}^0,l^*\tilde{R}^0)\quad\text{in } C^1(\bar{\Omega})\quad\text{as } d_S\to 0^+,$$

then (2.22) and (2.23) holds, which completes the proof of (i-1).

(i-2) Assume that $N=\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}$. In this case, there are two possibilities: either (7.1) holds or not. If (7.1) holds, then we can proceed as in the proof of (i-1) to derive that $\|E\|_\infty+\|I\|_\infty+\|R\|_\infty\to 0$ as $d_S\to 0$. In which case, we get $\int_\Omega S\to N$ as $d_S\to 0$. On the other hand, if (7.1) fails, then proceeding by the similar arguments leading to (7.3), we obtain again that $\int_\Omega S\to N$ as $d_S\to 0$, which in turn implies that $\int_\Omega (E+I+R)\to 0$ as $d_S\to 0$. The latter coupled with the regularity theory for elliptic equations implies that $\|E\|_\infty+\|I\|_\infty+\|R\|_\infty\to 0$ as $d_S\to 0$. Therefore, in any case, the desired result holds.

(ii) Suppose that $N>\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}$. By Theorem 5.7, for every $l>\mu^*$, (5.1) has a positive solution $(\tilde{E}^l,\tilde{I}^l,\tilde{R}^l)$. Moreover, by Lemma 5.3, as $l\to\infty$, $(\tilde{E}^l,\tilde{I}^l,\tilde{R}^l)\to(\tilde{E}^*,\tilde{I}^*,\tilde{R}^*)$ in $C^1(\Omega)$, where $(\tilde{E}^*,\tilde{I}^*,\tilde{R}^*)$ is the unique positive solution of (2.29) satisfying (2.30). Next, set $S^l=l(1-d_E\tilde{E}^l-d_I\tilde{I}^l-d_R\tilde{R}^l)$ and

$$d_{S,l}:=\frac{N-\int_\Omega S^l}{l\int_\Omega(\tilde{E}^l+\tilde{I}^l+\tilde{R}^l)}\quad\forall\, l>\mu^*.$$

Note that since $S^l\to\frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}$ as $l\to\infty$ in $C(\bar{\Omega})$ and $N>\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}$, there is $l^0>\mu^*$ such that $d_{S,l}>0$ for all $l>l^0$. It is clear that $d_{S,l}\to 0^+$ as $l\to\infty$. Finally, from the definition of $d_{S,l}$, it is easy to see that (5.30) holds for all $l>l^0$ with $d_S=d_{S,l}$. Therefore, defining (E^l,I^l,R^l) as in (5.31) with $d_S=d_{S,l}$, it follows from Lemma 5.8-(ii) that (S^l,E^l,I^l,R^l) is an EE solution of (1.3) for every $l>l^0$ and $d_S=d_{S,l}$. Furthermore, as $l\to\infty$,

$$E^l=d_{S,l}l\tilde{E}^l=\left(\frac{N-\int_\Omega S^l}{\int_\Omega(\tilde{E}^l+\tilde{I}^l+\tilde{R}^l)}\right)\tilde{E}^l\to\left(\frac{N-\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}}{\int_\Omega(\tilde{E}^*+\tilde{I}^*+\tilde{R}^*)}\right)\tilde{E}^*,$$

$$I^l=d_{S,l}l\tilde{I}^l=\left(\frac{N-\int_\Omega S^l}{\int_\Omega(\tilde{E}^l+\tilde{I}^l+\tilde{R}^l)}\right)\tilde{I}^l\to\left(\frac{N-\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}}{\int_\Omega(\tilde{E}^*+\tilde{I}^*+\tilde{R}^*)}\right)\tilde{I}^*,$$

and

$$R^l=d_{S,l}l\tilde{R}^l=\left(\frac{N-\int_\Omega S^l}{\int_\Omega(\tilde{E}^l+\tilde{I}^l+\tilde{R}^l)}\right)\tilde{R}^l\to\left(\frac{N-\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}}{\int_\Omega(\tilde{E}^*+\tilde{I}^*+\tilde{R}^*)}\right)\tilde{R}^*,$$

in $C^1(\bar{\Omega})$, which completes the proofs of (2.25)–(2.28).

We now suppose in addition that $\int_\Omega \frac{\alpha}{\beta}\frac{\varphi_R}{\varphi_I}<N<|\Omega|\mu^*$. Then, inequality (6.16) holds. Hence, there is $0<\varepsilon_0\ll 1$, $l_0\gg\mu^*$ such that (6.18) holds. Define d_S^* as in (6.19), we have that (6.20) holds for all $0<d_S<d_S^*$. Recalling the set $\Gamma_2^+:=\{(l,\mathbf{u})\in\Gamma^+:\mu^*\le l\le l_0\}$ introduced in the proof of Theorem 2.7 is connected. Hence, for every $0<d_S<d_S^*$, there is $l_2=l_2(d_S)\in(\mu^*,l_0)$ and $(l_2,\mathbf{u}^{l_2})\in\Gamma_2^+\subset\Gamma^+$ such that $(S_2,E_2,I_2,R_2):=(l_2(1-d_Eu_1^{l_2}-d_Iu_2^{l_2}-d_Ru_3^{l_2}),d_Sl_2u_1^{l_2},d_Sl_2u_2^{l_2},d_Sl_2u_3^{l_2})$ is an EE solution of (1.3). Hence

$$\|E_2\|_\infty+\|I_2\|_\infty+\|R_2\|_\infty\le d_Sl_0(\|u_1^{l_2}\|_\infty+\|u_2^{l_2}\|_\infty+\|u_3^{l_2}\|_\infty)\le d_Sl_0\left(\frac{1}{d_E}+\frac{1}{d_R}+\frac{1}{d_I}\right)\quad\forall\,0<d_S<d_S^*.$$

Therefore, (2.31) holds. If, up to a subsequence, $l_2\to\mu^*$ as $d_S\to 0$, then $N=\lim_{d_S\to 0}l_2\int_\Omega[(1-d_Eu_1^{l_2}-d_Iu_2^{l_2}-d_Ru_3^{l_2})+d_S(u_1^{l_2}+u_2^{l_2}+u_3^{l_2})]=\mu^*|\Omega|$, which contradicts our standing assumption. Thus, $\{l_2\}_{0<d_S<d_S^*}$ is bounded away from μ^*. Consequently, after passing to a subsequence, we may suppose that $l_2\to l^*$ as $d_S\to 0$ for some $l^*\in(\mu^*,l_0]$. Hence, $\|S_2-l^*(1-d_Eu_1^{l^*}-d_Iu_2^{l^*}-d_Ru_3^{l^*})\|_{C^1(\bar{\Omega})}\to 0$ as $d_S\to 0$, where $\mathbf{u}^{l^*}$ is a positive solution of (5.1) with $l=l^*$. Therefore, we can proceed by the similar arguments as in the proof of Theorem 2.11-(i-1) to conclude that S_2 has the same asymptotic profile as in (2.22) as $d_S\to 0$. □

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Next, we give a proof of Theorem 2.13.

Proof of Theorem 2.13. Fix $d_S > 0$, $d_R > 0$, and $d_E > 0$, and assume that (2.34) holds. Assume also that there is $d_* > 0$ such that system (1.3) has at least one EE solution (S, E, I, R) for all $0 < d_I \le d_*$. For each $0 < d_I < d_*$, let (S, E, I, R) be an EE of (1.3) and set $l_{d_I} = \frac{\kappa}{d_S}$, where $\kappa := d_S S + d_E E + d_I I + d_R R$. Also, let $(\tilde{E}^l, \tilde{I}^l, \tilde{R}^l)$ be as in (5.28). Observe from (5.30) that

$$\frac{N}{l_{d_I}} = |\Omega| + (d_S - d_I)\int_\Omega \tilde{I}^{l_{d_I}} + (d_S - d_R)\int_\Omega \tilde{R}^{l_{d_I}} + (d_S - d_E)\int_\Omega \tilde{E}^{l_{d_I}} \quad \forall\, 0 < d_I \le d_*. \tag{7.4}$$

Note also from Lemma 5.9 that

$$l_{d_I} \le \frac{2N}{|\Omega|}\max\left\{1, \frac{d_E}{d_S}, \frac{d_R}{d_S}, \frac{d_I}{d_S}\right\} \le \frac{2N}{|\Omega|}\max\left\{1, \frac{d_E}{d_S}, \frac{d_R}{d_S}, \frac{d_*}{d_S}\right\} := L^* \quad \forall\, 0 < d_I \le d_*. \tag{7.5}$$

Moreover, since

$$S^{l_{d_I}} = l_{d_I}\left(1 - d_E\tilde{E}^{l_{d_I}} - d_I\tilde{I}^{l_{d_I}} - d_R\tilde{R}^{l_{d_I}}\right) > 0 \quad \forall\, 0 < d_I < d_*,$$

then

$$0 < d_E\tilde{E}^{l_{d_I}} + d_I\tilde{I}^{l_{d_I}} + d_R\tilde{R}^{l_{d_I}} < 1 \quad \forall\, 0 < d_I < d_*. \tag{7.6}$$

However, by applying the comparison principle for elliptic equations to the second equation of (5.1), we get

$$\tilde{I}^{l_{d_I}} \le \left\| \frac{\sigma\tilde{E}^{l_{d_I}}}{\gamma} \right\|_\infty \quad \forall\, 0 < d_I < d_*.$$

This along with (7.6) implies that

$$\|\tilde{I}^{l_{d_I}}\|_\infty \le \left\| \frac{\sigma}{d_E\gamma} \right\|_\infty \quad \forall\, 0 < d_I < d_*. \tag{7.7}$$

Now, we complete the proof of the theorem. By (7.5) and (7.7), we have that

$$\sup_{0<d_I\le d_*} \|l_{d_I}\left(1 - d_E\tilde{E}^{l_{d_I}} - d_I\tilde{I}^{l_{d_I}} - d_R\tilde{R}^{l_{d_I}}\right)\tilde{I}^{l_{d_I}}\|_\infty < \infty.$$

Therefore, by the regularity theory for elliptic equations, it follows from the first equation of (5.1) that there is a positive number $c_1 > 0$ such that

$$\|\tilde{E}^{l_{d_I}}\|_{C^1(\bar{\Omega})} \le c_1 \quad \forall\, 0 < d_I < d_*. \tag{7.8}$$

Similarly, since (7.6) holds, by the regularity theory for elliptic equations, it follows from the third equation of (5.1) that there is a positive number $c_2 > 0$ such that

$$\|\tilde{R}^{l_{d_I}}\|_{C^1(\bar{\Omega})} \le c_2 \quad \forall\, 0 < d_I < d_*. \tag{7.9}$$

This shows that the set $\{(\tilde{E}^{l_{d_I}}, \tilde{R}^{l_{d_I}}) : d_I > 0\}$ is precompact in $C(\bar{\Omega})$.

Let $(\tilde{E}^*, \tilde{R}^*) \in [C(\bar{\Omega})]^2$ such that there is a sequence of positive numbers $\{d_{I,n}\}_{n\ge1}$ converging to zero such that $(\tilde{E}^{l_{d_{I,n}}}, \tilde{R}^{l_{d_{I,n}}}) \to (\tilde{E}^*, \tilde{R}^*)$ as $n \to \infty$. Applying the singular perturbation theory for elliptic equations to the second equation of (5.1), we have that

$$\lim_{n\to\infty} \left\| \tilde{I}^{l_{d_{I,n}}} - \frac{\sigma}{\gamma}\tilde{E}^* \right\|_\infty = 0. \tag{7.10}$$

Using this, and regularity theory for elliptic equations to the third equation of (5.1), we have that

$$\lim_{n\to\infty} \|\tilde{R}^{l_{d_I}} - \tilde{R}^*\|_{C^1(\bar{\Omega})} = 0, \tag{7.11}$$

where $\tilde{R}^* \ge 0$ is the unique solution to

$$\begin{cases} 0 = d_R\Delta\tilde{R}^* + \sigma\tilde{E}^* - \alpha\tilde{R}^* & x \in \Omega, \\ 0 = \partial_n\tilde{R}^* & x \in \partial\Omega. \end{cases} \tag{7.12}$$

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Recalling that (7.5) holds and $l_{d_I} > \mu^*$ for every $0 < d_I \le d_*$ by Lemma 5.5, where μ^* is the principal eigenvalue of (2.10). Thus, since $\mu^* \to \frac{1}{\mathcal{R}(d_E,\sigma,\frac{\sigma\beta}{\gamma})}$ as $d_I \to 0$ (see Proposition 2.9-(ii)), we may assume that there is $l^* \ge \frac{1}{\mathcal{R}(d_E,\sigma,\frac{\sigma\beta}{\gamma})}$ such that $l_{d_{I,n}} \to l^*$ as $n \to \infty$. Therefore, by (7.10) and (7.11), we can employ the regularity theory for elliptic equations and take limit in the first equation of (5.1) to conclude that $\tilde{E}^* \in C^2(\bar{\Omega})$ and solves the elliptic equation

$$\begin{cases} 0 = d_E \Delta \tilde{E}^* + l^* \frac{\sigma\beta}{\gamma}\left(1 - d_E \tilde{E}^* - d_R \tilde{R}^*\right)\tilde{E}^* - \sigma \tilde{E}^* & x \in \Omega, \\ 0 = \partial_n \tilde{E}^* & x \in \partial\Omega. \end{cases} \tag{7.13}$$

Note also from (7.4) and (7.10) that

$$l^* = \frac{N}{\int_\Omega \left(1 - d_E \tilde{E}^* - d_R \tilde{R}^*\right) + d_S \int_\Omega \left(\tilde{R}^* + \tilde{E}^* + \frac{\sigma}{\gamma}\tilde{E}^*\right)}. \tag{7.14}$$

Next, we show that

$$\tilde{E}^*(x) > 0 \quad \forall\, x \in \bar{\Omega}. \tag{7.15}$$

If (7.15) is false, then $\tilde{E}^* \equiv 0$ by the strong maximum principle principle for elliptic equations. This, in turn, with the fact that (7.12) holds, implies that $\tilde{R}^* \equiv 0$. Note also from (7.10) that $\|\tilde{I}^{l_{d_{I,n}}}\|_\infty \to 0$ as $n \to \infty$. Setting

$$\hat{E}^n = \frac{\tilde{E}^{l_{d_I},n}}{\|\tilde{E}^{l_{d_{I,n}}}\|_\infty}, \quad \hat{I}^n = \frac{\tilde{I}^{l_{d_I},n}}{\|\tilde{E}^{l_{d_{I,n}}}\|_\infty}, \quad \text{and} \quad \hat{R}^n = \frac{\tilde{R}^{l_{d_I},n}}{\|\tilde{E}^{l_{d_{I,n}}}\|_\infty} \quad \forall\, n \ge 1.$$

Then, $(\hat{E}^n, \hat{I}^n, \hat{R}^n)$ satisfies

$$\begin{cases} 0 = d_E \Delta \hat{E}^n + l_{d_{I,n}} \beta\left(1 - d_E \tilde{E}^{l_{d_{I,n}}} - d_I \tilde{I}^{l_{d_{I,n}}} - d_R \tilde{R}^{l_{d_{I,n}}}\right)\hat{I}^n - \sigma \hat{E}^n & x \in \Omega, \\ 0 = d_I \Delta \hat{I}^n + \sigma \hat{E}^n - \gamma \hat{I}^n & x \in \Omega, \\ 0 = d_R \Delta \hat{R}^n + \gamma \hat{I}^n - \alpha \hat{R}^n & x \in \Omega, \\ 0 = \partial_n \hat{E}^n = \partial_n \hat{I}^n = \partial_n \hat{R}^n & x \in \partial\Omega. \end{cases} \tag{7.16}$$

Note that $\|\hat{E}^n\|_\infty = 1$ for all $n \ge 1$. Hence, by the comparison principle for elliptic equations, we get from the second equation of (7.16) that $\|\hat{I}^n\|_\infty \le \|\sigma/\gamma\|_\infty$ for all $n \ge 1$. Using this fact, we get also from the third equation of (7.16) and the comparison principle for elliptic equations that $\|\hat{R}^n\|_\infty \le \|\sigma/\gamma\|_\infty \|\gamma/\alpha\|_\infty$ for all $n \ge 1$. Therefore, if possible after passing to a subsequence, we can employ the regularity theory for elliptic equation to conclude that there are $\hat{E}^*, \hat{I}^*$, and $\hat{R}^* \in C^+(\bar{\Omega})$ such that

$$\hat{E}^n \to \hat{E}^*, \quad \hat{I}^n \to \hat{I}^*, \quad \text{and} \quad \hat{R}^n \to \hat{R}^* \quad \text{as } n \to \infty, \tag{7.17}$$

uniformly on $\bar{\Omega}$. Moreover, from the first equation of (7.16) and the regularity theory for elliptic equations, $\hat{E}^* \in C^2(\Omega)$ and is a classical solution of

$$\begin{cases} 0 = d_E \Delta \hat{E}^* + l^* \frac{\sigma\beta}{\gamma} \hat{E}^* - \sigma \hat{E}^* & x \in \Omega, \\ 0 = \partial_n \hat{E}^* & x \in \partial\Omega. \end{cases} \tag{7.18}$$

Since $\hat{E}^* \ge 0$ and $\|\hat{E}^*\|_\infty = 1$, then by the strong maximum principle principle, it follows from (7.18) that $\hat{E}^* > 0$ on $\bar{\Omega}$. We can deduce from (7.18) that

$$l^* = \frac{1}{\mathcal{R}\left(d_E, \sigma, \frac{\sigma\beta}{\gamma}\right)}.$$

Noting also from (7.14) that $l^* = N/|\Omega|$, then

$$\frac{N}{|\Omega|}\mathcal{R}\left(d_E, \sigma, \frac{\sigma\beta}{\gamma}\right) = 1,$$

which contradicts with (2.34). Therefore, (7.15) holds.

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By (7.15) and (7.12), we also have that $\tilde{R}^* > 0$ on $\bar{\Omega}$. Furthermore, since S^l solves (5.12), then by the regularity theory for elliptic equations, we have that $S^{l_{d_{I,n}}} \to S^*$ as $n \to \infty$, uniformly on $\bar{\Omega}$, where $S^* \geq 0$ is a classical solution of

$$\begin{cases} 0 = \frac{1}{l^*}\Delta S^* + \alpha \tilde{R}^* - \frac{\beta\sigma \tilde{E}^*}{\gamma} S^* & x \in \Omega, \\ 0 = \partial_{\boldsymbol{n}} S^* & x \in \partial\Omega. \end{cases}$$

Thus, $S^* > 0$ on $\bar{\Omega}$ by the strong maximum principle principle for elliptic equations. Note also that

$$S^{l_{d_{I,n}}} \to l^*\big(1 - d_E\tilde{E}^* - d_R\tilde{R}^*\big) \quad \text{as } n \to \infty$$

uniformly on $\bar{\Omega}$. So, $1 - d_E\tilde{E}^* - d_R\tilde{R}^* = \frac{S^*}{l^*} > 0$ on $\bar{\Omega}$. Consequently, letting $n \to \infty$ in (5.31), we obtain that $(S^{l_{d_{I,n}}}, E^{l_{d_{I,n}}}, I^{l_{d_{I,n}}}, R^{l_{d_{I,n}}}) \to (l^*(1 - d_E\tilde{E}^* - d_R\tilde{R}^*), d_S l^*\tilde{E}^*, \frac{d_S l^*}{\gamma}\sigma\tilde{E}^*, d_S l^*\tilde{R}^*)$ uniformly in $x \in \bar{\Omega}$, which yields the desired result. □

Proof of Theorem 2.14. Fix $d_S > 0$, $d_R > 0$, and $d_I > 0$, and assume that (2.41) holds. Assume also that there is $d^* > 0$ such that system (1.3) has at least one EE solution (S, E, I, R) for all $0 < d_E \leq d^*$. For each $0 < d_E < d^*$, let (S, E, I, R) be an EE of (1.3) and set $l_{d_E} = \frac{\kappa}{d_S}$, where $\kappa := d_S S + d_E E + d_I I + d_R R$. Also, let $(\tilde{E}^{l_{d_E}}, \tilde{I}^{l_{d_E}}, \tilde{R}^{l_{d_E}})$ be as in (5.28). By Lemma 5.9, we have

$$l_{d_E} \leq \frac{2N}{|\Omega|}\max\left\{1, \frac{d_E}{d_S}, \frac{d_R}{d_S}, \frac{d_I}{d_S}\right\} \leq \frac{2N}{|\Omega|}\max\left\{1, \frac{d^*}{d_S}, \frac{d_R}{d_S}, \frac{d_I}{d_S}\right\} := l^0 \quad \forall\, 0 < d_E \leq d^*. \tag{7.19}$$

Note also that (7.6) holds with l_{d_I} replaced with l_{d_E}. Hence, $\|\tilde{I}^{l_{d_E}}\|_\infty < \frac{1}{d_I}$ for all $0 < d_E < d^*$. We can then apply the comparison principle to the first equation of (5.1) to obtain that

$$\|\tilde{E}^{l_{d_E}}\|_\infty \leq \frac{l^0}{d_I}\left\|\frac{\beta}{\sigma}\right\|_\infty \quad \forall\, 0 < d_E \leq d^*. \tag{7.20}$$

Hence, from the second and third equations of (5.1), we get

$$\sup_{0<d_E\leq d^*}\|\Delta\tilde{I}^{l_{d_E}}\|_\infty \leq \frac{l^0}{d_I^2}\left\|\frac{\beta}{\sigma}\right\|_\infty\|\sigma\|_\infty + \frac{\|\gamma\|_\infty}{d_I^2} \quad \text{and} \quad \sup_{0<d_E\leq d^*}\|\Delta\tilde{R}^{l_{d_E}}\|_\infty \leq \frac{\|\gamma\|_\infty}{d_I d_R} + \frac{\|\alpha\|_\infty}{d_R^2}.$$

We can then invoke the regularity theory for elliptic equations to conclude that the set $\{(\tilde{I}^{l_{d_E}}, \tilde{R}^{l_{d_E}}) : 0 < d_E \leq d^*\}$ is precompact in $C^1(\bar{\Omega})$. Let $(\tilde{I}^*, \tilde{R}^*) \in [C^1(\bar{\Omega})]^2$ such that, if possible after passing to a subsequence, $\|\tilde{I}^{l_{d_E}} - \tilde{I}^*\|_{C^1(\bar{\Omega})} \to 0$ and $\|\tilde{R}^{l_{d_E}} - \tilde{R}^*\|_{C^1(\bar{\Omega})} \to 0$ as $d_E \to 0$. Recall that (7.5) holds and $l_{d_E} > \mu^*$ for every $0 < d_E \leq d^*$ by Lemma 5.5, where μ^* is the principal eigenvalue of (2.10). Thus, since $\mu^* \to \frac{1}{\mathcal{R}(d_I,\gamma,\beta)}$ as $d_E \to 0$ (see Proposition 2.9-(i)), we may assume that there is $l^* \geq \frac{1}{\mathcal{R}(d_I,\gamma,\beta)}$ such that $l_{d_E} \to l^*$ as $d_E \to 0$. As a result, noting that $d_E\|\tilde{E}^{l_{d_E}}\|_\infty \to 0$ as $d_E \to 0$ by (7.20), we can employ the singular perturbation theory for elliptic equation to the first equation of (5.1) to conclude that

$$\lim_{d_E\to 0}\|\tilde{E}^{l_{d_E}} - l^*\frac{\beta}{\sigma}(1 - d_I\tilde{I}^* - d_R\tilde{R}^*)\tilde{I}^*\|_\infty = 0. \tag{7.21}$$

Hence, by the regularity theory for elliptic equations, we have that $(\tilde{I}^*, \tilde{R}^*)$ is a classical solution of (2.44). Now, we shall show that $\tilde{I}^* > 0$. Otherwise, we would have $\tilde{I}^* \equiv 0$ and $\tilde{R}^* \equiv 0$. Moreover, considering the scaling $\hat{R}^{l_{d_E}} = \frac{\tilde{R}^{l_{d_E}}}{\|\tilde{I}^{l_{d_E}}\|_\infty}$, $\hat{I}^{l_{d_E}} = \frac{\tilde{I}^{l_{d_E}}}{\|\tilde{I}^{l_{d_E}}\|_\infty}$, and $\hat{E}^{l_{d_E}} = \frac{\tilde{E}^{l_{d_E}}}{\|\tilde{I}^{l_{d_E}}\|_\infty}$ for $0 < d_E \leq d^*$. Then, $(\hat{E}^{l_{d_E}}, \hat{I}^{l_{d_E}}, \hat{R}^{l_{d_E}})$ also solves (7.16) with $l_{d_{I,n}}$ replaced with l_{d_E}. Since $\|\hat{I}^{l_{d_E}}\|_\infty = 1$, we can employ the comparison principle to the third equation of (7.16) to get $\|\hat{R}^{l_{d_E}}\|_\infty \leq \|\gamma/\alpha\|_\infty$ for all $0 < d_E \leq d^*$. Similarly, applying the comparison principle for elliptic equations to the first equation of (7.16), coupled with (7.19) and the fact that $\|\hat{I}^{l_{d_E}}\|_\infty = 1$, we get that $\|\hat{E}^{l_{d_E}}\|_\infty \leq l^0\|\beta/\sigma\|_\infty$ for all $0 < d_E \leq d^*$. We can now appeal to the regularity theory and singular perturbation theory for elliptic equations to conclude, after passing to a subsequence, that there is $(\hat{I}^*, \hat{R}^*) \in [C^2(\bar{\Omega})]^2$ such that

$$\|\hat{R}^{l_{d_E}} - \hat{R}^*\|_\infty + \|\hat{I}^{l_{d_E}} - \hat{I}^*\|_\infty + \left\|\hat{E}^{l_{d_E}} - l^*\frac{\beta}{\sigma}\hat{I}^*\right\|_\infty \to 0 \quad \text{as } d_E \to 0.$$

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Furthermore, $(\hat{I}^*, \hat{R}^*)$ solves

$$\begin{cases} 0 = d_I \Delta \hat{I}^* - \gamma \hat{I}^* + l^* \beta \hat{I}^* & x \in \Omega, \\ 0 = d_R \Delta \hat{R}^* - \alpha \hat{R}^* + \gamma \hat{I}^* & x \in \Omega, \\ 0 = \partial_n \hat{R}^* = \partial_n \hat{I}^* & x \in \partial\Omega. \end{cases} \tag{7.22}$$

Noting that $\max_{x\in\bar{\Omega}} \hat{I}^*(x) = 1$ and $\hat{I}^* \geq 0$, strong maximum principle principle applying to the second equation of (7.22) yields that $\hat{R}^* > 0$. Thus, by the first equation of (7.22), we get that $l^* = \frac{1}{\mathcal{R}(d_I,\gamma,\beta)}$. On the other hand, letting $d_E \to 0$, it follows from (7.4) that $l^* = \frac{N}{\Omega}$. Whence, $\frac{N}{|\Omega|}\mathcal{R}(d_I, \gamma, \beta) = 1$, which contradicts with (2.41). Therefore, $\tilde{I}^* > 0$ and hence $\tilde{R}^* > 0$.

Note from (5.30) and (7.21) that

$$N = l^* \left(\int_\Omega \left(1 - d_I \tilde{I}^* - d_R \tilde{R}^*\right) + d_S \int_\Omega \left(\tilde{R}^* + \tilde{I}^* + l^* \frac{\beta}{\sigma} \left(1 - d_I \tilde{I}^* - d_R \tilde{R}^*\right) \tilde{I}^* \right) \right).$$

Observe also that we can proceed as in the proof of Theorem 2.13 to show that $1 - d_I \tilde{I}^* - d_R \tilde{R}^* > 0$ on $\bar{\Omega}$. Finally, recalling that $S = l_{d_E}(1 - d_E \tilde{E}^{l_{d_E}} - d_I \tilde{I}^{l_{d_E}} - d_R \tilde{R}^{l_{d_E}})$, $E = d_S l_{d_E} \tilde{E}^{l_{d_E}}$, $I = d_S l_{d_E} \tilde{I}^{l_{d_E}}$, and $R = d_S l_{d_E} \tilde{R}^{l_{d_E}}$, we can let $d_E \to 0$ and derive that the conclusions (2.42) and (2.43) of the theorem hold. □

Proof of Theorem 2.15. Fix $d_S > 0, d_E > 0$, and $d_I > 0$, and assume that $\mathcal{R}_0 \neq 1$. Assume also that there is $d^{**} > 0$ such that system (1.3) has at least one EE solution (S, E, I, R) for all $0 < d_R \leq d^{**}$. For each $0 < d_R \leq d^{**}$, let (S, E, I, R) be an EE of (1.3) and set $l_{d_R} = \frac{\kappa}{d_S}$, where $\kappa := d_S S + d_E E + d_I I + d_R R$. Also, let $(\tilde{E}^{l_{d_R}}, \tilde{I}^{l_{d_R}}, \tilde{R}^{l_{d_R}})$ be as in (5.28). By Lemma 5.9, we have

$$l_{d_R} \leq \frac{2N}{|\Omega|} \max\left\{1, \frac{d_E}{d_S}, \frac{d_R}{d_S}, \frac{d_I}{d_S}\right\} \leq \frac{2N}{|\Omega|} \max\left\{1, \frac{d^{**}}{d_S}, \frac{d_E}{d_S}, \frac{d_I}{d_S}\right\} := l^0 \quad \forall\, 0 < d_R \leq d^{**}.$$

Thanks to (7.6), we have $\|\tilde{E}^{l_{d_R}}\|_\infty \leq \frac{1}{d_E}$ and $\|\tilde{I}^{l_{d_R}}\|_\infty \leq \frac{1}{d_I}$ for all $0 < d_R \leq d^{**}$. Hence, an application of the comparison principle for elliptic equations to the third equation of (5.1) implies that $\|\tilde{R}^{l_{d_R}}\|_\infty \leq \frac{1}{d_I}\|\gamma/\alpha\|_\infty$ for all $0 < d_R \leq d^{**}$. Therefore, we can proceed by the similar arguments as in the proof of Theorem 2.14 to conclude that, if possible after passing to a subsequence, there exist $l^{**} > 0$ and $(\tilde{E}^{**}, \tilde{I}^{**}) \in [C^1(\bar{\Omega})]^2$ such that

$$l_{d_R} \to l^{**}, \quad \|\tilde{E}^{l_{d_R}} - \tilde{E}^{**}\|_{C^1(\bar{\Omega})} + \|\tilde{I}^{l_{d_R}} - \tilde{I}^{**}\|_{C^1(\bar{\Omega})} \to 0 \quad \text{and} \quad \left\|\tilde{R}^{l_{d_R}} - \frac{\gamma}{\alpha}\tilde{I}^{**}\right\|_\infty \to 0 \quad \text{as } d_R \to 0.$$

This shows that (S, E, I, R) satisfies (2.45). Moreover, $(\tilde{E}^{**}, \tilde{I}^{**})$ and l^{**} solves (2.46). Since $\mathcal{R}_0 \neq 1$, we can proceed as in the proof of Theorem 2.14 and rule out the possibility for $\tilde{I}^{**} \equiv 0$. Therefore, we must have that $\tilde{I}^{**} > 0$ and $\tilde{E}^{**} > 0$ on $\bar{\Omega}$. □

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Appendix A: Proof of Theorem 5.7

This section is devoted to the proof of Theorem 5.7. Fix $p \gg n$ such that $W^{2,p}(\Omega)$ is compactly embedded in $C^{1,\nu_p}(\bar{\Omega})$ for some $0 < \nu_p < 1$. Set $X := \{u \in W^{2,p}(\Omega) : \partial_n u = 0 \text{ on } \partial\Omega\}$ and $Y = L^p(\Omega)$. We introduce the mapping

$$\begin{array}{rccc} \mathcal{F}: & \mathbb{R} \times X^3 & \to & Y^3 \\ & (l, \mathbf{u}) & \mapsto & \begin{pmatrix} d_E \Delta u_1 - \sigma u_1 + l\beta(1 - d_E u_1 - d_I u_2 - d_R u_3)u_2 \\ d_I \Delta u_2 - \gamma u_2 + \sigma u_1 \\ d_R \Delta u_3 - \alpha u_3 + \gamma u_2 \end{pmatrix}, \end{array}$$

where we used the notation $\mathbf{u} = (u_1, u_2, u_2)^T$. Note that $\mathcal{F}(l, \mathbf{0}) = \mathbf{0}$ for all $l \in \mathbb{R}$. It is clear that $\mathcal{F}$ is of class C^∞. Moreover, the Fréchet derivative of $\mathcal{F}$, denoted as $D_{\mathbf{u}}\mathcal{F}$, with respect to $\mathbf{u}$ is given by

$$D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u})(\mathbf{v}) = \begin{pmatrix} d_E \Delta v_1 - \sigma v_1 + l\beta((1 - d_E u_1 - 2d_I u_2 - d_R u_3)v_2 - d_E u_2 v_1 - d_R u_2 v_3) \\ d_I \Delta v_2 - \gamma v_2 + \sigma v_1 \\ d_R \Delta v_3 - \alpha v_3 + \gamma v_2 \end{pmatrix} \tag{A.1}$$

for all $l \in \mathbb{R}$, and $\mathbf{u}, \mathbf{v} \in X^3$. In particular,

$$D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0})(\mathbf{v}) = \begin{pmatrix} d_E \Delta v_1 - \sigma v_1 + \mu^* \beta v_2 \\ d_I \Delta v_2 - \gamma v_2 + \sigma v_1 \\ d_R \Delta v_3 - \alpha v_3 + \gamma v_2 \end{pmatrix} \quad \forall\, \mathbf{v} \in X^3, \tag{A.2}$$

where $\mu^* > 0$ is the principal eigenvalue of the weighted eigenvalue problem (2.10). In what follows, given a positive number $d > 0$ and positive Hölder continuous function η on $\bar{\Omega}$, we let $v := (\eta - d\Delta)^{-1}(u) \in X$ denote the unique solution of

$$\begin{cases} \eta v - d\Delta v = u & x \in \Omega, \\ \partial_n v = 0 & x \in \partial\Omega, \end{cases}$$

for every $u \in Y$. Note that $(\eta - d\Delta)^{-1} : Y \to X$ is a bounded linear isomorphism. From this point, we prove a few lemmas.

Lemma A.1. *Let* (Ψ^*, Φ^*) *be the positive eigenfunction pair associated with* μ^* *satisfying* $\|\Phi^* + \Psi^*\|_\infty = 1$ *and set*

$$\mathbf{v}^* = (\Psi^*, \Phi^*, (\alpha - d_R\Delta)^{-1}(\gamma\Phi^*)). \tag{A.3}$$

Then, $\mathbf{v}^* \in [C^{++}(\bar{\Omega})]^3$,

$$Ker(D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0})) = span(\mathbf{v}^*), \tag{A.4}$$

and

$$R(D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0})) \quad \text{is closed with} \quad codim(R(D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0}))) = 1. \tag{A.5}$$

where $Ker(D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0}))$ *and* $R(D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0}))$ *are the null space and range of* $D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0})$, *respectively.*

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Proof. By the strong maximum principle principle for elliptic equations, we have that $(\alpha - d_R\Delta)^{-1}(\gamma\Phi^*) \in C^{++}(\bar{\Omega})$ since $\gamma\Phi^* > 0$ on $\bar{\Omega}$. Hence, $\mathbf{v}^* \in [C^{++}(\bar{\Omega})]^3$. Next, from (A.2), we have that $\mathbf{v} \in \mathrm{Ker}(D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0}))$ if and only if

$$\begin{cases} 0 = d_E\Delta v_1 - \sigma v_1 + \mu^*\beta v_2 & x \in \Omega, \\ 0 = d_I\Delta v_2 - \gamma v_2 + \sigma v_1 & x \in \Omega, \\ 0 = \partial_n v_1 = \partial_n v_2 & x \in \partial\Omega, \end{cases}$$

and $v_3 = (\alpha - d_R\Delta)^{-1}(\gamma v_2)$. Hence, $\mathbf{v} \in \mathrm{Ker}(D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0}))$ if and only if $(v_1, v_2)^T$ is an eigenfunction associated with μ^* and $v_3 = (\alpha - d_R\Delta)^{-1}(\gamma v_2)$. Recalling that μ^* is a simple principal eigenvalue, we have that (A.4) holds.

Next, note that

$$\mathbf{w} \in R(D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0})) \Longleftrightarrow \mathcal{T}_0(\mathbf{w}) = (\mathcal{T}_1 - \mathcal{I})(\mathbf{v}) \quad \text{for some } \mathbf{v} \in X^3, \tag{A.6}$$

where $\mathcal{T}_0 : Y^3 \to X^3$ is the linear isomorphism defined by

$$\mathcal{T}_0(\mathbf{w}) = ((\sigma - d_E\Delta)^{-1}(w_1), (\gamma - d_I\Delta)^{-1}(w_2), (\alpha - d_R\Delta)^{-1}(w_3))^T, \tag{A.7}$$

$\mathcal{I}$ is the identity map on X^3, and $\mathcal{T}_1 : X^3 \to X^3$ is the linear map

$$\mathcal{T}_1(\mathbf{v}) = (\mu^*(\sigma - d_E\Delta)^{-1}(\beta v_2), (\gamma - d_I\Delta)^{-1}(\sigma v_1), (\alpha - d_R\Delta)^{-1}(\gamma v_2))^T \quad \forall\, \mathbf{v} \in X^3.$$

We note that, since βv_2, σv_1 and γv_2 are Hölder continuous functions on $\bar{\Omega}$ for every $\mathbf{v} \in X^3$. Then, by the regularity theory for elliptic equations, there is $0 < \nu \ll 1$ such that $\mathcal{T}_1(\mathbf{v}) \in [C^{2+\nu}(\bar{\Omega})]^3$ for all $\mathbf{v} \in X^3$, which shows that $\mathcal{T}_1$ is well defined. Moreover, there is $M_\nu > 0$ such that $\|\mathcal{T}_1(\mathbf{v})\|_{[C^{2+\nu}(\bar{\Omega})]^3} \le M_\nu\|\mathbf{v}\|_{X^3}$ for all $\mathbf{v} \in X^3$. Hence, $\mathcal{T}_1$ is a compact linear operator. This shows that $\mathcal{T}_1 - \mathcal{I}$ is a Fredholm operator with index zero. As a result, we have that $R(\mathcal{T}_1 - \mathcal{I})$ is closed and $\mathrm{codim}(R(\mathcal{T}_1 - \mathcal{I})) = 1$. Recalling that $\mathcal{T}_0$ is a linear isomorphism, we can then deduce from (A.6) that $R(D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0})) = \mathcal{T}_0^{-1}(R(\mathcal{T}_1 - \mathcal{I}))$ is also closed with $\mathrm{codim}(R(D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0}))) = 1$. □

Taking the derivative of (A.1) with respect to l, we get

$$D_{\mathbf{u},l}\mathcal{F}(l, \mathbf{u})(\mathbf{v}) = \begin{pmatrix} \beta((1 - d_Eu_1 - 2d_Iu_2 - d_Ru_3)v_2 - d_Eu_2v_1 - d_Ru_2v_3) \\ 0 \\ 0 \end{pmatrix} \quad \forall\, l \in \mathbb{R},\ \mathbf{u},\ \mathbf{v} \in X^3.$$

In particular, $D_{\mathbf{u},l}\mathcal{F}(\mu^*, \mathbf{0})(\mathbf{v}^*) = (\beta\Phi^*, 0, 0)^T$.

Lemma A.2. *It holds that*

$$D_{\mathbf{u},l}\mathcal{F}(\mu^*, \mathbf{0})(\mathbf{v}^*) = (\beta\Phi^*, 0, 0)^T \notin R(D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0})). \tag{A.8}$$

Proof. First, note from the Krein-Rutman theorem and the fact that μ^* is the principal eigenvalue of system (2.10), then $\mu^* > 0$ is also the principal eigenvalue of the adjoint weighted eigenvalue problem

$$\begin{cases} 0 = d_E\Delta\Psi - \sigma\Psi + \sigma\Phi & x \in \Omega, \\ 0 = d_I\Delta\Phi + \mu^*\beta\Psi - \gamma\Phi & x \in \Omega, \\ 0 = \partial_n\Psi = \partial_n\Phi & x \in \partial\Omega. \end{cases} \tag{A.9}$$

Let $(\tilde{\Psi}^*, \tilde{\Phi}^*)$ denote the positive eigenfunction of (A.9) satisfying $\|\tilde{\Psi}^* + \tilde{\Phi}^*\|_\infty = 1$.

Now, assume to the contrary that $D_{\mathbf{u},l}\mathcal{F}(\mu^*, \mathbf{0})(\mathbf{v}^*) = D_{\mathbf{u}}\mathcal{F}(\mu^*, \mathbf{0})(\mathbf{v}^0)$ for some $\mathbf{v}^0 \in X^3$. Then, it holds that

$$\begin{cases} \beta\Phi^* = d_E\Delta v_1^0 - \sigma v_1^0 + \mu^*\beta v_2^0 & x \in \Omega, \\ 0 = d_I\Delta v_2^0 - \gamma v_2^0 + \sigma v_1^0 & x \in \Omega, \\ 0 = \partial_n v_1 = \partial_n v_2 & x \in \partial\Omega, \end{cases} \tag{A.10}$$

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Multiplying the first and second equations of (A.10) by $\tilde{\Psi}^*$ and $\tilde{\Phi}^*$, respectively, and next integrate the resulting equation gives

$$\begin{aligned}\int_\Omega \beta\Phi^*\tilde{\Psi}^* &= \int_\Omega \left(d_E\Delta v_1^0 - \sigma v_1^0 + \mu^*\beta v_2^0\right)\tilde{\Psi}^* + \int_\Omega \left(d_I\Delta v_2^0 - \gamma v_2^0 + \sigma v_1^0\right)\tilde{\Phi}^* \\ &= \int_\Omega d_E v_1^0\Delta\tilde{\Psi}^* + \int_\Omega \left(-\sigma v_1^0 + \mu^*\beta v_2^0\right)\tilde{\Psi}^* + \int_\Omega d_I v_2^0\Delta\tilde{\Phi}^* + \int_\Omega \left(-\gamma v_2^0 + \sigma v_1^0\right)\tilde{\Phi}^* \\ &= \int_\Omega \left(d_E\Delta\tilde{\Psi}^* - \sigma\tilde{\Psi}^* + \sigma\tilde{\Phi}^*\right)v_1^0 + \int_\Omega \left(d_I\Delta\tilde{\Phi}^* - \gamma\tilde{\Phi}^* + \mu^*\beta\tilde{\Psi}^*\right)v_2^0 \\ &= 0,\end{aligned}$$

where we have used the fact that $(\tilde{\Psi}^*, \tilde{\Phi}^*)$ solves (A.9). Therefore, we get that $\int_\Omega \beta\Phi^*\tilde{\Psi}^* = 0$, which is impossible since $\tilde{\Psi}^* > 0$, $\Phi^* > 0$ and $\beta > 0$. Hence, we must have (A.8) holds. □

Lemma A.3. *It holds that*

$$D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u}) \text{ is a Fredholm operator with index zero for all } l \in \mathbb{R},\ \mathbf{u} \in X^3. \tag{A.11}$$

Furthermore, $kD_{\mathbf{u}}\mathcal{F}(l, \mathbf{0}) + (1-k)D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u})$ is a Fredholm operator with index zero for each $k \in (0, 1)$.

Proof. Fix $l \in \mathbb{R}$ and $\mathbf{u} \in X^3$. By the similar computations yielding (A.6), we have that

$$\mathbf{w} \in R(D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u})) \iff \mathcal{T}_0(\mathbf{w}) = (-\mathcal{I} + \mathcal{T}_1^{\mathbf{u}})(\mathbf{v}) \quad \text{for some } \mathbf{v} \in X^3 \tag{A.12}$$

where $\mathcal{T}_0$ is as in (A.7) and $\mathcal{T}_1^{\mathbf{u}} : X^3 \to X^3$ is given by

$$\mathcal{T}_1^{\mathbf{u}}(\mathbf{v}) = \begin{pmatrix} l(\sigma - d_E\Delta)^{-1}(\beta((1 - d_E u_1 - 2d_I u_2 - d_R u_3)v_2 - d_E u_2 v_1 - d_R u_2 v_3)) \\ (\gamma - d_I\Delta)^{-1}(\sigma v_1) \\ (\alpha - d_R\Delta)^{-1}(\gamma v_2) \end{pmatrix} \quad \forall\, \mathbf{v} \in X^3.$$

Noting that $\mathcal{T}_1^{\mathbf{u}}$ is also a compact linear operator, so that $\mathcal{T}_1^{\mathbf{u}} - \mathcal{I}$ is a Fredholm operator with index zero, we conclude from (A.12) that $R(D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u})) = \mathcal{T}_0^{-1}(R(\mathcal{T}_1^{\mathbf{u}} - \mathcal{I}))$ is closed with a finite codimension. Note again as in (A.12) that $\mathbf{v} \in \mathrm{Ker}(D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u}))$ if and only if $\mathbf{v} \in \mathrm{Ker}(\mathcal{T}_1^{\mathbf{u}} - \mathcal{I})$. Hence, $\mathrm{Ker}(D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u}))$ is also a finite dimensional space since so is $\mathrm{Ker}(\mathcal{T}_1^{\mathbf{u}} - \mathcal{I})$. Note that we have shown that $\mathrm{Ker}(D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u})) = \mathrm{Ker}(\mathcal{T}_1^{\mathbf{u}} - \mathcal{I})$ and $R(D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u})) = \mathcal{T}_0^{-1}(R(\mathcal{T}_1^{\mathbf{u}} - \mathcal{I}))$. As a result, we have that $D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u})$ is a Fredholm operator with index zero. Observe also that for fixed $0 < k < 1$, $l > 0$ and $\mathbf{u} \in X^3$, $kD_{\mathbf{u}}\mathcal{F}(l, \mathbf{0}) + (1-k)D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u})$ is a Fredholm operator with index zero. Indeed, we have from (A.1) that

$$kD_{\mathbf{u}}\mathcal{F}(l, \mathbf{0}) + (1-k)D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u}) = D_{\mathbf{u}}\mathcal{F}(l, (1-k)\mathbf{u}).$$

Hence, it follows from (A.11) that $kD_{\mathbf{u}}\mathcal{F}(l, \mathbf{0}) + (1-k)D_{\mathbf{u}}\mathcal{F}(l, \mathbf{u})$ is a Fredholm operator with index zero. □

Thanks to Lemmas A.1–A.3, the hypotheses of [35, Theorem 4.3] are satisfied. Define

$$Z := \left\{\mathbf{u} \in X^3 : \int_\Omega (u_1 v_1^* + u_2 v_2^* + u_3 v_3^*) = 0\right\}, \tag{A.13}$$

where $\mathbf{v}^*$ is defined by (A.3). Then, Z is a complement of $\mathrm{span}(\mathbf{v}^*)$, in the sense that $X^3 = Z \oplus \mathrm{span}(\mathbf{v}^*)$. We can now invoke [35, Theorem 4.3] or [10] to conclude the existence of $\varepsilon > 0$ and C^1 functions $l : (-\varepsilon, \varepsilon) \to \mathbb{R}^+$, and $\mathbf{z} : (-\varepsilon, \varepsilon) \to Z$ satisfying $l(0) = \mu^*$ and $\mathbf{z}(0) = \mathbf{0}$ such that $\mathcal{F}^{-1}(\{\mathbf{0}\})$ consists exactly of the two curves $C^0 := \{(l, \mathbf{0}) : l > 0\}$ and the curve $C^* := \{(l(s), s(\mathbf{v}^* + \mathbf{z}(s))) : -\varepsilon < s < \varepsilon\}$ near the bifurcation point $(\mu^*, \mathbf{0})$. Specifically, there is $0 < \tilde{\varepsilon} \ll \frac{\mu^*}{2}$ such that

$$\mathcal{F}^{-1}(\{\mathbf{0}\}) \cap \bar{\mathcal{B}}_{\mathbb{R}\times X^3}((\mu^*, \mathbf{0}), \tilde{\varepsilon}) \subset C^0 \cup C^*. \tag{A.14}$$

Here,

$$\mathcal{B}_{\mathbb{R}\times X^3}((\mu^*, \mathbf{0}), \tilde{\varepsilon}) = \{(l, \mathbf{u}) \in \mathbb{R} \times X^3 : |l - \mu^*| + \|\mathbf{u}\|_{X^3} < \tilde{\varepsilon}\}$$

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and

$$\bar{\mathcal{B}}_{\mathbb{R}\times X^3}((\mu^*,\mathbf{0}),\tilde{\varepsilon}) = \{(l,\mathbf{u}) \in \mathbb{R}\times X^3 : |l-\mu^*| + \|\mathbf{u}\|_{X^3} \le \tilde{\varepsilon}\}$$

is its closure. Note that since $0 < \tilde{\varepsilon} < \frac{\mu^*}{2}$, then $\mathcal{B}_{\mathbb{R}\times X^3}((\mu^*,\mathbf{0}),\tilde{\varepsilon}) \subset \mathbb{R}^+ \times X^3$.

Recalling that $\mathbf{v}^* \in [C^{++}(\bar{\Omega})]^3$, then possibly after decreasing ε, we may suppose that the curves l and $\mathbf{z}$ are defined on $[-\varepsilon,\varepsilon]$ and

$$C^+ := \{(l(s), s(\mathbf{v}^* + \mathbf{z}(s))) : 0 < s < \varepsilon\} \subset \mathbb{R}^+ \times [C^{++}(\bar{\Omega})]^3. \tag{A.15}$$

For convenience, we also set

$$C^- := \{(l(s), s(\mathbf{v}^* + \mathbf{z}(s))) : -\varepsilon < s < 0\}. \tag{A.16}$$

By Lemma 5.5, we have that $l(s) > \mu^*$ for all $0 < s < \varepsilon$. Next, applying [35, Theorem 4.3], there is a connected component Γ of the closure $\bar{\mathcal{S}}$ of the set $\mathcal{S} := \{(l,\mathbf{u}) \in \mathbb{R}\times X^3 : \mathcal{F}(l,\mathbf{u}) = \mathbf{0} \text{ and } \mathbf{u} \neq \mathbf{0}\}$ containing the curve C^*, with the properties that either Γ is not compact in $\mathbb{R}\times X^3$, or it contains an element $(\lambda,\mathbf{0})$ with $\lambda \neq \mu^*$. Moreover, letting Γ^+ denote the component of $\Gamma \setminus C^-$ in the direction of $\mathbf{v}^*$ which contains C^+, it follows from [35, Theorem 4.4] (or [20, Theorem 1.2]) that Γ^+ satisfies one of the following:

(a-i) Γ^+ is not compact;

(a-ii) Γ^+ contains $(\lambda_1,\mathbf{0})$ for some $\lambda_1 \neq \mu^*$;

(a-iii) Γ^+ contains a point $(\lambda_2,\mathbf{v})$ for some $\lambda_2 \in \mathbb{R}$ and $\mathbf{v} \in Z \setminus \{\mathbf{0}\}$.

Lemma A.4. *Introduce the open set*

$$\mathcal{V} := \mathcal{B}_{\mathbb{R}\times X^3}((\mu^*,\mathbf{0}),\tilde{\varepsilon}) \cup (\mathbb{R}^+ \times ([C^{++}(\bar{\Omega})]^3 \cap X^3)),$$

where $\tilde{\varepsilon} > 0$ is as in (A.14). Then

$$\Gamma^+ \subset \mathcal{V}. \tag{A.17}$$

Proof. Set

$$\tilde{\Gamma}^+ := \Gamma^+ \cap \mathcal{V}$$

Since Γ^+ is connected and closed set in $\mathbb{R}\times X^3$, it is enough to show that $\tilde{\Gamma}^+$ is both open and closed as a subset of Γ^+. It is clear that $\tilde{\Gamma}^+$ is an open subset of Γ^+. It is also clear that $l > 0$ whenever $(l,\mathbf{u}) \in \mathcal{V}$ since $0 < \tilde{\varepsilon} < \frac{\mu^*}{2}$. Now, let $(l^n,\mathbf{u}^n)$ be a convergent sequence of elements of $\tilde{\Gamma}^+$ to some $(l^\infty,\mathbf{u}^\infty) \in \mathbb{R}\times X^3$. Since Γ^+ is a closed set and $\tilde{\Gamma}^+ \subset [0,\infty)\times X^3$, then $(l^\infty,\mathbf{u}^\infty) \in \Gamma^+ \cap ([0,\infty)\times X^3)$.

If $(l^\infty,\mathbf{u}^\infty) \in \bar{\mathcal{B}}_{\mathbb{R}\times X^3}((\mu^*,\mathbf{0}),\tilde{\varepsilon})$, there is $s^\infty \in [0,\tilde{\varepsilon}]$ such that $(l^\infty,\mathbf{u}^\infty) = (l(s^\infty), s^\infty(\mathbf{v}^* + \mathbf{z}(s^\infty)))$ in view of (A.14). In the case that $s^\infty = 0$, then $(l^\infty,\mathbf{u}^\infty) = (\mu^*,\mathbf{0}) \in \mathcal{V}$. On the other hand, if $s^\infty > 0$, then $\mathbf{u}^\infty \in [C^{++}(\bar{\Omega})]^3$. It then follows from Lemma 5.5 that $l^\infty > \mu^* > 0$, which yields that $(l^\infty,\mathbf{u}^\infty) \in \mathcal{V}$.

If $(l^\infty,\mathbf{u}^\infty) \notin \bar{\mathcal{B}}_{\mathbb{R}\times X^3}((\mu^*,\mathbf{0}),\tilde{\varepsilon})$, then we must have that $l^\infty > 0$ and $l^n > \mu^*$ for large values of n. It then follows from Lemma 5.5 that $\mathbf{u}^n \in [C^{++}(\bar{\Omega})]^3$, which shows that $(l^\infty,\mathbf{u}^\infty) \in \mathbb{R}^+ \times [C^{++}(\bar{\Omega})]^3$.

From the two possible scenarios above, we see that it always holds that $(l^\infty,\mathbf{u}^\infty) \in \mathbb{R}^+ \times [C^{++}(\bar{\Omega})]^3$. Hence, $\tilde{\Gamma}^+$ is also closed in Γ^+, and hence $\tilde{\Gamma}^+ = \Gamma^+$, which yields the desired conclusion. □

Thanks to the above development, we can now proceed to prove Theorem 5.7.

Proof of Theorem 5.7. Let Γ^+ be connected set obtained above. Next, we proceed to show that the alternative (a-ii) is not possible. To this end, observe that

$$\Gamma \subset \underbrace{\mathcal{F}^{-1}(\{\mathbf{0}\}) \cap \bar{\mathcal{B}}_{\mathbb{R}\times X^3}((\mu^*,\mathbf{0}),\tilde{\varepsilon})}_{:=\mathcal{A}_1} \cup \underbrace{\overline{\mathcal{F}^{-1}(\{\mathbf{0}\}) \cap (\mathbb{R}^+ \times ([C^{++}(\bar{\Omega})]^3 \cap X^3)) \setminus \bar{\mathcal{B}}_{\mathbb{R}\times X^3}((\mu^*,\mathbf{0}),\tilde{\varepsilon})}}_{:=\mathcal{A}_2}.$$

From (A.14), we have

$$\mathcal{A}_1 \cap \Gamma \subset C^* \quad \text{and} \quad (C^* \setminus C^-) \cap \{(l,\mathbf{0}) : l \in \mathbb{R}\} = \{(\mu^*,\mathbf{0})\}. \tag{A.18}$$

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Hence, since $\Gamma^+ \subset \mathcal{V}$ by Lemma A.4, we have

$$\Gamma^+ \cap \mathcal{A}_1 \cap \{(l, \mathbf{0}) : l \in \mathbb{R}\} = \{(\mu^*, \mathbf{0})\}. \tag{A.19}$$

On the other hand, by Lemma 5.6, for every $\delta > 0$, there is $0 < \hat{\delta} < \delta$ such that

$$\|\tilde{E}\|_\infty + \|\tilde{I}\|_\infty + \|\tilde{R}\|_\infty < \delta \quad \text{whenever } \mu^* < l \le \mu^* + \hat{\delta} \text{ and } (\tilde{E}, \tilde{I}, \tilde{R}) \text{ is a positive solution of (5.1).} \tag{A.20}$$

Therefore, taking $\delta = \frac{\tilde{\varepsilon}}{2}$ and $\hat{\delta} < \frac{\tilde{\varepsilon}}{2}$ such that (A.20) holds, and using the fact that system (5.1) has no positive solution for $0 \le l \le \mu^*$ (see Lemma 5.5), we get that

$$\begin{aligned} &\mathcal{F}^{-1}(\{\mathbf{0}\}) \cap (\mathbb{R}^+ \times ([C^{++}(\bar{\Omega})]^3 \cap X^3)) \setminus \bar{\mathcal{B}}_{\mathbb{R}\times X^3}((\mu^*, \mathbf{0}), \tilde{\varepsilon}) \\ \subset &([\mu^* + \hat{\delta}, \infty) \times ([C^{++}(\bar{\Omega})]^3 \cap X^3)) \cap \mathcal{F}^{-1}(\{\mathbf{0}\}). \end{aligned} \tag{A.21}$$

Now taking $l^* = \mu^* + \hat{\delta}$ and let $\eta_* > 0$ such that (5.21) holds, we get from Lemma 5.4 that

$$\begin{aligned} &([\mu^* + \hat{\delta}, \infty) \times ([C^{++}(\bar{\Omega})]^3 \cap X^3)) \cap \mathcal{F}^{-1}(\{\mathbf{0}\}) \\ \subset &([\mu^* + \hat{\delta}, \infty) \times \{\mathbf{u} \in X^3 : u_i \ge \eta_*,\ i = 1, 2, 3\}) \cap \mathcal{F}^{-1}(\{\mathbf{0}\}). \end{aligned} \tag{A.22}$$

Note that $([\mu^* + \hat{\delta}, \infty) \times \{\mathbf{u} \in X^3 : u_i \ge \eta_*,\ i = 1, 2, 3\}) \cap \mathcal{F}^{-1}(\{\mathbf{0}\})$ is closed as the intersection of two closed sets. Hence

$$\mathcal{A}_2 \subset ([\mu^* + \hat{\delta}, \infty) \times \{\mathbf{u} \in X^3 : u_i \ge \eta_*,\ i = 1, 2, 3\}) \cap \mathcal{F}^{-1}(\{\mathbf{0}\}).$$

Thus, from (A.21) and (A.22), we have that $\mathcal{A}_2 \cap \{(l, \mathbf{0}) : l \in \mathbb{R}\} = \emptyset$. This along with (A.19) rules out scenario (a-ii).

Next, we proceed to rule out scenario (a-iii). To this end, we note from the first inclusion in (A.18) that

$$\Gamma^+ \cap \mathcal{A}_1 \subset \overline{C^+} = \{(l(s), s(\mathbf{v}^* + \mathbf{z}(s))) : 0 \le s \le \varepsilon\}.$$

Hence, since

$$\{(l(s), s(\mathbf{v}^* + \mathbf{z}(s))) : 0 \le s \le \varepsilon\} \cap [\mathbb{R} \times (Z \setminus \{\mathbf{0}\})] = \emptyset,$$

we get

$$\Gamma^+ \cap \mathcal{A}_1 \cap [\mathbb{R} \times (Z \setminus \{\mathbf{0}\})] = \emptyset. \tag{A.23}$$

Since $\mathbf{v}^* \in [C^{++}(\bar{\Omega})]^3$, then

$$\int_\Omega (u_1 v_1^* + u_2 v_2^* + u_3 v_3^*) > 0 \text{ whenever } (l, \mathbf{u}) \in ([\mu^* + \hat{\delta}, \infty) \times \{\mathbf{u} \in X^3 : u_i \ge \eta_*,\ i = 1, 2, 3\}) \cap \mathcal{F}^{-1}(\{\mathbf{0}\}).$$

As a result, we get

$$([\mu^* + \hat{\delta}, \infty) \times \{\mathbf{u} \in X^3 : u_i \ge \eta_*,\ i = 1, 2, 3\}) \cap \mathcal{F}^{-1}(\{\mathbf{0}\}) \cap [\mathbb{R} \times (Z \setminus \{\mathbf{0}\})] = \emptyset.$$

This in turn yields that

$$\Gamma^+ \cap \mathcal{A}_2 \cap [\mathbb{R} \times (Z \setminus \{\mathbf{0}\})] = \emptyset. \tag{A.24}$$

From (A.23) and (A.24), we conclude that scenario (a-iii) is not also possible.

Therefore, scenario (a-i) must occur. Finally, since $0 < d_E \tilde{E} + d_I \tilde{I} + d_R \tilde{R} < 1$ for every positive solution of (5.1), we can employ the regularity theory for elliptic equations to conclude that there exist $M_1 > 0$ and $0 < \nu_1 < 1$, such that

$$\|\mathbf{u}\|_{[C^{2+\nu_1}(\bar{\Omega})]} \le M_1 l \quad \forall\, (l, \mathbf{u}) \in \Gamma^+. \tag{A.25}$$

Therefore, thanks to the Sobolev compact embedding theorem, the Bolzano–Weierstrass theorem, and the regularity theory for elliptic equations, we have that any bounded subset of Γ^+ is precompact.

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Consequently, since Γ^+ is not compact in $\mathcal{V}$, it must be unbounded. This along with (A.25) implies that the projection of Γ^+ on the l-component is unbounded. In particular,

$$I := \{l \in \mathbb{R}^+ : (l, \mathbf{u}) \in \Gamma^+ \text{ for some } \mathbf{u} \in X^3\} = [\mu^*, \infty).$$

Moreover, since scenario (ii) is not possible, we have that $\mathbf{u} \in [C^{++}(\bar{\Omega})]^3$ whenever $l > \mu^*$ and $(l, \mathbf{u}) \in \Gamma^+$, which completes the proof of the theorem. □

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