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On polynomial groups

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1. *Introduction.* If M is a group, $Z(M)$ its integral group ring and A_M the augmentation ideal, then following Passi we can form the Abelian groups

$$P_n(M) = Z(M)/A_M^{n+1} \quad \text{and} \quad Q_n(M) = A_M^n/A_M^{n+1}.$$

These have been called *polynomial groups* (3). Let $SP^n(M)$ denote the nth symmetric product of an Abelian group M. Then there exists a natural epimorphism

$$\theta_n(M): SP^n(M) \to Q_n(M),$$

given by $$\theta_n(M)\Big(\sum_{x_{i_r}\in M} x_{i_1} \hat{\otimes} \dots \hat{\otimes} x_{i_n}\Big) = \sum_{x_{i_r}\in M} (x_{i_1}-e)\dots(x_{i_n}-e) + A_M^{n+1}.$$

While studying the functorial properties of P_n and Q_n Passi (4) proved that

1·1. *$\theta_n(M)$: $SP^n(M) \to Q_n(M)$ is an isomorphism if M is a free Abelian group.*

1·2. *$\theta_2(M)$: $SP^2(M) \to Q_2(M)$ is an isomorphism for every Abelian group M.*

However, $\theta_n(M)$, $n > 2$, is not always an isomorphism. For example, if

$$M = \underbrace{Z_p \oplus \dots \oplus Z_p}_{m \text{ copies}},$$

(Z_p the cyclic group of order p, p a prime) the polynomial groups $Q_n(M)$ have a constant structure for $n \geqslant (m-1)(p-1)+1$ ((3), Theorem 4.7) while this is certainly not the case for $SP^n(M)$ (1).

In this work we investigate further the relationship between the polynomial groups $Q_n(M)$ and the nth symmetric product $SP^n(M)$ of an Abelian group M. We here prove

1·3. THEOREM. *If M is an Abelian group, then the map $\phi_n: M \to SP^n(M)$ given by*

$$\phi_n(x) = \underbrace{x \hat{\otimes} \dots \hat{\otimes} x}_{n \text{ factors}}, \quad (x \in M),$$

is a polynomial map of degree $\leqslant n$.

The polynomial map ϕ_n induces a homomorphism, also denoted by ϕ_n,

$$\phi_n: Q_n(M) \to SP^n(M).$$

1·4. THEOREM. *If M is an Abelian group, then*

(i) *$\phi_n \circ \theta_n = n!\, I_{SP^n(M)}$, $I_{SP^n(M)}$ the identity map of $SP^n(M)$;*

(ii) *$\theta_n \circ \phi_n = n!\, I_{Q_n(M)}$, $I_{Q_n(M)}$ the identity map of $Q_n(M)$.*

We say that a group M is *k-torsion-free* if $kx = 0$, $x \in M$, implies $x = 0$.
We then have the following

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1·5. COROLLARY. $\operatorname{Ker}\theta_n \subseteq \{x \in SP^n(M)/n!x = 0\}$. *In particular, if* $SP^n(M)$ *is* $n!$*-torsion-free, then* θ_n *is an isomorphism.*

If M is k-torsion free we can prove (Proposition 4·1) that $SP^n(M)$ is also k-torsion-free. We thus obtain a generalization of the result (1·1):

1·6. THEOREM. *If an Abelian group* M *is* $n!$*-torsion-free, then* θ_n: $SP^n(M) \to Q_n(M)$ *is an isomorphism.*

We may remark that the results of this paper remain true when the integral group ring $Z(M)$ is replaced by the group ring $R(M)$ over any commutative ring R with unity.

2. *Preliminaries.* We recall (3) that if M and N are Abelian groups, a map $f: M \to N$ is called a polynomial map of degree $\leqslant n$, if the Z-linear extension $f^*: Z(M) \to N$ of f vanishes on A_M^{n+1}.

We also recall that if M is an Abelian group, then $SP^n(M)$, the nth symmetric product, is defined as follows:

Let F be the free Abelian group generated by the symbols

$$u(x_1, \ldots, x_n), x_i \in M, \quad i = 1, 2, \ldots, n.$$

Let R be the subgroup of F generated by all elements of the type

$$\begin{aligned}(a)\quad & u(x_1, \ldots, x_{i-1}, x_i x_i', x_{i+1}, \ldots, x_n) \\ & - u(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \\ & - u(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n).\end{aligned}$$

$$(b)\quad u(x_1, \ldots, x_n) - u(x_{\pi(1)}, \ldots, x_{\pi(n)}),$$

where π is a permutation of integers $1, 2, \ldots, n$. $SP^n(M)$ is defined to be the quotient group F/R. We write $x_1 \hat{\otimes} \ldots \hat{\otimes} x_n$ for the coset $u(x_1, \ldots, x_n) + R$. If $x_1 = x_2 = \ldots = x_n = x$, we write $x_1 \hat{\otimes} \ldots \hat{\otimes} x_n = x^{\hat{\otimes} n}$.

Let n be a fixed integer $\geqslant 2$ and M a given Abelian group. For sake of simplicity we write ϕ for ϕ_n, θ for θ_n, S for $SP^n(M)$ and Q for $Q_n(M)$.

3. *A polynomial map.* 3·1. *Proof of Theorem* 1·3. Extend the map $\phi: M \to S$ to $\phi: Z(M) \to S$ by linearity. If $x_1, \ldots, x_k \in M$, then

$$\begin{aligned}&\phi((e-x_1))\ldots(e-x_k)) \\ &\quad = \phi(e + \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant k} (-1)^r x_{i_1} \ldots x_{i_r}) \\ &\quad = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant k} (-1)^r \phi(x_{i_1} \ldots x_{i_r}) \\ &\quad = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant k} (-1)^r (x_{i_1} \ldots x_{i_r})^{\hat{\otimes} n} \\ &\quad = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant k} \Bigg[(-1)^r \sum_{j_1 + \ldots + j_r = n} x_{i_1}^{\hat{\otimes} j_1} \hat{\otimes} \ldots \hat{\otimes} x_{i_r}^{\hat{\otimes} j_r} \binom{n}{j_1} \binom{n-j_1}{j_2} \ldots \\ &\qquad\qquad \binom{n-j_1-\ldots-j_{r-1}}{j_r} \Bigg] \quad (*)\end{aligned}$$

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$$= \sum_{\substack{u_1+\ldots+u_s=n \\ 1\leqslant t_1<t_2<\ldots<t_s\leqslant k}} \left[x_{t_1}^{\hat{\otimes}u_1} \hat{\otimes} \ldots \hat{\otimes} x_{t_s}^{\hat{\otimes}u_s} \binom{n}{u_1}\binom{n-u_1}{u_2}\ldots \right.$$
$$\left. \binom{n-u_1-\ldots-u_{s-1}}{u_s} \sum_{s\leqslant r\leqslant k} (-1)^r \binom{k-s}{r-s} \right]$$

$$= \sum_{\substack{u_1+\ldots+u_s=n \\ 1\leqslant t_1<t_2<\ldots<t_s\leqslant k}} \left[x_{t_1}^{\hat{\otimes}u_1} \hat{\otimes} \ldots \hat{\otimes} x_{t_s}^{\hat{\otimes}u_s} \binom{n}{u_1}\binom{n-u_1}{u_2}\ldots \right.$$
$$\left. \binom{n-u_1-\ldots-u_{s-1}}{u_s} (-1)^s (1-1)^{k-s} \right].$$

If $k-s>0$, $(1-1)^{k-s}=0$. In particular if $k=n+1$, since s cannot be greater than n, $(1-1)^{n+1-s}=0$ for all s. Hence

$$\phi((e-x_1)\ldots(e-x_{n+1}))=0$$

and, therefore, ϕ is a polynomial map of degree $\leqslant n$.

3·2 COROLLARY. *$\phi((x_1-e)\ldots(x_n-e)) = n!(x_1 \hat{\otimes} \ldots \hat{\otimes} x_n)$, where*

$$x_i \in M, i = 1, 2, \ldots, n.$$

Proof. As proved above, we have

$$\phi((e-x_1)\ldots(e-x_n))$$
$$= \sum_{\substack{u_1+\ldots+u_s=n \\ 1\leqslant t_1<t_2<\ldots<t_s\leqslant n}} (-1)^s \left[x_{t_1}^{\hat{\otimes}u_1} \hat{\otimes} \ldots \hat{\otimes} x_{t_s}^{\hat{\otimes}u_s} \binom{n}{u_1}\binom{n-u_1}{u_2}\ldots \right.$$
$$\left. \binom{n-u_1-\ldots-u_{s-1}}{u_s} (1-1)^{n-s} \right].$$

Since $(1-1)^{n-s}=0$ whenever $s<n$, the only non-zero term in $\phi((e-x_1)\ldots(e-x_n))$ is that corresponding to $s=n$. In this case $t_1=1, t_2=2, \ldots, t_n=n$ and

$$u_1 = u_2 = \ldots = u_n = 1.$$

From (*) it is clear that the coefficient of $x_1 \hat{\otimes} \ldots \hat{\otimes} x_n$ in $\phi((e-x_1)\ldots(e-x_n))$ is

$$(-1)^n \binom{n}{1}\binom{n-1}{1}\ldots\binom{2}{1}\binom{1}{1} = (-1)^n n!.$$

Hence $\phi((x_1-e)\ldots(x_n-e)) = n!\,(x_1 \hat{\otimes} \ldots \hat{\otimes} x_n)$.

3·3. *Proof of Theorem* 1·4. Since $\phi: M \to S$ given by $\phi(x) = x^{\hat{\otimes}n}, x \in M$, is a polynomial map of degree $\leqslant n$, it defines (3) a homomorphism, again denoted by ϕ,

$$\phi: Z(M)/A_M^{n+1} \to S$$

given by $$\phi\Big(\sum_{x_i\in M} n_i x_i + A_M^{n+1}\Big) = \sum_{x_i\in M} n_i \phi(x_i) = \sum_{x_i\in M} n_i x_i^{\hat{\otimes}n}.$$

Consider the restriction of ϕ to Q. Denoting the restriction by ϕ again, we have a homomorphism

$$\phi: Q \to S.$$

We note that $$\phi((x_1-e)\ldots(x_n-e)+A_M^{n+1}) = n!.x_1 \hat{\otimes} \ldots \hat{\otimes} x_n.$$

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Now
$$\begin{aligned}\phi\circ\theta(x_1\hat{\otimes}\dots\hat{\otimes}x_n) &= \phi((x_1-e)\dots(x_n-e)+A_M^{n+1})\\ &= n!\,x_1\hat{\otimes}\dots\hat{\otimes}x_n\end{aligned}$$
and $x_1\hat{\otimes}\dots\hat{\otimes}x_n$, $x_i\in M$, $i=1,2,\dots,n$, generate S. Hence
$$\phi\circ\theta = n!\,I_S.$$
Again,
$$\begin{aligned}\theta\circ\phi((x_1-e)\dots(x_n-e)+A_M^{n+1}) &= \theta(n!x_1\hat{\otimes}\dots\hat{\otimes}x_n)\\ &= n!\,\theta(x_1\hat{\otimes}\dots\hat{\otimes}x_n)\\ &= n!((x_1-e)\dots(x_n-e)+A_M^{n+1}),\end{aligned}$$
and $(x_1-e)\dots(x_n-e)+A_M^{n+1}, x_i\in M$, generate Q. Therefore $\theta\circ\phi = n!I_Q$.

3·4. *Proof of Corollary* 1·5. Let $x\in\operatorname{Ker}\theta$. Then $\phi\circ\theta(x)=0$. Hence $n!x=0$. If, in particular, S is $n!$-torsion free, then θ is a monomorphism and hence an isomorphism.

3·5. *Remark.* It can be easily proved that ϕ is a natural homomorphism, i.e. if $\alpha: M\to N$ is a homomorphism of Abelian groups, then the diagram
$$\begin{array}{ccc} Q_n(M) & \xrightarrow{\phi(M)} & SP^n(M)\\ \downarrow & & \downarrow\\ Q_n(N) & \xrightarrow[\phi(N)]{} & SP^n(N)\end{array}$$
is commutative.

4. *Symmetric product.* In view of Corollary 1·5 we are naturally led to find the conditions on M under which S is $n!$-torsion-free. In this direction we have the following:

4·1. PROPOSITION. *If an Abelian group M is k-torsion-free then so is S.*

First, we have two lemmas about direct limit. The proofs of these lemmas are straightforward and, therefore, omitted.

4·2. LEMMA. *Let $\{M,\pi\}_\Lambda$ be a direct system of Abelian groups over a directed set Λ. Let $\varinjlim\{M,\pi\}$ denote the direct limit of the direct system $\{M,\pi\}$. Then $\{SP^n(M), SP^n(\pi)\}_\Lambda$ is also a direct system of Abelian groups and*
$$SP^n(\varinjlim\{M,\pi\}) \cong \varinjlim\{SP^n(M), SP^n(\pi)\}.$$

4·3. LEMMA. *Let $\{M,\pi\}_\Lambda$ be a direct system of k-torsion-free Abelian groups. Then $\varinjlim\{M,\pi\}$ is also k-torsion-free.*

To prove Proposition 4·1 we also need the following well known result (1):

4·4. If M and N are Abelian groups, then
$$SP^n(M\oplus N) \cong \sum_{r=0}^{n} SP^r(M)\otimes_Z SP^{n-r}(N).$$

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4·5. *Proof of Proposition* 4·1. Since every group is the direct limit of the direct system of its finitely generated subgroups directed by inclusion (2), it follows from Lemmas 4·2 and 4·3 that we need prove the proposition only when M is a finitely generated Abelian group.

First, let M be finite of order $m > 1$ (say). Since

$$m(x_1\hat{\otimes}\ldots\hat{\otimes}x_n) = x_1^m\hat{\otimes}x_2\hat{\otimes}\ldots\hat{\otimes}x_n = 0,$$

for every generator $x_1\hat{\otimes}\ldots\hat{\otimes}x_n$ of S, $mz = 0$ for every $z \in S$. M being k-torsion-free, m and k are co-prime and since the order of every element of S is a divisor of m, S is k-torsion-free.

Next, let M be a finitely Abelian group and T the torsion subgroup of M. Then (as M is finitely generated) $M = T \oplus N$, where N is a finitely generated free Abelian subgroup of M. By (4·4)

$$\begin{aligned} (4\cdot6) \qquad S &= SP^n(T \oplus N) \\ &\cong \sum_{r=0}^{n} SP^r(T) \otimes_Z SP^{n-r}(N) \\ &= SP^n(N) \oplus \left(\oplus \sum_{r=1}^{n-1} SP^r(T) \otimes_Z SP^{n-r}(N) \right) \oplus SP^n(T). \end{aligned}$$

N is a finitely generated free Abelian group and, therefore, $SP^r(N)$ is a finitely generated free Abelian group for every $r > 0$. Therefore, for every r, $1 \leqslant r \leqslant n-1$,

(4·7) $SP^r(T) \otimes_Z SP^{n-r}(N) \cong (SP^r(T))^{\lambda_r}$, for a positive integer λ_r, where $(SP^r(T))^s$ stands for direct sum of s copies of $SP^r(T)$.

Taking $\lambda_n = 1$, (4·6) and (4·7) give

$$(4\cdot8) \qquad S \cong SP^n(N) \oplus \left(\oplus \sum_{r=1}^{n} (SP^r(T))^{\lambda_r} \right).$$

T being the torsion subgroup of a finitely generated Abelian group is finite. Therefore $SP^r(T)$ and hence each of $(SP^r(T))^{\lambda_r}$ is k-torsion-free. Also $SP^n(N)$ being free Abelian is k-torsion-free. Hence S is k-torsion-free.

4·9. *Remark.* The converse of Proposition 4·1 is not true in general. For example, let

$$M = Z(p^\infty),$$

p a prime. $SP^n(M) = 0$ for $n > 1$ and, therefore, k-torsion-free for every positive integer k but M is not p-torsion-free. However, it is easy to prove that if M is a finitely generated Abelian group and $SP^n(M)$ is k-torsion-free, then so is M.

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